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A MATRIX EQUATION ARISING IN STATISTICAL FILTER THEORY

by James E. Potter

Prepared under Grant No. NsG-254 *by*
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Cambridge, Mass.
for

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ABSTRACT

This report describes the behavior of the solutions of a matrix Riccati differential equation arising in statistical filtering and optimal control theory. The statistical problem leading to the Riccati equation is outlined. Definiteness, ordering and boundedness properties of the solutions of the general differential equation with time varying coefficients are derived. When the differential equation has constant coefficients, it is shown that under a set of physically reasonable conditions there is a unique steady state solution to which all other solutions converge at an exponential rate.

by James E. Potter

February 1965

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SECTION 1
INTRODUCTION

This report discusses properties of the solutions of a Riccati differential equation

$$E' = A(t)\,E + EA^{*}(t) - EB(t)\,E + C(t) \qquad (1\text{-}1)$$

in which the solution E(t) and the coefficients A(t), B(t) and C(t) are n by n matrices. This equation arises in determining the best linear estimate of the solution of a linear differential equation driven by white noise. Matrix Riccati equations also arise in the calculus of variations and in various problems in applied mathematics. The assumptions made about the coefficients A(t), B(t) and C(t) and the particular properties of the solutions of (1-1) which are investigated in this report are those which are relevant to the statistical problem. Reid[1] and Levin[2] have discussed matrix Riccati equations in a general context. Many of the ideas which are treated in detail in this paper appear in reference 3.

The statistical problem leading to the Riccati equation is outlined in this section. The following sections examine the Riccati equation from the standpoint of differential equation theory and do not make use of probabilistic ideas. In Section 2, definiteness, ordering and boundedness properties are derived for solutions of equation (1-1). Section 3 is concerned with the autonomous case in which the coefficients of the Riccati equation are constant matrices. The results of Section 3 depend on the assumption that all of the unstable modes of the linear differential equation whose solution is being estimated are driven and are measurable. A Riccati equation with constant coefficients coming from such a statistical problem will be called regular. It is shown that a regular matrix Riccati equation has a unique positive semidefinite critical point or steady state solution and that any solution with a positive semidefinite initial value converges exponentially fast to the steady state solution.

Let the n-dimensional column vector x(t) be the solution of the linear differential equation

$$x' = A(t)\, x + w \qquad (1\text{-}2)$$

where A(t) is a deterministic n by n matrix. The driving function w in equation (1-2) will be assumed to be white noise, that is, an n-dimensional Schwartz distribution valued random process whose formal first and second moments are [4, 5]

$$\overline{w\ (t)} = 0 \qquad (1\text{-}3)$$

and

$$\overline{w\ (t) w^{*}(s)} = C(t)\, \delta (t - s) \qquad (1\text{-}4)$$

In equations (1-3) and (1-4) the horizontal bar represents the statistical mean or expected value, the asterisk represents the conjugate transpose operation and the n by n matrix C(t) is positive semidefinite for each value of t. The formal first and second moments of w are Schwartz distributions and determine the first and second moments of the random variables obtained when the random functional w acts on deterministic functions in its domain.

It is desired to estimate x (t) by filtering the observable k-dimensional stochastic process (measurements) m(t),

$$m(t) = H(t)\, x\, (t) + r(t) \qquad (1\text{-}5)$$

where H(t) is a k by n matrix and r(t) is k-dimensional white noise. The two white noise processes w (t) and r(t) will be assumed to be uncorrelated with each other and with the initial value of x.

Kalman and Bucy[6] have shown that, if only values of m(s) for $s \leqq t$ are to be used in forming the estimate, the minimum variance linear estimator of x(t) is the solution $\hat{x}(t)$ of the differential equation

$$\hat{x}' = A(t)\,\hat{x} + E(t)\,H^{*}(t)\left\{m - H(t)\,\hat{x}\right\} \qquad (1\text{-}6)$$

with the initial condition

$$\hat{x}(0) = \overline{x(0)}$$

The n by n weighting matrix E(t) in equation (1-6) is obtained by solving equation (1-1) with the initial condition

$$E(0) = \overline{\left\{x(0) - \overline{x(0)}\right\}\left\{x(0) - \overline{x(0)}\right\}^{*}}$$

The coefficient B(t) in equation (1-1) is given by the formula

$$B(t) = H^{*}(t)\,U^{-1}(t)\,H(t) \qquad (1\text{-}7)$$

where U(t) is the positive definite matrix associated with the formal second moment of r,

$$\overline{r(t)\,r^{*}(s)} = U(t)\,\delta(t-s)$$

Besides acting as a weighting factor for the measurements in equation (1-6), E(t) is the estimation error covariance matrix for the estimator $\hat{x}(t)$,

$$E(t) = \overline{\left\{x(t) - \hat{x}(t)\right\}\left\{x(t) - \hat{x}(t)\right\}^{*}}$$

As indicated above, the matrices E(0), B(t) and C(t) are positive semidefinite in the statistical application. Since E(t) is a covariance matrix, it should follow that E(t) is positive semidefinite for all values of t. This fact is established in Theorem 2-1. Since the right-hand side of equation (1-1) does not satisfy a global Lipschitz condition, its solution may escape to infinity at a finite time. For example, the Riccati equation

$$E' = E^2$$

has the solution

$$E(t) = \left\{ I - t\,E(0) \right\}^{-1} E(0)$$

which escapes to infinity when

$$t = \min_{\lambda > 0} \frac{1}{\lambda}$$

where the minimum is taken over the eigenvalues of E(0). This possibility is ruled out in the statistical application by Theorem 2-2.

The following three part regularity condition is assumed in proving the theorems in Section 3:

(a) B and C are positive semidefinite. The proofs of Theorems 3-4 and 3-5 could be considerably shortened if it were assumed that B and C are positive definite rather than positive semidefinite. However, definiteness is a fairly restrictive assumption since in applications the observed stochastic process m(t) is often a scalar so that the rank of B(t) is one. Furthermore, C has rank one when equation (1-2) is the system representation of a single n-th order differential equation driven by scalar white noise.

(b) No eigenvector of A whose eigenvalue has a non-negative real part is a null vector of B. Since U is positive definite, it follows from equation (1-7) that every null vector of B is a null vector of H. In view of equation (1-5), condition (b) may be interpreted as requiring that every unstable mode of equation (1-2) affect the measurements m(t). It is clear that if this condition is not satisfied, (1-1) cannot have a steady state solution since the estimation error variance for an unmeasurable unstable mode would be unbounded for large t.

(c) No eigenvector of A^* whose eigenvalue has a non-negative real part is a null vector of C. If e is an unstable eigenvector of A^*, the second moment of the scalar white noise u(t) driving the mode of equation (1-2) corresponding to e is

$$\overline{u(t)\, u(s)} = e^* C e\, \delta(t-s)$$

Thus, if e is a null vector of C, the unstable mode corresponding to e is undriven. Examples indicate that, if condition (c) does not hold, equation (1-1) has positive semidefinite critical points which are not stable in addition to a stable critical point. It is not assumed that A has a diagonal Jordan form.

Recent investigations[7, 8] have shown that the estimation problem in which the measurement noise r(t) is the solution of a linear differential equation driven by white noise leads to a matrix Riccati equation in which E(0), B(t) and C(t) are again positive semidefinite. This estimation problem contains the Wiener theory of filtering and prediction of stochastic processes with rational power spectra.[9]

Kalman and Bucy[5] have shown that the statistical problem described above is the dual of a problem in linear control theory. Therefore, the results of this report should apply to that problem.

SECTION 2

DEFINITENESS, ORDERING AND BOUNDEDNESS OF SOLUTIONS

If R and S are hermitian matrices, the inequality $R > S$ will be used to indicate that the matrix $(R-S)$ is positive definite. Similarly, $R \geqq S$ will indicate that the matrix $(R-S)$ is positive semidefinite. Any matrix which appears in an inequality will be assumed to be hermitian.

In the following analysis it will be assumed that $A(t)$, $B(t)$ and $C(t)$ are locally integrable functions of t and that $B(t)$ and $C(t)$ are hermitian for each value of t. The existence and uniqueness theorems for ordinary differential equations[10] imply that, for a given initial condition $E(0)$, equation (1-1) has a unique absolutely continuous solution on some interval (o, a). By the continuation theorem, this solution may be continued to $(o, a+\epsilon)$ for some $\epsilon > 0$ if

$$\overline{\lim_{t \to a-}} \max_{i,j} \left| E_{ij}(t) \right| < \infty$$

Since $B(t)$ and $C(t)$ are hermitian and $E(t)$ satisfies (1-1) on $(0, a)$, $E^*(t)$ also satisfies (1-1) and it follows by uniqueness that $E(t)$ is hermitian provided that $E(0)$ is hermitian.

<u>Theorem 2-1</u>

If $E(t)$ satisfies equation (1-1) for $t \in [o, a)$, $E(0) \geqq 0$ and $C(t) \geqq 0$ for almost every $t \in [o, a)$, then $E(t) \geqq 0$ for every $t \in [o, a)$.

Proof: Assume that $s \in [o, a)$, let x be an arbitrary n dimensional column vector and let $y(t)$ be the solution of the linear differential equation

$$y' = \left\{ \frac{1}{2} B(t)\, E(t) - A^*(t) \right\} y \qquad (2\text{-}1)$$

with

$$y(s) = x$$

Then by equation (1-1),

$$(y^* Ey)' = y^* Cy$$

and

$$x^* E(s)x = y^*(0)\, E(0)\, y(0) + \int_0^s y^*(t)\, C(t)\, y(t)\, dt \qquad (2\text{-}2)$$

The conclusion of the theorem follows immediately from equation (2-2). Equation (2-2) also implies the following corollary.

Corollary 1

If, in addition to the hypothesis of Theorem 2-1, $E(0) > 0$ or $C(t) > 0$ for almost every $t \in [0, a)$, then $E(t) > 0$ for every $t \in (0, a)$.

Corollary 2

If $E_1(t)$ and $E_2(t)$ are solutions of (1-1) on $[0, a)$ with coefficients $A(t)$, $B_1(t)$, $C_1(t)$ and $A(t)$, $B_2(t)$, $C_2(t)$ respectively, with $E_2(0) \geqq E_1(0)$ and $B_1(t) \geqq B_2(t)$ and $C_2(t) \geqq C_1(t)$ for almost every $t \in [0, a)$, then $E_2(t) \geqq E_1(t)$ for every $t \in [0, a)$.

Proof: The matrix

$$X(t) = E_2(t) - E_1(t)$$

satisfies the differential equation

$$X' = (A - E_1 B_2)\, X + X(A - E_1 B_2)^* - X\, B_2 X + E_1 (B_1 - B_2)\, E_1 + C_2 - C_1 \qquad (2\text{-}3)$$

The corollary follows by applying Theorem 2-1 to equation (2-3).

Theorem 2-2

If $E(0) \geqq 0$ and $B(t)$, $C(t) \geqq 0$ for almost every $t \in [0, \infty)$, then the solution of (1 - 1) does not escape to infinity at a finite time and the elements of E(t) satisfy inequality (2 - 7) below.

Proof: Suppose that the solution E(t) exists on the interval $[0, a)$. Since the hypothesis of Theorem 2-1 is satisfied, $E(t) \geqq 0$. This implies that

$$\left| E_{ij} \right|^2 \leqq E_{ii} E_{jj} \leqq \frac{1}{4} (E_{ii} + E_{jj})^2$$

and therefore

$$\left| E_{ij} \right| \leqq \frac{1}{2} \operatorname{tr} E$$

Let

$$a(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left| A_{ij}(t) \right|$$

Then

$$\left| \operatorname{tr} A E \right| = \left| \operatorname{tr} E A^* \right| = \left| \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} E_{ji} \right| \leqq \frac{1}{2} a \operatorname{tr} E \qquad (2-4)$$

Since $E B E \geqq 0$, its diagonal elements are non-negative, and

$$\operatorname{tr} E B E \geqq 0 \qquad (2-5)$$

Taking the trace of both sides of equation (1-1) and using (2-4) and (2-5) yields

$$(\operatorname{tr} E)' \leqq \frac{1}{2} a \operatorname{tr} E + \operatorname{tr} C \qquad (2-6)$$

and applying the method of proof of Gronwall's lemma yields

$$2\left|E_{ij}(t)\right| \leqq \operatorname{tr} E(t) \leqq \exp\left(\frac{1}{2}\int_0^t a(s)\, ds\right)\left\{\operatorname{tr} E(0) + \int_0^t \operatorname{tr} C(s) \exp\left(-\frac{1}{2}\int_0^s a(u)\, du\right) ds\right\} \quad (2\text{-}7)$$

In view of the continuation theorem, inequality (2-7) implies that the solution E(t) exists on the entire right half line.

This theorem may also be proved using the method of proof of Lemma 1 of Theorem 3-5 below.

SECTION 3

CRITICAL POINTS IN THE AUTONOMOUS CASE

In this section it will be assumed that the coefficients A, B and C in equation (1-1) are constant matrices and that B and C are hermitian. The matrix S is a critical point of equation (1-1) if it is a zero of the right-hand side, that is, if

$$AS + SA^* - SBS + C = 0 \tag{3-1}$$

It will always be assumed that S is a hermitian matrix. If S is a critical point of equation (1-1), then the matrix

$$X(t) = E(t) - S$$

satisfies the differential equation

$$X' = FX + XF^* - XBX \tag{3-2}$$

with

$$F = A - SB \tag{3-3}$$

Equation (3-2) may be formally transformed to a linear differential equation by letting

$$Y(t) = X(t)^{-1}$$

Then, if it exists, Y satisfies the differential equation

$$Y' = -F^*Y - YF + B$$

From equation (3-2) it follows that the linearization of (1-1) about the critical point S is

$$X' = FX + XF^* \tag{3-4}$$

The solution of equation (3-4) is

$$X(t) = \Phi(t) X(0) \Phi^*(t)$$

where $\Phi(t)$ is the solution of the differential equation

$$\Phi' = F \Phi$$

with

$$\Phi(0) = I$$

By a well known theorem on critical points of autonomous systems,(Ref. 10, p. 314) S is asymptotically stable if F is a stable matrix, that is, has only eigenvalues with negative real parts. If one of the eigenvalues of F has a positive real part, the critical point S cannot be asymptotically stable, but if some of the eigenvalues of F have zero real parts while the rest have negative real parts a more detailed analysis is required to determine whether S is asymptotically stable. If the matrix F is stable the critical point S will be called persistent.

The following theorem from matrix theory is needed as a tool in the following analysis.

Theorem 3-1

If the matrix R is stable, and $V \geqq 0$, then the equation

$$RU + UR^* = -V$$

has a unique solution and $U \geqq 0$. If $V > 0$, then $U > 0$.

This theorem is proved in reference 11 on pages 81 to 84 and in reference 12 on pages 220 to 226.

Theorem 3-2

If $B \geqq 0$, then equation (1-1) has at most one persistent critical point.

Proof: Let S_1 and S_2 be two persistent critical points of equation (1-1) and let $X = S_2 - S_1$. Then by (3-2), X satisfies

$$F_1X + XF_1^* = XBX$$

with

$$F_1 = A - S_1B$$

Since $XBX \geqq 0$ and F_1 is stable, Theorem 3-1 implies that $X \leqq 0$. Interchanging the roles of S_1 and S_2, it follows that $X \geqq 0$. Hence $X = 0$.

Definition: The set of coefficient matrices (A, B, C) in equation (1-1) will be called regular if

(a) $B, C \geqq 0$

(b) No eigenvector of A whose characteristic value has a non-negative real part is a null vector of B.

(c) No eigenvector of A^* whose characteristic value has a non-negative real part is a null vector of C.

Theorem 3-3

If $S \geqq 0$ is a critical point of (1-1) and (A, B, C) is regular, then S is persistent.

Proof: It will be shown that the eigenvalues of $F^* = A^* - BS$ have negative real parts. Equation (3-1) may be rewritten as

$$FS + SF^* + SBS + C = 0 \qquad (3\text{-}5)$$

Let e be an eigenvector of F^* corresponding to the eigenvalue $\lambda + i\mu$ and multiply equation (3-5) on the left by e^* and on the right by e.

Then

$$2\lambda\, e^* Se + e^* SBSe + e^* Ce = 0 \qquad (3\text{-}6)$$

Suppose $e^* Se = 0$. Since $S \geqq 0$, this implies that $Se = 0$, $F^*e = A^*e$ and hence e is an eigenvector of A^* with the same eigenvalue. Since each term in equation (3-6) is nonnegative, $e^*Ce = 0$ and e is a null vector of C. Therefore, λ is negative by regularity property (c).

Suppose $e^* SBSe = e^* Ce = 0$. Let B_1 be a matrix such that $B = B_1^* B_1$. B_1 can be found since $B \geqq 0$. Then $e^* SBSe = (B_1 Se)^*(B_1 Se) = 0$ or $B_1 Se = 0$ and finally $B_1^* B_1 Se = BSe = 0$. Again $F^* e = A^* e$, e is a null vector of C and hence λ is negative by regularity property (c).

Finally, if $e^* Se > 0$ and $e^* SBSe$ or $e^* Ce > 0$,

$$\lambda = -\frac{e^* SBSe + e^* Ce}{e^* Se} < 0$$

and the proof is complete.

<u>Remark:</u> Regularity property (b) need not be included in the hypothesis of Theorem 3-3 since it is never used in the proof. However, its inclusion results in no loss of generality since if regularity property (b) fails to hold, F cannot be stable. For, if e is an eigenvector of A corresponding to an eigenvalue with a nonnegative real part and e is also a null vector of B, then $Fe = Ae$ and F is not stable.

Physical reasoning leads to the conjecture that F is always more stable than A, that is that all of the eigenvalues of F have smaller real parts than the eigenvalue of A having the largest real part. This would mean that the transients in the estimation statistics die out at least as quickly as the transients in the equation (1-2). The following example shows that this is not always true.

$$A = \begin{bmatrix} -1 & 2 \\ -2 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 6 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 0 \\ 0 & 18 \end{bmatrix} \quad S = I$$

The eigenvalues of A are -2 and -5.

$$F = \begin{bmatrix} -1 & 2 \\ -2 & -12 \end{bmatrix}$$

The eigenvalues of F are

$$\frac{1}{2}(-13 \pm \sqrt{113}) = -1.18, \ -11.82$$

Theorem 3-4

If (A, B, C) is regular, then (1-1) has a critical point $S \geqq 0$.

Proof: The critical point will be obtained by solving equation (3-1) by Newton's method. For a hermitian matrix M, let

$$f(M) = -AM - MA^* + MBM - C \qquad (3\text{-}7)$$

To start the Newton's method iteration, it is necessary to find a matrix $S_0 \geqq 0$ such that $f(S_0) \geqq 0$.

Since it is not clear that such a matrix exists if B is singular, the Theorem will first be proved assuming $B > 0$. In this case $f(aI) \geqq 0$ for a sufficiently large positive number a.

If S is considered as an n^2 dimensional vector and the Newton's method iteration formula is applied to equation (3-1), the recursion formula

$$(A - S_k B)S_{k+1} + S_{k+1}(A - S_k B)^* + S_k B S_k + C = 0 \qquad (3\text{-}8)$$

is obtained. It will be proved by induction that there is a sequence of matrices

$$S_0 \geqq S_1 \geqq S_2 \ldots \geqq 0 \qquad (3\text{-}9)$$

satisfying equation (3-8).

First, $S_0 \geqq 0$ and $f(S_0) \geqq 0$. Suppose that there is a sequence

$$S_0 \geqq S_1 \geqq \ldots \geqq S_N \geqq 0$$

satisfying equation (3-8) for $k = 0, 1, \ldots, (N-1)$ and such that

$f(S_N) \geqq 0$. S_N satisfies the matrix quadratic

$$A\,S_N + S_N\,A^* - S_N\,B\,S_N + \hat{C} = 0$$

with

$$\hat{C} = C + f\,(S_N)$$

Since $\hat{C} \geqq C$, $(A,\ B,\ \hat{C})$ is regular and by Theorem 3-3,

$$\hat{F} = A - S_N\,B$$

is stable. By Theorem 3-1, equation (3-8) can be solved for $S_{N+1} \geqq 0$. By (3-7) and (3-8) it follows that

$$f(S_{N+1}) = (S_{N+1} - S_N)\,B\,(S_{N+1} - S_N) \geqq 0$$

Furthermore, let $D_N = S_N - S_{N+1}$. Then D_N satisfies the equation

$$(A - S_N B)\,D_N + D_N\,(A - S_N\,B)^* = -f(S_N)$$

By Theorem 3-1, $D_N \geqq 0$, or

$$S_{N+1} \leqq S_N$$

Thus, the existence of the sequence

$$S_0 \geqq S_1 \geqq \ldots \geqq S_{N+1} \geqq 0$$

with $f(S_{N+1}) \geqq 0$ has been proven. The existence of (3-9) follows by induction. Since $(S_k \mid k = 0, 1, 2, \ldots)$ is a decreasing sequence bounded below, it has a limit[13] $S \geqq 0$. Taking the limit of equation (3-8) as $k \to \infty$ yields

$$AS + SA^* - SBS + C = 0$$

and the theorem is proved for the case when $B > 0$.

The following lemmas are needed to handle the case when B is singular.

Lemma 1

If $S \geqq 0$ is a critical point of equation (1-1), (A, B, C) is regular and no eigenvalue of A has a negative real part, then $S > 0$.

Proof of lemma: Let N be the null space of S and let n be a null vector of S. Multiplying equation (3-1) on the left by n^* and on the right by n yields $n^* Cn = 0$ and since $C \geqq 0$,

$$Cn = 0 \qquad (3\text{-}10)$$

Now multiply equation (3-1) on the right by n. In view of (3-10) this yields $SA^* n = 0$ and hence $A^* N \subset N$. Thus, if N is not empty, A^* has an eigenvector e such that $e \in N$. By the hypothesis of the lemma, the eigenvalue corresponding to e must have a non-negative real part. This cannot be true since it violates regularity property (c) and therefore N must be empty.

Lemma 2

If (A, B, I) is regular, then there exists $U \geqq 0$ such that $(A - UB)$ is stable.

Proof of lemma: If A is stable the lemma follows by choosing $U = 0$. If A is not stable, choose T such that

$$T^{-1} A T = \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix}$$

where the submatrix A_1 has no stable eigenvalues and the submatrix A_4 is stable, and let

$$T^* B T = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

Since (A, B, I) is regular, it follows that no eigenvector of A_1 is a null vector of B_1 and therefore that $(-A_1^*, I, B_1)$ is regular.

Let $S_1 \geqq 0$ be a solution of the matrix quadratic

$$-A_1^* S_1 - S_1 A_1 - S_1^2 + B_1 = 0$$

S_1 exists since the coefficient of the second degree term in S_1 is definite. By the preceding lemma S_1 is nonsingular. Therefore

$$A_1 S_1^{-1} + S_1^{-1} A_1^* - S_1^{-1} B_1 S_1^{-1} + I = 0$$

and by Theorem 3-3, $A_1 - S_1^{-1} B_1$ is stable.

Finally, let

$$U = T \begin{bmatrix} S_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} T^* \qquad (3\text{-}11)$$

Then

$$T^{-1}(A - U B) T = \begin{bmatrix} A_1 - S_1^{-1} B_1 & -S_1^{-1} B_2 \\ 0 & A_2 \end{bmatrix} \qquad (3\text{-}11)$$

The matrix on the right-hand side of (3-11) is stable since its diagonal blocks are stable and therefore $(A - UB)$ is stable.

The existence of S_0 when B is singular may now be proved as follows. By the lemma, there is a matrix $U \geqq 0$ such that $G = A - UB$ is stable. Choose $M \geqq 0$ so that

$$M \geqq AU + UA^* - U B U + C$$

By Theorem 3-1, the equation

$$G H + HG^* = -M$$

has a solution $H \geqq 0$. Then, letting $S_0 = U + H$, it follows that $S_0 \geqq 0$ and

$$f(S_0) = M - AU - UA^* + UBU + HBH - C \geqq HBH \geqq 0$$

Remark: If A is stable, a suitable value of S_0 may be obtained by solving the equation

$$A S_0 + S_0 A^* = -C$$

When A is stable, S may also be found by the method of spectrum factorization due to Wiener.[Ref. 9, 14, 15] Although they are efficient from a computational standpoint, the spectrum factorization formulas look quite formidable when written in terms of the present notation, and the formal relationship between spectrum factorization and the matrix quadratic (3-1) is not clear. This might be an interesting area for further study. S may be written as an elementary function of A, B and C if A is hermitian and B or C is a multiple of the identity matrix. If $B = bI$, then

$$S = b^{-1} \left\{ A + (A^2 + bC)^{1/2} \right\}$$

and if $C = cI$, then

$$S = c \left\{ (A^2 + cB)^{1/2} - A \right\}^{-1}$$

In these formulas, the exponent 1/2 denotes the positive semidefinite square root operation.

Theorem 3-5

If (A, B, C) is regular, S is the positive semidefinite critical point of (1-1) and E(t) is a solution of (1-1) with $E(0) \geqq 0$, then $E(t) \to S$ exponentially fast as $t \to \infty$.

Proof: Let $E_0(t)$ be the solution of (1-1) with $E_0(0) = 0$. Also let $H > 0$ satisfy

$$H > S$$

and

$$H \geqq E(0)$$

and let $E_1(t)$ be the solution of (1-1) with $E_1(0) = H$. Then by Corollary 2 of Theorem 2-1 it follows that

$$E_0(t) \leqq E(t) \leqq E_1(t) \qquad (3\text{-}12)$$

It will be proved below that $E_0(t)$ and $E_1(t)$ approach S exponentially fast as $t \to \infty$. Thus E(t) is squeezed between $E_0(t)$ and $E_1(t)$ and approaches S exponentially fast. This may be shown formally as follows. By (3-12)

$$0 \leqq E - E_0 \leqq E_1 - E_0$$

and taking operator norms

$$\begin{aligned} \left\| E - E_0 \right\| &\leqq \left\| E_1 - E_0 \right\| \\ &\leqq \left\| E_1 - S \right\| + \left\| S - E_0 \right\| \end{aligned}$$

and finally

$$\left\| F - S \right\| \leqq \left\| E - E_0 \right\| + \left\| E_0 - S \right\|$$

or

$$\left\| E - S \right\| \leqq \left\| E_1 - S \right\| + 2 \left\| E_0 - S \right\| \tag{3-13}$$

If $\left\| E_1 - S \right\| = O(e^{-at})$ and $\left\| E_0 - S \right\| = O(e^{-at})$, then in view of (3-13), $\left\| E - S \right\| = O(e^{-at})$.

Lemma 1

If E(t) is a solution of (1-1) with $E(0) = H > S$ then $E(t) \to S$ exponentially fast as $t \to \infty$.

Proof of lemma: Let

$$X(t) = E(t) - S$$

X(t) satisfies the differential equation

$$X' = FX + XF^* - XBX \tag{3-2}$$

with the initial condition

$$X(0) = H - S > 0$$

By Theorem 2-1,

$$X(t) \geqq 0 \tag{3-14}$$

Let Y(t) be the solution of the differential equation

$$Y' = FY + YF^*$$

with initial condition

$$Y(0) = H - S$$

Then, assuming X is known, Y - X satisfies the differential equation

$$(Y - X)' = F(Y - X) + (Y - X)F^* + XBX$$

with the initial condition

$$Y(0) - X(0) = 0$$

By Theorem 2-1 and (3-14) it follows that

$$Y(t) \geqq X(t) \geqq 0$$

Finally, it may be verified by substitution that

$$Y(t) = e^{Ft} (H - S) e^{F^*t}$$

Thus

$$\left\| X(t) \right\| \leqq \left\| Y(t) \right\| \leqq \left\| e^{Ft} \right\|^2 \left\| H - S \right\|$$

and, since (by Theorem 3-3) F is stable,

$$X(t) = E(t) - S \to 0$$

exponentially fast as $t \to \infty$.

Lemma 2

If F(t) is a solution of (1-1) and E(0) = 0, then $E(t) \to S$ exponentially fast as $t \to \infty$.

Proof of lemma: The case when S > 0 will be treated first. The idea of the proof is derived from the fact that if equation (1-1) has a negative definite critical point $S^{(-)}$, then a hermitian initial condition $E(0) > S^{(-)}$ results in a solution of (1-1) which converges to S. Although equation (1-1) does not have a negative definite critical point in general, the formal matrix quadratic satisfied by ${S^{(-)}}^{-1}$ always has a negative semi-definite solution (which may be a singular matrix) if (A, B, C) is regular. This is the motivation for finding the solution of equation (3-15).

Consider the matrix quadratic equation

$$A^{*}U + UA - UCU + B = 0 \qquad (3\text{-}15)$$

(A^{*}, C, B) is regular since (A, B, C) is regular, and by Theorem 3-4, equation (3-15) has a solution $U \geqq 0$. Let

$$W = S^{-1} - (S + SUS)^{-1}$$

Since S and S + SUS are nonsingular, it follows that (I + US) and (I + SU) are nonsingular and that

$$W = U(I + SU)^{-1} = (I + US)^{-1}U \qquad (3\text{-}16)$$

Employing (3-1), (3-15) and (3-16) it may be verified that W satisfies the equation

$$F^{*}W + WF = -B \qquad (3\text{-}17)$$

Since F^{*} is stable,

$$W \geqq 0$$

by Theorem 3-1. From the definition of W it follows that

$$S^{-1} - W > 0 \qquad (3\text{-}18)$$

Finally, let

$$X(t) = F(t) - S \qquad (3\text{-}19)$$

X(t) satisfies equation (3-2) with the initial condition $X(0) = -S$. It may be verified by substitution, employing equation (3-17), that

$$X(t) = -e^{Ft} Z^{-1}(t)e^{F^{*}t} \qquad (3\text{-}20)$$

with

$$Z(t) = S^{-1} - W + e^{F^{*}t} We^{Ft}$$

Since $W \geqq 0$,

$$e^{F^{*}t} We^{Ft} \geqq 0 \qquad (3\text{-}21)$$

By (3-18) and (3-21), Z(t) is nonsingular for all values of t and

$$\left\| Z^{-1}(t) \right\| \leqq \left\| \left\{ S^{-1} - W \right\}^{-1} \right\| \tag{3-22}$$

Thus by (3-19), (3-20) and (3-22)

$$\left\| E(t) - S \right\| \leqq \left\| e^{-Ft} \right\|^{2} \left\| \left\{ S^{-1} - W \right\}^{-1} \right\|$$

and $E(t) \rightarrow S$ exponentially fast as $t \rightarrow \infty$ since F is a stable matrix.

The case when S is singular will be treated by transforming equation (1-1) to a more convenient form. Assume that the rank of S is k with $k < n$. Then there is a nonsingular n by n matrix T such that

$$T^{*} S T = \tilde{S} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \tag{3-23}$$

The partitioned matrix on the extreme right hand side of equation (3-23) consists of a k by k identity matrix in the upper left hand block and three other blocks containing zeros. Employing the same partitioning, let

$$\tilde{A} = T^{*} A T^{*-1} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \tag{3-24}$$

$$\tilde{B} = T^{-1} B T^{*-1} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \tag{3-25}$$

$$\tilde{C} = T^{*} C T = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \tag{3-26}$$

and define

$$\tilde{F} = T^{*} F T^{*-1} \tag{3-27}$$

and

$$\tilde{E}(t) = T^* E(t) T \tag{3-28}$$

Under this transformation, all of the preceeding equations retain the same form with tilde matrices replacing non-tilde matrices. In the statistical filtering application this transformation represents a change of basis in the domain of the stochastic process x(t) being estimated. Thus, the transformation does not change the qualitative aspects of the problem. In particular, F is stable if and only if $\tilde{F}$ is stable since they have the same Jordan normal form, and (A, B, C) is regular if and only if $(\tilde{A}, \tilde{B}, \tilde{C})$ is regular.

In terms of partitioned matrices, the tilde version of equation (3-1) takes the form

$$0 = \begin{bmatrix} A_1 + A_1^* - B_1 + C_1 & A_3^* + C_2 \\ A_3 + C_3 & C_4 \end{bmatrix} \tag{3-29}$$

Thus, $C_4 = 0$ and $C_2 = C_3 = 0$ since $\tilde{C} \geqq 0$. This in turn implies that $A_3 = 0$. Hence, $\tilde{A}$ and $\tilde{C}$ may be written as

$$\tilde{A} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix} \tag{3-30}$$

and

$$\tilde{C} = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}$$

It may be verified by substitution that, if $\tilde{E}(t)$ is the solution of the tilde version of (1-1) with $\tilde{E}(0) = 0$, then

$$\tilde{E}(t) = \begin{bmatrix} E_1(t) & 0 \\ 0 & 0 \end{bmatrix} \tag{3-31}$$

where $E_1(t)$ satisfies the differential equation

$$E_1' = A_1E_1 + E_1A_1^* - E_1B_1E_1 + C_1 \qquad (3\text{-}32)$$

with the initial condition

$$E_1(0) = 0$$

From the upper left hand block in equation (3-29), it follows that $S_1 = I$ is a critical point of equation (3-32). In order to apply the proof above for nonsingular S to equation (3-32) it is necessary to show that (A_1, B_1, C_1) is regular.

Regularity property (a) holds since principal submatrices of positive semidefinite matrices are positive semidefinite. If u is an eigenvector of A_1 whose eigenvalue has a non-negative real part, then

$$e = \begin{bmatrix} u \\ \\ 0 \end{bmatrix}$$

is an eigenvector of $\tilde{A}$ with the same eigenvalue. By regularity property (b) for $(\tilde{A}, \tilde{B}, \tilde{C})$, it follows that

$$e^*B\,e = u^*\,B_1\,u \neq 0$$

Therefore (A_1, B_1, C_1) has regularity property (b).

Let u be an eigenvector of A_1^* corresponding to an eigenvalue c with a non-negative real part. Since A_4 is stable, the matrix $cI - A_4^*$ is nonsingular. Let

$$v = (cI - A_4^*)^{-1}\,A_2^*\,u$$

and let

$$e = \begin{bmatrix} u \\ v \end{bmatrix}$$

Then

$$\tilde{A}^{*} e = c\, e$$

and by regularity property (c) for $(\tilde{A}, \tilde{B}, \tilde{C})$ it follows that

$$e^{*} \tilde{C} e = u^{*} C_1 u \neq 0$$

Therefore (A_1, B_1, C_1) has regularity property (c) and the proof of the regularity of (A_1, B_1, C_1) is complete.

By the proof above for a nonsingular S, it follows that

$$E_1(t) \to S_1 = I \tag{3-33}$$

exponentially fast as $t \to \infty$.

Thus

$$\tilde{E}(t) \to \tilde{S}$$

exponentially fast in view of (3-23), (3-31) and (3-33), and transforming back to non-tilde matrices the conclusion of the lemma follows.

The net result of Section 3 is the following:

<u>Theorem</u>

If (A, B, C) is regular, equation (1-1) has a unique positive semidefinite critical point S. This critical point is persistent and if E(t) is a solution of equation (1-1) with $E(0) \geqq 0$, then $E(t) \to S$ exponentially fast as $t \to \infty$.

REFERENCES

1. W. T. Reid, A Matrix Differential Equation of the Riccati Type, Am. J. Math., 68 (1946) pp. 237-246.

2. J. J. Levin, On the Matrix Riccati Equation, Proc. American Mathematical Society, Vol. 10, 1959, pp. 519-524.

3. R. E. Kalman, Contributions to the Theory of Optimal Control, Proceedings of the Conference on Ordinary Differential Equations, Mexico City, Mexico, 1959; Bol. Soc. Mat. Mex., 1961.

4. Kiyosi Ito, Stationary Random Distributions, Mem. Coll. Sci. Univ. Kyoto, Ser. A. Math 28, pp. 209-223 (1954).

5. I. M. Gel'fand, Generalized Random Processes, Dokl. Akad Nauk SSSR (N. S.) 100, pp. 853-856 (1955)(Russian).

6. R. E. Kalman and R. S. Bucy, New Results in Linear Filtering and Prediction Theory, Journal of Basic Engineering, March 1961, pp. 95-108.

7. A. E. Bryson, Jr. and D. E. Johansen, Linear Filtering for Time Varying Systems Using Measurements Containing Colored Noise, Joint AIAA - IMS - SIAM - ONR Symposium on Control and System Optimization, U. S. Naval Postgraduate School, Monterey, Calif., Jan. 27, 1964.

8. J. J. Deyst, Optimum Continuous Estimation of Nonstationary Random Variables, Master's Thesis, Department of Aeronautics and Astronautics, MIT, 1964.

9 N. Wiener, Extrapolation, Interpolation and Smoothing of Stationary Time Series, Technology Press - Wiley, 1949.

10. E.A. Coddington and N. Levinson, Ordinary Differential Equations, McGraw - Hill, New York, N.Y., 1955.

11. Joseph LaSalle and Solomon Lefschetz, Stability by Liapunov's Direct Method with Applications, Academic Press, New York and London, 1961.

12. F.R. Gantmacher, Applications of the Theory of Matrices, Interscience, New York and London, 1959.

13. B. v. Sz.Nagy, Spektraldarstellung Linearer Transformationen des Hilbertschen Raumes, Berlin, 1942.

14. D.C. Youla, On the Factorization of Rational Matrices, IRE Trans. on Information Theory, vol. IT-7, July, 1961.

15. M.C. Davis, Factoring the Spectral Matrix, IEEE Transactions on Automatic Control, October, 1963.