AN INTEGRAL EQUATION APPROACH TO THE PLASMA SELF CONSISTENT FIELD PROBLEM*

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ABSTRACT

The self consistent field problem of a "collisionless", fully ionized, single component plasma is treated by means of a Green's function technique. The latter describes the streaming motion of non-interacting electrons in a time dependent, homogeneous, applied electric field. A formal perturbation series solution to the Liouville equation, obtained by iteration, is then considered in the limit of large number and volume. For a problem time scale of the order of the inverse plasma frequency, only the terms in the perturbation series describing the collective interaction survive the limiting process, and therefore a time-reversible, hierarchy of integral equations is recovered for the various orders of distribution functions. The non-linear integral Vlassov equation follows from the first member of this hierarchy and a factorization assumption on the initial distribution function.

An approximate solution to the Vlassov equation is developed in terms of the solutions to the linearized form of this equation. In these calculations, the Green's function has been simplified

[redacted]

by averaging it over a period of the external field.

Finally, after defining the inverse dielectric function by the relationship between the "dressed" and "free" electron number densities, a correction term of order e^2 to the usual linear theory result is found. It is expected that this correction will be found useful in subsequent studies of non-linear electrical behavior in plasmas.

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An Integral Equation Approach to the Plasma

Self Consistent Field Problem

The collective motion of ionized gases has received much attention in recent years, both on its intrinsic merit as a fascinating aspect of matter, and from a utilitarian standpoint, in application to various plasma experiments. During this period, a number of plasma kinetic theories[1,2,3] have been suggested, and their results applied with varying degrees of success to the large body of experimental data that now exists. Both the inclusion of dissipative collisional mechanisms and the lack of a solution to the full self-consistent field problem have presented formidable barriers to complete theoretical understanding of a fully ionized plasma. However, the linearized self-consistent field problem, first obtained by Vlassov[4] in 1938, has provided great insight into the generation of plasma oscillations, and the stability of plasmas with respect to small perturbations.[5] It is upon this success that further investigation of the full non-linear self consistent field problem is predicated.

Three major results are developed in this paper. The first is the derivation of the full, self-consistent field expression (Vlassov equation) which describes the evolution of the singlet distribution function. The point of attack is based upon the Green's function technique so elegantly exploited by Balescu.[6] In this approach, a formal solution to the Liouville equation is developed in the form of an infinite perturbation series reminiscent of similar series in the quantum theory of scattering. With arguments on the form of this series in the limit of large confining volume and particle number and a choice of the initial distribution, the result may be summed in closed form. By the choice of the representation, however, we avoid the use of diagrammatic schemes to sum the perturbation series, and thus, (hopefully) preserve some clarity of the development. It should be noted, however, that the essential physical arguments of the derivation which limit the valid time regime and the choice of initial functions are identical with those of Balescu.[6]

The second result is the inclusion of a time dependent, externally applied electric field in the Green's function, that, in a certain approximation, may be carried into the solution of the problem.

Finally, the third result is a suggested approximate solution to the full Vlassov equation. This solution is written in terms of a correction to the standard dielectric function derived from the well-known linear Vlassov equation.

1. Derivation of the Integral Vlassov Equation

Let us consider a plasma confined in a large cubical box of volume V. This plasma is idealized as a completely ionized gas consisting of N free electrons and a fixed positive neutralizing background, all in the presence of an external electric field, $\underline{E}(t)$. We define the N particle distribution function in the usual manner in the phase space of the electrons to be:

i. bounded in V

ii. periodic with period $V^{1/3}$ in all coordinates

iii. symmetric under interchange of phases

iv. normalized to unity.

The set of lower order distribution functions are defined as contractions of $f^{(N)}$;

v. $$f^{(s)} = \frac{N!}{(N-s)!} \int \cdots \int d\underline{x}^{N-s} d\underline{v}^{N-s} f^{(N)}$$

The evolution of $f^{(N)}$ is given by the Liouville equation:

$$\left(\frac{\partial}{\partial t} + \sum_{j=1}^{N} \underline{v}_j \cdot \frac{\partial}{\partial \underline{x}_j} - \frac{e}{m_e} \sum_{j=1}^{N} \underline{E}(\underline{r},t) \cdot \frac{\partial}{\partial \underline{v}_j} \right) f^{(N)}$$

$$= \frac{e^2}{4\pi\epsilon_o m_e} \sum_{k<l} \sum \frac{\partial}{\partial \underline{x}_k} \frac{1}{|\underline{x}_k - \underline{x}_l|} \cdot \left(\frac{\partial}{\partial \underline{v}_k} - \frac{\partial}{\partial \underline{v}_l} \right) f^{(N)} \tag{1}$$

where MKS units are used.

The last equation has been written with the streaming terms on the left and the Coulomb interaction on the right. It is convenient to introduce operator notation and write this in the form;

$$L_o f^{(N)}(\tau,t) = \xi L_1 f^{(N)}(\tau,t) \tag{2}$$

where L_o and L_1 are identified with the streaming and scattering operators on the left and right side of Eq. 1 respectively; ξ is the "strength" of the Coulomb repulsion,

$$\xi = \frac{e^2}{4\pi\epsilon_o m_e}$$

and τ represents the set of phases, $\{x_i, v_i\}$, $i = 1, \ldots, N$.

The Green's function associated with the streaming operator, L_o, is defined as a bounded function of the relative velocities and periodic as previously described for the N particle distribution function. It satisfies the following equation;

$$L_o g(\tau,t|\tau',t') = \delta(\tau-\tau')\delta(t-t') \tag{3}$$

In addition the Green's function satisfies the important causal property;

$$g(\tau,t|\tau',t') = 0 \quad \text{if } t<t'$$

The solution to Eq. 2 is obtained by an application of Green's theorem, and is written in terms of the adjoint Green's function. The latter quantity is the solution to an equation similar to Eq. 3, but with the streaming operator replaced by its adjoint. In addition, the adjoint Green's function is anticausal, and hence is a so-called "advanced" solution which relates events to sources before they happen. For this particular case, the adjoint operator is simply the negative of L_o, or if the external electric field is symmetric in time, is equal to $L_o(-t)$. The boundary conditions and a further application of Green's theorem show that the adjoint Green's function is simply the transposed Green's function itself, and thus the solution to Eq. 2 may be written down:

$$f^{(N)}(\tau,t) = \int dt \int d\tau\, K^{-1}(\tau,t|\tau',0)\, f^{(N)}(\tau',0) \tag{4}$$

where the reciprocal kernel $\mathcal{K}^{-1}(\tau,t|\tau',0)$ is defined by the following integral equation:

$$\mathcal{K}^{-1}(\tau,t|\tau',0) = g(\tau,t|\tau',0) + \xi\int dt''\int d\tau'' g(\tau,t|\tau'',t'')\, L_1 \mathcal{K}^{-1}(\tau'',t''|\tau',0) \tag{5}$$

Eqs. 4 and 5 define a complete formal solution to the problem; however, in order that these results may be used to obtain equations for the lowest order distribution functions, we must integrate Eq. 4 over all but a few phases. This procedure requires a knowledge of the contractions of the reciprocal kernel, Eq. 5 and an explicit form of the streaming or "free particle" Green's function, defined by Eq. 3. Let us examine the latter point.

As is well known, the solution to the streaming equation:

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + F\frac{\partial}{\partial v}\right) y = 0 \tag{6}$$

is any function $y(u,w,t,t')$, where

$$u = \left(x' - x + \int_{t'}^{t} v(\tau)\,d\tau\right);\quad w = \left(v' - v + \int_{t'}^{t} F(\tau)\,d\tau\right);\quad (x',v') \text{ is the initial phase point.}$$

and the precise form of y is determined by the initial condition and the boundary conditions. In particular, the Green's function solution to Eq. 3 for a single particle, has the form;[7]

$$g(x,v,t|x',v',t') = \Theta(t-t')\,\delta\left(x'-x+\int_{t'}^{t} v(\tau)\,d\tau\right)\delta\left(v'-v+\int_{t'}^{t} F(\tau)\,d\tau\right) \tag{7}$$

where $\theta(t-t')$ is the Heavyside function. Since the streaming operator, by definition, does not contain interactions between particles, it is clear that the N particle Green's function is simply a product of the individual one particle functions. Thus the integrals of the N particle Green's function over N-s particle phases simply result in an s-particle Green's function.

There is another useful property of importance in the evaluation of Eq. 4. Eq. 1 shows that the Coulomb scattering operator is a sum over all pairs of particles. Let us denote one of these operators as L_{pq}. Now this quantity has the property that if ψ is any function obeying the boundary conditions on $f^{(N)}$;

$$\int d\tau_{i,j,k,\ldots,m} L_{pq} \psi(i,j,k,\ldots,m) = 0 \quad \text{if } p,q \in \{i,j,k,\ldots,m\}$$

where $d\tau_{i,j,k,\ldots,m}$ is the integration over $i,j,k,\ldots,m$ phases. This result follows from the velocity boundary conditions and the form of L_{pq} as a divergence in velocities.

By a straight-forward calculation, using the properties discussed above, and the definition of a contracted distribution function, Eq. v , one may show that the pair distribution function may be written in the form of the following infinite series;

$$f^{(2)}(\alpha,\beta,t) = N(N-1)\int d\tau''_{\alpha\beta}\, g(\alpha,\beta;t,0)\frac{f^{(2)}(\alpha,\beta,0)}{N(N-1)} + N(N-1)\int d\tau''_{\alpha\beta}\int dt''\, g(\alpha,\beta;t,t'')$$

$$\times\Big\{\xi\sum_{j\neq\alpha,\beta}^{N}\int d\tau''_j\,(L_{\alpha\beta;j}+L_{\alpha\beta})\int d\tau'_{\alpha\beta j}\, g(\alpha\beta j;t'',0)\frac{f^{(3)}(\alpha\beta j;0)}{N(N-1)(N-2)} + \xi^2\sum_{j\neq\alpha,\beta}^{N}\int d\tau'_j\,(L_{\alpha\beta;j}+L_{\alpha\beta})$$

$$\times\int dt'''\int d\tau'_{\alpha\beta j}\, g(\alpha\beta j;t'',t''')\sum_{k\neq\alpha,\beta,j}\int d\tau''_k\,(L_{\alpha\beta j;k}+L_{\alpha\beta;j}+L_{\alpha\beta})\int d\tau'_{\alpha\beta jk}$$

$$\times\, g(\alpha,\beta,j,k;t''',0)\frac{f^{(4)}(\alpha,\beta,j,k,0)}{N(N-1)(N-2)(N-3)} + \xi^3\cdots\Big\} \tag{8}$$

The notation used in this equation is the following: Greek letters label the pair of particles, Roman letters are dummy indices; the Green's functions are written in contracted form, and any integration to the immediate left of the latter is understood to operate on the second set of indices. The pair scattering operator notation, $L_{abc\ldots d;g}$ symbolizes

$$L_{abc\ldots d;g} = L_{ag} + L_{bg} + L_{cg} + \cdots + L_{dg}$$

A similar infinite series representation for $f^{(1)}$ is easily obtained in the same manner.

To this point, all manipulations have been formal, and none of the information contained in Eqs. 4 and 5 has been lost. In particular, the set of N expansions for all of the distribution functions is completely equivalent to the original pair of integral equations. In fact, upon comparison of the various series expansions for the different distribution functions, one easily recovers the BBGKY hierarchy equations.[8] In particular, one find that

$$f^{(1)}(\alpha,t) = \int d\tau_\alpha'' \, g(\alpha;t,0)\, f^{(1)}(\alpha,0) + \frac{\xi}{(N-1)} \int d\tau_\alpha'' \int dt'' \, g(\alpha;t,t'') \sum_j \int d\tau_j'' L_{\alpha j} \, f^{(2)}(\alpha j, t') \tag{9}$$

This result, of course, also follows from a straightforward partial integration of the Liouville equation.

Information loss ~~and the~~ eventual summation of the perturbation series representation for the distribution functions follow from consideration of the limit $N \to \infty$, $V \to \infty$, $N/V = n$, and from a statement on the form of the distribution function at $t = 0$. Examination of Eq. 8 shows that each coefficient of ξ^p consists of a series of terms containing no sums over dummy particles, one sum, two sums, etc., up to the order p. Each sum is of order N, and thus the only contribution to the coefficient of ξ^p that survives the limiting process is the one term with the maximum number of sums. After replacing the sums by N minus the number of excluded dummy particles, the remaining infinite series now represents the contribution to the distribution function from the collective interaction, and is the dominant effect in a dilute plasma. The validity of the limiting procedure does depend on the time scale however; in particular it is valid for times short with respect to a hydrodynamic time scale. Terms which give the growth and decay of correlations are not accounted for, as they contribute only to the perturbation series at later times and are lost in the limiting procedure.

Let us digress briefly on the matter of time scales. The collective motion of a plasma is associated with a characteristic time made up of the constants; the electron mass, the number density and the square of the charge. The time, which is the period of the longitudinal oscillations of the electrons against the fixed ion background is

$$t_P = \left(\frac{m_e \epsilon_0}{n e^2}\right)^{1/2} 2\pi$$

On the other hand, the characteristic diffusion (or hydrodynamic) time for electrons executing Brownian motion in a positive ion lattice is

$$t_h \sim m_e^{1/2} (kT)^{3/2} \epsilon_0 / (n e^4)$$

Now, from these two time scales, we may define the plasma temperature and mean number density such that the ordering

$$t_p < t_h$$

holds which, in turn, implies that

$$1/(n\lambda_D^3) < 1$$

where

$$\lambda_D = \left(\frac{\epsilon_0 kT}{ne^2}\right)^{1/2}$$

is the characteristic length (Debye length) for separation of the electrons from the ion background.

The dimensionless cluster, $1/n\,\lambda_D^3$ is the conventional plasma coupling parameter and corresponds to ξ in a dimensionless theory. Since we assume the ordering given above as defining the mean state of our plasma, we shall also consider ξ to be a "small" parameter, even though in a true sense this is meaningless. We note also that ξ is the only parameter explicitly appearing in our treatment, which again is a result of the choice of the particular perturbation treatment.

The growth and decay of naturally occurring correlations during the evolution of a plasma from its initial state gradually obliterates the "memory" of this state, and determines the plasma's irreversible behavior. A description of the plasma is simple only at times before correlations have developed or changed; or when a gas is very near equilibrium and the correlation pattern is changing very slowly. In the present treatment, we consider the former case. This analysis then leads to time reversible equations which do not approach equilibrium. At most such a description gives the streaming of electrons in an external and self consistent field. In fact, our perturbation series may be interpreted as the perturbation of the streaming trajectories of electrons in an applied field due to the self consistent contribution. Under the restrictions discussed above, the most general self-consistent field equation describing the evolution of the singlet distribution function is the non-linear Vlassov equation, and it is our next task to prove that this equation follows from a summation of the perturbation expansion.

Let us turn now to the results of our limit procedure. We may write the solution for an s-order distribution function in the following compact form:

$$f^{(S)}(\alpha_1,\ldots,\alpha_S,t) = \int dt' \int d\tau'_{\alpha_1,\ldots,\alpha_S}\, g(\alpha_1,\ldots,\alpha_S;t,t') \sum_{p=0}^{\infty} \xi^p K^{(p)}(\alpha_1,\ldots,\alpha_s|\ell_1,\ldots,\ell_p;t'_p)\, f^{(s+p)}(\alpha_1,\ldots,\alpha_s,\ell_1,\ldots,\ell_p,0)$$
$$\text{for all } s\geq 1 \qquad (10)$$

where the sum over p is the expansion of the reciprocal kernel in powers of the coupling parameter, and connects particle states $\alpha_1,\ldots,\alpha_s$ at time t' with states $\alpha_1,\ldots,\alpha_s,\ell_1,\ldots,\ell_p$ at t = 0. The kernel iterates, in turn are defined by the following:

$$K^{(0)}(\alpha_1,\ldots,\alpha_s|t,t') = \delta(t-t') \qquad (11\text{-a})$$

$$K^{(1)}(\alpha_1,\ldots,\alpha_s|\ell_1;t,t') = \mathcal{U}(\alpha_1,\ldots,\alpha_s;\ell_1|\alpha_1,\ldots,\alpha_s\,\ell_1;t,t') \qquad (11\text{-b})$$

$$K^{(p)}(\alpha_1,\ldots,\alpha_s|\ell_1,\ldots,\ell_p;t,t') = \int dt''\, \mathcal{U}(\alpha_1,\ldots,\alpha_s;\ell_1|\alpha_1,\ldots,\alpha_s\ell_1;t,t'')\, K^{(p-1)}(\alpha_1,\ldots,\alpha_s\ell_1|\ell_2,\ldots,\ell_p;t'',t') \qquad (11\text{-c})$$

The quantity $U(\alpha_1,\ldots,\alpha_s;\ell_1|\alpha_1,\ldots,\alpha_s\ell_1;t,t'')$ appearing in the latter equation is defined as

$$\mathcal{U}(\alpha_1,\ldots,\alpha_s;\ell_1|\alpha_1,\ldots,\alpha_s\ell_1;t,t'') = \int d\tau_{\ell_1}\, L_{\alpha_1,\ldots,\alpha_s;\ell_1} \int d\tau_{\alpha_1,\ldots,\alpha_s\ell_1}\, g(\alpha_1,\ldots,\alpha_s\ell_1;t,t'') \qquad (12)$$

and may be interpreted as the propagation of the cluster from time t'' to time t and then interaction of particle ℓ_1

with the remaining $\alpha_1,\ldots,\alpha_s$ particles as read from right to left. The properties 11-a), 11-c) and Eq. 12 may be used once in Equation 10 to give, after the relabelling $p-1\to p$!

$$f^{(s)}(\alpha_1,\ldots,\alpha_s,t)=f_0^{(s)}(\alpha_1,\ldots,\alpha_s,t)+\xi\int d\tau_{\alpha_1,\ldots,\alpha_s}\int dt'\,g(\alpha_1,\ldots,\alpha_s;t,t')\int d\tau_{l_1}\,L_{\alpha_1,\ldots,\alpha_s;l_1}$$

$$\times\left\{\int dt''\int d\tau_{\alpha_1,\ldots,\alpha_s l_1}\,g(\alpha_1,\ldots,\alpha_s l_1;t',t'')\sum_{p=0}^{\infty}\xi^p K^{(p)}(\alpha_1,\ldots,\alpha_s l_1|l_2,\ldots,l_{p+1};t'')\,f^{(p+s+1)}(\alpha_1,\ldots,\alpha_s l_1,\ldots,l_{p+1},0)\right\} \tag{13}$$

where $f_0^{(s)}$ is the unperturbed distribution function obtained by integrating the s-particle Green's function multiplied by the initial distribution. In the above equation, the quantity in the braces is just $f^{(S+1)}$ from Eq. 10. Thus we have shown that the choice of perturbation expansion retains the BBGKY hierarchy. This result is consistent with the time reversibility of the plasma description.

We shall now consider the pair of equations for the singlet and pair distribution functions as follows:

$$f^{(1)}(\alpha,t)=f_0^{(1)}(\alpha,t)+\xi\int d\tau_\alpha'\int dt'\,g(\alpha;t,t')\int d\tau_{l_1}\,L_{\alpha l_1}\,f^{(2)}(\alpha l_1;t') \tag{14}$$

and

$$f^{(2)}(\alpha l_1;t)=f_0^{(2)}(\alpha l_1,t)+\xi\int dt'\int d\tau'_{\alpha l_1}\,g(\alpha l_1;t,t')\int d\tau'_{l_2}\,L_{\alpha l_1;l_2}\int dt''\int d\tau''_{\alpha l_1 l_2}$$

$$\times\, g(\alpha l_1 l_2;t',t'')\sum_{q=0}^{\infty}\xi^q K^{(q)}(\alpha l_1 l_2|l_3,\ldots,l_{q+2};t'')\,f^{(q+3)}(\alpha l_1,\ldots,l_{q+2};0) \tag{15}$$

As stands, this set of equations is not closed; but becomes so if an additional argument on the form of the initial distribution function is made. By analogy with an equilibrium cluster expansion, $f^{(q+3)}$ may be expanded in powers of ξ as follows:

(16)

$$f^{(q+3)}(0) = \prod_{k=0}^{q+3} f^{(1)}(l_k, 0) + \xi \sum_{\text{all pairs } i,j} g^{(2)}(l_i, l_j; 0) \prod_{k=0 \neq i,j}^{q+3} f^{(1)}(l_k, 0) + \xi^2 \cdots \quad (l_0 \equiv \alpha)$$

Higher terms represent all possible patterns of three, four, etc., particle correlations. Thus to lowest order in the coupling parameter;

$$f^{(q+3)}(0) = \prod_{k=0}^{q+3} f^{(1)}(l_k, 0) \qquad (17)$$

The rejection of terms involving correlations is consistent with our assumption that ξ is a "small" parameter which in turn, rests upon the time scale, number density and temperature of the plasma as described above.

Now from Eqs. 15 and Eq. 17, we may prove that for the choice of the perturbation series valid for times short with respect to hydrodynamic times, the pair distribution function factors if the initial distribution function is a product of singlet functions. Eq. 15 may be written in terms of the U operators explicitly, but for our purposes it is more convenient to regroup the perturbation series as:

(gL)(gL)(gL) ....

We indicate this grouping by the letter W and also define:

$$f^{(2)}(l_0 l_1, t) = \lim_{M \to \infty} {}^M f^{(2)}(l_0 l_1, t)$$

where:

$$ {}^M f^{(2)}(l_0, l_1, t) = f_0^{(1)}(l_0, t) f_0^{(1)}(l_1, t) + \sum_{q=1}^{M} \xi^q \int dt_1 \cdots \int dt_q \, W(l_0 l_1 | l_0 l_1; l_2; t, t_1) \qquad (t_0 \equiv t)$$

$$\times W(l_0 l_1 l_2 | l_0 l_1 l_2; l_3; t_1, t_2) \cdots W(l_0 l_1 \cdots l_q | l_0 l_1 \cdots l_q; l_{q+1}; t_{q-1}, t_q) \prod_{k=0}^{q+1} f_0^{(1)}(l_k, t_q) \tag{18}$$

and $f_o^{(1)}$ is given by:

$$f_0^{(1)}(\alpha, t) = \int d\tau_\alpha' g(\alpha; t) f^{(1)}(\alpha, 0) \tag{19}$$

We now show that W operating on the product of $f^{(1)}$'s preserves their factorization.

To begin with, we observe that Eq. 18 gives $f^{(2)}$ as a limit in M where the latter indicates the total number of scattering events (maximum power of ξ) among the progenitor (dummy) particles giving rise to $f^{(2)}$ and does not depend on the order in which these events take place. Eq. 19 is the special case of a distribution function unmodified by coulomb scattering, and may be generalized in definition in the following way: If m is an integer giving the total number of scattering events among particle α's progenitors (including the interaction explicitly shown), then:

$$f_{(m)}^{(1)}(\alpha, t) = m^{-1} \sum_{p=0}^{m-1} \int dt' \int d\tau_\alpha' g(\alpha; t, t') \int d\tau_l L_{\alpha l} f_{(m-p-1)}^{(1)}(\alpha, t') f_{(p)}^{(1)}(l, t') \qquad m \geq 1 \tag{20}$$

The sum with the constant weight of 1/m in the definition gives each way of forming $f^{(1)}_{(m)}$ from the previous m-1 interactions an equal contribution. This expression is a natural extension of Eq. 19 in light of the independence of the limit M on the particular sequence of interactions giving rise to $f^{(2)}$. Since the integer m counts the number of coulomb interactions, it is the power of ξ that appears in the infinite series expansion of $f^{(1)}$:

$$f^{(1)}(\alpha,t) = \sum_{q=0}^{\infty} \xi^{q} f^{(1)}_{q}(\alpha,t) \tag{21}$$

The proof of the theorem then consists in showing that $f^{(2)}$ factors in a product of two such series.

Consider the term in Eq. 18 for q = p. From right to left, the last W operating on the product of initial distribution functions is:

$$\int dt_p\, W(l_0,\dots,l_p \,|\, l_0\, l_1,\dots,l_p; l_{p+1}; t_{p-1}, t_p) \prod_{k=0}^{p+1} f^{(1)}_{0}(l_k, t_p)$$

written out in full, this term becomes:

$$\int dt_p \int d\tau'_{l_0,\dots,l_p}\, g(l_0,\dots,l_p; t_{p-1}, t_p) \int d\tau_{p+1}\, L_{l_0\dots l_{p+1}} \prod_{k=0}^{p+1} f^{(1)}_{0}(l_k, t_p)$$

$$= \int dt_p \int d\tau'_{l_0,\dots,l_p}\, g(l_0,\dots,l_p; t_{p-1}, t_p) \sum_{m=0}^{p} L_{l_m l_{p+1}} \prod_{k=0}^{p} f^{(1)}_{0}(l_k, t_p)\, f^{(1)}_{0}(l_{p+1}, t_p)$$

$$= \sum_{j=0}^{p} \prod_{\substack{k=0\\ \neq j}}^{p} f^{(1)}_{0}(l_k, t_{p-1}) \int dt_p \int d\tau'_{l_j}\, g(l_j; t_{p-1}, t_p) \int d\tau_{p+1}\, L_{l_j l_{p+1}}\, f^{(1)}_{0}(l_k, t_p)\, f^{(1)}_{0}(l_{p+1}, t_p)$$

$$= \sum_{j=0}^{p} \prod_{\substack{k=0\\ \neq j}}^{p} f^{(1)}_{0}(l_k, t_p)\, f^{(1)}_{(1)}(l_j, t_{p-1})$$

Thus we see that the distribution function remains factored under the operation of W. By employing the definition, Eq. 20, we may repeat the operation r times to give:

$$\int dt_1 \cdots \int dt_{p-r}\, W(l_0 l_1 | l_0 l_1; l_2; t, t_1) W(l_0 l_1 l_2 | l_0 l_1 l_2; l_3; t_1, t_2) \cdots$$

$$W(l_0 l_1, \ldots, l_{p-r} | l_0 l_1, \ldots, l_{p-r}; l_{p-r+1}; t_{p-r+1}, t_{p-r}) \sum_{(\Sigma a_k = r)} \prod_{k=0}^{p-r+1} f^{(1)}_{(a_k)}(l_k, t_{p-r})$$

where the quantity

$$\sum_{(\Sigma a_k = r)}' \prod_{k=0}^{p-r+1} f^{(1)}_{(a_k)}(l_k, t_{p-r})$$

is the sum of all products of $f^{(1)}$ such that in each term the sum of subscripts equals r. Since r is arbitrary, we may take r = p to give the final result:

$$p\text{-term} = \sum_{(a_0+a_1=p)}' \prod_{k=0}^{1} f^{(1)}_{(a_k)}(l_k, t) = \sum_{j=0}^{p} f^{(1)}_{(p-j)}(l_0, t) f^{(1)}_{(j)}(l_1, t) \qquad (22)$$

Therefore Eq. 18 may be written:

$$\begin{aligned} {}^{M}f^{(2)}(l_0, l_1, t) &= f^{(1)}_0(l_0, t) f^{(1)}_0(l_1, t) + \sum_{q=1}^{M}{}' \xi^q \sum_{j=0}^{q}{}' f^{(1)}_{(q-j)}(l_0, t) f^{(1)}_{(j)}(l_1, t) \\ &= \sum_{q=0}^{M}{}' \xi^q \sum_{j=0}^{q}{}' f^{(1)}_{(q-j)}(l_0, t) f^{(1)}_{(j)}(l_1, t) \end{aligned}$$

or

$$\begin{aligned} f^{(2)}(l_0 l_1, t) &= \lim_{M\to\infty} {}^{M}f^{(2)}(l_0 l_1 t) = \lim_{M\to\infty} \sum_{q=0}^{M}{}' \xi^q \sum_{j=0}^{q}{}' f^{(1)}_{(q-j)}(l_0, t) f^{(1)}_{(j)}(l_1, t) \\ &= \left(f^{(1)}_0(l_0, t) + \xi f^{(1)}_{(1)}(l_0, t) + \xi^2 f^{(1)}_{(2)}(l_0, t) + \cdots + \right)\left(f^{(1)}_0(l_1, t) + \xi f^{(1)}(l_1, t) + \cdots + \right) \\ &= f^{(1)}(l_0, t) f^{(1)}(l_1, t) \end{aligned} \qquad (23)$$

and the theorem is proved. Although we do not use the fact, it may also be shown that this theorem, along with the general BBGKY equation, Eq. 13, shows that the distribution function to any order is factored for all times in which the treatment is valid.

2. An Approximate Solution of the Non-linear Vlassov Equation.

The Vlassov equation follows upon inserting the factored $f^{(2)}$ into the first BBGKY equation, Eq. 14, to give:

$$f^{(1)}(\alpha,t) = f_0^{(1)}(\alpha,t) + \xi\int dt'\int d\tau'_\alpha\, g(\alpha;t,t')\int d\tau_{\ell_1} L_{\alpha\ell_1} f^{(1)}(\alpha,t') f^{(1)}(\ell_1,t') \qquad (24)$$

The solution of this equation is obtained formally by summing the series, Eq. 21, with all the iterates explicitly written. However, there is no known way of representing this series in closed form. Therefore, the Vlassov equation is usually linearized in $f^{(1)}$ by replacing $f^{(1)}(\alpha,t')$ in the integral on the r.h.s. of Eq. 23 by $n\varphi(v_\alpha)$, where $\varphi(v_\alpha)$ is an arbitrary function of the velocity magnitude and is normalized to unity. The resulting linearized equation

$$f^{(1)}(\alpha,t) = f_0^{(1)}(\alpha,t) + n\xi\int dt'\int d\tau'_\alpha\, g(\alpha;t,t')\int d\tau_{\ell_1} L_{1\alpha\ell_1}\varphi(v_\alpha) f^{(1)}(\ell_1,t') \qquad (25)$$

has been studied extensively.[5]

The linearized Vlassov equation may be obtained less arbitrarily by linearizing the recursion relation, Eq. 20. If we suppose that the plasma is nearly spacially homogeneous, we may write

$$f^{(1)}_{(m-1)}(x,v,t) = n\varphi(v) + u_{(m-1)}(x,v,t)$$

where $\mathcal{U}_{m-1}(\underline{x},\underline{v},t)$ is a "small" quantity. Then if we omit terms of second order in u on the right of Eq. 20, we may define $\tilde{f}^{(1)}_{(m)}$ by the recursion

$$\tilde{f}^{(1)}_{(m)}(\alpha,t) = n\int dt\int d\tau_\alpha' g(\alpha;t,t')\int d\tau_\ell L_{1\alpha\ell}\,\varphi(\underline{v}_\alpha)\,\tilde{f}^{(1)}_{(m-1)}(\ell,t') \qquad (m\geq 1) \tag{26}$$

But this equation is just the iterated form of Eq. 25, and it is clear therefore, that the sum

$$\tilde{f}^{(1)}(\alpha,t) = \sum_{q=0}' \xi^q \tilde{f}^{(1)}_{(q)}(\alpha,t)$$

is the solution to the linear Vlassov equation.

As a second approximation to the solution of the full Vlassov equation (regarding the linear equation as the first), it is suggestive to replace the recursion for $f^{(1)}_{(m)}$, Eq. 20, by its linearized relative, Eq. 26. The proof of the factorization proceeds as before, and we find that the pair of equations

$$\tilde{f}^{(1)}(\alpha,t) = f_0^{(1)}(\alpha,t) + n\xi\int dt'\int d\tau_\alpha' \, g(\alpha;t,t')\int d\tau_\ell \, L_{\alpha\ell}\,\varphi(v_\alpha)\,\tilde{f}^{(1)}(\ell,t') \tag{27}$$

and

$$f^{(1)}(\alpha,t) = f_0^{(1)}(\alpha,t) + \xi\int dt'\int d\tau_\alpha' g(\alpha;t,t')\int d\tau_\ell \, L_{\alpha\ell}\,\tilde{f}^{(1)}(\alpha,t')\,\tilde{f}^{(1)}(\ell,t') \tag{28}$$

give the new result. Since the second equation is explicitly dependent on the solution of the first, the set is uncoupled if solved in order.

There is one additional difficulty that prevents a solution of the pair of equations given above by standard methods; namely, the presence of the time dependent electric field in the Green's function. However, if we assume the incident field frequency to be higher than the plasma frequency, we may average the motion of the electrons over a period of the former, and recover a Green's function in tractable form. From Appendix A, we find that the average Green's function for a harmonic external electric field may be written;

$$g(\underline{x},\underline{v}|\underline{x}',\underline{v}';t-t')_{AV} = \Theta(t-t')\,\delta(\underline{v}-\underline{v}')\int d\underline{k}\, J_0\left(\frac{e(t-t')\,\underline{k}\cdot\underline{E}}{m_e\,\omega_0}\right) \exp i\underline{k}\cdot(\underline{x}'-\underline{x}+\underline{v}(t-t')) \tag{29}$$

where ω_0 is the frequency of the incident field, and J_0 is the zero order Bessel function. Since we also have assumed that the incident field is spacially homogeneous at the outset, we are limited to a frequency range of a decade or so beginning at the plasma frequency. This restriction insures that, for the most part, the scale length of plasma disturbances are small with respect to the wavelength of the electric field.

The final approximation in solving the pair of equations, Eqs. 27, and 28, is to replace the Green's functions by the average expression given above. The set of equations then becomes;

$$\tilde{f}^{(1)}(\alpha,t) = f_0^{(1)}(\alpha,t) + n\xi\int dt'\int d\tau_\alpha'\, g(\alpha;t-t')_{AV}\int d\tau_\ell\, L_{\alpha\ell}\,\varphi(v_\alpha)\,\tilde{f}^{(1)}(\ell,t') \tag{30}$$

and

$$f^{(1)}(\alpha,t) = f_0^{(1)}(\alpha,t) + \xi \int dt' \int d\tau_\alpha' g(\alpha;t-t')_{AV} \int d\tau_\ell L_{\alpha\ell} \tilde{f}^{(1)}(\alpha,t')\tilde{f}^{(1)}(\ell,t') \tag{31}$$

We may turn to a Fourier-Laplace analysis of the above.

3. Development of a Generalized Dielectric Function.

The linear integral equation for $\tilde{f}^{(1)}$ is of the Volterra type in the time variable (a consequence of causality), and because it is an inhomogeneous equation, possesses a unique solution for each value of the parameter $(n\xi)$. Since the equation is a convolution in time, the Laplace transform of Eq. 30 is a simple algebraic expression in the Laplace variable. In Appendix A, it is also shown that the Fourier transform of the averaged Green's function is diagonal in the reciprocal coordinate vector, and is also diagonal in the velocities; the latter as a result of the averaging procedure. Therefore, it Eq. 30 were integrated over the velocities, and a Fourier-Laplace transform taken, the resultant expression could be solved algebraically. However, the full distribution function and not simply its first moment is required for the solution of Eq. 31. Thus a solution by iteration is desirable.

The Fourier-Laplace transform of Eq. 30 is:

$$\tilde{f}^{(1)}(\underline{k},\underline{v},p) = \tilde{f}_0^{(1)}(\underline{k},\underline{v},p) + \frac{-4\pi n \xi i}{\left((p - i\underline{k}\cdot\underline{v})^2 + \left(\frac{e\,\underline{k}\cdot\underline{E}}{m_e \omega_0}\right)^2\right)^{1/2}} \left(\frac{\underline{k}}{k^2}\cdot\frac{\partial \varphi(\underline{v})}{\partial \underline{v}}\right) \int d\underline{v}'' \tilde{f}^{(1)}(\underline{k},\underline{v}'',p) \tag{32}$$

$$Re\,p > 0$$

We shall look for the solution to this equation in the form of a power series in the plasma frequency:

$$\tilde{f}^{(1)}(\underline{k},\underline{v},p) = \tilde{f}_0^{(1)}(\underline{k},\underline{v},p) + \sum_{q=1}^{\infty} (\omega_p)^{2q}\, \tilde{f}^{(1)}_{(q)}(\underline{k},\underline{v},p)$$

where ω_p is related to ξ by:

$$\omega_p^2 = 4\pi n \xi$$

The coefficients of various powers of the plasma frequency in the above sum are given by the Fourier-Laplace transform of the linearized recursion relation, Eq. 26 (without the n);

$$\tilde{f}^{(1)}_{(q)}(\underline{k},\underline{v},p) = -\frac{i\underline{k}}{k^2}\cdot\frac{\partial\varphi(\underline{v})}{\partial\underline{v}}\left((p-i\underline{k}\cdot\underline{v})^2+\left(\frac{e\,\underline{k}\cdot\underline{E}}{m_e\omega_0}\right)^2\right)^{-\frac{1}{2}}\int d\underline{v}''\, \tilde{f}^{(1)}_{(q-1)}(\underline{k},\underline{v}'',p) \tag{33}$$

Because of the simple velocity dependence of the right side of this equation, the indicated recursion may be written explicitly with the aid of the following definition:

$$\underline{J}(\underline{k},p,\underline{k}\cdot\underline{E}) = \int d\underline{v}\,\frac{\partial\varphi(\underline{v})}{\partial\underline{v}}\left((p-i\underline{k}\cdot\underline{v})^2+\left(\frac{e\,\underline{k}\cdot\underline{E}}{m_e\omega_0}\right)^2\right)^{-\frac{1}{2}}, \quad Re\,p>0$$

The result is

$$\tilde{f}^{(1)}(\underline{k},\underline{v},p) = f_0^{(1)}(\underline{k},\underline{v},p) - \frac{i\underline{k}}{k^2}\cdot\frac{\partial}{\partial\underline{v}}\varphi(\underline{v})\,\omega_p^2\, n_0(\underline{k},p)\left((p-i\underline{k}\cdot\underline{v})^2+\left(\frac{e\,\underline{k}\cdot\underline{E}}{m_e\omega_0}\right)^2\right)^{-\frac{1}{2}} \times \sum_{q=0}^{\infty}\omega_p^{2q}\left(-\frac{i\underline{k}}{k^2}\cdot\underline{J}(\underline{k},p,\underline{k}\cdot\underline{E})\right)^q \tag{34}$$

But the sum is just a geometric series, and therefore may be summed explicitly to give:

$$\tilde{f}^{(1)}(\underline{k},\underline{v},p) = f_0^{(1)}(\underline{k},\underline{v},p) - \frac{i\underline{k}}{k^2}\cdot\frac{\partial}{\partial\underline{v}}\varphi(\underline{v})\, n_0(\underline{k},p)\,\omega_p^2\left((p-i\underline{k}\cdot\underline{v})^2+\left(\frac{e\,\underline{k}\cdot\underline{E}}{m_e\omega_0}\right)^2\right)^{-\frac{1}{2}} D(\underline{k},p,\underline{k}\cdot\underline{E})^{-1} \tag{35}$$

where $n_0(\underline{k},p)$ is the number density (in k,p space) that would exist if there were no coulomb interactions:

$$n_0(\underline{k},p) = \int d\underline{v}\, f_0^{(1)}(\underline{k},\underline{v},p)$$

and $D(\underline{k},p,\underline{k}\cdot\underline{E})$ is the function

$$D(\underline{k},p,\underline{k}\cdot\underline{E}) = 1 + \omega_p^2 \frac{i\underline{k}}{k^2} \cdot \underline{\mathcal{J}}(\underline{k},p,\underline{k}\cdot\underline{E}) \tag{36}$$

If Eq. 35 is integrated over velocities, an equation for the perturbed number density results:

$$\tilde{n}(\underline{k},p) = n_0(\underline{k},p)\left[1 - \omega_p^2 \frac{i\underline{k}}{k^2} \cdot \underline{\mathcal{J}}(\underline{k},p,\underline{k}\cdot\underline{E}) / D(\underline{k},p,\underline{k}\cdot\underline{E})\right]$$

$$= n_0(\underline{k},p) / D(\underline{k},p,\underline{k}\cdot\underline{E})$$

Therefore the function D plays the role of a dielectric function (and is one in the linear theory) by "dressing" the free particle number density with the global influence of the plasma as a whole. (A polarization effect).

The integral expression, Eq. 31, may be simplified somewhat if $f^{(1)}$ is considered to be the sum of $\tilde{f}^{(1)}$ and a correction term, $f^{(1)*}$.

$$f^{(1)} = \tilde{f}^{(1)} + \xi f^{(1)*} \tag{38}$$

Further, let us rewrite Eq. 35 with $\tilde{f}_0^{(1)}$ written explicitly;

$$\tilde{f}^{(1)}(\underline{k},\underline{v},p) = (2\pi)^{3/2} n\,\delta(\underline{k})\,\varphi(v_\alpha)/p \quad + U_0(\underline{k},\underline{v},p)$$

$$- \frac{i\underline{k}}{k^2} \cdot \frac{\partial\varphi(\underline{v})}{\partial\underline{v}}\, n_0(\underline{k},p)\,\omega_p^2\, \mathcal{A}(\underline{k}\cdot\underline{v},\underline{k}\cdot\underline{E},p) / D(\underline{k},p,\underline{k}\cdot\underline{E}) \tag{39}$$

where $\mathcal{G}$ is the Fourier-Laplace transform of the average Green's function;

$$\mathcal{G}(\underline{k}\cdot\underline{v}, \underline{k}\cdot\underline{E}, p) = \left((p - i\underline{k}\cdot\underline{v})^2 + \left(\frac{e\,\underline{k}\cdot\underline{E}}{m_e\omega_0}\right)^2\right)^{-1/2}$$

and $\mathcal{U}_0(\underline{k},\underline{v},p)$ is the inhomogeneous part of $f_0^{(1)}(\underline{k},\underline{v},p)$ and is a known function. We now consider the Fourier-Laplace transform of Eq. 31. Let Eq. 39 be substituted for the transform of $\tilde{f}(\alpha,t)$ that appears on the right in the transformed equation, and also replace $f^{(1)}$ by $\tilde{f}^{(1)} + \xi f^{(1)*}$ on the left. If the indicated multiplications and integrations are carried out, the equation for $\tilde{f}^{(1)}$, Eq. 32, is recovered, and the remainder gives an expression for $f^{(1)*}$;

$$f^{(1)*}(\underline{k},\underline{v},p) = \frac{1}{(2\pi)^{3/2}\, i\pi}\, \mathcal{G}(\underline{k}\cdot\underline{v}, \underline{k}\cdot\underline{E}, p) \int_{-i\infty+}^{+i\infty+} dp' \int d\underline{k}'\, D\left((\underline{k}-\underline{k}'), (\underline{k}-\underline{k}')\cdot\underline{E}, p-p'\right)^{-1}$$

$$\times \left[\frac{i\omega_p^2 (\underline{k}-\underline{k}')\underline{k}'}{|\underline{k}-\underline{k}'|^2 k'^2} : \frac{\partial}{\partial \underline{v}}\left(\frac{\partial}{\partial \underline{v}}\varphi(\underline{v})\, \mathcal{G}(\underline{k}'\cdot\underline{v}, \underline{k}'\cdot\underline{E}, p')\right) \int d\underline{v}''\, f_0^{(1)}(\underline{k}', \underline{v}'', p) \right. \tag{40}$$

$$\left. \times D(\underline{k}', \underline{k}'\cdot\underline{E}, p')^{-1} - \frac{(\underline{k}-\underline{k}')}{|\underline{k}-\underline{k}'|^2} \cdot \frac{\partial}{\partial \underline{v}}\, \mathcal{U}_0(\underline{k}', \underline{v}, p') \right] \int d\underline{v}'''\, f_0^{(1)}(\underline{k}-\underline{k}', \underline{v}''', p-p')$$

In order to obtain an expression that permits comparison with Eq. 37, we integrate Eq. 40 over $\underline{v}$. Let this quantity be added to a similar integration of Eq. 39. Then the final result may be written as:

$$n(\underline{k},p) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\underline{k}'\int_{-i\infty+}^{+i\infty+} dp'\,\mathcal{D}^{-1}(\underline{k},\underline{k}',p,p';n_0,\underline{E})\,n_0(\underline{k}-\underline{k}',p-p') \tag{41}$$

where $\mathcal{D}^{-1}$ is a generalized dielectric function;

$$\mathcal{D}^{-1}(\underline{k},\underline{k}',p,p',n_0,\underline{E}) = \frac{\delta(\underline{k}')}{p'\,D(\underline{k},\underline{k}\cdot\underline{E},p)} + \frac{2\xi}{(2\pi)^{3/2}}\int d\underline{v}\,\frac{\mathcal{G}(\underline{k}\cdot\underline{v},\underline{k}\cdot\underline{E},p)}{D(\underline{k}-\underline{k}',(\underline{k}-\underline{k}')\cdot\underline{E},p-p')}$$

$$\times\left\{\frac{i\omega_p^2(\underline{k}-\underline{k}')\underline{k}'}{|\underline{k}-\underline{k}'|^2k'^2}:\frac{\partial}{\partial\underline{v}}\left(\frac{\partial}{\partial\underline{v}}\varphi(\underline{v})\,\mathcal{G}(\underline{k}'\cdot\underline{v},\underline{k}'\cdot\underline{E},p')\right)\frac{n_0(\underline{k}',p')}{D(\underline{k}',\underline{k}'\cdot\underline{E},p')}\right. \tag{42}$$

$$\left.-\frac{(\underline{k}-\underline{k}')}{|\underline{k}-\underline{k}'|^2}\cdot\frac{\partial}{\partial\underline{v}}u_0(\underline{k}',\underline{v},p')\right\}$$

This quantity is the sum of the linear theory dielectric function given by Eq. 36, and a correction term of order ξ . The correction term is simply second order modification of the charge number density arising from the Coulomb interaction based upon the perturbed charge density given by the first term, and a prescribed initial inhomogeneity, u_o . The integrals over the dummy variables follow from the form of the non-linear Vlassov equation, or, in other words, there are no non-trivial stationary, homogeneous perturbations! Therefore, of necessity, the convolutional relationship in space-time between the displacement and electric fields which would be given by the first term of Eq. 42, alone, is destroyed in second order. The inclusion of the correction term in any subsequent scattering problem, would, in principle, give harmonic and sideband generation in some approximation, or if the first term in the braces is discarded, the remaining one may be used to base a discussion of instability growth rates pertinent to many plasma configurations today.

Three main points are discussed in the present work. The first of these is that the Vlassov equation is the most general description of plasma phenomena whose frequency spectrum encompasses the plasma frequency. Lower frequency (longer time) effects are rejected both in the choice of perturbation series and in the assumption of a completely factored initial distribution function. The theorem, which is the core of this derivation, shows that this factorization persists in distribution functions of all orders at the current time. Thus the full Vlassov equation follows from the first member of the BBGKY hierarchy.

The second point is the advantage of dealing with Green's functions instead of streaming operators when external field effects are included. It is the choice of a representation of the streaming Green's function which permitted the averaging procedure (correspondent to a random phase approximation) over a period of the incident field as shown in Appendix A. While it is true that the averaged Green's function may no longer be an exact "reciprocal" of the streaming operator, later solutions of the equations do not require this fact, and therefore an expression for a plasma dielectric function, anisotropic in the external field direction, may be derived.

Lastly, the approximate solution to the non-linear Vlassov equation gives a generalized dielectric function for the plasma which shows that the relation between electric and displacement fields in a plasma are no longer completely convolutional in space-time. The two terms that modify this behaviour describe a second perturbation of the already perturbed electron density (as derived in the linear theory), and a perturbation of the initial inhomogeniety in the electron distribution. Thus, subsequent calculations based upon this generalized dielectric function might describe both electrical non-linear effects such as side-band and harmonic generation, and instability growth in a plasma.

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APPENDIX A

Let us consider an assembly of electrons moving freely in the influence of an external, sinusoidal, homogeneous, electric field, but not interacting with each other. For classical electrons, the equation of motion is:

$$\frac{d\underline{v}}{dt} = \underline{\varepsilon} \sin\omega_0 t \quad \text{where} \quad \underline{\varepsilon} = -\frac{e\underline{E}}{m_e} \tag{A1}$$

For this specification of the field, the one-particle Green's function becomes;

$$g(\underline{x},\underline{v}|\underline{x}',\underline{v}';t,t') = \theta(t-t')\,\delta\!\left(\underline{x}'-\underline{x}+\left(\underline{v}+\frac{\underline{\varepsilon}}{\omega_0}\cos\omega_0 t\right)(t-t')-\frac{\underline{\varepsilon}}{\omega_0}(\sin\omega_0 t-\sin\omega_0 t')\right)$$
$$\times\,\delta\!\left(\underline{v}'-\underline{v}-\frac{\underline{\varepsilon}}{\omega_0}(\cos\omega_0 t-\cos\omega_0 t')\right) \tag{A2}$$

which describes the evolution of the electron from the state $(\underline{x}',\underline{v}')$ at time t' to the state $(\underline{x},\underline{v})$ at time t.

Unfortunately, the path coupling the electron's initial and final state depends on the phase of the external electric field, and thus g depends on the origin of time. This effect is important, however, only for the electron motion in a time interval comparable to the period of the external field. Therefore, after finding an appropriate representation for the Green's function, we may average it over a period of the external field, and use this result in our subsequent development. Such an average corresponds to a "random phase" approximation in that the averaged Green's function describes the motion of a typical electron picked at random from the assembly of electrons whose initial phases are arbitrary. The information loss, embodied in this assumption, is reasonable, provided the kinetic theory is not called upon to describe phenomena close to the external field frequency. In particular, we shall assume $\omega_0 > \omega_p$.

Now the Fourier transform of Eq. A2 is given by:

$$g(\underline{k},\underline{m}|\underline{k}',\underline{m}';t,t') = \frac{1}{(2\pi)^6}\int d\underline{x}\int d\underline{x}'\int d\underline{v}\int d\underline{v}'\, e^{i\underline{k}\cdot\underline{x}+i\underline{m}\cdot\underline{v}}\, g(\underline{x},\underline{v}|\underline{x}',\underline{v}';t,t')\, e^{-i\underline{k}'\cdot\underline{x}'-i\underline{m}'\cdot\underline{v}'}$$

$$= \theta(t-t')\,\delta(\underline{k}-\underline{k}')\,\delta(\underline{m}-\underline{m}'+\underline{k}(t-t'))\exp\Big\{-i\frac{\underline{m}\cdot\underline{\varepsilon}}{\omega_0}(\cos\omega_0 t-\cos\omega_0 t') \tag{A3}$$

$$+i\frac{\underline{k}\cdot\underline{\varepsilon}}{\omega_0}\cos\omega_0 t\,(t-t') - i\frac{\underline{k}\cdot\underline{\varepsilon}}{\omega_0}(\sin\omega_0 t-\sin\omega_0 t')\Big\}$$

where it is seen that g is diagonal only in the reciprocal length, $\underline{k}$. Let us change the time variables from the set t,t' to the set T, η , defined by:

$$T = \tfrac{1}{2}(t+t')$$

and

$$\eta = \tfrac{1}{2}(t-t')$$

From the previous equation we then have;

$$g(\underline{k},\underline{m}|\underline{k}',\underline{m}';\eta,T) = \theta(\eta)\,\delta(\underline{k}-\underline{k}')\,\delta(\underline{m}-\underline{m}'+2\underline{k}\eta) \tag{A4}$$

$$\times\exp i\left[\frac{2\underline{\varepsilon}}{\omega_0}\cdot(\underline{m}'-\underline{k}\eta)\sin\omega_0\eta\,\sin\omega_0 T + \frac{2(\underline{k}\cdot\underline{\varepsilon})}{\omega_0}\Big(\eta\cos\omega_0\eta-\frac{\sin\omega_0\eta}{\eta}\Big)\cos\omega_0 T\right]$$

The complex exponentials of the trigonometric functions may be represented by Bessel functions through the definition:

$$\exp(iz\sin\varphi) = \sum_{-\infty}^{\infty}{}' \exp(in\varphi)J_n(z) \tag{A5}$$

Thus Eq. A4 may be written:

$$g(\underline{k},\underline{m}|\underline{k}',\underline{m}';\eta,T) = \Theta(\eta)\,\delta(\underline{k}-\underline{k}')\,\delta(\underline{m}-\underline{m}'+2k\eta)\sum_{l=-\infty}^{\infty} e^{il\omega_0 t} J_l\left(\frac{2\underline{\varepsilon}}{\omega_0}\cdot(\underline{m}'-\underline{k}\eta)\sin\omega_0\eta\right)$$
$$\times\sum_{m=-\infty}^{\infty} e^{im(\omega_0 T+\pi/2)} J_m\left(\frac{2\underline{k}\cdot\underline{\varepsilon}}{\omega_0}\left(\eta\cos\omega_0\eta - \frac{\sin\omega_0\eta}{\omega_0}\right)\right) \tag{A6}$$

The average over T is defined by:

$$g(\underline{k},\underline{m}|\underline{k}',\underline{m}';\eta)_{Av} = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} g(\underline{k},\underline{m}|\underline{k}',\underline{m}';\eta,T)\,dT$$
$$=\Theta(\eta)\,\delta(\underline{k}-\underline{k}')\,\delta(\underline{m}-\underline{m}'+2\underline{k}\eta)\sum_{l=-\infty}^{\infty} J_l\left(\frac{2\underline{\varepsilon}}{\omega_0}\cdot(\underline{m}'-\underline{k}\eta)\sin\omega_0\eta\right) J_{-l}\left(\frac{2\underline{k}\cdot\underline{\varepsilon}}{\omega_0}\left(\eta\cos\omega_0\eta-\frac{\sin\omega_0\eta}{\omega_0}\right)\right)e^{i\pi l/2} \tag{A7}$$

Now on the other hand, we have the sum rule for Bessel functions;

$$J_0\left((a^2+b^2+2ab\cos\alpha)^{1/2}\right) = \sum_{\nu=-\infty}^{\infty} J_\nu(a)\,J_\nu(b)\cos\nu\alpha \tag{A8}$$

From Eq. A7, we see that g is even in λ , and thus only $\cos\pi l/2$ contributes to the sum. Therefore, with the sum rule just given, we may write g as;

$$g(\underline{k},\underline{m}|\underline{k}',\underline{m}';\eta)_{Av} = \Theta(\eta)\,\delta(\underline{k}-\underline{k}')\,\delta(\underline{m}-\underline{m}'+2k\eta)$$
$$\times J_0\left(\left[4\frac{(\underline{k}\cdot\underline{\varepsilon})^2\eta^2}{\omega_0^2} - 8\eta(\underline{k}\cdot\underline{\varepsilon})\frac{\sin\omega_0\eta}{\omega_0^2}\left((\underline{m}'\cdot\underline{\varepsilon})\sin\omega_0\eta+\frac{\cos\omega_0\eta}{\omega_0}\right)\right.\right. \tag{A9}$$
$$\left.\left.+\frac{4}{\omega_0^2}\left((\underline{m}'\cdot\underline{\varepsilon})^2+\frac{(\underline{k}\cdot\underline{\varepsilon})^2}{\omega_0^2}\right)\sin^2\omega_0\eta\right]^{1/2}\right)$$

The argument of the Bessel function may be simplified by noting that for large η , the first term dominates. Since the average does not represent the correct behavior for small η , we introduce no additional error by neglecting all but the first term. We may write the Green's function as:

$$g(\underline{k},\underline{m}|\underline{k}',\underline{m}';\eta)_{AV} = \Theta(\eta)\,\delta(\underline{k}-\underline{k}')\,\delta(\underline{m}-\underline{m}'+2\underline{k}\eta)\,J_0\left(\frac{2\underline{k}\cdot\underline{\varepsilon}}{\omega_0}\eta\right) \tag{A10}$$

For our purposes, a more useful representation is obtained by transforming back to velocity space. The result of this operation and its time Laplace transform is:

$$g(\underline{k},\underline{v}|\underline{k}',\underline{v}';t-t')_{AV} = \Theta(t-t')\,\delta(\underline{k}-\underline{k}')\,\delta(\underline{v}-\underline{v}') \times e^{i\underline{k}\cdot\underline{v}(t-t')}\,J_0\left(\frac{\underline{k}\cdot\underline{E}\,e\,(t-t')}{m_e\omega_0}\right) \tag{A11}$$

and

$$g(\underline{k},\underline{v}|\underline{k}',\underline{v}',p) = \delta(\underline{k}-\underline{k}')\,\delta(\underline{v}'-\underline{v})\left((p-i\underline{k}\cdot\underline{v})^2+\left(e\frac{\underline{k}\cdot\underline{E}}{m_e\omega_0}\right)^2\right)^{-1/2} \tag{A12}$$

These two equations are the desired result.

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