View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by NASA Technical Reports Server

N65-28099
(ACCESSION NUMBER)
30
(PAGES)
CR 63783
(NASA CR OR TMX OR AD NUMBER)
(THRU)
1
(CODE)
23
(CATEGORY)
FACILITY FORM 602

Antenna Laboratory Report No. 65-12

Scientific Report No. 6

# PRINCIPLE OF SCALING IN A UNIAXIAL MEDIUM

C. P. Bates and R. Mittra

June 1965

Sponsored by the National Aeronautics
and Space Administration under Grant NsG-395

Department of Electrical Engineering
Engineering Experiment Station
University of Illinois
Urbana, Illinois

GPO PRICE $ ________

OTS PRICE(S) $ ________

Hard copy (HC) 2.00

Microfiche (MF) .50

Antenna Laboratory Report No. 65-12

Scientific Report No. 6

# PRINCIPLE OF SCALING IN A UNIAXIAL MEDIUM

C. P. Bates and R. Mittra

June 1965

Sponsored by the National Aeronautics
and Space Administration under Grant NsG-395

Department of Electrical Engineering
Engineering Experiment Station
University of Illinois
Urbana, Illinois

## ABSTRACT

28099

Solution to Maxwell's equations is given for source currents in an unbounded magneto-ionic medium for which the dielectric tensor is uniaxial, that is, diagonal with two elements equal. Three-dimensional Fourier transforms technique is used because it gives the field solutions in a form which shows that the uniaxial fields may be obtained by a scaling of certain quantities. These quantities are related to fields in free space due to an equivalent source current. The choice of the quantities to be scaled and their respective equivalent source currents depend on how the uniaxial field solution is arranged. For example, two methods of scaling are given for the case of the source current perpendicular to the magnetostatic field. The problems of an electric dipole parallel and perpendicular to the magnetostatic field are given as examples of such scaling. Author

## TABLE OF CONTENTS

# 1. INTRODUCTION

The solution of Maxwell's equations in a uniaxial medium is important because it forms a useful basis for the field solution in an unbounded magneto-ionic medium when Y, the ratio of gyro-frequency to signal frequency becomes very large. The fields in a uniaxial medium are also important as a means of determining the reactive component of the impedance of an antenna embedded in a magneto-ionic medium. It has been shown in a recent paper[1] that the very near field of a dipole in a magneto-ionic medium is independent of the off-diagonal term of the dielectric tensor. This suggests that the very near field, which is required for such impedance calculations, of a current source in a magneto-ionic medium is the same as the very near field of the corresponding uniaxial medium.

It is an advantage when trying to solve for the fields in a uniaxial medium if they can be related to certain quantities that occur in the calculation of fields in free space due to certain related current sources. These quantities may already be known and then the uniaxial fields can be obtained simply by a scaling of the free space quantities. Therefore, we wish, given any general volume source in the uniaxial medium, to be able to find equivalent sources in free space to which a scalar potential may be associated such that the scalar potential may be conveniently scaled to give the scalar potential in the uniaxial medium. Three-dimensional Fourier transforms method is used as it was found that the form of the solution for the uniaxial fields obtained by this method permitted relationships between certain free space quantities and the uniaxial quantities to be found.

A general volume source in this paper is treated as a superposition of two components. One component of the source is taken parallel, the other perpendicular, to the magnetostatic field. In Section 2 the formal solution of this general volume source is obtained.

The scaling, which is required when the current is parallel to the magnetostatic field, is derived in Section 3. This turns out to be a relatively simple case giving only one logical method for the scaling.

When the current is perpendicular to the magnetostatic field, the method of scaling is no longer simple. Two possible scaling procedures are given for this case. The first method is given in Section 4 and this corresponds to a partial fraction expansion suggested by the form of the general solution. Another method is given in Section 5 by representing the uniaxial field as a superposition of a TE (H-wave) and a TM field (E-wave). The TE field corresponds to $E_x = 0$ and the TM field corresponds to $H_x = 0$.

The method of representing the uniaxial field as a superposition of a TE and a TM field is also used by Clemmow.[4] He confines himself to surface currents and finds the uniaxial field by using the fact that to every vacuum field, represented as the superposition of a TE and a TM field, there corresponds a uniaxial field, represented as the superposition of a TE and a TM field. The method in Section 5 is applicable to the more general case of volume sources and stresses the converse: the uniaxial fields are found by reference to the corresponding vacuum fields. No explicit use is made of a TE and TM representation for the vacuum field.

The two examples of a dipole parallel and perpendicular to the magnetostatic field are solved by the scaling methods of Sections 3 and 4. Original solution of these two problems is not claimed but rather they are used simply as examples of the scaling that occurs, and reference is given to alternative solutions in the literature.

## 2. FORMULATION OF THE SOLUTION TO MAXWELL'S EQUATION IN A UNIAXIAL MEDIUM

Maxwell's equations for a uniaxial medium, using an $e^{j\omega t}$ time connection, may be written as

$$\nabla \times \bar{H} = j\omega\epsilon_o \; \bar{\bar{\epsilon}} \, \bar{E} + \bar{J} \tag{1}$$

$$- \nabla \times \bar{E} = j\omega\mu_o \; \bar{H} \tag{1-A}$$

where[3]

$$\bar{\bar{\epsilon}} = (1 - \frac{X}{U}) \; \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} \tag{2}$$

$$X = \frac{N \; e^2}{\epsilon_o \; m \; \omega^2} \; , \quad U = 1 - jZ, \quad Z = \frac{\nu}{\omega} \tag{3}$$

Equation (2) implies that the magnetostatic field is taken to be in the x-direction. On eliminating $\bar{E}$ from Equation (1) using Equation (1-A), we have

$$\nabla \times \bar{\bar{\epsilon}}^{-1} \; \nabla \times \bar{H} - k_o^2 \; \bar{H} = \nabla \times \bar{\bar{\epsilon}}^{-1} \; \bar{J} \tag{4}$$

where

$$\bar{\bar{\epsilon}}^{-1} = \beta \; \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} \tag{5}$$

$$\beta = \frac{U}{U - X} \tag{6}$$

A solution for $\bar{H}$ satisfying Equation (4) is desired.

Let the three-dimensional Fourier transform of $\bar{H}$ be defined by $\tilde{\bar{H}}$ where

$$\tilde{H}_{x,y,z} \; (\ell, m, n) = \frac{k_o^3}{(2\pi)^3} \iiint_{-\infty}^{\infty} H_{x,y,z} \; (x,y,z) \; e^{-jk_o(\ell x + my - nz)} \; dxdydz \tag{7}$$

and let the transforms $\tilde{J}_{x,y,z}$ of $J_{x,y,z}$ be similarly defined. Taking the transforms of Equation (4) yields

$$\bar{k} \times \bar{\bar{\epsilon}}^{-1} \bar{k} \times \tilde{\bar{H}} + \tilde{\bar{H}} = \frac{-j}{k_o} (\bar{k} \times \bar{\bar{\epsilon}}^{-1} \tilde{\bar{J}}) \tag{8}$$

$$\bar{k} = \ell\hat{x} + m\hat{y} - n\hat{z} \tag{9}$$

This gives in matrix form

$$\begin{bmatrix} (1-n^2 - m^2) & \ell m & -\ell n \\ +\ell m & (1-n^2\beta - \ell^2) & -mn\beta \\ -\ell n & -mn\beta & (1-m^2\beta - \ell^2) \end{bmatrix} \begin{bmatrix} \tilde{H}_x \\ \tilde{H}_y \\ \tilde{H}_z \end{bmatrix} = \tag{10}$$

$$\frac{-j}{k_o} \tilde{J}_x \begin{bmatrix} o \\ -n\beta \\ -m\beta \end{bmatrix} \quad \frac{-j}{k_o} \tilde{J}_y \begin{bmatrix} n \\ o \\ \ell \end{bmatrix} \quad \frac{-j}{k_o} \tilde{J}_z \begin{bmatrix} m \\ -\ell \\ o \end{bmatrix}$$

Solving for $\tilde{H}_{x,y,z}\ (\ell,m,n)$ gives

$$\tilde{H}_x(\ell,m,n) = \frac{-j}{k_o\Delta} (1-\beta(m^2+ n^2) -\ell^2) (n\ \tilde{J}_y + m\ \tilde{J}_z) \tag{11}$$

$$\tilde{H}_y(\ell,m,n) = \frac{-j}{k_o\Delta} \left\{ \tilde{J}_x\beta n\ (-1+\ell^2+m^2+n^2) + \tilde{J}_y(\beta-1)\ n\ell m + \tilde{J}_z\ell\ (-1+m^2\beta+\ell^2+n^2) \right\} \tag{12}$$

$$\tilde{H}_z(\ell,m,n) = \frac{-j}{k_o\Delta} \left\{ \tilde{J}_x\beta m\ (-1+\ell^2+m^2+n^2) + \tilde{J}_y\ell(1-n^2\beta-\ell^2-m^2) + \tilde{J}_z m\ell n(1-\beta) \right\} \tag{13}$$

where

$$\Delta = \beta\ (n^2-n_1^{\ 2})\ (n^2 - n_2^{\ 2}) \tag{14}$$

$$n_1^2 = 1-\ell^2 - m^2 \ , \ n_2^2 = \beta^{-1} (1-\ell^2) - m^2 \tag{15}$$

The fields as a function of position are obtained by inverting Equations (11), (12) and (13), using the Fourier inversion integral

$$H_{x,y,z} (x,y,z) = \iiint_{-\infty}^{\infty} (\tilde{H}_x, \tilde{H}_y, \tilde{H}_z) e^{jk_o (\ell x + my - nz)} d\ell dm dn \tag{16}$$

## 3. FIELDS DUE TO THE CURRENT COMPONENT PARALLEL TO THE DIRECTION OF THE MAGNETOSTATIC FIELD

Since the magnetostatic field is in the x-direction, this case corresponds to the fields due to $\tilde{J}_x$. By referring to Equations (11) to (16), the fields are

$$H_x = 0, \; H_y = \frac{1}{jk_o} \frac{\partial \pi}{\partial z} , \; H_z = \frac{1}{jk_o} \frac{\partial \pi}{\partial y} \tag{17}$$

where

$$\pi(x,y,z) = \frac{-j}{k_o} \iiint_{-\infty}^{\infty} \frac{\tilde{J}_x(\ell,m,n)\, e^{jk_o(\ell x + my - nz)}}{n^2 - \beta^{-1}(1-\ell^2) + m^2} \, d\ell dmdn \tag{18}$$

In Equation (18) make the variable change

$$m = \frac{m_o}{\sqrt{\beta}} , \quad n = \frac{n_o}{\sqrt{\beta}} \tag{19}$$

$$\pi(x,y,z) = \frac{-j}{k_o} \iiint_{-\infty}^{\infty} \frac{\tilde{J}_x\left(\ell, \frac{m_o}{\sqrt{\beta}}, \frac{n_o}{\sqrt{\beta}}\right) e^{jk_o\left(\ell x + \frac{m_o}{\sqrt{\beta}} y - \frac{n_o}{\sqrt{\beta}} z\right)}}{n_o^2 - 1 + \ell^2 + m_o^2} \, d\ell dm_o dn_o \tag{20}$$

Let

$$\frac{y}{\sqrt{\beta}} = y_o , \; \frac{z}{\sqrt{\beta}} = z_o \tag{21}$$

$$\pi^o(x,y_o,z_o) = \frac{-j}{k_o} \iiint_{-\infty}^{\infty} \frac{\tilde{J}_x^{\,o}(\ell, m_o, n_o)\, e^{jk_o(\ell x + m_o y_o - n_o z_o)}}{n_o^2 - 1 + \ell^2 + m_o^2} \, d\ell dm_o dn_o \tag{22}$$

where

$$\pi^o(x,y_o,z_o) = \pi\left(x, y_o\sqrt{\beta}\ ,\ z_o\sqrt{\beta}\right) \tag{23}$$

$$\tilde{J}_x^{\,o}(\ell,m_o,n_o) = \tilde{J}_x\left(\ell, \frac{m_o}{\sqrt{\beta}}, \frac{n_o}{\sqrt{\beta}}\right)$$

$$= \frac{k_o^{\,3}}{(2\pi)^3}\iiint_{-\infty}^{\infty} J_x(x,y,z)\ e^{-jk_o\left(\ell x + \frac{m_o}{\sqrt{\beta}} y - \frac{n_o}{\sqrt{\beta}} z\right)} dxdydz$$

$$= \frac{k_o^{\,3}}{(2\pi)^3}\iiint_{-\infty}^{\infty} J_x^{\,o}(x,y_o,z_o)\ e^{-jk_o(\ell x + m_o y_o - n_o z_o)}\ dxdy_o dz_o \tag{24}$$

So

$$J_x^{\,o}(x,y_o,z_o) = \beta\ J_x\left(x, y_o\sqrt{\beta}, \ z_o\sqrt{\beta}\right) \tag{25}$$

$\pi^o(x,y_o,z_o)$ in Equation (22) is recognizable as the functional form of the vector potential in free space with coordinates $(x,y_o,z_o)$ due to the equivalent source current $J_x^{\,o}(x,y_o,z_o)$. That is, $\pi^o(x,y_o,z_o)$ satisfies

$$\nabla^2\ \bar{A} + k_o^{\,2}\,\bar{A} = jk_o\ J_x^{\,o}(x,y_o,z_o)\hat{x} \tag{26}$$

where

$$\bar{A} = \pi^o\ (x,y_o,z_o)\ \hat{x} \tag{27}$$

$$\overline{H} = -\frac{\nabla \times \overline{A}}{jk_o} \tag{28}$$

The fields in the uniaxial medium are obtained once the $\pi^o(x,y_o,z_o)$ due to $J_x^{\,o}(x,y_o,z_o)$ in free space is known since

$$\pi(x,y,z) = \pi^o(x,y_o,z_o)\Bigg|_{\substack{y_o = \frac{y}{\sqrt{\beta}} \\ z_o = \frac{z}{\sqrt{\beta}}}} \tag{29}$$

As an example of this case consider an infinitesimal dipole source

$$J_x(x,y,z) = I\ \delta(x)\delta(y)\delta(z) \tag{30}$$

Then

$$J_x^{\,o}(x,y_o,z_o) = \beta\ I\ \delta(x)\ \delta(y_o\sqrt{\beta})\ \delta(z_o\sqrt{\beta}) = I\ \delta(x)\delta(y_o)\delta(z_o)^{*} \tag{31}$$

The $\pi^o(x,y_o,z_o)$ in free space due to $J_x^{\,o}$ is well-known to be

$$\pi^o(x,y_o,z_o) = \frac{-jk_oI}{4\pi}\ \frac{e^{-jk_o\sqrt{x^2+y_o^2+z_o^2}}}{\sqrt{x^2+y_o^2+z_o^2}} \tag{32}$$

So $\pi(x,y,z)$ for the uniaxial medium is

$$\pi(x,y,z) = \frac{-jk_oI\ e^{-jk_oR}}{4\pi R} \tag{33}$$

where

$$R = \sqrt{x^2 + \left(1 - \frac{X}{U}\right)\ (y^2 + z^2)} \tag{34}$$

---

$^{*}\beta$ is taken as positive real for preciseness.

The actual field components are obtainable from Equations (17) and (1).

This simple example has been solved elsewhere in the literature, for example, Clemmow[2], on page 463.

## 4. FIELDS DUE TO THE CURRENT COMPONENT PERPENDICULAR TO THE DIRECTION OF THE MAGNETOSTATIC FIELD

In this case we want to find the fields due to the current in the y-direction. Referring to Equations (11) to (16), we may write

$$H_x = \frac{-j}{k_o} \frac{\partial}{\partial z} \left[ \frac{-j}{k_o} \iiint_{-\infty}^{\infty} \frac{\tilde{J}_y(\ell,m,n)\, e^{+jk_o(\ell x + my - nz)}}{n^2 - 1 + \ell^2 + m^2} \, d\ell dm dn \right] \tag{35}$$

$$H_y = \frac{(\beta-1)}{\beta k_o^2} \frac{\partial^2}{\partial z \partial y} \left[ \frac{-j}{k_o} \iiint_{-\infty}^{\infty} \frac{\ell\, \tilde{J}_y(\ell,m,n)\, e^{jk_o(\ell x + my - nz)}}{(n^2 - n_1^2)(n^2 - n_2^2)} \, d\ell dm dn \right] \tag{36}$$

$$H_z = \frac{j}{k_o} \frac{\partial}{\partial x} \left[ \frac{-j}{k_o} \iiint_{-\infty}^{\infty} \frac{\tilde{J}_y(\ell,m,n) e^{jk_o(\ell x + my - nz)}}{(n^2 - 1 + \ell^2 + m^2)} \, d\ell dm dn \right] - \frac{(\beta-1)}{\beta k_o^2} \frac{\partial^2}{\partial y^2} \left[ \frac{-j}{k_o} \iiint_{-\infty}^{\infty} \frac{\ell\, \tilde{J}_y(\ell,m,n) e^{jk_o(\ell x + my - nz)}}{(n^2 - n_1^2)(n^2 - n_2^2)} \, d\ell dm dn \right] \tag{37}$$

Obviously $H_{x,y,z}$ cannot be derived from one single $\pi$ as was done in the previous case. However, note that $H_x$ is already related to a free space $\pi$. Equations (36) and (37) can also be written by a simple factorization as

$$H_y = \frac{1}{k_o^2} \frac{\partial^2}{\partial z \partial y} \left[ \frac{-j}{k_o} \iiint_{-\infty}^{\infty} \frac{\ell}{(1-\ell^2)} \tilde{J}_y(\ell,m,n) \left[ \frac{1}{(n^2-1+\ell^2+m^2)} - \frac{1}{n^2-\beta^{-1}(1-\ell^2)+m^2} \right] e^{jk_o(\ell x+my-nz)} \, d\ell dm dn \right] \tag{38}$$

$$H_z = \frac{j}{k_o} \frac{\partial}{\partial x} \left[ \frac{-j}{k_o} \iiint_{-\infty}^{\infty} \frac{\tilde{J}_y(\ell,m,n) e^{jk_o(\ell x+my-nz)}}{(n^2-1+\ell^2+m^2)} \, d\ell dm dn \right] - \frac{\partial^2}{k_o^2 \partial y^2} \left[ \frac{-j}{k_o} \iiint_{-\infty}^{\infty} \frac{\ell \tilde{J}_y(\ell,m,n)}{1-\ell^2} \left[ \frac{1}{n^2-1+\ell^2+m^2} - \frac{1}{n^2 - \beta^{-1}(1-\ell^2) + m^2} \right] e^{jk_o(\ell x+my-nz)} \, d\ell dm dn \right] \tag{39}$$

Or, using a simpler notation,

$$H_x = \frac{-j}{k_o} \frac{\partial}{\partial z} \pi_1 \tag{40}$$

$$H_y = \frac{1}{k_o^2} \frac{\partial^2}{\partial z \partial y} (\pi_2 - \pi_3) \tag{41}$$

$$H_z = \frac{j}{k_o} \frac{\partial}{\partial x} \pi_1 - \frac{1}{k_o^2} \frac{\partial^2}{\partial y^2} (\pi_2 - \pi_3) \tag{42}$$

where $\pi_1$, $\pi_2$, and $\pi_3$ are defined by comparing Equations (40) and (41) with Equations (35) and (38), respectively. Equation (42) merely verifies that the $\nabla \cdot \overline{H} = 0$.

The function $\pi_1(x,y,z)$ is obviously the solution of the scalar wave equation in free space for a current $J_y(x,y,z)$, that is,

$$\nabla^2 \pi_1 + k_o^2 \pi_1 = jk_o J_y(x,y,z) \tag{43}$$

No scaling is required for $\pi_1$.

Now $\pi_2$ is given by

$$\pi_2(x,y,z) = \frac{-j}{k_o} \iiint_{-\infty}^{\infty} \frac{\tilde{J}_{y2}^{o}(\ell,m,n)}{n^2-1+\ell^2+m^2} e^{jk_o(\ell x+my-nz)} d\ell dm dn \tag{44}$$

where

$$\tilde{J}_{y2}^{o}(\ell,m,n) = \frac{\ell}{1-\ell^2} \tilde{J}_y(\ell,m,n) \tag{45}$$

Therefore, $\pi_2(x,y,z)$ is the solution of the scalar wave equation in free space for an equivalent source current $J_{y2}^{o}(x,y,z)$, that is

$$\nabla^2 \pi_2 + k_o^2 \pi_2 = jk_o J_{y2}^{o}(x,y,z) \tag{46}$$

and

$$J_{y2}^{o}(x,y,z) = \iiint_{-\infty}^{\infty} \frac{\ell}{1-\ell^2} \tilde{J}_y(\ell,m,n) e^{jk_o(\ell x+my-nz)} d\ell dm dn \tag{47}$$

or

$$J_{y2}^{o}(x,y,z) = \int_{-\infty}^{\infty} \frac{\ell}{1-\ell^2} \quad \tilde{\mathcal{J}}_y(\ell,y,z)\ e^{jk_o \ell x}\ d\ell \tag{48}$$

where

$$\tilde{\mathcal{J}}_y = \frac{k_o}{2\pi} \int_{-\infty}^{\infty} J_y(x,y,z)\ e^{-jk_o \ell x}\ dx \tag{48-A}$$

The function $\pi_3$ is given by

$$\pi_3(x,y,z) = \frac{-j}{k_o} \iiint_{-\infty}^{\infty} \frac{\ell}{(1-\ell^2)} \quad \frac{\tilde{J}_y(\ell,m,n)\ e^{jk_o(\ell x+my-nz)}}{(n^2-\beta^{-1}(1-\ell^2)+m^2)}\ d\ell dmdn \tag{49}$$

In Equation (49) make the variable change

$$m = \frac{m_o}{\sqrt{\beta}} \quad , \quad n = \frac{n_o}{\sqrt{\beta}} \tag{50}$$

$$\pi_3(x,y,z) = \frac{-j}{k_o} \iiint_{-\infty}^{\infty} \frac{\ell}{(1-\ell^2)} \quad \frac{\tilde{J}_y(\ell, \frac{m_o}{\sqrt{\beta}}, \frac{n_o}{\sqrt{\beta}})\ e^{jk_o(\ell x+m_o \frac{y}{\sqrt{\beta}} - n_o \frac{z}{\sqrt{\beta}})}}{(n_o^2 - 1 + \ell^2 + m_o^2)}\ d\ell dm_o dn_o \tag{51}$$

Let $\frac{y}{\sqrt{\beta}} = y_o$, $\frac{z}{\sqrt{\beta}} = z_o$ so that we may write

$$\pi_3^{o}(x,y_o,z_o) = \frac{-j}{k_o} \iiint_{-\infty}^{\infty} \frac{\tilde{J}_{y3}^{o}(\ell,m_o,n_o)\ e^{jk_o(\ell x+m_o y_o-n_o z_o)}}{(n_o^2-1+\ell^2+m_o^2)}\ d\ell dm_o dn_o \tag{52}$$

where

$$\pi_3^{o}(x,y_o,z_o) = \pi_3(x,y_o\sqrt{\beta}\,,\ z_o\sqrt{\beta}\,) \tag{53}$$

and

$$\tilde{J}_{y3}^{o}(\ell,m_o,n_o) = \frac{\ell}{1-\ell^2}\ \tilde{J}_y\left(\ell,\frac{m_o}{\sqrt{\beta}},\frac{n_o}{\sqrt{\beta}}\right)$$

$$= \frac{\ell}{1-\ell^2}\ \frac{k_o^{\ 3}}{(2\pi)^3}\iiint_{-\infty}^{\infty} J_y(x,y,z)\ e^{-jk_o\left(\ell x+\frac{m_o y}{\sqrt{\beta}}-\frac{n_o z}{\sqrt{\beta}}\right)}\ dxdydz$$

$$= \frac{\ell}{1-\ell^2}\ \frac{k_o^{\ 3}}{(2\pi)^3}\iiint_{-\infty}^{\infty} J_y^{\ 1}(x,y_o,z_o)\ e^{-jk_o(\ell x+m_o y_o-n_o z_o)}\ dxdy_o dz_o \tag{54}$$

if

$$J_y^{\ 1}(x,y_o,z_o) = \beta\ J_y(x,y_o\sqrt{\beta}\,,\ z_o\sqrt{\beta}\,) \tag{55}$$

Therefore $\pi_3^{o}(x,y_o,z_o)$ is the solution of the scalar wave equation in free space with coordinates $(x,y_o,z_o)$ due to the equivalent source current $J_{y3}^{o}(x,y_o,z_o)$, that is

$$\nabla^2\pi_3^{o} + k_o^{\ 2}\pi_3^{o} = jk_o\ J_{y3}^{o}(x,y_o,z_o) \tag{56}$$

and

$$J_{y3}^{o}(x,y_o,z_o) = \iiint_{-\infty}^{\infty} \frac{\ell}{1-\ell^2}\ \tilde{J}_y^{\ 1}(\ell,m_o,n_o)\ e^{jk_o(\ell x+m_o y_o-n_o z_o)}\ d\ell dm_o dn_o \tag{57}$$

So

$$J^{o}_{y3}(x,y_o,z_o) = \beta \int_{-\infty}^{\infty} \frac{\ell}{1-\ell^2} \quad \tilde{\mathcal{J}}_y(\ell, y_o\sqrt{\beta}, z_o\sqrt{\beta}) e^{jk_o \ell x} d\ell \tag{58}$$

$$\tilde{\mathcal{J}}_y(\ell, y_o\sqrt{\beta}, z_o\sqrt{\beta}) = \frac{k_o}{2\pi} \int_{-\infty}^{\infty} \tilde{J}_y(x, y_o\sqrt{\beta}, z_o\sqrt{\beta}) e^{-jk_o \ell x} dx \tag{58-A}$$

This gives $\pi_3(x,y,z)$ for the uniaxial medium

$$\pi_3(x,y,z) = \pi^{o}_3(x,y_o,z_o) \Bigg| \begin{array}{l} y_o = \frac{y}{\sqrt{\beta}} \\ z_o = \frac{z}{\sqrt{\beta}} \end{array} \tag{59}$$

As an example of this case consider* an infinitesimal dipole source

$$J_y(x,y,z) = I\ \delta(x)\ \delta(y)\ \delta(z) \tag{59-A}$$

The solution for $\pi_1(x,y,z)$ is immediate from Equation (43)

$$\pi_1(x,y,z) = \frac{-jk_o I}{4\pi} \frac{e^{-jk_o\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}} \tag{60}$$

The equivalent current needed to calculate $\pi_2$ is obtained from Equation (48)

$$\begin{aligned} J^{o}_{y2}(x,y,z) &= \frac{-jk_o I}{2}\ \delta(y)\ \delta(z)\ e^{-jk_o x}, \quad x > 0 \\ &= \frac{jk_o I}{2}\ \delta(y)\ \delta(z)\ e^{jk_o x}, \quad x < 0 \end{aligned} \tag{61}$$

*$\beta$ is again taken as positive real.

Solving Equation (46) for $\pi_2(x,y,z)$ with the current as given in Equation (61) means we have to solve the scalar wave equation for a phased line current and this yields

$$\pi_2(x,y,z) = \pm \frac{k_o^2 I}{4\pi}\left[\ln(\rho)e^{\mp jk_o x} - \int_o^y \frac{\cos\alpha}{\rho^1}\left[e^{\mp jk_o x} \mp \frac{x}{r^1} e^{-jk_o r^1}\right] dy^1\right] \quad (62)$$

where: + for $x > 0$, - for $x < 0$,

$$\rho^2 = y^2 + z^2,\ (\rho^1)^2 = (y^1)^2 + z^2,\ \alpha^1 = \mathrm{Tan}^{-1}(z/y^1),$$

$$(r^1)^2 = x^2 + (y^1)^2 + z^2$$

The equivalent current needed to calculate $\pi_3^o(x,y_o,z_o)$ is obtained from Equation (58)

$$J_{y3}^o(x,y_o,z_o) = \frac{-jk_o}{2} I\ \delta(y_o)\ \delta(z_o)\ e^{-jk_o x},\quad x > 0$$

$$= \frac{+jk_o}{2} I\ \delta(y_o)\ \delta(z_o)\ e^{jk_o x},\quad x < 0 \quad (63)$$

Solving Equation (56) for $\pi_3^o(x,y_o,z_o)$ with the current as given in (63) gives

$$\pi_3^o(x,y_o,z_o) = \pi_2(x,y,z)\Big|_{\substack{y = y_o \\ z = z_o}} \quad (64)$$

from which we obtain

$$\pi_3(x,y,z) = \pm \frac{k_o^2 I}{4\pi}\left[\ln(\rho_3)e^{\mp jk_o x} - \frac{1}{\sqrt{\beta}}\int_o^y \frac{\cos\alpha_3^1}{\rho_3^1}\left[e^{\mp jk_o x} \mp \frac{x}{r_3^1} e^{-jk_o r_3^1}\right] dy^1\right] \quad (65)$$

where: + for $x > 0$, - for $x < 0$

$$\rho_3^{\,2} = \beta^{-1}(z^2+y^2), \quad (\rho_3^{\,1})^2 = \beta^{-1}((y^1)^2+z^2), \quad \alpha_3^{\,1} = \mathrm{Tan}^{-1}(z/y^1),$$

$$(r_3^{\,1})^2 = x^2 + (\rho_3^{\,1})^2$$

The actual field components are obtained from Equations (40), (41), (42) and (1). For example, the H components are

$$H_x = \frac{I}{4\pi}\, jk_o\, e^{-jk_o r}\, \frac{z}{r^2} \left[1 - \frac{j}{k_o r}\right] \tag{66-A}$$

$$H_y = \frac{-Ik_o xyz}{4\pi(y^2+z^2)} \left[ \frac{2}{k_o(y^2+z^2)} \left( \frac{e^{-jk_o r}}{r} - \frac{e^{-jk_o r_3}}{r_3} \right) + \right.$$

$$\left. + \frac{j}{r}\left(1 - \frac{j}{k_o r}\right) \frac{e^{-jk_o r}}{r} - \frac{j}{\beta r_3}\left(1 - \frac{j}{k_o r_3}\right) \frac{e^{-jk_o r_3}}{r_3} \right] \tag{66-B}$$

$$H_z = \frac{Ix}{4\pi(y^2+z^2)} \left[ \frac{-(z^2-y^2)}{(z^2+y^2)} \left( \frac{e^{-jk_o r}}{r} - \frac{e^{-jk_o r_3}}{r_3} \right) - jk_o z^2 \left(1 - \frac{j}{k_o r}\right) \frac{e^{-jk_o r}}{r^2} + \right.$$

$$\left. + \frac{-jk_o y^2}{\beta r_3} \left(1 - \frac{j}{k_o r_3}\right) \frac{e^{-jk_o r_3}}{r_3} \right] \tag{66-C}$$

where:

$$r^2 = x^2 + y^2 + z^2, \quad r_3^{\,2} = x^2 + \beta^{-1}(y^2 + z^2)$$

The H components as given in Equations (66-A), (66-B) and (66-C) may be compared with expressions obtained by Clemmow[4] on page 105. When comparing, it is necessary to relate the coordinate system as used by Clemmow to the coordinate system used here and also to relate the symbols used for the dielectric tensors. It is seen that the $H_y$ and $H_z$ components given here differ from Clemmow's by a $k_o$ in one of the terms suggesting that a printing error occurred in his paper.

## 5. SCALING OBTAINED BY USE OF A TM AND TE REPRESENTATION FOR THE FIELD

The transformed fields in the uniaxial medium, given by Equations (11), (12) and (13), may also be represented as the superposition of a transverse magnetic (TM) field and a transverse electric (TE) field to the direction of the magnetostatic field. For the TM field this implies $H_x = 0$ and $E_x = 0$ for the TE field. We wish to arrange Equations (11), (12) and (13) so as to yield this representation.

For the case of a current component in the x-direction, the uniaxial field is already TM as $\tilde{H}_x = 0$ which implies $H_x = 0$. The scaling for this case has already been given in Section 3.

Let the current component perpendicular to the magnetostatic field be in the y-direction without loss of generality. We now wish to express

$$\bar{J} = J_y \hat{y} = \bar{J}_1 + \bar{J}_2 \tag{67}$$

where $\bar{J}_1$ will generate a TM field and $\bar{J}_2$ will generate a TE field. Let

$$\bar{J}_1 = J_{y1} \hat{y} + J_z \hat{z} \tag{68-A}$$

$$\bar{J}_2 = J_{y2} \hat{y} - J_z \hat{z} \tag{68-B}$$

Obviously

$$J_{y1} + J_{y2} = J_y \tag{69}$$

Using the fact that $H_x = 0$ due to $\bar{J}_1$ and $E_x = 0$ due to $\bar{J}_2$ in the field equations for the uniaxial medium and using Equation (69), we obtain

$$\tilde{J}_z = \frac{-nm}{n^2 + m^2} \tilde{J}_y (\ell, m, n) \tag{70}$$

$$\tilde{J}_{y1} = \frac{m^2}{n^2+m^2} \tilde{J}_y(\ell,m,n) \tag{71}$$

$$\tilde{J}_{y2} = \frac{n^2}{n^2+m^2} \tilde{J}_y(\ell,m,n) \tag{72}$$

So the TM component of the uniaxial field may be written as

$$H_{xe} = 0 \tag{73}$$

$$H_{ye} = \frac{-j}{k_o\beta} \frac{\partial}{\partial z} \pi_4 \tag{74}$$

$$H_{ze} = \frac{j}{k_o\beta} \frac{\partial}{\partial y} \pi_4 \tag{75}$$

The subscript e refers to an E-wave or TM field.

Also

$$H_{xh} = \frac{1}{k_o^2} \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \pi_5 \tag{76}$$

$$H_{yh} = \frac{1}{k_o^2} \frac{\partial^2}{\partial x \partial y} \pi_5 \tag{77}$$

$$H_{zh} = -\frac{1}{k_o^2} \frac{\partial^2}{\partial x \partial z} \pi_5 \tag{78}$$

The subscript h refers to a H-wave or a TE field.

The potentials $\pi_4$ and $\pi_5$ are given by

$$\pi_4(x,y,z) = \frac{-j}{k_o}\iiint_{-\infty}^{\infty} \frac{m\ell}{(n^2+m^2)} \frac{\tilde{J}_y(\ell,m,n)}{(n^2-\beta^{-1}(1-\ell^2)+m^2)} e^{jk_o(\ell x+my-nz)} d\ell dm dn \quad (79)$$

and

$$\pi_5(x,y,z) = \frac{-j}{k_o}\iiint_{-\infty}^{\infty} \frac{n\,\tilde{J}_y(\ell,m,n)\, e^{jk_o(\ell x+my-nz)}}{(n^2+m^2)(n^2-1+\ell^2+m^2)} d\ell dm dn \quad (80)$$

$\pi_5(x,y,z)$ is recognizable as the solution of the scalar wave equation in free space due to an equivalent source $J^o_{y5}(x,y,z)$, that is,

$$\nabla^2 \pi_5 + k_o^2 \pi_5 = jk_o J^o_{y5}(x,y,z) \quad (81)$$

and

$$J^o_{y5}(x,y,z) = \iiint_{-\infty}^{\infty} \frac{n}{n^2+m^2} \tilde{J}_y(\ell,m,n)\, e^{+jk_o(\ell x+my-nz)} d\ell dm dn \quad (82)$$

The scalar potential $\pi_4$ can be written in a more convenient form by making the following variable changes:

$$n = \sqrt{\beta^{-1}}\, n_o \ , \quad m = \sqrt{\beta^{-1}}\, m_o \quad (83)$$

$$y_o = \sqrt{\beta^{-1}}\, y \ , \quad z_o = \sqrt{\beta^{-1}}\, z \quad (84)$$

$$\pi_4^{o}(x,y_o,z_o) = \frac{-j}{k_o}\iiint_{-\infty}^{\infty} \frac{\tilde{J}_{y4}^{o}(\ell,m_o,n_o)\, e^{jk_o(\ell x+m_o y_o-n_o z_o)}}{n_o^2-1+\ell^2+m_o^2}\, d\ell\, dm_o\, dn_o \tag{85}$$

where

$$\pi_4^{o}(x,y_o,z_o) = \pi_4(x,y_o\sqrt{\beta},\, z_o\sqrt{\beta}) \tag{86}$$

and

$$\tilde{J}_{y4}^{o}(\ell,m_o,n_o) = \sqrt{\beta}\,\frac{m_o\ell}{n_o^2+m_o^2}\,\tilde{J}_y\left(\ell,\frac{m_o}{\sqrt{\beta}},\frac{n_o}{\sqrt{\beta}}\right) \tag{87}$$

Therefore, $\pi_4^{o}(x,y_o,z_o)$, is the solution of the scalar wave equation in free space with coordinates $(x,y_o,z_o)$ due to the equivalent source current $J_{y4}^{o}(x,y_o,z_o)$, that is

$$\nabla^2\pi_4^{o} + k_o^2\,\pi_4^{o} = jk_o\,J_{y4}^{o}(x,y_o,z_o) \tag{88}$$

where:

$$J_{y4}^{o}(x,y_o,z_o) = \sqrt{\beta}\iiint_{-\infty}^{\infty}\frac{m_o\ell}{n_o^2+m_o^2}\,\tilde{J}_y\left(\ell,\frac{m_o}{\sqrt{\beta}},\frac{n_o}{\sqrt{\beta}}\right) e^{jk_o(\ell x+m_o y_o-n_o z_o)}\, d\ell\, dm_o\, dn_o \tag{89}$$

The uniaxial potential is, therefore,

$$\pi_4(x,y,z) = \pi_4^{o}(x,y_o,z_o)\Bigg|_{\substack{y_o=\frac{y}{\sqrt{\beta}}\\ z_o=\frac{z}{\sqrt{\beta}}}} \tag{90}$$

The TE and TM components of the H-field are given by Equations (73) to (78). The total field is then the superposition of the TE and TM components and most, of course, agree with the original representation given by Equations (35), (36) and (37).

## 6. COMMENTS ON EQUIVALENT FREE SPACE CURRENTS CORRESPONDING TO A GIVEN UNIAXIAL CURRENT

In Sections 3, 4 and 5, it was seen that the uniaxial fields could be obtained by a simple coordinate scaling of a scalar function which was a solution of the inhomogeneous scalar wave equation in free space with a source term that was related to the given uniaxial current. In Section 3 it was seen that the uniaxial current parallel to the magnetostatic field transformed into the free space source current by merely multiplying the uniaxial current by a constant and scaling its coordinates. (Refer to Equation (25).) This means that a dipole, for example, parallel to the magnetostatic field in the uniaxial medium transformed into the free space source current that was also a dipole (Equation 31).

The situation for the current perpendicular to the magnetostatic field is more complex. In Section 4 the equivalent source currents $J_{y2}^{o}$ and $J_{y3}^{o}$, associated with the scalar functions $\pi_2$ and $\pi_3^{o}$, respectively, for free space, are related to the uniaxial source current by more than just a coordinate scaling. The functional form of both $J_{y2}^{o}$ and $J_{y3}^{o}$ may be different than that of the uniaxial source as seen in Equations (48) and (58). For example, if the uniaxial source is a dipole, the equivalent free space currents become phased line elements (Equations 61 and 63). It is interesting to note that the equivalent free space current $J_{y4}^{o}$ and $J_{y5}^{o}$, obtained in Section 5 by representing the uniaxial field as a superposition of TE and TM fields, are functionally different than those of Section 4. In Section 4 the equivalent free space currents were functionally different from the uniaxial current in one dimension only, namely x. In Section 5 the equivalent free space currents are functionally different than the uniaxial source in at least two dimensions as seen from Equations (83) and (89). Recalling that the uniaxial fields are obtained from the solution of the inhomogeneous scalar wave equation in free space with a source term given by these equivalent currents, the functional difference between the equivalent currents becomes important. For instance, it may be simpler to solve for the field by the method of Section 4 rather than 5 because of the difference in the nature of the equivalent source terms. This would depend on the functional form of the given uniaxial current.

# 7. CONCLUSIONS

Given a uniaxial medium and a current source, the uniaxial fields are expressed in terms of uniaxial scalar potentials that can be determined by scaling the solutions of the inhomogeneous wave equations in free space with source terms that are related to the current source in the uniaxial medium. Two possible methods of scaling are presented for the case of the uniaxial current source perpendicular to the magnetostatic field. One is suggested by the form of the solution obtained by the three-dimensional Fourier transforms technique. The other method is obtained by representing the uniaxial field as a superposition of a TE and TM field. The choice of which method to use depends on the given uniaxial current. This is a result of the fact that the equivalent currents that occur in each method are functionally dependent on the uniaxial source but in a different manner. Therefore, one equivalent source may be easier to work with than the other.

The advantage of being able to use the solution of the inhomogeneous scalar wave equation in free space, to obtain the uniaxial fields, is because of our previous knowledge gained from working with this equation. It may be that the solution of the inhomogeneous scalar wave equation with a source term given by the equivalent current may already be known. However, if this is not the case, then a direct attack on the uniaxial fields, as given by Equations (35), (36) and (37), may be best.

It is clear from the work in Sections 3, 4 and 5 that the scaling is possible only because two of the three terms of the diagonal tensor are equal. Thus, the scaling principle outlined here cannot be extended to the general magneto-ionic medium.

One final comment about the possibility of $\beta$ being negative or complex. The latter case occurs if the plasma is lossy and the former occurs if X (Equation (3)) is greater than unity (plasma frequency is greater than wave frequency). Reference to the literature suggests that the solutions obtained for $\beta$ real and positive still hold if $\beta$ takes on these values, although to date this point is still controversial.

## REFERENCES

1. Mittra, R. and Deschamps, G. A., "Field Solution for a Dipole in an Anisotropic Medium," Symposium on Electromagnetic Theory and Antennas, Copenhagen, June 25-30, 1962, p. 497.

2. Clemmow, P. C., "On the Theory of Radiation from a Source in a Magneto-ionic Medium," Symposium on Electromagnetic Theory and Antennas, Copenhagen, June 25-30, 1962.

3. Budden, K. G., Lectures on Magneto-ionic Theory, Gordon and Breach.

4. Clemmow, P. C., "The Theory of Electromagnetic Waves in a Simple Anisotropic Medium," Proceedings I.E.E., Vol. 110, No. 1, January 1963.