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Solutions of Euler's Equations Illustrating Effects of Finite Eddies

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Abstract

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The two problems treated in this report are intended to illustrate mathematical problems occurring in the solution of Euler's equations for stationary, two-dimensional flows containing one or more closed, finite regions where the vorticity is finite and non-zero (eddies). An approximate procedure derived formally by expansion for slender eddies is used to simplify the analysis, which then reduces to the solution of a non-linear, singular integral equation. In the present report, solutions of this equation corresponding to (i) a class of wake bubbles reducing to Riabouchinsky's solution when the vorticity in the eddy is zero, and (ii) a class of cusped eddies attached to the base of a slender wedge immersed in a uniform flow are derived. In the Case (i) it is found that the non-linear interaction between the eddy and the exterior, irrotational flow can result in the branching of solutions at critical parameter values, and consequently that in general the bubble is not uniquely determined by the parameters.

Author

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I. Introduction

In this report we shall derive and discuss several new approximate solutions of Euler's equations

$$\vec{q}\cdot\nabla\vec{q} + \nabla p = 0 \tag{1a}$$

$$\nabla\cdot\vec{q} = 0 \tag{1b}$$

where $\vec{q}$ is the velocity, p the pressure, and density is constant and equal to unity. These solutions have certain properties which are believed to be of importance in physical problems of greater complexity than the mathematical problems to be considered. In particular, the study of stationary laminar flows in the limit of infinite Reynolds number, i.e., the mathematical problem of solving the full Navier-Stokes equations

$$\vec{q}\cdot\nabla\vec{q} + \nabla p - \frac{1}{Re}\nabla^2\vec{q} = 0 \tag{1a*}$$

$$\nabla\cdot\vec{q} = 0 \tag{1b*}$$

[redacted]

for $Re \gg 1$ leads us to consider the merits of (1) as a "reduced" form of (1), and the possible emergence of a solution of (1) as the limit of a one parameter family of solutions of $(1)^*$, obtained over a certain region (depending in general upon Re) when the parameter Re becomes infinitely large. However, the present report is concerned solely with the theory of solutions of (1) and not of $(1)^*$. (However, see Chapter VI.)

The solutions of (1) considered below share the property that there exist one or more closed, finite regions where the vorticity is equal to a constant $\omega > 0$; any such region will be called an eddy. Our discussion will be limited to two-dimensional problems and will consist of the approximate analysis of two boundary-value problems. The first represents a logical extension to flows containing eddies of the classical theory of discontinuous, irrotational solutions of (1), while the second is a special case of a recently proposed model for the inviscid limit.

The main new analytical problem which occurs in the study of solutions of Euler's equations containing an eddy arises when (for example) there exists a region R_0 of irrotational motion which is separated from the eddy R_1 by a streamline γ (Fig. 1). Supposing for simplicity that the points A and B are fixed, the problem is then to choose the free streamline so that on it the pressure is a continuous function of x and y. Note that because of the rotational flow γ is no longer a constant-pressure line. Clearly the process of fitting in such a γ involves a non-linear <u>interaction</u> between a rotational (interior) flow and an irrotational (exterior) flow, which is essentially different from, e.g., the choice of a constant pressure curve in an irrotational discontinuous flow. At the same time an inessential but troublesome technical difficulty occurs. Because of the appearance of an inhomogeneous differential equation in the computation of the interior flow, mapping procedures (which leave Laplace's equation invariant but not its inhomogeneous form) can not be used in any straightforward manner. Thus it has been found necessary in the examples discussed herein to solve for γ by a direct attack on the non-linear interaction problem. That is, we restrict attention to the physical plane and use the pressure condition on γ as the determining one.

We shall study various aspects of this non-linear interaction, using two models. The first, considered in Chapter IV, is an extension of

Riabouchinsky's solution for uniform flow past two symmetrical obstacles, one the mirror image of the other (Fig. 2a). Here the attachment points A and B are fixed and the eddy replaces the stagnant cavity of Riabouchinsky's model. The second example (sketched in Fig. 2b) represents the separated flow past a symmetrical obstacle, the eddies forming an attached, cusped region of closed streamlines to the rear of the base of the body. This model, which was proposed as a stationary inviscid limit for the flow past a bluff obstacle by Batchelor (Ref. 1), is formulated and solved approximately in Chapter V. In either case the principal simplifying assumption, introduced in Chapter III, will be that the eddy is <u>slender</u>. This is accomplished, in the first case, by increasing the separation of the obstacles, and in the second case by assuming the obstacle itself is slender. The resulting equations and boundary conditions constitute a "slender-eddy" approximation fully analogous to the slender-body approximation of conventional thin-airfoil theory. On the other hand, the essential interaction terms, which remain non-linear in the slender-eddy theory, are in each case retained by an appropriate choice of the order of ω relative to the parameter that is small. This non-linear interaction constitutes the main theme of our investigations.

The principal results of this report are, first, that solutions of the type sketched in Fig. 2 exist in the slender-eddy approximation. This will be established by a combination of analytical and numerical steps. Since closed cusped cavities are known not to exist behind wedges for the irrotational case $\omega = 0$, this first result shows that the addition of a rotational eddy can enlarge the class of possible solutions of a given problem. Second, we have found (in the Riabouchinsky model) that the non-linear interaction leads to branching of solutions and a possible indeterminacy for "reasonable" boundary conditions. This behavior is not altogether unexpected in non-linear elliptic problems, and is related to the branching of various "inviscid modes" in linear stability theory. The present solution does, however, provide a specific example which can be worked out through a range of ω where the interaction between vorticity and flow is non-linear. It also suggests that related models of non-stationary flows may be of interest in the investigation of global stability in the presence of <u>finite</u> disturbances. We shall, however, not consider the non-stationary problem in this report.

For the case in which the velocity is continuous on γ , Gol'dshtik (Ref. 2) has established the existence of at least two exact solutions of a general problem involving a single eddy, provided only that ω is sufficiently large. The extension of the mathematical techniques used in Ref. 2 to the discontinuous problem considered in the present report would probably prove very fruitful. However (with the exception of Gol'dshtik's one-dimensional model, which we extend in Chapter II) we shall restrict our analysis to the slender-eddy approximation. At the same time, it should be pointed out that the discontinuous case involves several definite departures from the methods and results of Ref. 2. For example, there is a difference in the manner in which branching occurs in the two problems; in our first example one branch contains the "classical" Riabouchinsky solution, while in the continuous case ω is positive, uniformly over both branches.

II. The Exact Problem

2.1. General considerations. Consider a two-dimensional solution of (1) which is bounded and single-valued over a simply-connected region R having boundary Γ. Suppose further that structure of the solution is topologically equivalent to the sketch shown in Fig. 3. In the region R_0 (which contains the point at infinity) the flow is irrotational. In the regions R_i, $i = 1, 2, \ldots, n$ the vorticity of the flow is constant and equal to ω_i, $i = 1, 2, \ldots, n$, respectively. The curves γ_i, $i = 1, 2, \ldots, n$ are the free streamlines and are attached to Γ at the points A_i and B_i. In general both the velocity and the vorticity are discontinuous on the γ_i, but the pressure is always continuous there. Such a flow is to be called a Euler flow containing n-eddies. That is, an eddy will here mean a closed finite region bounded by a streamline, over which the vorticity is constant. In this report we shall study two simple examples of Euler flows of the above type containing a single eddy, which are therefore equivalent to the special case shown in Fig. 1.*

* Note that not all Euler flows in a simply connected region which contain n closed, finite regions of constant vorticity can be mapped onto Fig. 3. This is because of the possibility of two free streamlines meeting at an interior point to form a cusp (or, if the velocity is continuous on the free streamlines, a stagnation point). In the examples considered the cusp will occur at the dividing streamline and therefore effectively on Γ, so that this situation does not arise.

If we introduce the streamfunction ψ ,

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x}, \quad \vec{q} = u\,\vec{i} + v\,\vec{j} \tag{2}$$

then ψ is a constant on Γ and each γ_i , and ψ will always be normalized so that this constant is equal to zero. Using (2) we may partially integrate (1) to obtain the well known relations

$$\frac{1}{2}(u^2+v^2) + p = \text{Bernoulli's Function} = \mathcal{H}(\psi) \tag{3a}$$

$$-\nabla^2\psi = \text{Vorticity} = -\frac{d\mathcal{H}}{d\psi} \tag{3b}$$

The function $\mathcal{H}(\psi)$ has the form

$$\mathcal{H}(\psi) = -\omega_i\psi + h_i = \mathcal{H}_i(\psi) \tag{4}$$

in R_i , where the h_i, $i = 1, 2, \dots, n$ are constants, and $\mathcal{H}_0 = h_0$. We shall assume that $\psi = 0$ on the segments A_1C and B_1D of Γ , while on the open segment CD we will have $\psi > 0$. Since ψ is harmonic in R_0 , we therefore have $\psi \geq 0$ on the closure of R_0. In a region

R_i , $\nabla^2\psi = -\omega_i$ and $\psi = 0$ on the boundary, so that $\psi \geqslant 0$ or $\psi \leqslant 0$ in the closure of R_i depending upon whether or not $\omega_i > 0$ or $\omega_i < 0$, respectively. We shall say that the flow in these cells is <u>kinematically possible</u> if

$$\omega_1 < 0 \ , \quad \omega_i \, \omega_{i+1} < 0 \ , \quad i = 1, 2, \ldots, n \tag{5}$$

It follows from this definition that the flow is kinematically possible if and only if for any two adjacent regions R_i , R_{i+1} , $i = 0, 1, 2, \ldots$ there exists a (possibly discontinuous) function $\mathcal{H}(\psi)$ which through (3) generates a solution of (1) in $R_i + R_{i+1}$, defined by $\mathcal{H} = \mathcal{H}_i$ for $\psi \geqslant 0$ (or $\psi \leqslant 0$, as the case may be) and by $\mathcal{H} = \mathcal{H}_{i+1}$ for $\psi < 0$ ($\psi > 0$) . Note that a flow which is kinematically possible is one for which $\omega_1 < 0$ and the vorticity of adjacent eddies is of opposite signs. It is conjectured that (3) are necessary conditions for the existence of multiple-eddy solutions of (1) which are at the same time at interior points limits in the sense used in Chapter I. However, this possibility will not be investigated here and for the purposes of the present paper the question is of no importance.*

It follows from the preceding remarks that a flow pattern containing n eddies is fully and uniquely determined by fixing the free streamlines and the $2n$ constants ω_i, h_i, $i = 1, 2, \ldots, n$. To obtain a continuous pressure and therefore a solution of (1) having the required properties, we must choose the γ_i so that

$$\frac{1}{2}\left[q_i^2 - q_{i+1}^2\right]_{\gamma_i} = h_i - h_{i+1}, \qquad i = 0, 1, \ldots, n \tag{6}$$

where q_i denotes the fluid speed in R_i; the constant h_0 always will be known in advance. The most important problem of this general type is the <u>problem of fixed attachment</u>, which may be formulated as follows:

<u>Given the vectors $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ and</u>

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* In fact, in the final equations only the square of ω_i appears.

† It should be noted that this *a priori* condition leads to a topological orientation of eddies which is the same as what would be assigned intuitively by considering the probable action of viscous stresses, and which is in agreement with observation of certain flows. It is possible to show (by considering the dissipation in an eddy) that (3) are necessary provided that every region of closed streamlines has finite area in the limit of infinite Reynolds number, and provided also that the limit is sufficiently well behaved.

$2n-m$ $(0<m<2n)$ of the $2n$ constants $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$, $h=(h_1,h_2,\ldots,h_n)$, to find $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)$ and m remaining constants such that the n equations (6) and the m contraints

$$\mathcal{F}_j[\gamma_{:(j)}]=0 \quad , \qquad j=1,2,\ldots,m \tag{7}$$

are satisfied. The constraints are typically associated with the geometry of attachment of γ, and in the above definition each of the $\mathcal{F}_j$ presumably determines exactly one of the m unknown constants. Examples illustrating the cases $(n,m)=(1,1)$ and (1, 2) will be worked out in the slender-eddy approximation in Chapters IV and V.

A final remark of a general nature concerns the possibility of trivial solutions of the above problem. Clearly one solution of the n-eddy problem is obtained when γ_n collapses onto Γ and a solution of the $n-1$ -eddy problem exists for the same $\omega_1,\omega_2,\ldots,\omega_{n-1}$. Therefore we may say that solutions containing $n-1$ are trivial in the n-eddy problem, and unless otherwise noted we shall mean by a "solution" of the problem a non-trivial solution. However, it will be seen through the examples that a given problem of the above type may possess only trivial solutions, or it may have more than one non-trivial solution.

2.2. The case n = 1. Reduction to an integral equation. Restricting attention henceforth to the situation shown in Fig. 1, we write (6) in the form

$$\frac{1}{2}\,(\nabla\psi_0)^2 - \frac{\omega^2}{2}\,(\nabla\psi_1)^2 = h_0 - h \tag{8}$$

In (7) ψ_0 denotes the streamfunction in R_0, ψ_1 the streamfunction in R_1 when the vorticity is unity. Now (8) may be expressed as a non-linear integral equation for γ in the following way: First, regarding γ as fixed, ψ_0 is expressed in terms of the Green's function for Laplace's equation in R_0. Analogously ψ_1 can be expressed in terms of the Green's function for Laplace's equation in R_1. The relations are of the form

$$\psi_0 = -\frac{1}{2\pi}\int_{\Gamma_{CD}} \psi(\xi,\eta)\, G_{R_0}(x,y;\xi,\eta)\, ds(\xi,\eta) \tag{9a}$$

$$\psi_1 = -\frac{1}{2\pi}\iint_{R_1} G_{R_1}(x,y;\xi,\eta)\, d\xi\, d\eta \tag{9b}$$

where G_{R_i} denotes the Green's function, and in (9a) Γ is traversed in the positive sense (R_0 to the left). Since the values of ψ on Γ_{CD} are known, we may express the left-hand side of (7) in terms of the and the region of integration in (9b), i.e., in terms of γ alone.

2.3. <u>The one-dimensional problem</u>. For the purpose of illustrating the use of (7) in the simplest possible form, we consider briefly the one-dimensional single-eddy problem. A function $\psi(x)$, defined in two adjacent intervals on the x-axis, is considered, and we require that satisfy a certain jump condition at the common end point. Using our previous notation, we require

$$\psi_{xx} = \omega \geq 0 \quad , \quad \psi(0) = \psi(x_\gamma) = 0 \quad \text{for} \quad 0 \leq x < x_\gamma \tag{10a}$$

$$\psi_{xx} = 0 \quad , \quad \psi(x_\gamma) = 0 \ , \ \psi(1) = 1 \quad \text{for} \quad x_\gamma \leq x \leq 1 \tag{10b}$$

$$\frac{1}{2}(\psi_x)^2_{x=x_\gamma+} - \frac{1}{2}(\psi_x^2)_{x=x_\gamma-} = h \tag{10c}$$

Solving for x_γ, (10c) leads to the algebraic equation

$$\frac{1}{2}(1-x_\gamma)^{-2} - \frac{1}{8}\omega^2 x_\gamma^2 = h \tag{10d}$$

which is the one-dimensional version of (8). The solutions of (11) are sketched in Fig. 4. In the absence of constraints (7) we determine X_γ by fixing ω and h. We then observe two properties of the solutions: (i) For all values of ω and h there exists a solution which lies outside the basic interval $0 \leq x \leq 1$. The corresponding ψ is therefore necessarily a multivalued function of x. In the planar case multivalued solutions $\psi(x,y)$ can also occur, but are sometimes (and will here be) rejected on physical grounds. (ii) X_γ is not necessarily uniquely determined by ω and h. In fact, for the same values of the parameters, there may exist one, two, or three solutions in the interval $0 \leq x \leq 1$. The critical values of ω and h where the number of solutions changes determine the <u>bifurcation points</u> of the solution.[*] Thus in Fig. 4a there is one bifurcation point at a, and in Fig. 4b there are two bifurcation points, at b and c. We shall find that a bifurcation point analogous to the point c occurs in ~~the planar case~~ plane flows in the slender eddy approximation. (iii) For a certain range of ω and $h < \frac{1}{2}$ there are no solutions in the interval $0 \leq x \leq 1$.

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[*] See, e.g., Ref. 3, Chapter IV.

III. Approximations for Slender Eddies

3.1. The exterior and interior problems. In order to discuss the passage from the exact problem formulated in the preceding chapter, to the approximate problem valid for slender eddies, we shall consider the geometry shown in Fig. 5. The boundary Γ consists of the curve.

$$y = \left\{ \begin{array}{lll} \varepsilon Y_A(x) & , & -1-\lambda \leqslant x \leqslant -1 \\ 0 & , & -1 < x < +1 \\ \varepsilon Y_B(x) & , & +1 \leqslant x \leqslant 1+\kappa \end{array} \right\} = \varepsilon Y_\Gamma(x)$$

where ε is a small number, $\varepsilon << 1$, and the free streamline γ consists of the curve

$$y = \varepsilon Y_\gamma(x) \, , \quad -1 < x < +1$$

where Y_γ , Y_A , Y_B are positive, continuous functions of x and

$$Y_A'(-1) = Y_\gamma'(-1) \; , \quad Y_B'(+1) = Y_\gamma'(+1)$$

$$Y_A'(-1-\lambda) = Y_B(1+\kappa) = 0$$

At infinity the conditions are

$$p = 0, \quad u = 1, \quad v = 0 \quad ; \quad (h_0 = 1/2) \tag{11}$$

We propose to represent ψ_0 and ψ_1 by approximate solutions valid for small ε, using conventional thin-airfoil and boundary-layer procedures, and to use these expressions to derive an approximation to the integral equation obtained from the pressure condition (8). The <u>exterior problem</u> is therefore to find the streamfunction of the irrotational flow, assuming that the boundary conditions imposed on this flow may be written approximately as

$$v(x,0) = \begin{cases} \varepsilon Y_A'(x) & -1-\lambda \leq x \leq -1 \\ \varepsilon Y_\gamma'(x) & -1 < x < +1 \\ \varepsilon Y_B'(x) & +1 \leq x \leq 1+\kappa \end{cases} \tag{12}$$

i.e., in their thin-airfoil form. The <u>interior problem</u> is to be solved through the use of a boundary-layer approximation, that is, by the replacement of the exact problem $\nabla^2\psi = \omega$, ψ zero on the boundary, by the approximate form

$$\frac{\partial^2\psi}{\partial y^2} = \omega \quad \text{and} \quad \psi(x,0) = \psi(x, \varepsilon Y_\gamma) = 0 \; ; \; -1 < x < +1 \tag{13}$$

The solutions of each of these problems is easily found, and we obtain the following expressions for the streamfunction:

Exterior: $$\psi = \psi_0(x,y) = y - \frac{\varepsilon}{\pi}\left\{ \int_{-1-\lambda}^{-1} \frac{y Y_A'(\xi)}{(x-\xi)^2+y^2}\,d\xi + \int_{-1}^{+1} \frac{y Y_\gamma'(\xi)}{(x-\xi)^2+y^2}\,d\xi + \int_{+1}^{+1+\kappa} \frac{y Y_B'(\xi)}{(x-\xi)^2+y^2}\,d\xi \right\} \tag{14}$$

Interior: $$\frac{1}{\omega}\psi = \psi_1(x,y) = \frac{1}{2}\left[y^2 - \varepsilon y Y_\gamma(x) \right] \tag{15}$$

3.2. Reduction to a non-linear integral equations. Using (14) and (15) in (7) there results

$$\frac{1}{2} + \frac{\varepsilon}{\pi}\left\{ \int_{-1-\lambda}^{-1} \frac{Y_A'(\xi)}{x-\xi}\,d\xi + \int_{-1}^{+1} \frac{Y_\gamma'(\xi)}{x-\xi}\,d\xi + \int_{+1}^{+1+\kappa} \frac{Y_B'(\xi)}{x-\xi}\,d\xi \right\} - \frac{\omega^2\varepsilon^2}{8} Y_\gamma^2(x) = \frac{1}{2} - h + o(\varepsilon) \tag{16}$$

for $-1 < x < +1$, the integrals now being given by the Cauchy Principal Value. In order that the contribution from interior and exterior solutions be of the same order, and of the order of , we therefore must put

$$\omega = \frac{\omega^*}{\sqrt{\varepsilon}}, \quad h = -\varepsilon h^* \tag{17}$$

where ω^* and h^* are now numbers of order unity. The physical meaning of (17) should be noted here; the interior velocity must be small if the pressure perturbation caused by a slender eddy is to be balanced by the interior flow, and consequently h jumps by almost the exterior value as γ is crossed. On the other hand, the width of the eddy is small, of order ε, so that if the velocity perturbations are of order $\sqrt{\varepsilon}$, the vorticity must be <u>large</u>, of order $1/\sqrt{\varepsilon}$.

Collecting the terms of order ε in (16), we then obtain the asymptotic form for a slender eddy

$$\frac{1}{\pi} \int_{-1}^{+1} \frac{Y_\gamma'(\xi)}{x-\xi} d\xi = h^* + \frac{\omega^{*2}}{8} Y_\gamma^2(x) - \int_{-1-\lambda}^{-1} \frac{Y_A'(\xi)}{x-\xi} d\xi - \int_{+1}^{+1+\varkappa} \frac{Y_D'(\xi)}{x-\xi} d\xi \tag{18}$$

which is a non-linear singular differential-integral equation for Y_γ. In order to convert (18) into a pure integral equation, we first make use of the well known inversion*

$$f(x) = \frac{1}{\pi} \int_{-1}^{+1} \frac{g(\xi)}{x-\xi} d\xi \qquad (19a)$$

$$g(x) = -\frac{1}{\pi} \int_{-1}^{+1} \frac{\sqrt{1-\xi^2}}{\sqrt{1-x^2}} \frac{f(\xi)}{x-\xi} d\xi + \frac{1}{\pi\sqrt{1-x^2}} \int_{-1}^{+1} g(x)\, dx \qquad (19b)$$

together with the identity

$$\frac{1}{\pi} \int_{-1}^{+1} \frac{\sqrt{1-\xi^2}}{(\xi - x_1)(x_2-\xi)} d\xi = 1 + \frac{\sqrt{x_1^2-1}}{x_2-x_1} \operatorname{sgn} x_1 \ , \quad |x_1| > 1 \, , |x_2| < 1 \qquad (20)$$

to put (18) into the form

$$Y_\gamma'(x) = -\frac{1}{\pi} \int_{-1}^{+1} \frac{\sqrt{1-\xi^2}}{\sqrt{1-x^2}} \frac{(h^* + \frac{1}{8}\omega^{*2} Y_\gamma^2)}{x-\xi} d\xi - \frac{1}{\pi} \int_{-1-\lambda}^{-1} \frac{\sqrt{\xi^2-1}}{\sqrt{1-x^2}} \frac{Y_A'(\xi)}{x-\xi} d\xi$$

$$- \frac{1}{\pi} \int_{1}^{1+\lambda} \frac{\sqrt{\xi^2-1}}{\sqrt{1-x^2}} \frac{Y_B'(\xi)}{x-\xi} d\xi \ , \qquad -1 < x < +1 \qquad (21)$$

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* In the form (19) it is necessary that $(1-x^2) f^2(x)$ be integrable on $[-1,+1]$, a condition which is always met below.

The conditions given previously on the derivative of Y_γ at the attachment points next give two conditions which will determine h^* and ω^*. These are

$$\frac{1}{\pi}\int_{-1}^{+1}\sqrt{\frac{1-\zeta}{1+\zeta}}\left(h^*+\frac{1}{8}\omega^{*2}Y_\gamma^2\right)d\zeta-\frac{1}{\pi}\int_{-1-\lambda}^{-1}\sqrt{\frac{1-\zeta}{-1-\zeta}}\,Y_A'(\zeta)\,d\zeta-\frac{1}{\pi}\int_{+1}^{+1+\kappa}\sqrt{\frac{\zeta-1}{\zeta+1}}\,Y_B'(\zeta)\,d\zeta=0 \quad (22a)$$

$$-\frac{1}{\pi}\int_{-1}^{+1}\sqrt{\frac{1+\zeta}{1-\zeta}}\left(h^*+\frac{1}{8}\omega^{*2}Y_\gamma^2\right)d\zeta-\frac{1}{\pi}\int_{-1-\lambda}^{-1}\sqrt{\frac{-1-\zeta}{1-\zeta}}\,Y_A'(\zeta)\,d\zeta-\frac{1}{\pi}\int_{+1}^{+1+\kappa}\sqrt{\frac{\zeta+1}{\zeta-1}}\,Y_B'(\zeta)\,d\zeta=0 \quad (22b)$$

If now the change of variables

$$x=\cos\pi\theta \quad , \quad \zeta=\cos\pi\theta' \quad , \quad Y_\gamma(x)=w^*(\theta) \qquad (23)$$

is made in (21) and the result integrated once, we obtain the integral equation for $w^*(\theta)$,

$$w^*(\theta)=h^*\sin\pi\theta+\frac{1}{8}\omega^{*2}\int_0^1 K(\theta,\theta')\sin\pi\theta'\,w^{*2}(\theta')\,d\theta' \qquad (24a)$$
$$+\,w_A^*(\theta)-w_B^*(\theta)$$

where

$$K(\theta,\theta') = \log\left|\frac{\tan\frac{\pi}{2}\theta + \tan\frac{\pi}{2}\theta'}{\tan\frac{\pi}{2}\theta - \tan\frac{\pi}{2}\theta'}\right| = 2\sum_{n=1}^{\infty}\frac{\sin\pi n\theta\,\sin\pi n\theta'}{n} \tag{24b}$$

$$W_A^*(\theta) = \frac{3}{\pi}\int_0^{\lambda}\tan^{-1}\left[\sqrt{\frac{t}{2+t}}\,\tan\frac{\pi}{2}\theta\right] Y_A'(-1-t)\,dt \tag{24c}$$

$$W_B^*(\theta) = \frac{2}{\pi}\int_0^{\kappa}\tan^{-1}\left[\sqrt{\frac{t}{2+t}}\,\cot\frac{\pi}{2}\theta\right] Y_B'(1+t)\,dt \tag{24d}$$

The constraints (22) become

$$\int_0^1 (1-\cos\pi\theta)\left(h^* + \tfrac{1}{8}\omega^{*2}W^{*2}\right)d\theta = \frac{1}{\pi}\int_0^{\lambda}\sqrt{\frac{2+t}{t}}\,Y_A'(-1-t)\,dt + \frac{1}{\pi}\int_0^{\kappa}\sqrt{\frac{t}{2+t}}\,Y_B'(1+t)\,dt \tag{25a}$$

$$\int_0^1 (1+\cos\pi\theta)\left(h^* + \tfrac{1}{8}\omega^{*2}W^{*2}\right)d\theta = -\frac{1}{\pi}\int_0^{\lambda}\sqrt{\frac{t}{2+t}}\,Y_A'(-1-t)\,dt - \frac{1}{\pi}\int_0^{\kappa}\sqrt{\frac{2+t}{t}}\,Y_B'(1+t)\,dt \tag{25b}$$

These together with the non-linear integral equation (24) constitute the final form of the general slender-eddy approximation, and we now consider various special cases.

IV. Riabouchinsky's Model

4.1. Preliminary remarks. We shall apply the slender-eddy theory and equations (24) and (25) first to the case obtained by putting

$$Y_A = c\sqrt{1+\lambda+X} \quad , \quad -1-\lambda \leq X \leq -1 \tag{26a}$$

$$Y_B = c\sqrt{1+\varkappa-X} \quad , \quad 1 \leq X \leq 1+\varkappa \tag{26b}$$

and letting λ and $\varkappa$ tend to zero. By this limit process the "obstacles" defined by (26) are reduced to the two points at $X = \pm 1$; the residual effect of the obstacles is, however, finite and connected with the constant $c > 0$. A physical interpretation of the solutions obtained under these conditions emerges from considering Riabouchinsky's model in the limit of large separation. Suppose that the uniform stream of speed unity impinges upon an isolated obstacle, and suppose that there exists an irrotational (discontinuous) flow for this obstacle for which the drag is D (per unit length). Then it is known* that such a flow pattern contains

* See Ref. 4 , p. 68 .

a free streamline whose asymptotic behavior is given by

$$y' \sim \sqrt{\frac{4D}{\pi}} \sqrt{x'} \qquad \text{as} \quad x' \longrightarrow \infty \tag{27}$$

where x' , y' are dimensional. Consider now two such obstacles, placed as shown in Fig. 2a. (We consider below only the half-plane $y \geq 0$). If the separation between the obstacles is $2L$, and if ℓ is a characteristic body dimension, (27) may be written

$$y \sim \sqrt{\frac{4D}{\pi}} \sqrt{\frac{\ell}{L}} \sqrt{x} \tag{28}$$

Comparing (26) and (28), we see that the slender eddy solutions obtained at $\lambda = x = 0$ for a given c approach near $x = \pm 1$ the asymptotic behavior of the streamlines for the irrotational flow provided that

$$c = \sqrt{\frac{4D}{\pi \ell U^2}} \quad , \quad \varepsilon = \sqrt{\frac{\ell}{L}} \tag{29}$$

The corresponding vorticity in the eddy, when scaled with ℓ instead of L , is then equal to

$$\frac{\ell}{L}\,\frac{L^{1/4}}{\ell^{1/4}}\,\omega^* = \frac{\ell^{3/4}}{L^{3/4}}\,\omega^* = \epsilon^{3/2}\,\omega^* \qquad (30)$$

and is small. Specifying c is moreover equivalent in the slender eddy approximation to specifying*

$$\frac{1}{\pi}\,\frac{dw^*(0)}{d\theta} = -\frac{1}{\pi}\,\frac{dw^*(1)}{d\theta} = \frac{c}{\sqrt{2}} \qquad (31)$$

The governing equation for the free streamline in Riabouchinsky's model is therefore found to be

$$w^*(\theta) = h^*\,\sin\pi\theta + \frac{1}{8}\,\omega^{*2}\int_0^1 K(\theta,\theta')\,\sin\pi\theta'\,w^{*2}(\theta')\,d\theta' \qquad (32)$$

The two constraints (25) become

$$\int_0^1 (1-\cos\pi\theta)\,\left[h^* + \frac{1}{8}\,\omega^{*2}\,w^{*2}\right]\,d\theta = c/\sqrt{2} \qquad (33a)$$

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* Note that (31) states that near $x = \pm 1$ the slender-eddy solution matches, in the sense of Ref. 5, in an overlap domain which lies in a small neighborhood, with a flow of discontinuous type past an isolated obstacle. Therefore, in our approximation, all such flows for which the obstacle experiences the same drag are equivalent.

$$\int_0^1 (1+\cos\pi\theta)\,[h^* + \tfrac{1}{8}\,\omega^{*2} w^{*2}]\,d\theta = c/\sqrt{2} \tag{33b}$$

The two sets of constraints, (31) following from the physical model for large separation, and (33) following from the substitution (26) are seen to be equivalent if w^* is a solution of (32). Adding and subtracting, (33a) and (33b), we obtain

$$\tfrac{1}{8}\,\omega^{*2} \int_0^1 \cos\pi\theta \; w^{*2}(\theta)\,d\theta = 0 \tag{34}$$

$$h^* + \tfrac{1}{8}\,\omega^{*2} \int_0^1 w^{*2}(\theta)\,d\theta = c/\sqrt{2} \tag{35}$$

The first of these is satisfied identically if w^* is a symmetric function of $\theta - 1/2$, i.e., if w^* is a symmetric solution. Moreover it is seen that the symmetric functions can solve (32) (in the sense that the equation is invariant under $\theta \to 1-\theta$). However there remains the possibility that there exist solutions of (32) which are not symmetric.

We shall in fact find solutions of (32), (34), (35) among the symmetric solutions, so that (34) is the sole constraint and one and only one free parameter is determined.* We shall take ω^* to be the parameter which is prescribed in advance, and write the equations in a form involving a single unknown parameter by the change of variables

$w^* = 8/\omega^{*2}\, w$, There results

$$w(\theta) = k \sin\pi\theta + \int_0^1 K(\theta,\theta') \sin\pi\theta' \, w^2(\theta')\, d\theta' \tag{36a}$$

$$\frac{1}{\pi} w'(0) = r \tag{36b}$$

$$k = \frac{1}{8}\omega^{*2} h^* , \quad r = \frac{c\,\omega^{*2}}{8\sqrt{2}} \tag{36c}$$

---

* Our conclusion that (34), (35) are in a sense redundant is supported by the following physical argument. Consider the drag force exerted on the surrounding fluid by the closed contour containing both obstacles and consisting of segments of the streamlines $\psi = 0$. Clearly this quantity must be exactly zero. To order unity, however, the contributions from integrals over γ are zero (by virtue of the fact that $Y(-1) = Y(+1) = 0$) so that to the same order the contributions from the wetted surfaces of the obstacles must vanish. But this is equivalent to the statement that the drag of the upstream obstacle is balanced by an equal thrust at the downstream one, or that $\frac{dw^*}{d\theta}(0) = -\frac{dw^*}{d\theta}(1)$.

where r is the given, and k the unknown parameter.

4.2. The existence of solutions. The aim of the next two paragraphs is to demonstrate, for given $r \geq 0$, the existence and possible non-uniqueness of solutions of (32), (35). No further approximations are made, although our analysis will involve a combination of analytical and numerical steps. Our main result is contained in the following theorem: For given positive constant c, there exists (i) if $\omega^* = 0$, a unique solution (Riabouchinsky's solution), (ii) if $\omega^* > 0$ and r is less than a certain constant $r^* \doteq .71$, at least two distinct solutions. By solution we here mean a solution of (32), (35) which is continuous and positive if $0 < \theta < 1$. The solutions exhibited below are symmetric solutions.

In our discussion of the theorem the general reference will be Tricomi (Ref. 3). The following properties of the kernel function $K(\theta,\theta')$ will first be noted: (i) K is symmetric in θ and θ', and is positive over the interior of the unit square S consisting of the points $0 \leq \theta, \theta' \leq 1$, vanishes on the boundary of S excluding the points $(0,0)$ and $(1,1)$, and behaves like $-\log|\theta-\theta'|$ near $\theta = \theta'$, $\theta \neq 0, 1$. (ii) The iterated kernel K_2

defined by

$$K_2(\theta,\theta') = \int_0^1 K(\theta,\theta'')\ K(\theta',\theta'')\ d\theta''$$

is bounded (and therefore continuous) on S. In particular K^2 is integrable over S. (iii) The functions $\sin\pi n\theta$, $n=1,2,\ldots$ are eigenfunctions with respective eigenvalues $1/n$:

$$\int_0^1 K(\theta,\theta')\ \sin\pi n\theta\ d\theta = \frac{1}{n}\ \sin\pi n\theta \tag{37}$$

The proofs of (i)-(iii) are either elementary or well known and will not be given here. An immediate consequence is that $w-h\sin\pi\theta$, and hence any solution of (36a) may be represented by an eigenfunction expansion

$$w(\theta) = \sum_{n=0}^{\infty} a_n\ \sin\pi n\theta \tag{38}$$

which converges absolutely and uniformly on $0 \leq \theta \leq 1$ *.

For the case $\omega^* = 0$, the unique solution is

$$w^* = h^* \sin\pi\theta \quad , \quad h^* = \frac{c}{\sqrt{2}} \tag{39}$$

---

* This is the Hilbert-Schmidt Theorem, Ref. 3, page 110.

This is Riabouchinsky's solution in the slender-eddy approximation. We propose to continue this solution to $\omega^* > 0$ by formal expansion of (36) in powers of k :

$$w = \sum_{i=0}^{\infty} k^{i+1} w_i \tag{40}$$

Substituting in (36) and collecting terms of like order there results

$$w_0 = \sin \pi\theta \tag{41a}$$

$$w_i = \sum_{j=0}^{i-1} \int_0^1 K(\theta,\theta') \; w_{i-j}(\theta') \; w_j(\theta') \; \sin\pi\theta' \, d\theta', \quad i = 1, 2, \cdots \tag{41b}$$

We now prove that (41) provides a (symmetric) solution if $k < 1/4$. To do this, we first generate the numbers m_i by

$$1 - \sqrt{1-4k} = 2 \sum_{i=0}^{\infty} m_i \, k^{i+1}$$

If now (37) is used, it is seen from (41) that the m_i provide bounds in the form

$$\max_{0\leq\theta\leq 1} |w_i(\theta)| \leq m_i \quad , \; i = 0, 1, 2, \ldots$$

and the result follows. The first few terms of the series are

$$w_1(\theta) = \frac{3}{4}\sin\pi\theta - \frac{1}{12}\sin 3\pi\theta$$

$$w_2(\theta) = \frac{7}{6}\sin\pi\theta - \frac{11}{72}\sin 3\pi\theta + \frac{1}{120}\sin 5\pi\theta$$

$$w_3(\theta) = \frac{263}{144}\sin\pi\theta - \frac{12721}{17280}\sin 3\pi\theta + \frac{1321}{86400}\sin 5\pi\theta - \frac{1}{1680}\sin 7\pi\theta$$

and, as is easily shown, in general

$$w_i(\theta) = \sum_{n=1}^{2i+1} a_n^{(i)} \sin\pi n\theta \qquad \text{where} \quad a_{2n}^{(i)} = 0 \;, \; n = 1, 2, \ldots$$

and

$$a_1^{(i)} > 0, \; a_{2n+1}^{(i)} \, a_{2n-1}^{(i)} < 0, \; n = 1, 2, \ldots$$

The number of solutions of (36) depends, however, upon k. We show now that <u>(36) has no solutions (38) if $k > 1/2$; moreover the formal series (40) diverges when $k \geq 1/3$</u>. Actually the proof is quite simple.

Consider the equation ~~for~~ satisfied by the first Fourier coefficient a_1 :

$$a_1 = 2\int_0^1 \sin\pi\theta \, w(\theta)\, d\theta = k + 2\int_0^1 [\sin\pi\theta \; w(\theta)]^2 \, d\theta$$

$$= k + \tfrac{1}{4}\left[2a_1^2 + a_2^2 + (a_1 - a_3)^2 + (a_2 - a_4)^2 + \cdots\right]$$

Thus in general

$$a_1^2 - 2a_1 + 2k < 0 \qquad (42)$$

and this is possible only if $k < 1/2$. The second part follows from the fact that in the series solution the even-numbered coefficients vanish and a_1 and a_3 are of opposite sign, as noted earlier, so that the coefficient of a_1^2 in (42) can be replaced by 3/2.

Therefore as k is decreased from $1/3$, there occurs a critical value k^*, $1/4 \leq k^* < 1/3$, where solutions first appear. There are two possibilities to be considered. Either k^* is a singular point of the integral equation, in which case solutions become unbounded there, or k^* is a bifurcation point and there is at least one branch of solutions distinct from (40) for some $k < k^*$. Now we can eliminate the first possibility by observing that (42) implies a_1 , and for non-negative solutions the maximum value of W as well, is bounded if k is bounded. It follows that k^* is a bifurcation point and there an upper branch of symmetric solutions should be sought. We shall obtain a new branch of solutions by means of an iterative method.

Specifically a sequence $\{w^{(n)}, n = 1, 2, \dots\}$ of iterates will be found which converges to a solution of (36). Consider any non-negative, symmetric, continuous function $f(\theta)$ defined on $0 \leq \theta \leq 1$, and atisfying $f(0) = f(1) = 0$; then for any positive number $\varkappa$ we define the quantity $F_\varkappa[f^2]$ by

$$F_\varkappa[f^2] = F_\varkappa(\theta)$$
$$= \int_0^1 K(\theta,\theta') \sin\pi\theta' \, f^2(\theta')\, d\theta' + \varkappa \sin\pi\theta \quad , \quad \varkappa > 0$$

The function $F_\varkappa(\theta)$ has the following properties: (i) It is continuous on $0 \leq \theta \leq 1$. (ii) If the coefficients of f alternate in sign, beginning with a positive coefficient of $\sin\pi\theta$, then $F_\varkappa(\theta)$ has the same property.

Consider now the iterates

$$f^{(0)} = \sin\pi\theta \tag{43a}$$

$$f^{(n)} = \tau^{(n)} F_\varkappa[f^{(n)^2}] = \tau^{(n)} F_\varkappa^{(n)}(\theta) \tag{43b}$$

$$1/\tau^{(n)} = F_\varkappa^{(n)}(1/2)$$

The sequences $\{f^{(n)}\}$ and $\{\tau^{(n)}\}$ have the uniform bounds 1 and $1/\varkappa$, respectively. Therefore $\{f^{(n)}\}$ is an equicontinuous family of functions, and we may accordingly select a subsequence which converges to a non-negative, continuous limit function f.* It can be shown also that

$$F_\varkappa[f^{(n)^2}] \to F_\varkappa[f^2] \quad , \quad \tau^{(n_i)} \to \tau$$

say, where $\tau > 1/_{1+\varkappa}$. The function $W = \tau f$ is seen to be a solution of (36a) provided $k = \tau^2 \varkappa$. We can therefore generate a family of solutions of the required type.

These solutions contain those obtained by expansion in k (occurring here for $\varkappa$ sufficiently large), but our aim is to exhibit an upper branch. Consider then the choice $\varkappa = \epsilon << 1$. We have seen that τ is bounded, and therefore we may choose ϵ so that a solution is obtained for k arbitrarily small, but for which $\tau > 1/_{1+\epsilon}$. Therefore this

* See, e.g., Ref. 6, Chapter II.

solution is distinct from any solution on the lower branch which is close to the Riabouchinsky's solution, and this proves the result.

It is evident from the continuous dependence of $F_{\varkappa}$ upon $\varkappa$ that we can, starting from a solution for k small and positive, continue it analytically into a certain range of negative k. However, then the term $\varkappa \sin \pi\theta$ tends to decrease the values of $F_{\varkappa}(\theta)$, with the result that the iterates may cease to be non-negative functions. For this reason it appears to be difficult to find a negative lower bound for $\varkappa'$ without more refined estimates than those used above. The numerical results discussed in the following paragraph bear out the existence of a limit point $k_c \doteq -1.87$ $(\varkappa_c \doteq -.18)$ below which solutions contain negative regions near the end points.

..3. Numerical results. We now complete the proof of the theorem stated in the previous paragraph by describing the results of numerical computations of both branches of solutions.* Figures 7, 8, and 9 contain the essential results. The curve of the maximum eddy thickness τ versus

* See Appendix A for a brief description of the numerical procedure.

xx

k separates into the upper and lower branches of solutions at the bifurcation point $k^* \doteq .32$. The limit point $k_c \doteq -1.87$ occurs at the cusped eddy. The behavior of γ along the upper branch is shown in Fig. 8, which also shows the manner in which the cusped solution appears. Strictly speaking, the limit point k_c is excluded by the way in which the flow field is considered to be set up, that is, by the separation of two obstacles with positive drag, since $\gamma = 0$ now corresponds to zero drag instead of zero ω^* . However, the cusped eddy is of interest as a possible model for separation and reattachment on a wall.**

The analytic continuation of the upper branch of solutions past $k = k_c$ (and indeed the continuation of the lower branch past $k = 0$) must be rejected in the present model since γ will have a negative region near the end points and (~~since~~ recalling that the flow considered is symmetric in y) must then be multivalued in either half-plane. However, there may exist physical problems where this is not an objection and where the solutions

** The possibility of cusped solutions in the present problem is not associated entirely with the presence of distributed vorticity, since examples are known in the classical theory. Indeed, since in the slender-eddy approximation the velocity in the eddy vanishes at a cusp, insofar as the local behavior is concerned the strength of the vorticity is immaterial and the arguments advanced by Lighthill (Ref. 7) regarding the existence of stagnant, cusped cavities behind obstacles are fully applicable here.

~~physical problems where this is not an objection and where the solutions~~ for $k < k_c$ on the upper branch and $k < 0$ on the lower branch (that is, the full range $k < k^*$) may be admissible solutions.

The curve shown in Fig. 9 sums up the solution for h^*. For non-negative $r < r^* \doteq .71$ there are again two branches of solutions, the bifurcation point r^* being the image of the point on the upper branch at $k = 0$ in Fig. 7. (That is, with respect to r the two branches of solutions are not the same as before.) Corresponding to the two possible values of τ there are two distinct values of k , and therefore two distinct values of h^*. This curve suggests that r might be preferred to k as the parameter for expansion on the lower branch, since more solutions (namely those originally on the upper branch in the interval $0 < k \lesssim k^*$) are obtained for $r < r^*$. However there appears to be no essential advantage in this change of parameters, since in either case only part of the possible solutions is involved.

V. Flow Past a Slender Obstacle. Batchelor's Model

5.1. Formulation. Referring once again to Fig. 5, we now consider the special case

$$Y_A = \eta\left(\frac{1+\lambda+x}{\lambda}\right), \quad \eta(0) = 0, \quad \eta(1) = 1, \quad Y_B = 0, \quad x = 0 \qquad (44)$$

where λ is a positive parameter, corresponding to separated flow past a one-parameter family of geometrically similar pointed, blunt-based, slender obstacles. The eddy downstream of the base terminates at a cusp at $x = +1$. It is evident from (44) that by appropriate changes of scale this family on flows can be reduced to flows past a single obstacle, and the parameter which varies may be then taken conveniently to be the position of the cusp. Therefore, even if we could establish the existence of a unique solution ω^*, h^*, γ for each λ, the theory still does not provide more than a one-parameter family of flows past a given slender obstacle.*

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* This freedom should not be regarded as unexpected, since it is highly unlikely that a purely inviscid theory could yield for a given obstacle a unique value for ω^*.

The governing equations for this case follow from (24) and (25) and are

$$w^*(\theta) = h^* \sin\pi\theta + \frac{1}{8}\omega^{*2}\int_0^1 K(\theta,\theta') \sin\pi\theta' \, w^{*2}(\theta')\, d\theta' + f(\theta;\lambda) \tag{45a}$$

where

$$f(\theta;\lambda) = \frac{2}{\pi\lambda}\int_0^\lambda \tan^{-1}\left(\sqrt{\frac{t}{2+t}}\,\tan\frac{\pi}{2}\theta\right) \eta'\left(\frac{\lambda-t}{\lambda}\right) dt$$

and

$$h^* + \frac{1}{8}\omega^{*2}\int_0^1 w^{*2}(\theta)\, d\theta = \frac{1}{\pi\lambda}\int_0^\lambda \frac{\eta'\left(\frac{\lambda-t}{\lambda}\right)}{\sqrt{t^2+2t}}\, dt = \rho(\lambda) \tag{45b}$$

$$\frac{1}{8}\omega^{*2}\int_0^1 \cos\pi\theta \; w^{*2}(\theta)\, d\theta = -\frac{1}{\pi\lambda}\int_0^\lambda \frac{1+t}{\sqrt{t^2+2t}}\, \eta'\left(\frac{\lambda-t}{\lambda}\right) dt = -\nu(\lambda) \tag{45c}$$

If the transformation $w^* \to w$, $h^* \to k$ introduced in paragraph 4.1 is carried out here, and if (45b) and (45c) are used to eliminate ω^* and k from the transformed (45a), a single equation for w is obtained in the form

$$w = F[w^2] \tag{46a}$$

where

$$F[w^2] = \int_0^1 K(\theta,\theta') \sin\pi\theta' \, w^2(\theta')\, d\theta' - \sin\pi\theta \int_0^1 w^2(\theta')\, d\theta'$$

$$- \frac{1}{\nu(\lambda)} \left[\int_0^1 \cos\pi\theta' \, w^2(\theta')\, d\theta' \right] \left[\rho(\nu) \sin\pi\theta + f(\theta;\lambda) \right] \tag{46b}$$

Note that the equations for Riabouchinsky's model can be put in a form similar to (46), and can be obtained from (46b) by replacing the last term by $r \sin\pi\theta$. A general equation containing both cases takes the form

$$w = F[w^2] + r = F_r[w^2] \tag{47}$$

where r is a given non-negative, continuous function of θ, vanishing at $\theta = 0, 1$. Note that (46) and (47) differ from Riabouchinsky's problem in that the value of w at $\theta = 1$ is no longer known <u>a priori</u>.

5.2. <u>Comments on the existence of solutions</u>. Putting aside the possibility of physically unrealistic (multiply-valued) solutions, the main mathematical model which emerges from our study is now clear: to establish under as general conditions as possible the number of solutions of the nonlinear integral equation $w = F_r[w^2]$ and whenever possible provide a constructive existence proof. Apart from the specific examples worked out in Chapter IV, and the elementary general result stated below, we shall not attempt to provide a rigorous answer to these questions in this report.* However, for the purpose of justifying the usefulness of our numerical investigation, at least heuristically, a few remarks are in order.

As is clear from the construction of the lower branch of solutions in Chapter IV, the finding of solutions sufficiently "small," in the sense of the norm $\|w\| = \max |w|$ $(0 \le \theta \le 1)$, is a simple matter. In fact, if $w = F_r[w^2]$ then $\|w\| \le \|F\|\,\|w\|^2 + \|r\|$, $\|F\|$ being a certain constant, and if we construct a series

$$w = r(\theta) + \sum_{i=1}^{\infty} \|r\|^{i}\, w_i(\theta) \tag{48}$$

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* A mathematical study of the integral equation is in preparation.

in the usual way, it is not difficult to see (proceeding as in § 4.2) that (48) will converge so long as $\|F\|\,\|r\| < 1/4$.

Of course, if r = 0, (48) provides us with only the trivial solution w = 0. We have seen, however, that a second solution may exist and it is the finding of non-trivial solutions that leads us to consider a somewhat more involved mathematical theory. The way in which this occurs, at least in our examples, is analogous to the dependence of roots of the quadratic equation $w = w^2 + r$, upon the constant r. If $0 < r < 1/4$ there are exactly two real roots. For $r < 0$ there is one positive and one negative root, and for $0 < r < 1/4$ two positive roots. For r sufficiently large, all roots are imaginary. Thus we are led to conjecture that the existence of one solution of $w = F_r[w^2]$ (trivial or not) guarantees the existence of at least one other solution (which, of course, may or may not be physically acceptable). In particular the equation (46a) could then possess a non-trivial solution. More complicated examples suggest similar conjectures, even in cases when the kernel of the integral equation is not positive. For example, the linear equation

$$w(\theta) = \int_0^1 \sin \pi(\theta - \theta') \; w(\theta') \, d\theta'$$

has only the trivial solution $w = 0$, while the quadratic form

$$w(\theta) = \int_0^1 \sin \pi(\theta - \theta') \; w^2(\theta') \, d\theta'$$

has in addition the solution $w = -\frac{3}{2\pi} \cos \pi \theta$.

Equations such as the last can be reduced to a finite system of n quadratic equations to be solved for n unknown numbers. Purely algebraic considerations will then always lead to a proof of existence or nonexistence. If n is not finite (such as in our problem where the unknown numbers are the Fourier coefficients) this procedure involves in addition the problem of establishing the convergence of a sequence of truncated series as well as, from a practical point of view, the question of how rapidly the sequence converges.*

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*. These considerations do, however, provide the following interesting result: There are always an even number of real solutions appearing by pairs at points of bifurcation.

It should be observed that, on the basis of the numerical results of Chapter IV, equation (46a) always has one exact solution, namely that corresponding to the symmetric, cusped eddy. In this solution the obstacle has disappeared entirely, and it is clearly applicable only to the limiting case $\lambda = 0$. In this limit the product of bracketed terms in (46b) vanishes and (in effect) the cusp has moved infinitely far downstream. Therefore, in spite of the fact that we have not shown the symmetric solution to be unique among solutions which are cusped at the endpoints, it is plausible that the symmetric cusped eddy always provides an asymptotic development of a one parameter family of solutions of (46a). Our numerical results for the wedge are indeed consistent with this last conjecture.

5.3. A numerical example. We have solved numerically a system of equations equivalent to (46), for the case $\eta(x) = x$, corresponding to a family of eddies attached to a wedge. The method of iteration used to obtain these solutions is straightforward and is summarized in the Appendix. In the present paragraph we describe the results of the calculations.

Values of $Y(x)$, h^*, and ω^* were computed for $\lambda = 1/2$, $\frac{3}{4}$, 1, 2, 5, 10, and ∞ and are presented in Fig. 10 and Table 1.

The variation of k with λ is shown in Fig. 11.* The main feature to be noted concerning the development of the eddy as the reattachment point move aft is that there is a definite trend toward the symmetric, cusped eddy obtained in Chapter IV. This is suggested in Fig. 10, and supported in Fig. 11, by the apparent convergence of k toward k_c. There is therefore reason to believe that the conjecture of §5.2, concerning the asymptotic behavior of long eddies, is valid in the more general way suggested there.

We shall consider the structure of the flow in more detail for the particular case of flow down a step in a wall, for which $\lambda = \infty$, which is sketched in Fig. 12. In particular, the streamline pattern within the eddy and the pressure distribution on the wall downstream of the step computed from the slender-eddy approximation are shown. These are defined by

$$\psi = \frac{\omega}{2}\left(y^2 - \varepsilon y Y\right) \tag{51a}$$

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* Some calculations at $\lambda = 1/4$ were also tried, but convergence in γ was not sufficiently rapid and the results were rejected. However, k converged rapidly and the limiting value was used in Fig. 11.

$$p = -\varepsilon\left(h^* + \tfrac{1}{8}\omega^{*2}Y^2\right) = \frac{\varepsilon\,\omega^{*2}}{24}\left(1-3Y^2\right) \tag{51b}$$

where we have used the identity $h^* = -\dfrac{\omega^{*2}}{24}$ *. We may use (51a) and (51b) in $-1 < x \leq +1$, but neither is uniformly valid at $x = -1$, since at that point the base interferes with essentially a parallel sheared velocity profile. The tangency condition on the base requires that we add to (51) a contribution depending upon $x+1/\varepsilon$, which is negligible when this quantity is large, but which otherwise contains essential terms.

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* This relation can be proved as follows: Consider the force which acts on the wall. The total source strength needed to represent the flow in the harmonic region is just $-\varepsilon$ so that the total (dimensionless) force is exactly ε and acts in the direction of increasing x . Computing now the force on γ , to order ε^2 inclusive, there is obtained

$$\varepsilon\int_{-1}^{+1} Y'(x)\left[1-\varepsilon\left(h^*+\tfrac{1}{8}Y^2\omega^{*2}\right)\right]dx + o(\varepsilon^2) = \left[\varepsilon - \varepsilon^2\left(h^* + \frac{\omega^{*2}}{24}\right) + o(\varepsilon^2)\right]$$

and the identity follows. This direct relation between the two parameters, independent of γ , seems to be peculiar to the case $\lambda = \infty$, since if $\lambda < \infty$ the force on γ depends upon the pressure distribution caused by sources used to represent the obstacle, in a way which retains a dependence upon Y .

The computation of this correction is similar to a "boundary layer" analysis, although it should be noted that the "boundary-layer variables" are now $\bar{x} = {}^{x+1}/_{\epsilon}$, $\bar{y} = {}^{y}/_{\epsilon}$ and therefore the coordinates are stretched in the same way.

Consider then the following problem:

$$\nabla^2 \bar{\psi} = 1 \tag{52a}$$

$$\psi(0,\bar{y}) = 0 \quad , \quad 0 \leq \bar{y} \leq 1 \tag{53b}$$

$$\psi(\bar{x},0) = \psi(\bar{x},1) = 0, \quad \bar{x} \geq 0 \tag{54c}$$

The solution of (51) is

$$\bar{\psi}(\bar{x},\bar{y}) = \frac{4}{\pi^3} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi\bar{y}}{(2n+1)^3} e^{-(2n+1)\pi\bar{x}} + \frac{1}{2}(\bar{y}^2 - \bar{y}) \tag{55}$$

Therefore a solution which is uniformly valid *in* x, in the sense that it contains both (51a) and (55) is

$$\psi_{unit} = \omega \epsilon^2 \bar{\psi}(\bar{x},\bar{y}) + \frac{\omega}{2}(\epsilon y^2 - \epsilon Y y) \tag{56a}$$

The corresponding pressure variation on $y = 0$ or Y is

$$p_{unit} = \frac{1}{8}\omega^{*2}\epsilon^2\left[\frac{1}{3} - \left(Y - \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{e^{-(2n+1)\pi\bar{x}}}{(n+1)^2}\right)^2\right] \tag{56b}$$

while on the rear face we obtain

$$p_{unit} = \frac{\omega^2\epsilon^2}{8}\left[\frac{1}{3} - \frac{64}{\pi^4}\left(\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi y/\epsilon}{(2n+1)^2}\right)^2\right] \tag{56c}$$

Upstream of the point A, and downstream of the point B, the wall pressure decays from $\frac{1}{24}\omega^2\epsilon^2$ to 0, this following from the fact that the pressure varies continuously at these points.

It is clear that along Y near $X = -1$, the slender-eddy theory cannot give $Y(x)$ accurately, the error arising from the exterior pressure field which balances the variation from $-\frac{1}{12}\omega^2\epsilon^2$ to $\frac{1}{24}\omega^2\epsilon^2$ given by (56b). A calculation of the necessary correction is straightforward and will not be dealt with here. We note, however, that the linearized theory will again be sufficient, since the changes in $Y(x)$ will be $O(\varepsilon)$.*

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* The effect on y is therefore nominally $O(\varepsilon^2)$ and if the flow field is to be studied to within terms of this order, we must study as well the second-order terms in the basic slender-eddy expansion, defined in the region $-1 < x \leq +1$.

VI. Discussion

The approximate solutions of Euler's equations considered in this report suggest that the following general conclusions can be drawn concerning the role of finite eddies: (i) The class of flow problems which are solvable can be increased if the structure of the flow is refined by the introduction of an eddy. (ii) The basic new fluid-dynamical problem arises in the non-linear interaction between the eddy and an exterior flow. This interaction can lead to questions which are peculiar to rotational flows, for example, the possibility of a new kind of indeterminacy arising from the branching of the solutions.

In view of the known inadequacies of the theory of irrotational, discontinuous solutions of Euler's equations in the representation of laminar viscous flow at high Reynolds numbers, it is believed that an extended class of solutions which are not necessarily irrotational, and which therefore ~~are associated with properties~~ fall within conclusions (i) and (ii), might be useful in the construction of ~~realistic~~ models of the laminar wake in the limit of infinite Reynolds numbers.

The way in which flows having a new structure are generated has been illustrated by solution in the slender eddy approximation for a family of flows past a given slender wedge. These approximate solutions suggest the existence of exact solutions having the same structure, and to this extent confirm the possibility of the inviscid model proposed by Batchelor (Ref. 1). There remains the prospect that this could actually be demonstrated rigorously by constructing for slender wedges a series which converges to an exact solution for a non-slender wedge. On the other hand, our results say nothing of the relevance of Batchelor's model as limit. The drag experienced by the wedge is necessarily zero and this remains one ~~serious~~ objection which can be raised against this proposal. We have also not considered the values of ω and h which would actually be obtained in the limit. As Batchelor notes in Ref. 1, considered as a limit the length of the wake would most probably be bounded from above by the action of viscous stresses in shear and boundary layers, so that the slender-eddy solutions themselves could provide (if in fact the eddy is sufficiently long) no more than a numerical approximation to the limit.

The generalization of Riabouchinsky's model presented in this report provides us with an example of non-linear interaction leading to branching, although the effect of the eddy does not in this case remove the image obstacle downstream, which is the major objection to the classical model.*

The branching phenomenon itself ~~probably~~ deserves closer attention. It is possible to formulate a variational problem for the symmetric eddies, in which the extrema are maxima or minima depending upon the branch. However, this does not necessarily imply a "neutral stability" at the bifurcation point, since the "neutral disturbance" may not be consistent with constraints imposed by the Euler equations. In our case, it is possible to show that for ω^* fixed the eddy volume is not conserved during deformation from the bifurcation point, unless the distance between the obstacles is simultaneously varied in a suitable way. This means that the only "neutral disturbances" consistent with the

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* Note that in the slender-eddy approximation Riabouchinsky's model is equivalent to the well known re-entr~~r~~ant-jet model, since the discrepancies between them are contained within an infinitesimal neighborhood of the endpoints of the eddy. Interpreted in this way the flow remains two-valued. In either case these inconsistencies at the end-points are to be considered distinct from the global effects of non-linear eddy interaction.

constraint of incompressibility involve a change of separation between the obstacle and its image. Despite the artificiality of this arrangement of flow and obstacles, these results suggest nevertheless that branching will be of importance in any sufficiently general theory of solutions of Euler's equations, and is likely to be a problem of great complexity in instances where a large number of distinct eddies are involved.

An attendant problem in the stationary case centers on the possibility of indeterminacy.* It is not unexpected that this problem arises here, since even in the classical theory uniqueness can be achieved for a given obstacle only by imposing connections at points of attachment, and without such conditions there may exist several and even an infinite number of "acceptable" solutions. However, the indeterminacy we refer to here is different from the classical one. In fact, in the present slender-eddy theory the details of attachment are absorbed into a single matching condition at the obstacle, and the indeterminacy arises not from the local behavior of γ at the obstacle but rather from the global properties of the eddy.

* The corresponding possibility in the non-stationary case is the occurrence of "buckling," i.e., the transition between two stationary flows at the same parameter values, through a continuous sequence of accessible flows.

But can this question of indeterminacy arise in other models of the wake which do not require an image obstacle? We cannot give a firm answer, although our numerical investigation indicates that in Batchelor's model the solution is uniquely determined once the position of the cusp is fixed. Moreover even if a model could be found exhibiting different behavior, it could very well be that the values of the parameters for which there exists more than one branch could never be realized in practice.

We close with a final remark concerning the relation of our work to the Navier-Stokes theory. The point of view of the present report has been that simple examples of inviscid flows containing eddies are useful in the understanding of a large class of solutions of Euler's equations, in spite of the fact that they may not be limits of viscous flows which are of physical interest. Nevertheless, apart from any relation they may have to the limits themselves, there remains the possibility that these or similar solutions could illustrate, in the sense of a mathematical model, properties of the Navier-Stokes solutions which are of definite physical interest. A notable possibility is that the branching of solutions of the Navier-Stokes equations, with respect to the Reynolds number of the flow, could be modeled in this way. Indeed, observation suggests the presence of elongated, almost symmetric eddies (relaxing now our definition) behind bluff bodies at moderate Reynolds numbers. It is therefore plausible that the inviscid model can provide insight into the dynamics of the distributed vorticity (but of course not into the diffusion of vorticity, which however may not be a serious shortcoming in the study of branching). The questions which we raise here are currently under study.

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Appendix. The Numerical Procedure

The numerical solutions of the two problems treated in this report were obtained as follows.* All integrations were performed using Simpson's procedure over 51 points in the interval $0 \leqslant \theta \leqslant 1$. In the case of the upper branch of symmetric eddies, the iterative procedure was precisely that of § 4.2 above, for various values of $\varkappa$, starting with the function $w = w_0 = \sin \pi\theta$. The n^{th} iterate w_n was considered a solution if $\left\| \frac{w_n - w_{n-1}}{w_n} \right\|$ was smaller than 10^{-3}. The value of n depended upon $\varkappa$, and decreased from n = 4 at bifurcation to n = 12 at $\varkappa = -.1$. To obtain the cusped eddy the value of $\varkappa$ was free and determined at each iteration by the cusp condition. This improved convergence and 10 iterations were required. An inspection of the iterates indicated a monotonic approach to a limit, and suggested that more rapid convergence could be realized by arbitrarily adding differences to the iterate and of course by a closer choice of the starting function. These improvements would probably be

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* The author is indebted to W. L. Elkington for the coding and programming of the numerical computations.

useful if the criterion for convergence were decreased to 10^{-4}, and would certainly be essential in order to obtain solutions very near the cusped solution.

The manner in which the wedge flows were obtained is similar, and can be illustrated by considering the case of flow down a step. The iterates were computed from the equation

$$w_{n+1}(\theta) = h^*_{n+1} \sin \pi\theta + \frac{\omega^{*2}_{n+1}}{8} \int_0^1 K(\theta,\theta') \sin\pi\theta' \left[w_n^2(\theta') - w_n^2(\theta)\right] d\theta' + w_n^2(\theta) \sin\pi\theta + \theta$$

where the two parameters ω^*_{n+1} and h^*_{n+1} are obtained from two conditions on w_{n+1} expressing the tangency of γ at the points A and B. Starting with $w_0 = \frac{1}{2}(1-\cos\pi\theta)$ a solution was reached (using the same criterion as before) at the end of seven iterations. In this way the values $h^* = -.418$, $\frac{\omega^{*2}}{8} = 1.252$ were obtained, in close agreement with our independent exact result $h^* = -\frac{\omega^{*2}}{24}$. For finite λ the method was essentially the same, and again it was found that the number of iterations needed to achieve convergence increased as the cusped eddy was approached (i.e., as λ decreased), a maximum number of 22 iterations being required in the most extreme case. It was therefore again concluded that improvements would be needed in order to obtain solutions near the limit.

λ	$-h^*$	$\frac{1}{8}\omega^{*2}$
.50	.79	1.11
.75	.46	1.49
1.00	.38	1.59
2.00	.33	1.55
5.00	.34	1.40
10.00	.36	1.33
∞	.42	1.25

Table 1

Figure Captions

Fig. 1 Euler flows containing one eddy.

Fig. 2 The principal boundary-value problems:

(a) Riabouchinsky's model

(b) Batchelor's model.

Fig. 3 The multiple eddy problem.

Fig. 4 Solutions of the one-dimensional problem.

Fig. 5 A flow containing a slender eddy.

Fig. 6 Riabouchinsky's model in the slender-eddy approximation.

Fig. 7 Maximum thickness of slender eddies.

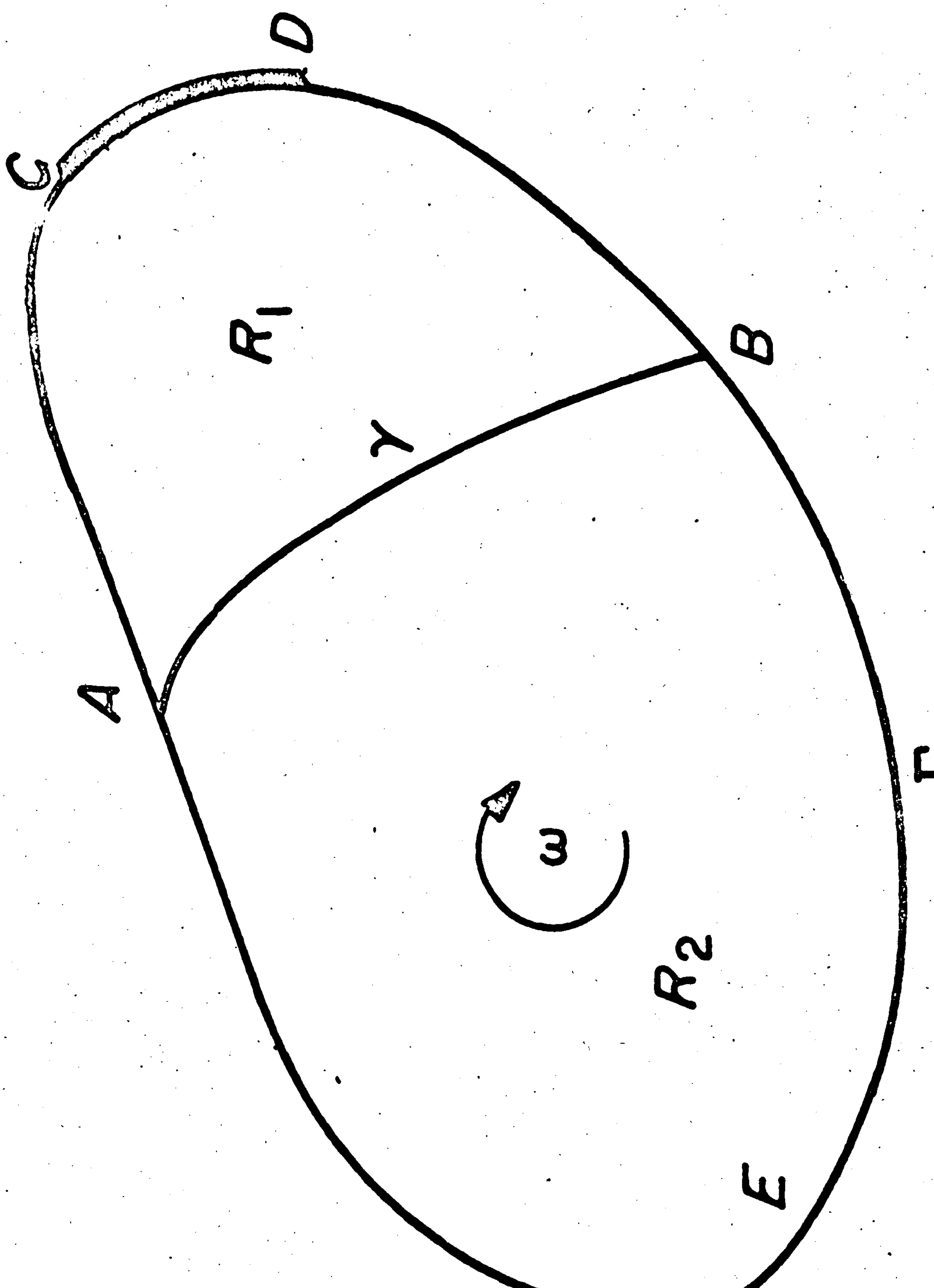
Fig. 8 Eddy boundaries on the upper branch, normalized with respect to maximum thickness.

Fig. 9 Non-uniqueness for $r < r^*$.

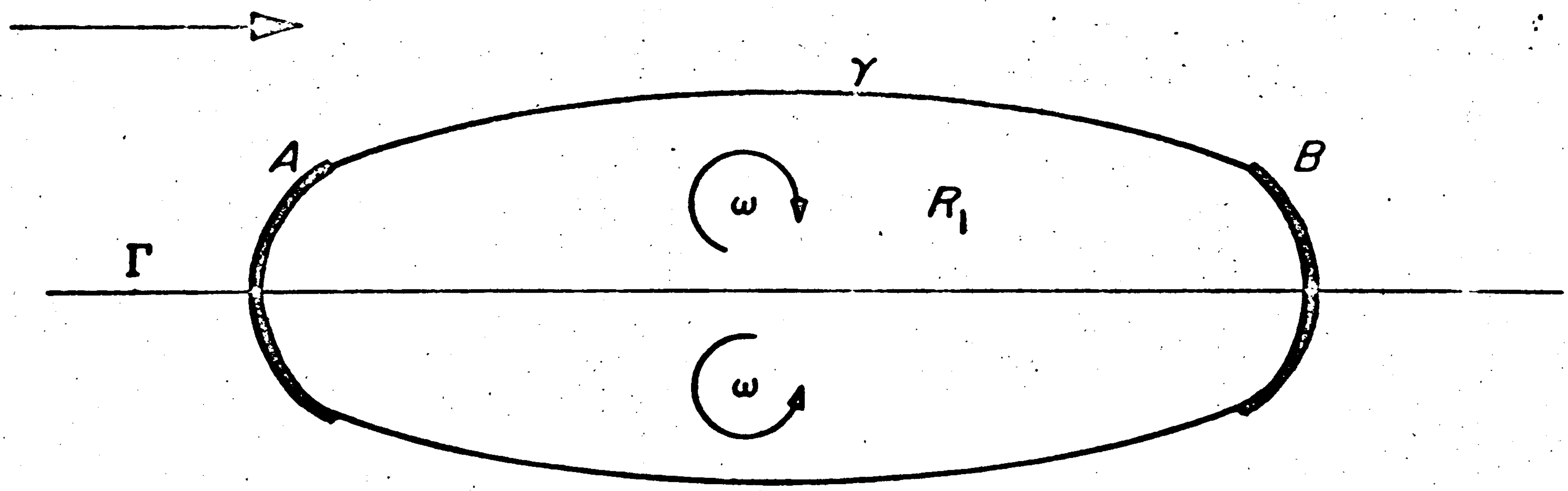
Fig. 10 Eddy boundaries for a wedge.

Fig. 11 Variation of k with λ.

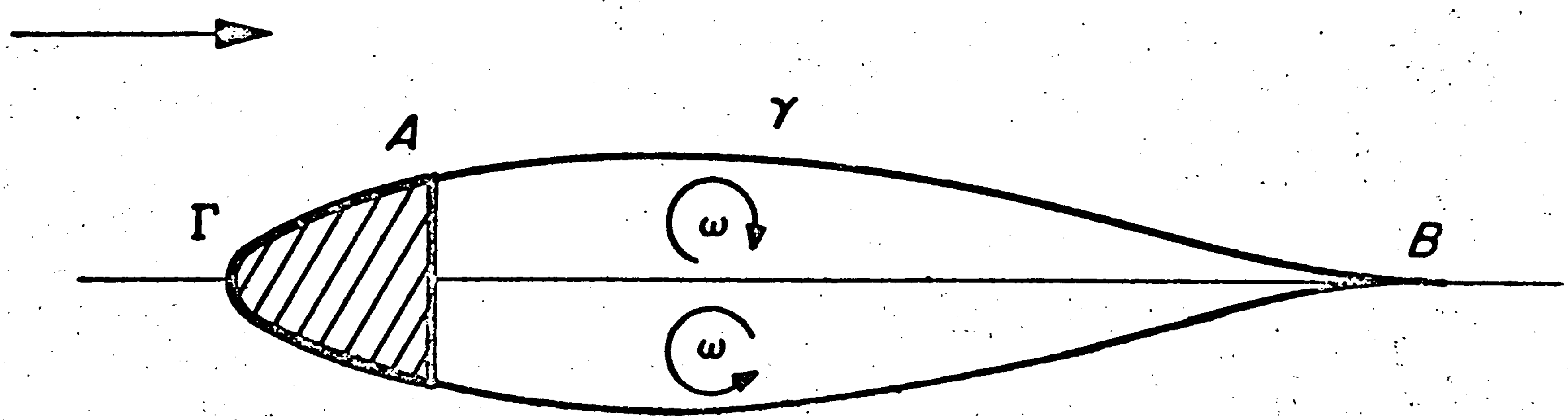
Fig. 12 Flow over a step. The unbroken lines represent the pressure distribution and streamlines derived from the slender-eddy approximation; the dotted lines indicate the effect from the step for an ϵ of about 1/5, and are qualitative only.

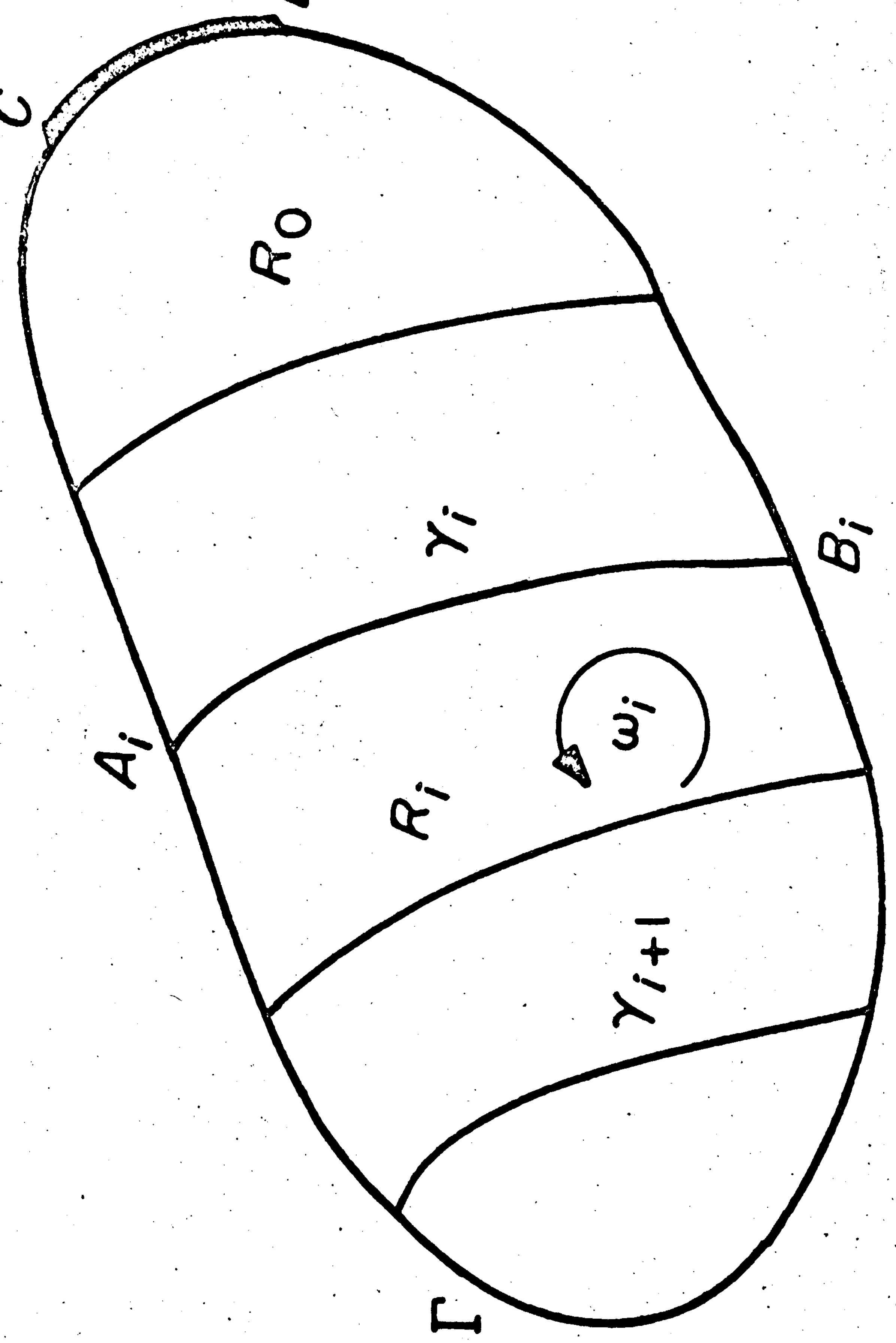

C
D
R_1
B
γ
A
Γ
ω
R_2
E

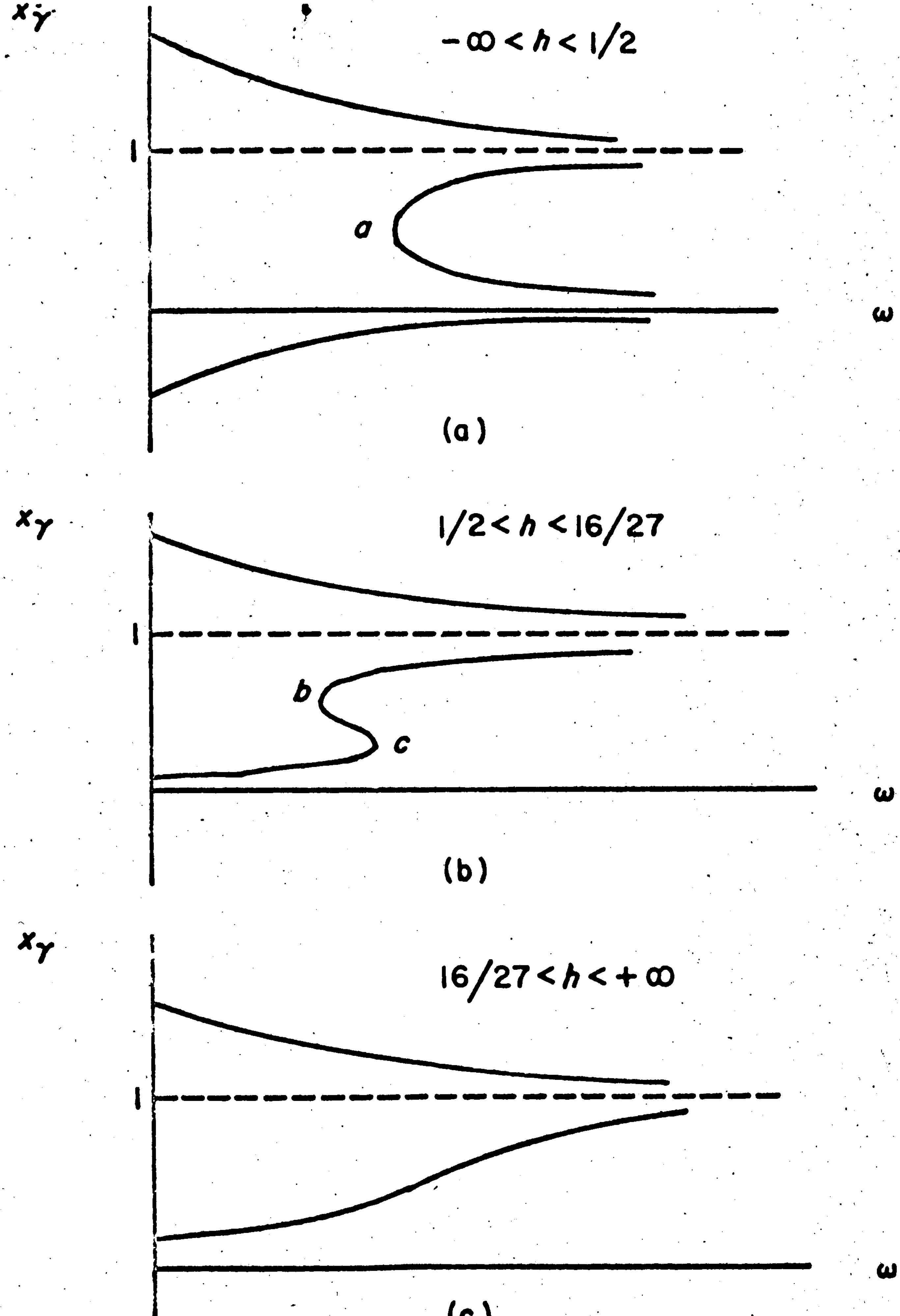
(a)

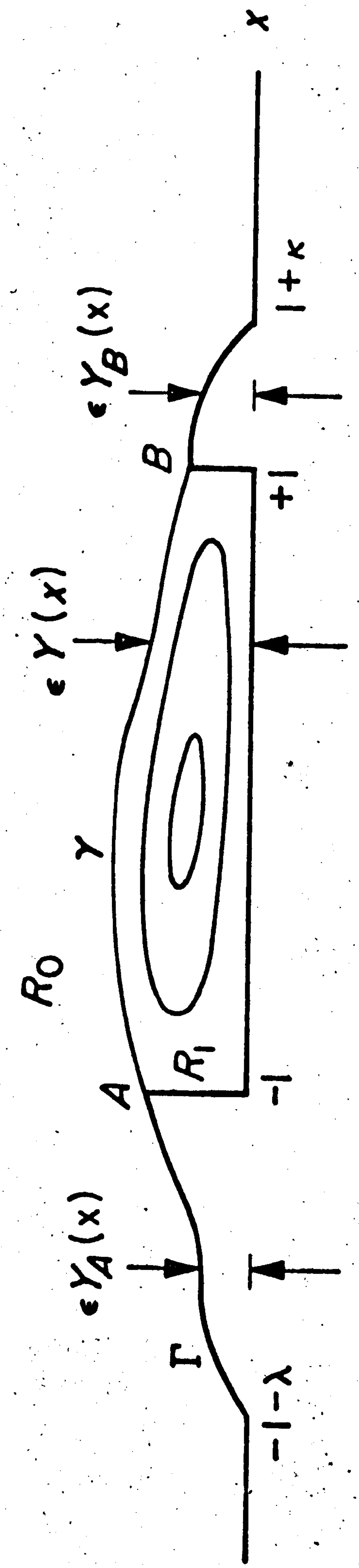


(b)

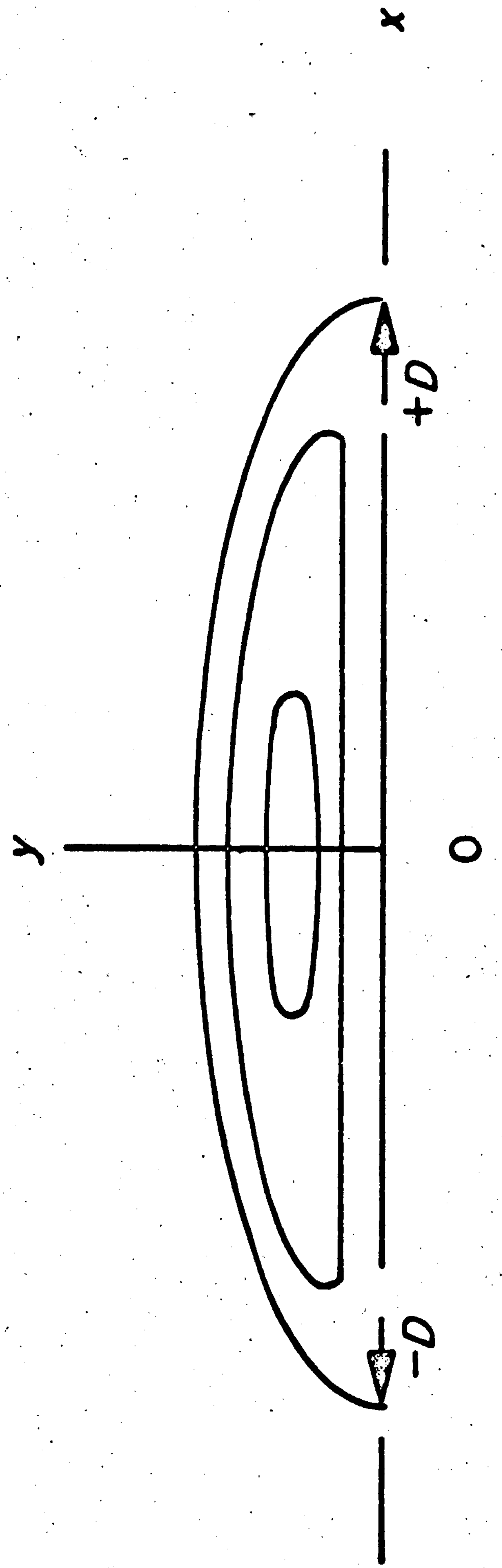


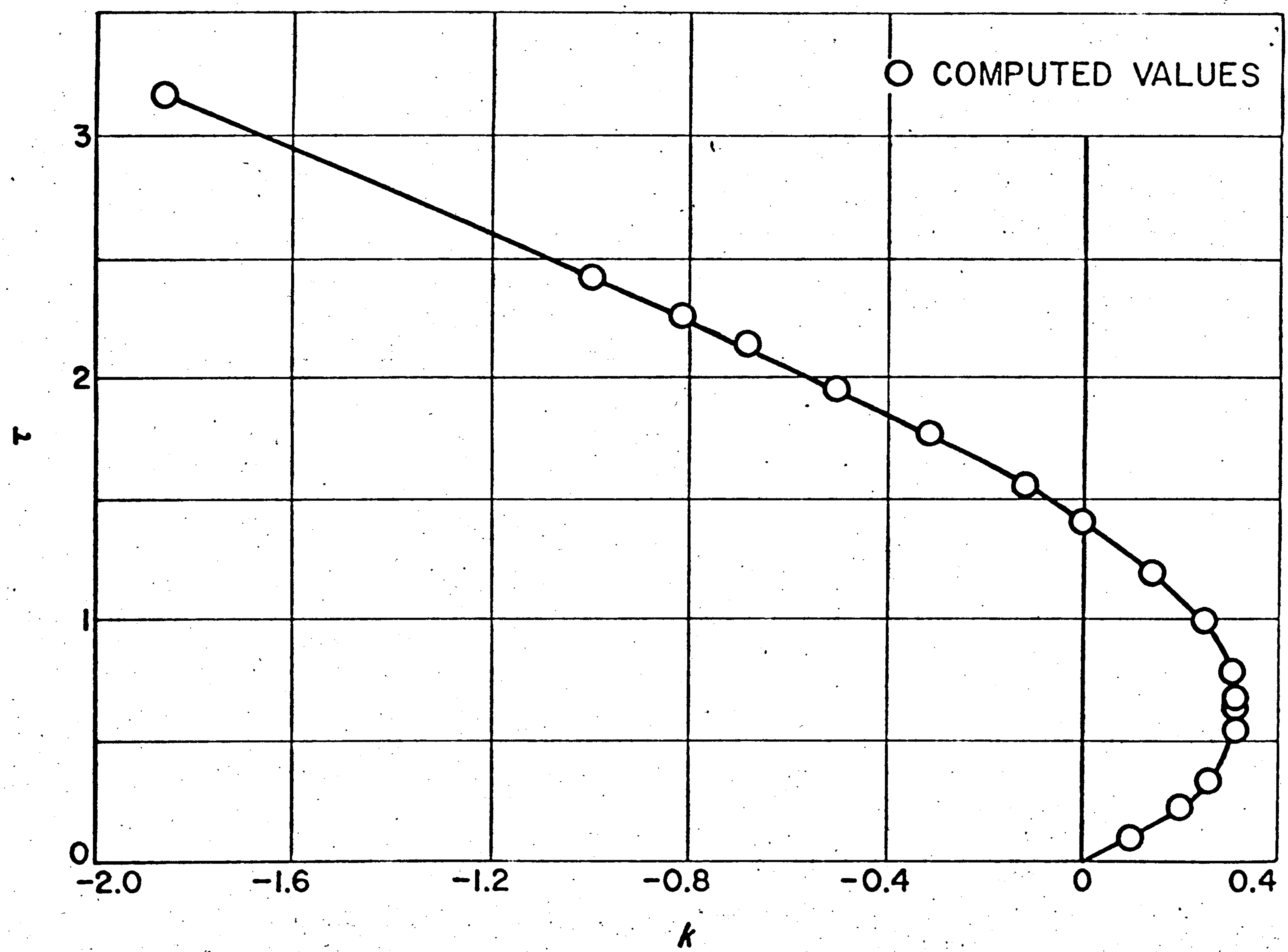

C
D
R_0
γ_i
B_i
A_i
R_i
ω_i
γ_{i+1}
Γ


x_γ
$-\infty < h < 1/2$
1
a
ω
(a)
x_γ
$1/2 < h < 16/27$
1
b
c
ω
(b)
x_γ
$16/27 < h < +\infty$
1
ω
(c)

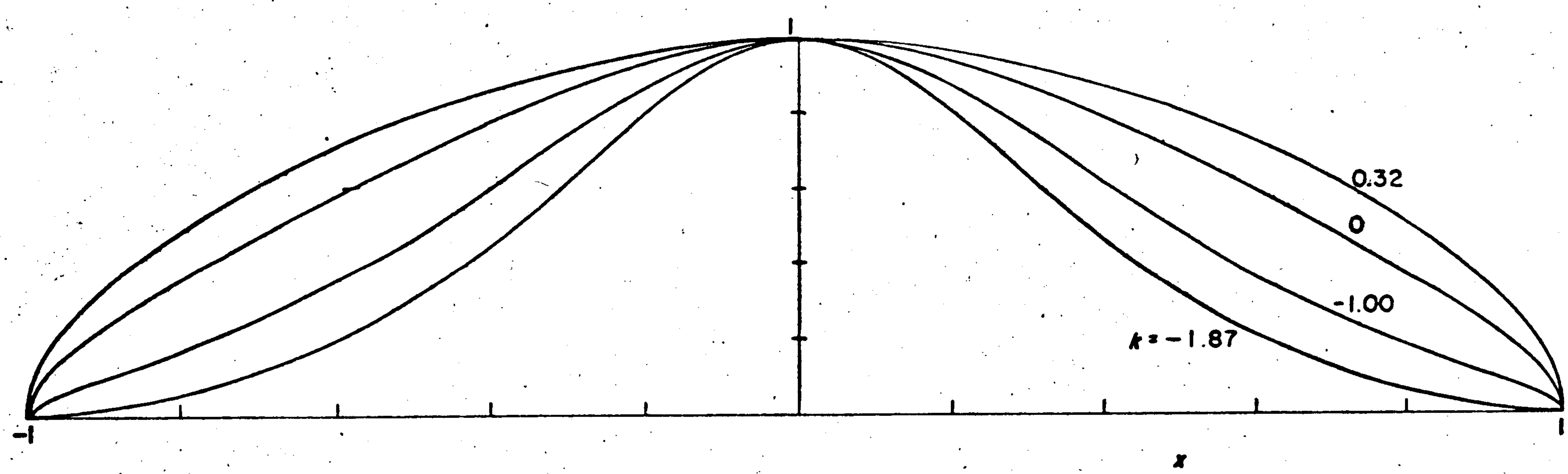


x
1+κ
εY_B(x)
B
+1
εY(x)
Y
R_0
R_1
A
-1
εY_A(x)
Γ
-1-λ

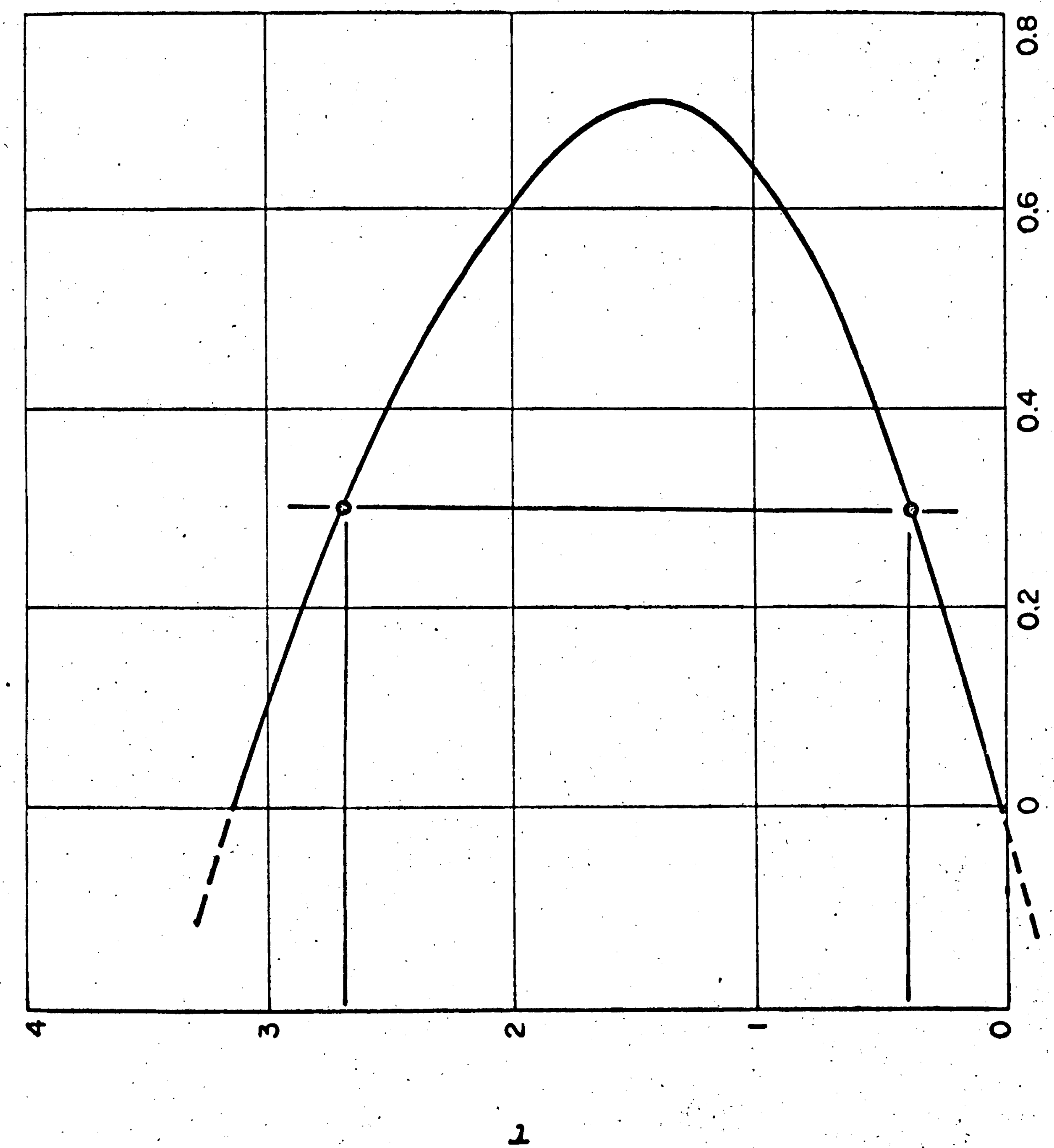

x
y
O
+D
-D

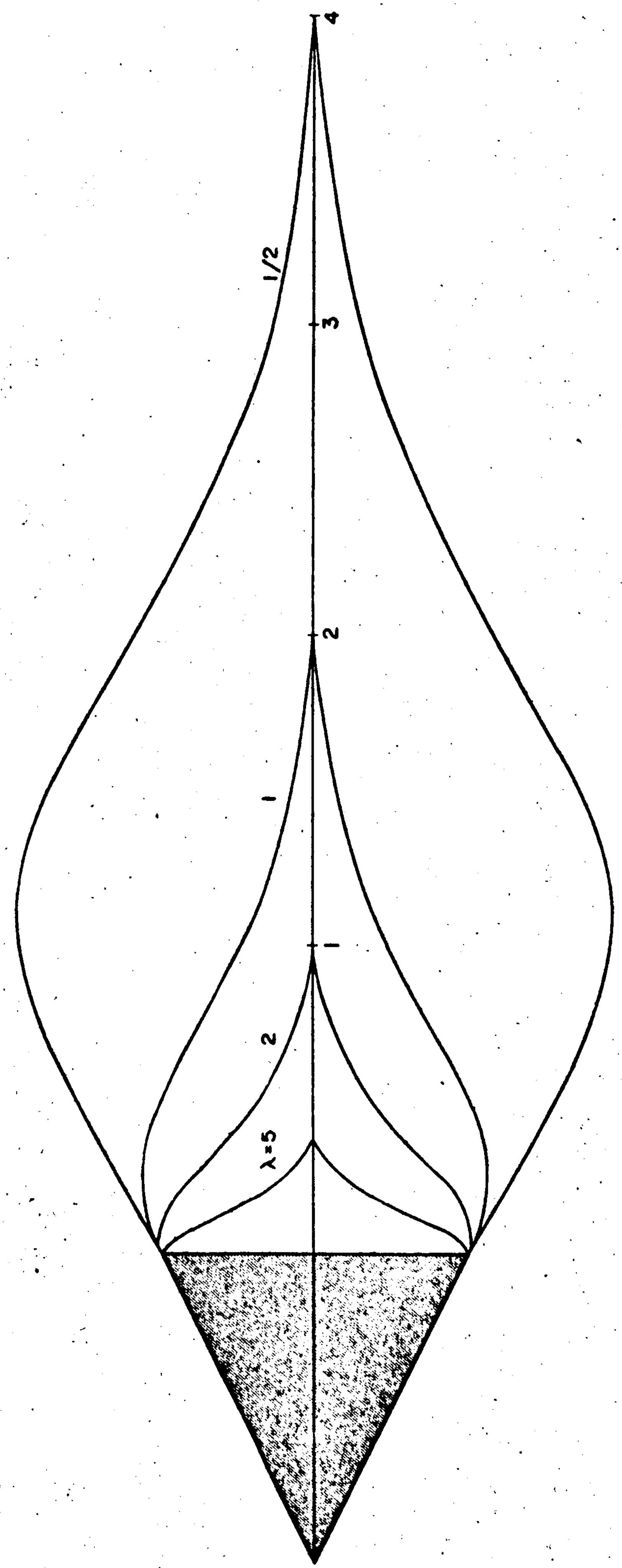


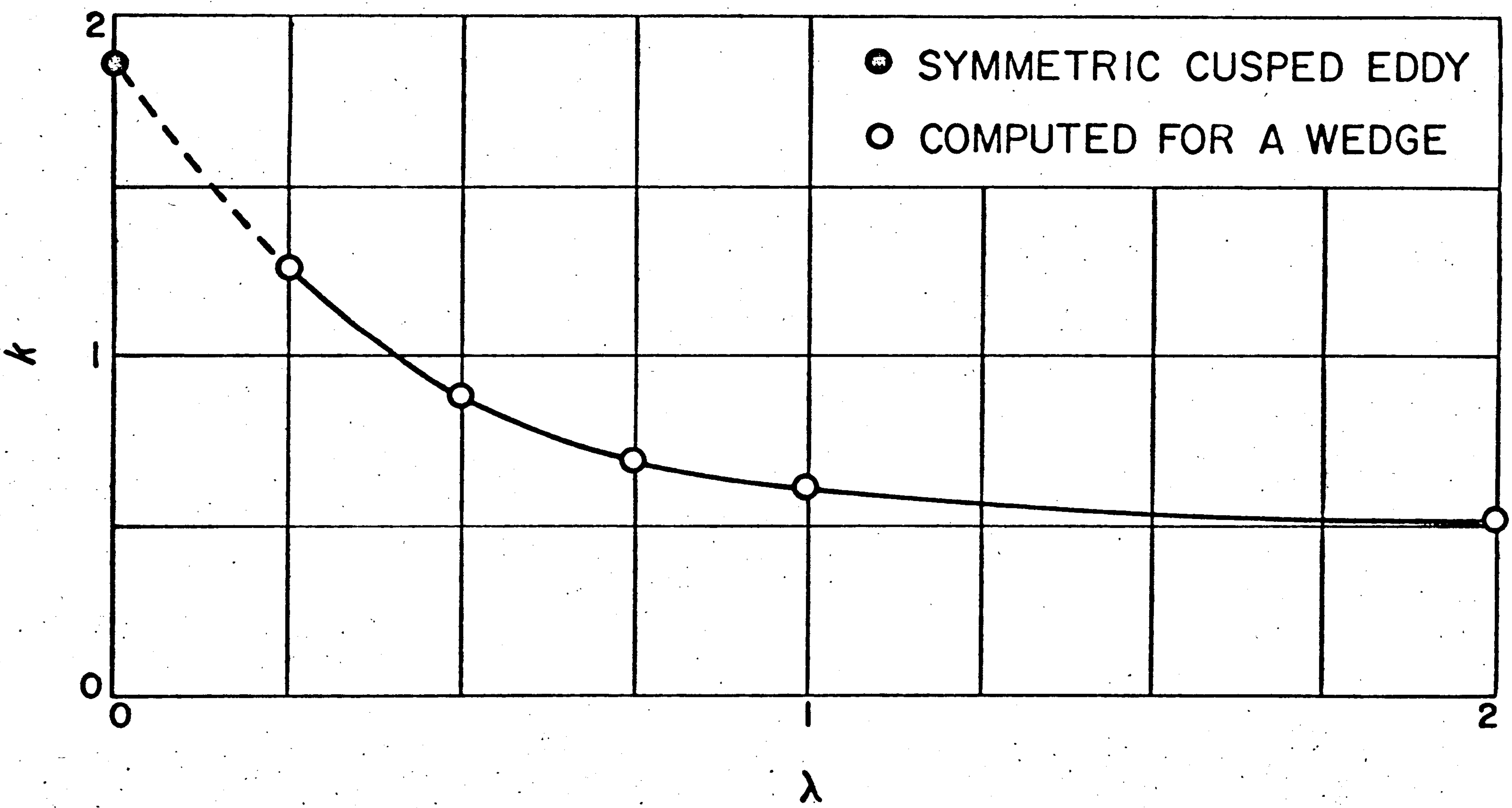
COMPUTED VALUES
3
2
1
0
τ
-2.0
-1.6
-1.2
-0.8
-0.4
0
0.4
k



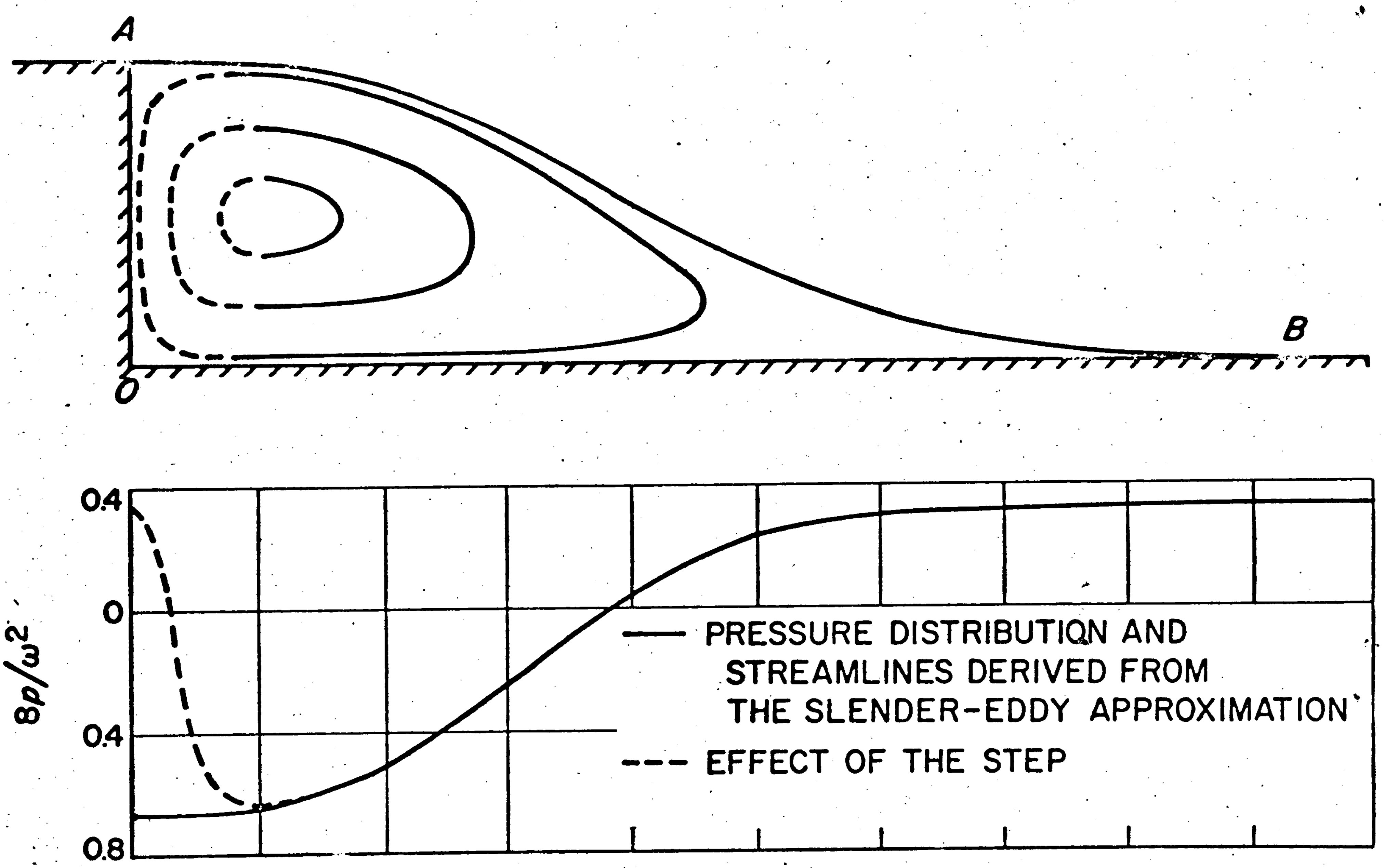
1
0.32
0
-1.00
k = -1.87
-1
1
x


0
0.2
0.4
0.6
0.8
r
0
1
2
3
4


4
3
2
1
1/2
1
2
λ=5



SYMMETRIC CUSPED EDDY
COMPUTED FOR A WEDGE
k
λ
0
1
2


A
B
O
0.4
0
0.4
0.8
8p/ω²
PRESSURE DISTRIBUTION AND STREAMLINES DERIVED FROM THE SLENDER-EDDY APPROXIMATION
EFFECT OF THE STEP