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A LYAPUNOV METHOD FOR THE ESTIMATION OF STATISTICAL AVERAGES

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1. INTRODUCTION

We consider a randomly perturbed dynamical system described by the equation

$$dx/dt = f(x) + G(x)\xi(t) \ , \ t \geq 0, \tag{1}$$

where x,f are n-vectors, G is an $n \times n$ matrix and $\xi(t)$ is n-dimensional Gaussian white noise. Such equations arise in control theory [1], and the theory of random vibrations [2]. In these applications it is of interest to know under what conditions the process

$$X = \{x(t), \ t \geq 0\}$$

generated by (1) is stable, in the sense that X admits a unique invariant probability distribution. If X is stable then it is often desirable to estimate various stationary averages $\mathcal{E}\{L(x)\}$, when these averages exist.

In a previous paper [3] a criterion of Lyapunov type was given for stability in the sense described. In the present note a Lyapunov criterion (Theorem 3.1) is obtained for the existence (finiteness) of the stationary average $\mathcal{E}\{L(x)\}$ where L is an arbitrary nonnegative function. This result is applied to show that algebraic moments of all orders exist when, in (1), G is bounded and the unperturbed system $dx/dt = f(x)$ is of Lur'e type.

The existence criterion is extended to yield an effective method of calculating an upper bound for $\mathcal{E}\{L(x)\}$ (Theorem 4.1). The method is illustrated by an example from control theory.

2. STATEMENT OF THE PROBLEM

We start with a precise version of (1), namely Itô's equation

$$dx(t) = f(x(t))dt + G(x(t))dw(t) \qquad t \geq 0 \qquad (2)$$

$$x(0) = x_0$$

The following assumptions are made with respect to (2):

(i) x, f are vectors in Euclidean n-space E ($n \geq 2$) and G is an $n \times n$ matrix.

(ii) $\{w(t) \; ; \; t \geq 0\}$ is a Wiener process in E

(iii) x_0 is a random variable independent of the process $w(t)$.

(iv) There is a constant $c > 0$ such that

$$|f(x)-f(y)| + |G(x)-G(y)| < c|x-y|$$

for all $x, y \in E$. (Here $|\cdot|$ denotes Euclidean norm of a vector or matrix.)

(v) There is a constant $\epsilon > 0$ such that

$$y'G(x)\;G(x)'y \geq \epsilon|y|^2$$

for all $x,y \in E$. (A prime denotes transpose of a vector or matrix).

Under these assumptions it is known (cf. [3]) that (2), interpreted in the sense of Itô, defines a continuous, strongly Feller process

$$X = \{x(t), \; t \geq 0\} \; .$$

The differential generator of X will be denoted by $\mathcal{L}$, where

$$\mathcal{L}[u(x)] = \tfrac{1}{2}\,\mathrm{tr}[G(x)G(x)'u_{xx}(x)] + f(x)'u_x(x) \tag{3}$$

whenever the indicated derivatives exist. (In (3), u_x is the vector of first partial derivatives of u and u_{xx} is the matrix of second partial derivatives).

In the following we shall always assume that X is positive [4]. Under these conditions it is known [4] that there exists a unique invariant probability measure μ defined on the Borel sets $B \subset E$: that is, if P denotes probability measure on the paths of X, and if

$$P(x_0 \in B) = \mu(B)$$

then

$$P(x(t) \in B) = \mu(B), \quad t > 0 .$$

An effective criterion for positivity of X is given in [3].

Let $L(x) \geq 0$ be Hölder continuous on the compact subsets of E. The main problem is to obtain a sufficient condition that

$$\mathcal{E}\{L(x)\} = \int_E L(x)\mu(dx)$$

be finite. Subsequently we shall describe a method for deriving an upper bound on $\mathcal{E}\{L(x)\}$.

In the following, the terms smooth, and normal domain, have the same meaning as in [3].

3. A CRITERION FOR EXISTENCE OF $\mathcal{E}\{L(x)\}$

A Lyapunov criterion for the existence of $\mathcal{E}\{L(x)\}$ can be derived by arguments very similar to those of [3] and [4]. The result is given in Theorem 3.1. We start with some preliminary lemmas.

Let D be a normal domain with boundary Γ, and let τ_Γ be the first time X hits Γ. Let $\mathcal{E}_x$ denote expectation on the paths of X when $x(0) = x \in E$. Since X is positive, $\mathcal{E}_x(\tau_\Gamma) < \infty$, $x \in E - D$.

Lemma 3.1

Let

$$u(x) = \mathcal{E}_x\{ \int_0^{\tau_\Gamma} L[x(t)]dt\}$$

If $u(x_o) < \infty$ for some point $x_o \in E - \bar{D}$ then $u(x) < \infty$ for all $x \in E - \bar{D}$.

Furthermore

$$\mathcal{L}[u(x)] = - L(x), \quad x \in E - \bar{D}$$

$$u(x) = 0, \quad x \in \Gamma. \tag{4}$$

Proof.

The proof closely follows that of Lemma 5.3 of [4]. Let $\{D_n; n=1,2,\ldots\}$ be an increasing sequence of normal domains such that $D \subset D_1$, $x_o \in D_1 - D$ and $\lim D_n = E$ $(n \to \infty)$. Let τ_n be the first time X hits the boundary $\Gamma \cup \Gamma_n$ of $D_n - D$, and define

$$u_n(x) = \mathcal{E}_x\{ \int_0^{\tau_n} L[x(t)]dt\}, \quad x \in \bar{D}_n - D$$

$$u(x) = \mathcal{E}_x\{ \int_0^{\tau_\Gamma} L[x(t)]dt\}, \quad x \in E - D.$$

Since X is positive (and hence, regular [4]) $\tau_n \uparrow \tau_\Gamma$ $(n \to \infty)$ and therefore $u_n(x) \uparrow u(x)$ $(n \to \infty)$. For fixed $m \geq 1$

$$u(x) = u_m(x) + \sum_{n=m}^{\infty}[u_{n+1}(x) - u_n(x)], \quad x \in \bar{D}_m - D, \tag{5}$$

with convergence at least for $x = x_o$. We now use the fact that $u_n(x)$ is the unique smooth solution of

$$\mathcal{L}[u_n(x)] = - L(x), \quad x \in D_n - \bar{D}.$$

$$u_n(x) = 0, \quad x \in \Gamma \cup \Gamma_n \quad .$$

(see e.g. [5], Ch. 5, §5). Let $v_n(x) = u_{n+1}(x) - u_n(x)$, $x \in \bar{D}_n - D$. Then $\mathcal{L}[v_n(x)] = 0$, $x \in D_n - \bar{D}$ and (since $u_n(x) \geqq 0$) $v_n(x) \geqq 0$, $x \in \Gamma \cup \Gamma_n$. By the maximum principle $v_n(x) \geqq 0$, $x \in D_n - \bar{D}$. It follows that all terms of the series (5) except the first are positive functions for which $\mathcal{L}[v(x)] = 0$, and the series converges for $x = x_o$. From the generalized Harnack inequality [11] it follows that the series converges for all $x \in \bar{D}_m - D$ and $u(x)$ satisfies (4) for $x \in D_m - \bar{D}$. Since m is arbitrary the result follows.

Lemma 3.2

A necessary and sufficient condition that

$$\mathcal{E}\{L(x)\} < \infty$$

is that

$$\mathcal{E}_x\{ \int_0^{\tau_\Gamma} L[x(t)]dt\} < \infty, \quad x \in E. \tag{6}$$

Proof.

We use the construction and notation of [4]. Let D_1 be a normal domain with boundary Γ_1 such that $D \subset D_1$ and $\Gamma \cap \Gamma_1 = \phi$. Let τ denote the length of a cycle, namely, in obvious notation,

$$\tau = \min \{t : x(t) \in \Gamma \mid x(0) \in \Gamma \text{ and } x(s) \in \Gamma_1 \text{ for some } s, 0 < s < t\}.$$

Let $\tilde{\mu}$ be the finite invariant measure (see [4]) induced on the Borel sets of Γ. Then if K is an arbitrary compact subset of E we have, within a constant of normalization,

$$\mu(K) = \int_\Gamma \tilde{\mu}(dx)\mathcal{E}_x\{\tau^K\} \qquad (7)$$

where τ^K = meas $\{t : 0 \leq t \leq \tau,\ x(t) \in K\}$.

Let $L_n(x)$ be an increasing sequence of simple functions (constructed on compact sets) such that $L_n(x) = 0$ $(|x| > n)$ and $L_n(x) \uparrow L(x)$ $(n \to \infty)$. From (7)

$$\int_E \mu(dx)L_n(x) = \int_\Gamma \tilde{\mu}(dx)\ \mathcal{E}_x\{\int_0^\tau L_n[x(t)]dt\},$$

$$n = 1,2,\ldots;$$

and by monotone convergence

$$\mathcal{E}\{L(x)\} = \int_\Gamma \tilde{\mu}(dx)\ \mathcal{E}_x\{\int_0^\tau L[x(t)]dt\}. \qquad (8)$$

Let $\tau_1 = \min\{t : x(t) \in \Gamma_1 | x(0) \in \Gamma\}$. By the strong Markov property

$$\mathcal{E}_x\{\int_0^\tau L[x(t)]dt\} = \mathcal{E}_x\{\int_0^{\tau_1} L[x(t)]dt\} + \mathcal{E}_x\{\mathcal{E}_{x(\tau_1)}\{\int_0^{\tau_\Gamma} L[x(t)]dt\}\},$$

$$x \in \Gamma. \qquad (9)$$

Since $\bar{D}_1$ is compact the first expectation on the right side of (9) is bounded for $x \in \Gamma$. If

$$u(y) = \mathcal{E}_y\{\int_0^{\tau_\Gamma} L[x(t)]dt\} < \infty, \qquad y \in \Gamma_1,$$

then, by Lemma 3.1, $u(y)$ is smooth, and therefore bounded on Γ_1. By the strong Feller property, $\mathcal{E}_x\{u[x(\tau_1)]\}$ is continuous, hence bounded on Γ; it follows from (8) that $\mathcal{E}\{L(x)\} < \infty$.

Conversely if (6) fails for some $x \in E - D$ then by Lemma 3.1 (6) fails for all $x \in E - D$, and by (8) and (9), $\mathcal{E}\{L(x)\} = \infty$.

Lemma 3.3

If the equation

$$\mathcal{L}[v(x)] = -L(x) \ , \ x \in E - \bar{D}$$

has a smooth positive solution $v(x)$ in $E - D$

then

$$\mathcal{E}_x\{\int_0^{\tau_\Gamma} L[x(t)]dt\} < \infty .$$

Proof

Let $\{D_n \ ; \ n = 1,2,..\}$ be a sequence of normal domains constructed as in the proof of Lemma 3.1, and let $u_n(x)$ be the corresponding sequence of smooth functions such that

$$\mathcal{L}[u_n(x)] = -L(x), \ x \in D_n - \bar{D}$$

$$u_n(x) = 0, \ x \in \Gamma \cup \Gamma_n.$$

Since $\mathcal{L}[v(x) - u_n(x)] = 0, \ x \in D_n - \bar{D}$, and $v(x)-u_n(x) \geq 0, \ x \in \Gamma \cup \Gamma_n$, we have $u_n(x) \leq v(x), \ x \in \bar{D}_n - D$; therefore

$$\mathcal{E}_x\{\int_0^{\tau_\Gamma} L[x(t)]dt\} = \lim u_n(x)$$

$$\leq v(x), \ x \in E.$$

This completes the proof.

Before stating Theorem 3.1 we introduce a class of real-valued functions V, analogous to Lyapunov functions, with the following properties.

P_1 : V is defined for $x \in \bar{D}_v$ where

$$D_v = \{x : |x| > R\} \qquad (R < \infty \text{ is arbitrary})$$

P_2 : V is continuous in $\bar{D}_v$ and is twice continuously differentiable in D_v.

P_3 : $V(x) \to +\infty$ as $|x| \to \infty$.

Theorem 3.1

Let X be positive. If there exists a function V with properties P_1-P_3 and if

$$\mathcal{L}[V(x)] \leqq - L(x), \quad x \in D_V$$

then

$$\mathcal{E}\{L(x)\} < \infty .$$

We remark that if X is positive and L(x) is bounded then $\mathcal{E}\{L(x)\}$ is obviously finite. If L(x) is bounded away from zero for $x \in D_V$ then, by Theorem 2 of [3], the existence of V already implies that X is positive.

Proof

By Lemmas 3.2 and 3.3 we have that $\mathcal{E}\{L(x)\} < \infty$ if and only if there exists a normal domain D such that the equation

$$\mathcal{L}[u(x)] = - L(x) \tag{10}$$

has a smooth positive solution u(x) defined for $x \in E - D$. Let $D = E - \overline{D}_V$ and define a sequence $\{D_n\}$ of normal domains as in the proof of Lemma 3.1. The remainder of the proof follows that of Lemma 3.3, with v(x) replaced by V(x). By adding a constant to V if necessary we can arrange that $V(x) \geqq 0$, $x \in \overline{D}_V$. If $\mathcal{L}[u_n(x)] = - L(x)$ $(x \in D_n - \overline{D})$, $u_n(x) = 0$ $(x \in \Gamma \cup \Gamma_n)$, then $0 \leqq u_n(x) \leqq u_{n+1}(x) \leqq V(x)$, $x \in \overline{D}_n - D$. It follows by a compactness theorem ([6] p.344) that lim $u_n(x)$ exists and is a solution of (10) for $x \in E - \overline{D}$.

Remark. The proof of Theorem 3.1 remains unchanged if property P_3 of V is replaced by

$$P_3' : \quad V(x) \geqq 0, \quad x \in \overline{D}_V .$$

4. ESTIMATION OF $\mathcal{E}\{L(x)\}$.

In this section we assume that $\mathcal{E}\{L(x)\} < \infty$ and derive an upper bound for this quantity. The result is given in Theorem 4.1.

For $x \in E$ and $t > 0$ define

$$u(t,x) = \mathcal{E}_x\{ \int_0^t L[x(s)]ds\}.$$

Lemma 4.1

If $\mathcal{E}\{L(x)\} < \infty$ *then* $u(t,x) < \infty$ *for all* $t > 0$, $x \in E$.

Proof.

We use the notation and construction of the proof of Lemma 3.2 and assume $x \in \Gamma$. If τ is the length of a cycle which starts at x then (cf. (9))

$$\mathcal{E}_x\{ \int_0^\tau L[x(s)]ds\}$$

is bounded for $x \in \Gamma$. With $t < \infty$ and fixed, let $\nu(x)$ denote the number of complete cycles which occur in the interval $[0,t)$ when $x(0) = x \in \Gamma$. Obviously $u(t,x) < \infty$ if $\mathcal{E}_x\{\nu(x)\} < \infty$. Let $\rho = \min\{|x-y| : x \in \Gamma, y \in \Gamma_1\}$. By our assumptions, $\rho > 0$; and if $\epsilon > 0$

$$P_x\{\tau < \epsilon\} \leq P_x\{ \max_{0 \leq s \leq \epsilon} |x(s) - x| \geq \rho\}$$
$$= 0(\epsilon^{3/2}) \text{ as } \epsilon \to 0,$$

uniformly for $x \in \Gamma$. (The last estimate can be derived as in [7],VI,§ 3.). Consider a chain of n cycles starting at x with lengths $\tau_1,\dots,\tau_n$. By the strong Markov property

$$P_x\{\nu(x) = n\} \leq P_x\{\tau_1 + \dots + \tau_n < t\}$$

$$\leq \left[\sup_{x \in \Gamma} P_x\{\tau < t\} \right]^n \leq (ct^{3/2})^n$$

where c is independent of t, and $t > 0$ is sufficiently small. Therefore $\sup[\mathcal{E}_x\{\nu(x)\} : x \in \Gamma\} < \infty$ for some $t > 0$, hence (by continuation over a finite number of subintervals) for every $t > 0$.

Lemma 4.2

If $\mathcal{E}\{L(x)\} < \infty$ *then*

$$\mathcal{E}\{L(x)\} = \lim_{t \to \infty} t^{-1} u(t,x) \tag{11}$$

Proof

Let $L_n(x)$ $(n = 1,2,\ldots)$ be a sequence of nonnegative simple functions such that $L_n(x) \uparrow L(x)$ $(n \to \infty)$ and $L_n(x) = 0$, $|x| > n$. By the corollary to Theorem 3.1 of [4],

$$\begin{aligned} \lim_{n \to \infty} \lim_{t \to \infty} t^{-1} \mathcal{E}_x\{ \int_0^t L_n[x(s)]ds\} &= \lim_{n \to \infty} \mathcal{E}\{L_n(x)\} \\ &= \mathcal{E}\{L(x)\} \end{aligned} \tag{12}$$

Let $P(t,x,B)$ be the transition function of X. If μ is the invariant measure of X then, by repeated applications of Fubini's Theorem,

$$\begin{aligned} &\mathcal{E}\{\mathcal{E}_x\{t^{-1} \int_0^t L_n[x(s)]ds\}\} \\ &= \int_E \mu(dx) t^{-1} \int_0^t \int_E P(s,x,dy) L_n(y) ds \\ &= t^{-1} \int_0^t \int_E \mu(dy) L_n(y) ds \\ &= \mathcal{E}\{L_n(x)\}. \end{aligned}$$

Passing to the limit $(n \to \infty)$ we have by monotone convergence

$$\mathcal{E}\{t^{-1}u(t,x)\} = \mathcal{E}\{L(x)\} . \tag{13}$$

Now let $\chi_n(x) = 1,\ |x| \geq n\ ;\ = 0$, otherwise. Suppose that for some $\epsilon > 0$ there exists a sequence $t_\nu \uparrow \infty$ and a subsequence $n(\nu)$ of positive integers such that

$$\mathcal{E}_x\{t_\nu^{-1}\int_0^{t_\nu} \chi_{n(\nu)}[x(s)]\ L[x(s)]ds\} > \epsilon,\ \nu = 1,2,\ldots \tag{14}$$

From (13) and (14) it follows that

$$\mathcal{E}\{\chi_{n(\nu)}(x)\ L(x)\} > \epsilon\ ,\ \nu = 1,2,\ldots,$$

which contradicts the fact that $\mathcal{E}\{L(x)\} < \infty$. Hence for each fixed $x \in E$,

$$\mathcal{E}_x\{t^{-1}\int_0^t L_n[x(s)]ds\} \to \mathcal{E}_x\{t^{-1}\int_0^t L[x(s)]ds\}$$

as $n \to \infty$, uniformly in t for t sufficiently large. We can therefore interchange limits in the left side of (12), and the result (11) follows by monotone convergence.

We consider functions V with properties $\overline{P}_1$-$\overline{P}_3$, where these differ from properties P_1-P_3 of section 3 only in that now we require $D_V = E$.

Theorem 4.1

Let X be positive. If there exist a function V with properties $\overline{P}_1$-$\overline{P}_3$ and a positive constant k such that

$$\mathcal{L}[V(x)] \leq k - L(x),\ x \in E,$$

then

$$\mathcal{E}\{L(x)\} \leq k .$$

Proof

We first show that $\mathcal{E}\{L(x)\} < \infty$. Indeed if D is a normal domain with boundary Γ and if $v(x) = \mathcal{E}_x\{\tau_\Gamma\}$ then $\mathcal{L}[v(x)] = -1$ $(x \in E - D)$ and $v(x) = 0$ $(x \in \Gamma)$. It follows that the function $V(x) + kv(x)$ satisfies the conditions of Theorem 3.1.

Let $D_n = \{x : |x| < n\}$ and put $\tau_n = \min\{t : |x(t)| = n |\ x(0) = x \in D_n\}$. Let $t_n = \min(t, \tau_n)$ and define

$$u_n(t,x) = \mathcal{E}_x\{\int_0^{t_n} L[x(s)]ds\}$$

$t > 0,\ x \in D_n$ $(n = 1,2,\ldots)$. Since $\tau_n \uparrow \infty$ $(n \to \infty)$ we have $u_n(t,x) \uparrow u(t,x)$. We now use the fact that $u_n(t,x)$ is the unique smooth solution of the problem

$$\mathcal{L}[u_n(t,x)] - \partial u_n(t,x)/\partial t = -L(x), \qquad t > 0,\ x \in D_n$$

$$u_n(0,x) = 0, \quad x \in D_n$$

$$u_n(t,x) = 0,\ t > 0,\ |x| = n$$

(see e.g.[5], Ch. 5). We can assume that $V(x) \geq 0$, $x \in E$. If $W_n(t,x) = kt + V(x) - u_n(t,x)$ $(t \geq 0,\ x \in \overline{D}_n)$ then

$$\mathcal{L}[W_n(t,x)] - \partial W_n(t,x)/\partial t \leq 0 ;$$

$W_n(0,x) \geq 0$; and $W_n(t,x) \geq 0$, $|x| = n$. By the maximum priciple for parabolic equations $W_n(t,x) \geq 0$ $(t \geq 0,\ x \in \overline{D}_n)$; that is $u_n(t,x) \leq kt + V(x)$; hence

$$u(t,x) \leq kt + V(x)\ ,\ t \geq 0,\ x \in E.$$

The result now follows from Lemma 4.2.

5. APPLICATIONS

EXAMPLE 1

Let X satisfy the Itô equation

$$dx = Fxdt - b\phi(\sigma)dt + G(x)dw$$

$$\sigma = c'x \qquad (15)$$

In (15), F is a constant matrix, b and c are constant n-vectors, and ϕ is a scalar-valued, in general nonlinear, function of σ. The non-stochastic differential equation, obtained from (15) by setting G = 0, has been studied extensively in connection with the Lur'e problem [8].

Theorem 5.1

Let the system (15) satisfy the following conditions:

(i) All the eigenvalues of F have negative real parts

(ii) $\sigma\,\phi(\sigma) > 0$ for all $|\sigma|$ sufficiently large; $\phi(\sigma)$ is continuously differentiable; and $d\phi(\sigma)/d\sigma$ is bounded $(-\infty < \sigma < \infty)$.

(iii) There exist two nonnegative constants α and β such that

$\alpha + \beta > 0$

and

$$\mathrm{Re}(\alpha + i\omega\beta)\; c'\; (i\omega I - F)^{-1}\, b > 0$$

for all real ω.

(iv) G(x) satisfies the conditions of section 2 and, in addition, $|G(x)|$ is bounded for $x \in E$.

Then X is positive and

$$\mathcal{E}\{|x|^{\nu}\} < \infty$$

for every $\nu > 0$.

Proof.

The positivity of X was proved in [3]. To satisfy the conditions of Theorem 3.1 we introduce a function $\tilde{V}(x)$ of the form

$$\tilde{V}(x) = x'Px + \beta \int_0^{c'x} \phi(\sigma)d\sigma$$

and define

$$V(x) = \exp(\gamma \tilde{V}(x))$$

where $\gamma > 0$ will be chosen later. By a result of Meyer [9] there exist positive definite matrices P and Q such that

$$[Fx-b\phi(c'x)]' \tilde{V}_x(x) \leq -x'Qx \tag{16}$$

for all $|x|$ sufficiently large. Moreover

$$\tfrac{1}{2}\mathrm{tr}[G(x)G(x)'\tilde{V}_{xx}(x)]$$
$$= \mathrm{tr}[G(x)G(x)'P] + \tfrac{1}{2}\beta|G(x)'c|^2 d\phi(c'x)/d\sigma \tag{17}$$

Since the right side of (17) is bounded it follows on adding (16) and (17) that, for arbitrary $\delta > 0$,

$$\mathcal{L}[\tilde{V}(x)] \leq -(1-\delta)x'Qx \tag{18}$$

for all $|x|$ sufficiently large. Let $\delta\epsilon(0,1)$ be fixed. Now

$$\exp(-\gamma \tilde{V}(x)) \mathcal{L}[V(x)]$$
$$= \gamma \mathcal{L}[\tilde{V}(x)] + \tfrac{1}{2}\gamma^2 |G(x)' \tilde{V}_x(x)|^2$$
$$= \gamma \mathcal{L}[\tilde{V}(x)] + \tfrac{1}{2}\gamma^2|G(x)' [2Px + \beta\phi(c'x)c]|^2$$
$$\leq -\gamma (1-\delta)x'Qx + \gamma^2 x'Rx \tag{19}$$

for some positive definite constant matrix R. Since Q is positive definite the matrix $(1-\delta)Q-\gamma R$ is positive definite for $\gamma > 0$ sufficiently small.

Then, for $|x|$ sufficiently large

$$\mathcal{L}[V(x)] \leq -\exp(\gamma \tilde{V}(x))$$

$$\leq -|x|^{\nu}.$$

The result now follows by Theorem 3.1.

Remark.

It is clear from the proof that, under the conditions of Theorem 5.1, $\mathcal{E}\{L(x)\} < \infty$ provided

$$L(x) = O[\exp(\theta|x|^2)] \quad (|x| \to \infty)$$

for $\theta > 0$ sufficiently small.

EXAMPLE 2

We shall illustrate the application of Theorem 4.1 to the analysis of a simple control system. Suppose

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 - \psi(x_1 + x_2) \tag{20}$$

where

$$\psi(y) = \begin{cases} 1, & y \geq 1 \\ y, & |y| \leq 1 \\ -1, & y \leq -1 \end{cases}$$

The null solution $x_1 = x_2 = 0$ is asymptotically stable. If the system is perturbed by Gaussian white noise it is of interest to estimate the mean square error $\mathcal{E}\{x_1^2\}$. The prior verification that X is positive will be omitted. Introducing perturbation terms and making the change of variables

$x_1 = x$, $x_1 + x_2 = y$, we obtain

$$dx = -(x-y)dt + a_{11}dw_1 + a_{12}dw_2$$
$$dy = -\psi(y)dt + a_{21}dw_1 + a_{22}dw_2 \qquad (21)$$

where w_1, w_2 are independent 1-dimensional Wiener processes and the coefficients a_{ij} are constants. The differential generator of the (x,y) process is

$$\mathcal{L}[u] = Au_{xx} + 2Bu_{xy} + Cu_{yy} - (x-y)u_x - \psi(y)u_y$$

where

$$A = (a_{11}^2 + a_{12}^2)/2$$
$$B = (a_{11}a_{21} + a_{12}a_{22})/2$$
$$C = (a_{21}^2 + a_{22}^2)/2$$

To satisfy condition (v) of section 2 we assume that $a_{11}a_{22} - a_{12}a_{21} \neq 0$; in applications such a restriction is clearly not significant.

To estimate $\mathcal{E}\{x^2\}$ we try to construct a positive function $V(x,y)$ with continuous second derivatives such that

$$\mathcal{L}[V(x,y)] \leq k - x^2 \qquad (x,y \in E)$$

for some positive constant k. As a first step we assume that the perturbation terms are absent from (21) and evaluate

$$V^\circ(x,y) = \int_0^\infty x(t)^2 dt \qquad (x(0) = x,\ y(0) = y).$$

The result is

$$\begin{aligned} V^\circ(x,y) &= x^2/2 + xy/2 + y^2/4\ , \quad |y| \leq 1 \\ &= x^2/2 - x + xy - y^2/2 + y^3/3 \\ &\qquad + e^{1-y}(x-y-1)/2 + 17/12\ , \quad y \geq 1 \\ &= V^\circ(-x,-y)\ , \quad y \leq -1\ . \end{aligned} \qquad (22)$$

From (22) we find that V°_{xx} , V°_{xy} are continuous, but

$$V^\circ_{yy}(x,1-0) = 1/2 \quad , \qquad V^\circ_{yy}(x,1+0) = 1 + x/2 \; .$$

To achieve the required smoothness replace V° by V^1 , where

$$V^1(x,y) = V^\circ(x,y) - (1/4)(1+x)(y-1)^2 e^{-\alpha(y-1)} \; , \qquad y \geqq 1$$

$$= V^\circ(x,y) \; , \; |y| \leqq 1$$

$$= V^1(-x,-y) \; , \; y \leqq -1 \qquad (23)$$

where $\alpha > 0$ is arbitrarily large. From (23),

$$\mathcal{L}[V_1] \sim 2C|y| - x^2 \qquad (|x| \to \infty \; , \; |y| \to \infty)$$

To cancel the term $2C|y|$ for large $|y|$, define

$$V^{(2)}(x,y) = V^1(x,y) + C(|y| - 1)^2 \exp[-\beta(|y| - 1)^{-1}], \qquad |y| \geqq 1$$

$$= V^\circ(x,y) \; , \; |y| \leqq 1$$

where $\beta > 0$ is arbitrarily small. Finally, let

$$V(x,y) = (1+\gamma)V^{(2)}(x,y)$$

where $\gamma > 0$ will be chosen later. Then

$$\mathcal{L}[V(x,y)] \leqq K - x^2 \qquad (x,y \in E) \qquad (24)$$

if K is sufficiently large. By straightforward estimation of the individual terms of $\mathcal{L}[V]$ we can obtain a value k_γ of K for which (24) is true; we then choose

$$k = \min\{k_\gamma : \gamma > 0\} \; .$$

Carrying out the estimates for $|y| \le 1$ and $|y| \ge 1$ separately, we find

$$\mathcal{E}\{x^2\} < \max\,(k', k'')$$

where

$$k' = (A + B + C/2)\,[1 + C(9C^2 + 4D)^{-\frac{1}{2}}]$$

$$k'' = (5/2)C^2 + D + (C/2)(9C^2 + 4D)^{\frac{1}{2}}$$

and

$$D = A + 2B + |B| + 3C/2$$

To obtain a rough idea of how conservative the bound may be in this case, suppose that $A \simeq 0$, $B \simeq 0$, $C \to \infty$. Then

$$\mathcal{E}\{x^2\} < k'' \sim 4C^2 \qquad (25)$$

Analysis of the system (21) based on 'statistical linearization' [10] of the nonlinear function ψ yields

$$\mathcal{E}\{x^2\} \simeq (\pi/2)C^2 \qquad (C \to \infty)\,. \qquad (26)$$

The qualitative agreement between the results (25) and (26) is due to the special choice of the function V_o. We should emphasize that the upper bound (25) was derived rigorously; the estimate (26), although probably reliable, was obtained by a heuristic procedure.

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