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by Elisabeth A. Cooper

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UNIVERSITY OF CALIFORNIA
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Elisabeth A. Cooper

1. Introduction

A study has been carried out on the propagation of magnetohydrodynamic waves in a plasma bounded by a vacuum or a neutral gas, in order to improve understanding of the propagation of these waves in the magnetosphere and their characteristics as observed on the surface of the earth.

For simplicity a two-dimensional problem, consisting of a magnetized plasma bounded by two planes, is considered here. Previous relevant work is discussed in Section 2 and the basic equations and boundary conditions are derived in Section 3. In Section 4 expressions are found for the field variables of waves in a plasma, a vacuum and a neutral gas, and in Sections 5 and 6 these waves are matched at the plasma boundaries. The plasma waves separate into principal modes which may propagate in the plasma without inducing any external fields, and modes labelled TE which generate fields outside the plasma. Under the requirement that the waves outside the plasma should be outgoing or damped a consistency condition is obtained which, together with the dispersion relation, determines the possible TE modes. It is shown that for these modes propagation is only possible for a limited range of values of ω/k, the phase velocity along the plasma slab.

These results show that the waves which may propagate in a bounded plasma must satisfy more stringent conditions than waves in an infinite plasma, and that it may be necessary to use bounded plasma theory in interpretation of magnetohydrodynamic wave observations.

2. Previous Work

It is well known that the types of waves which may propagate in a magnetized plasma vary with the ratio of the wave frequency ω to the plasma frequencies and the gyrofrequencies. We are here concerned with waves having frequencies much less than the ion gyrofrequency ω_{ci}, so that terms of the order of $^{\omega}/\omega_{ci}$ may be neglected. This is known as the magnetohydrodynamic frequency range, and a general discussion of magnetohydrodynamic waves may be found in FERRARO and PLUMPTON (1961). Many authors have considered the propagation of these waves in infinite media, their reflection at an interface between two semi-infinite media (PRIDMORE-BROWN, 1963), their excitation by an incident electromagnetic wave (TURCOTTE and SCHUBERT, 1961) etc. The propagation of these waves in the magnetosphere and ionosphere has been discussed by FEJER (1960), MACDONALD (1961), and KARPLUS et al (1962), among others, but without inclusion of any possible guided waves.

Previous work on the modes possible in bounded plasmas has often been concerned with the high frequency range which is important in laboratory investigations of plasmas. For instance DAWSON and OBERMAN (1959) investigated the possible modes in a plasma slab and a cylinder under the assumption that the ion motion was negligible. BERS (1963) considered in great detail the propagation of waves in plasma wave guides, but mentioned only briefly the magnetohydrodynamic limit of his results, referring for more details to NEWCOMB (1957) and GAJEWSKI (1959).

NEWCOMB (1957) considered the problem of magnetohydrodynamic waves propagating along an axially magnetized circular cylinder of infinitely conducting plasma bounded by conducting rigid walls. He assumed that the plasma particle pressure was much less than the magnetic pressure and included particle pressure effects only through a perturbation treatment. Three types of modes appeared, named TE (transverse electric), principal and sound-like respectively. The effects of a finite plasma conductivity were discussed briefly by NEWCOMB, and in more detail by SHMOYS and MISHKIN (1960), who identified the principal modes as the limiting form of TM (transverse magnetic) modes.

LUDFORD (1959) discussed resonant magnetohydrodynamic waves in an infinitely conducing plasma confined in a rectangular cavity with conducting rigid walls. His solutions separate into two sets, one of which could be named principal modes and the other of which appears to be a combination of TE and sound-like modes.

GAJEWSKI (1959) considered magnetohydrodynamic waves in a cylinder of arbitrary cross-section with generators parallel to the constant magnetic field. He used the general boundary conditions

$$\underline{n} \cdot \underline{v} \big|_C = 0, \quad \underline{n} \times \underline{v} \big|_C = 0$$

where $\underline{n}$ is the unit normal to the wave-guide boundary C and $\underline{v}$ is the velocity, arguing that the solution in any particular physical situation is an appropriate combination of these two sets of solutions. For either boundary condition the solutions divide into "inhomogeneous" modes, which correspond to NEWCOMB's principal modes, and "homogeneous" modes which are either longitudinal (L), transverse and longitudinal acoustic (TLA) or transverse and longitudinal magnetic (TLM). The L and TLA modes correspond to NEWCOMB's sound-like modes, and the TLM to NEWCOMB's TE modes. GAJEWSKI discussed briefly the form taken by these modes for rigid, perfectly conducting walls and for rigid insulating walls.

GAJEWSKI and MAWARDI (1960) extended these results to a cylindrical cavity with rigid ends and found the possible resonant modes, which consisted of a set of principal modes, and a set of combined TLM and TLA modes, in agreement with LUDFORD. This combination arises because a TLA or TLM wave reflected from a boundary excites both a TLA and a TLM reflected wave unless the constant magnetic field is parallel to the boundary (PRIDMORE-BROWN, 1960).

WOODS (1962, 1964) considered the possible modes in a cylindrical wave-guide with rigid walls which could be either conducting or insulating, and included the effects of neutral gas collisions and of viscosity and finite conductivity. In his 1964 paper particular attention is given to the boundary conditions when the plasma has large but finite conductivity and the walls are insulators.

In summary, the general types of magnetohydrodynamic waves which may exist in a cylindrical wave guide of arbitrary cross-section have been determined, but applications have been made generally to wave guides with rigid walls, as required for laboratory experiments, and little attention has been paid to the form of the fields generated outside the wave guide.

3. Basic Equations and Boundary Conditions

We consider a fully ionized gas in the presence of a static magnetic field, bounded by a vacuum or a neutral gas. Within the plasma we assume infinite conductivity, zero viscosity and a scalar pressure p; ρ, $\underline{v}$, σ, and $\underline{j}$ denote the material density and velocity, the charge density and the current density, and $\underline{E}$ and $\underline{B}$ are the components of the electromagnetic field. The basic equations are then

$$\nabla \times \underline{B} = \frac{4\pi}{c} \underline{j} + \frac{1}{c} \frac{\partial \underline{E}}{\partial t} \tag{1}$$

$$\nabla \cdot \underline{B} = 0 \tag{2}$$

$$\nabla \times \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t} \tag{3}$$

$$\nabla \cdot \underline{E} = 4\pi\sigma \tag{4}$$

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\underline{v}) = 0 \tag{5}$$

$$\rho \frac{dv}{dt} = -\nabla p + \sigma\underline{E} + \frac{1}{c}\underline{j} \times \underline{B} \tag{6}$$

$$\underline{E} + \frac{1}{c}\underline{v} \times \underline{B} = 0 \tag{7}$$

In addition as equation of state we use the adiabatic law

$$\frac{1}{p}\frac{dp}{dt} = \frac{\gamma}{\rho}\frac{d\rho}{dt} \tag{8}$$

In a vacuum $p = 0$, $\rho = 0$ and equations (1) - (4) are valid with $\underline{j} = 0$, $\sigma = 0$.

In a neutral gas equations (1) - (6) and (8) are valid with $\underline{j} = 0$, $\sigma = 0$.

The plasma boundary is assumed to have zero thickness and to contain possible surface charges and currents. We are thus neglecting the various skin depths such as the ion Larmor radius (STIX, 1962, pg. 72) which would appear in an exact theory.

Let $\underline{n}$ denote the unit normal at the plasma boundary, directed into the plasma, and let σ^*, $\underline{j}^*$ denote the surface charge and current densities respectively. By integration of equations (1) - (4) across the plasma boundary the following boundary conditions may be obtained (KRUSKAL and SCHWARZSCHILD, 1954; STIX, 1962, pg. 73)

$$\begin{aligned}
\underline{n} \times (\underline{B}^p - \underline{B}^v) &= \frac{4\pi}{c}\underline{j}^* - \frac{u}{c}(\underline{E}^p - \underline{E}^v) \\
\underline{n} \cdot (\underline{B}^p - \underline{B}^v) &= 0 \\
\underline{n} \times (\underline{E}^p - \underline{E}^v) &= \frac{u}{c}(\underline{B}^p - \underline{B}^v) \\
\underline{n} \cdot (\underline{E}^p - \underline{E}^v) &= 0
\end{aligned} \tag{9}$$

where

$$u = \underline{n} \cdot \underline{v}^p = \underline{n} \cdot \underline{v}^v \tag{10}$$

and the superscripts p and v refer to the plasma and vacuum (or neutral gas) quantities respectively.

Equations (5) and (7) add nothing, but equation (6) gives

$$\frac{1}{c} \underline{j}^* \times \left(\frac{\underline{B}^p + \underline{B}^v}{2} \right) + \sigma^* \left(\frac{\underline{E}^p + \underline{E}^v}{2} \right) - \underline{n} (p^p - p^v) = 0 \tag{11}$$

In addition it is easily shown that

$$\frac{d\underline{n}}{dt} = \underline{n} \times (\underline{n} \times \nabla u) \tag{12}$$

For a vacuum $p^v = 0$, and $\underline{v}^v$ is undefined. Boundary conditions (9) - (12) are quite general and apply to any motion of the plasma. They simplify considerably when applied to a static plasma undergoing small perturbations.

Let $\underline{B} = \underline{B}_o + \underline{B}_1$, $p = p_o + p_1$, $\rho = \rho_o + \rho_1$, $\underline{n} = \underline{n}_o + \underline{n}_1$, $\underline{v} = \underline{v}_1$, and and $\underline{E} = \underline{E}_1$, where $\underline{B}_o$, p_o, ρ_o, and $\underline{n}_o$ are constants in either medium. To zero order equations (9) - (12) become

$$\begin{aligned} &\underline{n}_o \times (\underline{B}_o^p - \underline{B}_o^v) = \frac{4\pi}{c} \underline{j}_o^* \\ &\underline{n}_o \cdot (\underline{B}_o^p - \underline{B}_o^v) = 0 \\ &\frac{1}{c} \underline{j}_o^* \times \left(\frac{\underline{B}_o^p + \underline{B}_o^v}{2} \right) = \underline{n}_o (p_o^p - p_o^v) \\ &\sigma_o^* = 0 \\ &u_o = 0 \end{aligned} \tag{13}$$

From these we deduce that

either

$$\underline{B}_o^p = \underline{B}_o^v, \; p_o^p = p_o^v, \; \underline{j}_o^* = 0 \tag{14}$$

or

$$\underline{n}_o \cdot \underline{B}_o^p = \underline{n}_o \cdot \underline{B}_o^v = 0$$

$$p_o^p + \frac{(B_o^p)^2}{8\pi} = p_o^v + \frac{(B_o^v)^2}{8\pi} \qquad (15)$$

For a plasma-vacuum boundary it is impossible to satisfy (14) if $p_o^p \neq 0$, and equations (15) must therefore apply. For a plasma-neutral gas boundary either equations (14) or (15) may apply.

To first order equations (9) - (12) become

$$\underline{n}_o \times (\underline{B}_1^p - \underline{B}_1^v) + \underline{n}_1 \times (\underline{B}_o^p - \underline{B}_o^v) = \frac{4\pi}{c} \underline{j}_1^*$$

$$\underline{n}_o \cdot (\underline{B}_1^p - \underline{B}_1^v) + \underline{n}_1 \cdot (\underline{B}_o^p - \underline{B}_o^v) = 0$$

$$\underline{n}_o \times (\underline{E}_1^p - \underline{E}_1^v) = \frac{u_1}{c} (\underline{B}_o^p - \underline{B}_o^v)$$

$$\underline{n}_o \cdot (\underline{E}_1^p - \underline{E}_1^v) = 4\pi \sigma_1^*$$

$$\frac{1}{c} \underline{j}_1^* \times \left(\frac{\underline{B}_o^p + \underline{B}_o^v}{2}\right) + \frac{1}{c} \underline{j}_o^* \times \left(\frac{\underline{B}_1^p + \underline{B}_1^v}{2}\right) = \underline{n}_o (p_1^p - p_1^v) + n_1 (p_o^p - p_o^v)$$

$$u_1 = \underline{n}_o \cdot \underline{v}_1^p = \underline{n}_o \cdot v_1^v$$

$$\frac{\partial \underline{n}_1}{\partial t} = \underline{n}_o \times (\underline{n}_o \times \nabla u_1)$$

If equations (14) hold and $\underline{n}_o \cdot \underline{B}_o \neq 0$, these equations simplify to give

$$\underline{B}_1^p = \underline{B}_1^v$$

$$p_1^p = p_1^v$$

$$\underline{n}_o \times \underline{E}_1^p = \underline{n}_o \times \underline{E}_1^v \qquad (16)$$

$$\underline{n}_o \cdot (\underline{E}_1^p - \underline{E}_1^v) = 4\pi \sigma_1^*$$

$$u_1 = \underline{n}_o \cdot \underline{v}_1^p = \underline{n}_o \cdot \underline{v}_1^v$$

$$\frac{\partial \underline{n}_1}{\partial t} = \underline{n}_o \times (\underline{n}_o \times \nabla u_1)$$

whereas if $\underline{n}_o \cdot \underline{B}_o^p = \underline{n}_o \cdot \underline{B}_o^v = 0$, we obtain

$$
\begin{aligned}
&\underline{n}_o \cdot \underline{B}_1^p = - \underline{n}_1 \cdot \underline{B}_o^p \\
&\underline{n}_o \cdot \underline{B}_1^v = - \underline{n}_1 \cdot \underline{B}_o^v \\
&p_1^p + \frac{\underline{B}_o^p \cdot \underline{B}_1^p}{4\pi} = p_o^v + \frac{\underline{B}_o^v \cdot \underline{B}_1^v}{4\pi} \\
&\underline{n}_o \times (\underline{E}_1^p - \underline{E}_1^v) = \frac{u_1}{c} (\underline{B}_o^p - \underline{B}_o^v) \\
&\underline{n}_o \cdot (\underline{E}_1^p - \underline{E}_1^v) = 4\pi \sigma_1^* \\
&u_1 = \underline{n}_o \cdot \underline{v}_1^p = \underline{n}_o \cdot \underline{v}_1^v \\
&\frac{\partial \underline{n}_1}{\partial t} = \underline{n}_o \times (\underline{n}_o \times \nabla u_1)
\end{aligned}
\tag{17}
$$

4. Waves in a Plasma, a Vacuum and a Neutral Gas

We now consider the propagation of small perturbations along a plasma slab bounded by the planes $x = 0, a$. This plasma slab may be regarded as a cylindrical waveguide having rectangular cross-section of infinite width. Expressions are to be obtained for the field variables corresponding to the possible magnetohydrodynamic waves. Substitution from (13) into the plasma equations (1) - (8) yields the first order equations

$$
\begin{aligned}
&\nabla \times \underline{B}_1^p = \frac{4\pi}{c} \underline{j}_1^p + \frac{1}{c} \frac{\partial \underline{E}_1^p}{\partial t} \\
&\nabla \cdot \underline{B}_1^p = 0 \\
&\nabla \times \underline{E}_1^p = - \frac{1}{c} \frac{\partial \underline{B}_1^p}{\partial t} \\
&\nabla \cdot \underline{E}_1^p = 4\pi \sigma_1^p \\
&\frac{\partial \rho_1^p}{\partial t} + \rho_o^p \nabla \cdot \underline{v}_1^p = 0 \\
&\rho_o^p \frac{\partial \underline{v}_1^p}{\partial t} = \nabla p_1^p + \frac{1}{c} \underline{j}_1^p \times \underline{B}_o^p
\end{aligned}
\tag{18}
$$

$$\underline{E}_1^p + \frac{1}{c}\, \underline{v}_1^p \times \underline{B}_o^p = 0$$

$$\frac{1}{p_o^p} \frac{\partial p_1^p}{\partial t} = \frac{\gamma}{\rho_o^p} \frac{\partial \rho_1^p}{\partial t} \qquad (18) \text{ cont.}$$

We assume that $\underline{B}_o^p$ is directed along the z-axis

$$\underline{B}_o^p = (0,\ 0,\ \underline{B}_o^p)$$

and that $\frac{\partial}{\partial y} = 0$, i.e. that all propagation is in the z-direction.

In component form, dropping the subscript 1, equations (18) may be written as

$$\begin{aligned} E_x^p &= -\frac{B_o^p}{c} v_y^p \\ E_y^p &= \frac{B_o^p}{c} v_x^p \\ E_z^p &= 0 \end{aligned} \qquad (19)$$

$$\begin{aligned} j_x^p &= -\frac{c}{4\pi} \frac{\partial B_y^p}{\partial z} + \frac{B_o^p}{4\pi c} \frac{\partial v_y^p}{\partial t} \\ j_y^p &= \frac{c}{4\pi} \left(\frac{\partial B_x^p}{\partial z} - \frac{\partial B_z^p}{\partial x} \right) - \frac{B_o^p}{4\pi c} \frac{\partial v_x^p}{\partial t} \\ j_z^p &= \frac{c}{4\pi} \frac{\partial B_y^p}{\partial x} \end{aligned} \qquad (20)$$

$$\begin{aligned} \frac{\partial B_x^p}{\partial t} &= B_o^p \frac{\partial v_x^p}{\partial z} \\ \frac{\partial B_y^p}{\partial t} &= B_o^p \frac{\partial v_y^p}{\partial z} \\ \frac{\partial B_z^p}{\partial t} &= -B_o^p \frac{\partial v_x^p}{\partial x} \end{aligned} \qquad (21)$$

$$\frac{1}{\gamma p_o^p} \frac{\partial p^p}{\partial t} + \frac{\partial v_x^p}{\partial x} + \frac{\partial v_z^p}{\partial z} = 0 \qquad (22)$$

$$\frac{\partial v_x^p}{\partial t} = -\frac{1}{\rho_o^p}\frac{\partial p^p}{\partial x} + \frac{B_o^p}{4\pi\rho_o^p}\left(\frac{\partial B_x^p}{\partial z} - \frac{\partial B_z^p}{\partial x}\right) - \frac{(B_o^p)^2}{4\pi\rho_o^p c^2}\frac{\partial v_x^p}{\partial t}$$

$$\frac{\partial v_y^p}{\partial t} = \frac{B_o^p}{4\pi\rho_o^p}\frac{\partial B_y^p}{\partial z} - \frac{(B_o^p)^2}{4\pi\rho_o^p c^2}\frac{\partial v_y^p}{\partial t} \qquad (23)$$

$$\frac{\partial v_z^p}{\partial t} = -\frac{1}{\rho_o^p}\frac{\partial p^p}{\partial z}$$

Our variables separate into two independent sets (v_y^p, B_y^p) and $(p^p, v_x^p, v_z^p, B_x^p, B_z^p)$.

For the first set, from equations (21) and (23)

$$\left[(1 + V_A^2/c^2)\frac{\partial^2}{\partial t^2} - V_A^2\frac{\partial^2}{\partial z^2}\right](v_y^p, B_y^p) = 0$$

where

$$V_A^2 = \frac{(B_o^p)^2}{4\pi\rho_o^p}$$

is the square of the Alfvén velocity. In general $V_A^2/c^2 << 1$ and is often neglected. It appears here through the inclusion of the displacement current in equation (1). For brevity we write

$$V_A' = V_A\left(1 + V_A^2/c^2\right)^{-1/2}$$

The solutions for v_y^p, B_y^p, and, from (19) and (20), for E_x^p, j_x^p, and j_z^p have the form

$$v_y^p = f(x)\, e^{-i\omega\left[t - z/V_A'\right]}$$

$$B_y^p = -B_o^p/V_A'\, f(x)\, e^{-i\omega\left[t - z/V_A'\right]}$$

$$E_x^p = -B_o^p/c\, f(x)\, e^{-i\omega\left[t - z/V_A'\right]} \qquad (24)$$

$$j_x^p = \frac{i\omega c\, B_o^p}{4\pi V_A^2}\, f(x)\, e^{-i\omega\left[t - z/V_A'\right]}$$

$$j_z^p = -\frac{c\, B_o^p}{4\pi V_A'}\, f'(x)\, e^{-i\omega\left[t - z/V_A\right]}$$

where f(x) is an arbitrary function. The remaining variables are all zero for this set of modes which we call principal modes (NEWCOMB, 1957). Both the velocity and the electromagnetic components of these modes are transverse to the direction of propagation, and all the modes propagate at the modified Alfvén velocity V_A'.

The remaining solutions are given by setting $v_y^p = B_y^p = 0$, and using equations (21), (22) and (23). Elimination of p^p, and B_x^p and B_z^p from these equations gives

$$\left(1 + V_A^2/c^2\right) \frac{\partial^2 v_x^p}{\partial t^2} = c_o^2 \left(\frac{\partial^2 v_x^p}{\partial x^2} + \frac{\partial^2 v_z^p}{\partial x \partial z}\right) + V_A^2 \left(\frac{\partial^2 v_x^p}{\partial x^2} + \frac{\partial^2 v_x^p}{\partial z^2}\right)$$

$$\frac{\partial^2 v_z^p}{\partial t^2} = c_o^2 \left(\frac{\partial^2 v_x^p}{\partial x \partial z} + \frac{\partial^2 v_z^p}{\partial z^2}\right)$$

where

$$c_o^2 = \frac{\gamma p_o^p}{\rho_o^p}$$

is the square of the acoustic velocity.

Hence

$$\left\{\left(1 + V_A^2/c^2\right) \frac{\partial^4}{\partial t^4} - \left[\left(c_o^2 + V_A^2\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{c_o^2 V_A^2}{c^2} \frac{\partial^2}{\partial z^2}\right] \frac{\partial^2}{\partial t^2} + c_o^2 V_A^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) \frac{\partial^2}{\partial z^2}\right\}$$

$$(v_x^p, v_z^p) = 0$$

Thus solutions of the form

$$e^{i(kz - \omega t \pm rx)}$$

exist where

$$\left(1 + V_A^2/c^2\right)\omega^4 - \left[(c_o^2 + V_A^2)(k^2 + r^2) + \frac{c_o^2 V_A^2 k^2}{c^2}\right]\omega^2 + c_o^2 V_A^2 k^2 (k^2 + r^2) = 0 \quad (25)$$

The two roots of this equation for ω^2 correspond to the slow and fast magneto-hydrodynamic waves. One root is sometimes called magneto-acoustic since it becomes a sound wave in the limit $\frac{V_A}{c_o} \longrightarrow 0$.

From (25)

$$r^2 = \frac{\left[\left(1 + V_A^2/c^2\right)\omega^2 - V_A^2 k^2\right]\left[\omega^2 - c_o^2 k^2\right]}{\omega^2 (c_o^2 + V_A^2) - c_o^2 V_A^2 k^2} \qquad (26)$$

Thus r = 0 when $\omega^2 = V_A'^2 k^2,\ c_o^2 k^2$. For real ω and k, r is real if

a) $\omega^2/k^2 >$ maximum $\left[c_o^2,\ V_A'^2\right]$

or

b) minimum $\left[c_o^2,\ V_A'^2\right] > \omega^2/k^2 > \frac{c_o^2 V_A^2}{c_o^2 + V_A^2}$

For all other positive values of ω^2/k^2, r is pure imaginary.

The values of the field components corresponding to solutions of (25) are as follows, where A_r is a constant:

$$v_x^p = A_{o1} e^{-i\omega(t - z/V_A')} + \sum_{r\neq 0} (A_r e^{irx} + A_{-r} e^{-irx}) e^{i(kz - \omega t)}$$

$$v_z^p = A_{oz} e^{-i\omega(t - z/c_o)} + \sum_{r\neq 0} \frac{krc_o^2}{\omega^2 - c_o^2 k^2} (A_r e^{irx} - A_{-r} e^{-irx}) e^{i(kz - \omega t)}$$

$$B_x^p = -\frac{B_o^p}{V_A'} A_{o1} e^{-i\omega(t - z/V_A')} - \sum_{r\neq 0} \frac{B_o^p k}{\omega} (A_r e^{irx} + A_{-r} e^{-irx}) e^{i(kz - \omega t)}$$

$$B_z^p = \sum_{r\neq 0} \frac{B_o^p r}{\omega} (A_r e^{irx} - A_{-r} e^{-irx}) e^{i(kz - \omega t)}$$

$$E_y^p = \frac{B_o^p}{c} A_{o1} e^{-i\omega(t - z/V_A')} + \sum_{r\neq 0} \frac{B_o^p}{c} (A_r e^{irx} + A_{-r} e^{-irx}) e^{i(kz - \omega t)}$$

$$p^p = c_o \rho_o^p A_{oz} e^{-i\omega(T - z/c_o)} + \sum_{r\neq 0} \frac{r\omega\rho_o^p c_o^2}{\omega^2 - c_o^2 k^2} (A_r e^{irx} - A_{-r} e^{-irx}) e^{i(kz - \omega t)}$$

$$j_y^p = -\frac{i\omega B_o^p c}{4\pi V_A^2} A_{o1} e^{-i\omega(t - z/V_A')} + \sum_{r\neq 0} \frac{iB_o^p}{4\pi\omega c} (\omega^2 - c^2k^2 - c^2r^2)(A_r e^{irx} + A_{-r} e^{-irx}) e^{i(kz - \omega t)}$$

(27)

We call these modes transverse electric (TE modes), since for $r \neq 0$ the magnetic field has a longitudinal component B_z^p. These modes include Newcomb's TE and sound-like modes. Note that the mode for which $r = 0$, $\omega^2 = V_A^2 k^2$ has only transverse velocity and electromagnetic components, while the mode for which $r = 0$, $\omega^2 = c_o^2 k^2$ is a pure acoustic wave.

The electromagnetic waves in a vacuum or a neutral gas satisfy equations (1) - (4) with $\underline{j} = 0$, $\sigma = 0$. If $\frac{\partial}{\partial y} = 0$, these equations separate into two sets for (E_x^v, E_z^v, B_y^v), and (E_y^v, B_x^v, B_z^v) which have solutions

$$
\begin{aligned}
E_x^v &= C e^{i(kz - \omega t + \alpha x)} \\
E_z^v &= -\frac{\alpha}{k} C e^{i(kz - \omega t + \alpha x)} \\
B_y^v &= \frac{\omega}{ck} C e^{i(kz - \omega t + \alpha x)}
\end{aligned}
\tag{28}
$$

and

$$
\begin{aligned}
E_y^v &= D e^{i(kz - \omega t + \alpha x)} \\
B_x^v &= -\frac{ck}{\omega} D e^{i(kz - \omega t + \alpha x)} \\
B_z^v &= \frac{c\alpha}{\omega} D e^{i(kz - \omega t + \alpha x)}
\end{aligned}
\tag{29}
$$

where C and D are constants and

$$\alpha^2 = \omega^2/c^2 - k^2 \tag{30}$$

In the neutral gas acoustic waves may propagate in addition to electromagnetic waves. These waves satisfy equations (5), (6), and (8) with $\underline{j} = 0$, $\sigma = 0$. To first order these equations may be written

$$
\begin{aligned}
\frac{1}{\gamma p_o^v} \frac{\partial p^v}{\partial t} &= -\frac{\partial v_x^v}{\partial x} - \frac{\partial v_z^v}{\partial z} \\
\frac{\partial v_x^v}{\partial t} &= -\frac{1}{\rho_o^v} \frac{\partial p^v}{\partial x} \\
\frac{\partial v_z^v}{\partial t} &= -\frac{1}{\rho_o^v} \frac{\partial p^v}{\partial z}
\end{aligned}
$$

and have the solutions

$$p^v = d\, e^{i(kz - \omega t + \beta x)}$$

$$v_x^v = \frac{\beta}{\omega \rho_o^v}\, d\, e^{i(kz - \omega t + \beta x)} \qquad (31)$$

$$v_z^v = \frac{k}{\omega \rho_o^v}\, d\, e^{i(kz - \omega t + \beta x)}$$

where d is a constant and

$$\beta^2 = \omega^2 / c_o^2 - k^2 \qquad (32)$$

The signs chosen for α and β in solutions (28), (29), and (31) depend upon the requirements of the problem.

5. Modes in a Plasma Bounded by a Vacuum

We now assume that the regions $x \geq a$, $x \leq 0$ are vacuum regions, and apply the boundary conditions obtained in Section 3 to match the magnetohydrodynamic waves with the vacuum waves.

The unit normal to the static boundaries is

$$\underline{no} = (\underline{+} 1, 0, 0) \quad \text{on } x = \begin{cases} 0 \\ a \end{cases} \qquad (33)$$

Since $p_o^p \neq 0$, boundary conditions (15) must be satisfied. Therefore

$$B_{ax}^p = B_{ax}^v = 0,$$
$$(B_a^v)^2 = 8\pi p_o^p + (B_o^p)^2$$

We have already chosen

$$\underline{B}_o^p = (0, 0, B_o^p)$$

For simplicity we choose $\underline{B}_o^v$ parallel to $\underline{B}_o^p$

$$\underline{B}_o^v = \left(0, 0, \left[8\pi p_o^p + (B_o^p)^2\right]^{1/2}\right) \qquad (34)$$

The plasma particle pressure is thus supported by additional magnetic pressure in the vacuum.

The first order boundary conditions are obtained from equations (17). For variations of the form $e^{i(kz - \omega t)}$ on $x = \begin{cases} 0 \\ a \end{cases}$ these become

$$u_1 = \pm v_x^p$$

$$\underline{n}_1 = (0, 0, \pm \frac{k v_x^p}{\omega})$$

$$B_x^p = -\frac{k B_o^p}{\omega} v_x^p$$

$$B_x^v = -\frac{k B_o^v}{\omega} v_x^p$$

$$p^p + \frac{B_o^p}{4\pi} B_z^p = \frac{B_o^v}{4\pi} B_z^v$$

$$E_z^p - E_z^v = 0$$

$$E_y^p - E_y^v = \frac{(B_o^p - B_o^v)}{c} v_x^p$$

Using equation (19) for E_y^p and E_y^v, and noting from (21) that $B_x^p = -\frac{kB_o^p}{\omega} v_x^p$ throughout the plasma, and from (29) that $B_x^v = \frac{-ck}{\omega} E_y^v$ throughout the vacuum, we see that these conditions reduce to

$$E_z^v = 0$$

$$E_y^v = \frac{B_o^v}{c} v_x^p \qquad (35)$$

$$B_z^v = \frac{4\pi p^p + B_o^p B_z^p}{B_o^v}$$

on x = 0, a.

The principal modes of equations (24) and the TE modes of equations (27) are to be matched to the vacuum modes of equations (28) and (29), under the requirement that the vacuum waves are to be outgoing or damped. Therefore in the region $x \geq a$ the sign of α must be chosen such that if $\omega^2/k^2 > c^2$, $\alpha > 0$ while if $\omega^2/k^2 < c^2$, $i\alpha < 0$. Under this sign convention the electromagnetic waves in the region $x \geq a$ vary as $e^{i(kz - \omega t + \alpha x)}$, while those in the region $x \leq 0$ vary as $e^{i(kz - \omega t - \alpha x)}$

For the principal modes

$$v_x^p = p^p = B_z^p = 0$$

throughout the plasma. Therefore on $x = 0, a$, from (35)

$$E_z^v = E_y^v = B_z^v = 0$$

From (28) and (29) for the vacuum fields we see that this implies

$$E_x^v = E_z^v = B_y^v = 0$$

since $\alpha^2 = \omega^2/c^2 - k^2 \neq 0$, and that

$$E_y^v = B_x^v = B_z^v = 0$$

Thus there are no electromagnetic waves in the vacuum associated with the principal modes.

Let

$$f(x) = \sum_{n=0}^{\infty} \left(a_n \sin \frac{n\pi x}{a} + b_n \cos \frac{n\pi x}{a}\right)$$

where a_n and b_n are arbitrary constants. The principal modes may then be written as

$$v_y^p = \sum_{n=0}^{\infty} \left(a_n \sin \frac{n\pi x}{a} + b_n \cos \frac{n\pi x}{a}\right) e^{-i\omega(T - z/V_A')}$$

$$B_y^p = -\sum_{n=0}^{\infty} \frac{B_o^p}{V_A'} \left(a_n \sin \frac{n\pi x}{a} + b_n \cos \frac{n\pi x}{a}\right) e^{-i\omega(t - z/V_A')}$$

$$E_x^p = -\sum_{n=0}^{\infty} \frac{B_o^p}{c} \left(a_n \sin \frac{n\pi x}{a} + b_n \cos \frac{n\pi x}{a}\right) e^{-i\omega(t - z/V_A')} \tag{36}$$

$$j_x^p = \sum_{n=0}^{\infty} \frac{i\omega c B_o^p}{4\pi V_A^2} \left(a_n \sin \frac{n\pi x}{a} + b_n \cos \frac{n\pi x}{a}\right) e^{-i\omega(t - z/V_A')}$$

$$j_z^p = -\sum_{n=0}^{\infty} \frac{cB_o^p}{4\pi V_A'} \cdot \frac{n\pi}{a} \left(a_n \cos \frac{n\pi x}{a} - b_n \sin \frac{n\pi x}{a}\right) e^{-i\omega(t - z/V_A')}$$

for $0 \leq x \leq a$, while $\underline{E}_1 \equiv 0$, $\underline{B}_1 \equiv 0$ for $x \geq a$, $x \leq 0$.

Next we require the vacuum fields associated with the TE modes in the plasma.

Since $E_z^v = 0$ on $x - 0, a$, from (35), we see from (28) that

$$E_x^v = E_z^v = B_y^v = 0$$

throughout the vacuum unless $\omega^2 = c^2k^2$. Since boundary conditions (35) place no restrictions on E_x^v and B_y^v, any field of the form

$$E_x^v = C e^{-i\omega(t - z/c)}$$

$$E_z^v = 0$$

$$B_y^v = C e^{-i\omega(t - z/c)}$$

may exist in the vacuum regions without affecting the plasma fields.

The values of E_y^v, B_x^v, and B_z^v are more strictly determined. Consider fir the region $x \geq a$. From (27) and (35) we find that on $x = a$

$$E_y^v = \frac{B_o^v}{c} A_{o1} e^{-i\omega(t - z/V_A')} = \sum_{r \neq 0} \left(A_r e^{ira} + A_{-r} e^{-ira}\right) \frac{B_o^v}{c} e^{i(kz - \omega t)} \tag{37}$$

and

$$B_z^v = \frac{4\pi c_o \rho_o^p}{B_o^v} A_{o2} e^{-i\omega(t - z/c_o)} + \sum_{r \neq 0} \frac{4\pi \rho_o^p r}{\omega B_o^v} \left[\frac{\omega^2(c_o^2 + V_A^2) - c_o^2 V_A^2 k^2}{\omega^2 - c_o^2 k^2}\right] (A_r e^{ira} - A_{-r} e^{-ira}) e^{i(kz - \omega t)} \tag{38}$$

From (29) the vacuum fields for $x \geq a$ may be written as

$$E_y^v = D_{o1} e^{-i\omega(t - z/V_A') + i\alpha_{o1}x} + D_{oz} e^{-i\omega(t - z/c_o) + i\alpha_{o2}x} + \sum_{r \neq 0} D_r e^{i(kz - \omega t + \alpha_r x)} \tag{39}$$

and

$$B_z^v = \frac{c\,a_{o1}}{\omega} D_{o1} e^{-i\omega(t - z/V_A') + i a_{o1} x} + \frac{c\,a_{o2}}{\omega} D_{o2} e^{-i\omega(t - z/c_o) + i a_{o2} x} + \sum_{r \neq 0} \frac{c a_r}{\omega} D_r e^{i(kz - \omega t + a_r x)} \tag{40}$$

where

$$a_{o1}^2 = -\omega^2/V_A^2$$

$$a_{o2}^2 = \omega^2 \left(\frac{1}{c^2} - \frac{1}{c_o^2}\right)$$

$$a_r^2 = \omega^2/c^2 - k^2$$

and the constants D_r are to be determined. Comparison of coefficients of $e^{i(kz - \omega t)}$ on $x = a$, from (37), (38), (39), and (40) then yields

$$\begin{aligned} D_{o1} &= 0 \\ D_{o2} &= 0 \\ D_r &= \frac{B_o^v}{c} (A_r e^{ira} + A_{-r} e^{-ira}) e^{-i a_r a} \end{aligned} \tag{41}$$

and for consistency

$$A_{o1} = 0$$

$$A_{02} = 0$$

$$\frac{A_r e^{ira} + A_{-r} e^{-ira}}{A_r e^{ira} - A_{-r} e^{-ira}} = \frac{r}{a_r V_{AV}^2} \left[\frac{\omega^2 (c_o^2 + V_A^2) - c_o^2 V_A^2 k^2}{\omega^2 - c_o^2 k^2}\right] \tag{42}$$

where

$$V_{AV}^2 = (B_o^v)^2 / 4\pi \rho_o^p$$

We have thus shown that the two TE modes given by $r = 0$ cannot propagate in the plasma-vacuum system, while the modes for $r \neq 0$ are subject to the restriction of equation (42). When the matching process is repeated for the $x = 0$ boundary, the field in the vacuum region $x \leq 0$ becomes

$$E_y^v = \sum_{r \neq 0} \frac{B_o^v}{c} (A_r + A_{-r}) e^{i(kz - \omega t - a_r x)} \tag{43}$$

etc., while the consistency condition for this boundary is

$$\frac{A_r + A_{-r}}{A_r - A_{-r}} = - \frac{r}{a_r V_{AV}^2} \left[\frac{\omega^2 (c_o^2 + V_A^2) - c_o^2 V_A^2 k^2}{\omega^2 - c_o^2 k^2} \right] \tag{44}$$

Combination of (42) and (44) yields

$$A_r^2 e^{ira} = A_{-r}^2 e^{-ira}$$

We therefore set

$$A_r = \frac{p_r}{2} e^{-ira/2}, A_{-r} = \pm \frac{p_r}{2} e^{ira/2} \tag{45}$$

where p_r is a constant. Equation (42) (or (44)) may then be written as

$$ir \begin{Bmatrix} \tan \\ -\cot \end{Bmatrix} ra/2 = \frac{a_r V_{AV}^2 (\omega^2 - c_o^2 k^2)}{\omega^2 (c_o^2 + V_A^2) - c_o^2 V_A^2 k^2} \tag{46}$$

which, together with (26), determines the possible propagating modes. From (26) we see that for real ω and k, r is either real or pure imaginary, and $r \begin{Bmatrix} \tan \\ -\cot \end{Bmatrix} ra/2$ is therefore always real. Consistent solutions of (26) and (46) are thus only possible if a_r is imaginary or zero, i.e., if

$$\omega^2/k^2 \leq c^2$$

The fields corresponding to these transverse electric modes are as follows:

Within the plasma, for $0 \leq x \leq a$,

$$\begin{aligned}
v_x^p &= \sum_{r \neq 0} p_r \begin{Bmatrix} \cos \\ i \sin \end{Bmatrix} r(x - a/2) \, e^{i(kz - \omega t)} \\
v_z^p &= \sum_{r \neq 0} \frac{rkc_o^2}{\omega^2 - c_o^2 k^2} \left\{ p_r \begin{matrix} i \sin \\ \cos \end{matrix} r(x - a/2) \right\} e^{i(kz - \omega t)} \\
B_x^p &= -\sum_{r \neq 0} \frac{B_o^p k}{\omega} p_r \begin{Bmatrix} \cos \\ i \sin \end{Bmatrix} r(x - a/2) \, e^{i(kz - \omega t)} \\
B_z^p &= \sum_{r \neq 0} \frac{rB_o^p}{\omega} p_r \begin{Bmatrix} i \sin \\ \cos \end{Bmatrix} r(x - a/2) \, e^{i(kz - \omega t)}
\end{aligned} \tag{47}$$

$$E_y^{\,p} = \sum_{r\neq 0} \frac{B_o^{\,p}}{c}\; p_r \begin{Bmatrix} \cos \\ i \sin \end{Bmatrix} r\,(x - a/2)\; e^{\,i\,(kz - \omega t)}$$

$$p^p = \sum_{r\neq 0} \frac{r\omega \rho_o^{\,p}\, c_o^{\,2}}{\omega^2 - c_o^{\,2} k^2}\; p_r \begin{Bmatrix} i \sin \\ \cos \end{Bmatrix} r(x - a/2)\; e^{\,i(kz - \omega t)} \qquad (47)\ \text{cont.}$$

$$j_y^{\,p} = \sum_{r\neq 0} \frac{iB_o^{p}}{4\pi\omega c} \left(\omega^2 - c^2 \overline{k^2 + r^2}\right) p_r \begin{Bmatrix} \cos \\ i \sin \end{Bmatrix} r(x - a/2)\; e^{\,i(kz - \omega t)}$$

In the vacuum region $x \geq a$

$$E_y^{\,v} = \sum_{r\neq 0} \frac{B_o^{\,v}}{c}\; p_r \begin{Bmatrix} \cos \\ i \sin \end{Bmatrix} ra/2 \; e^{\,i\left[kz - \omega t + \alpha_r (x - a)\right]}$$

$$B_x^{\,v} = -\sum_{r\neq 0} \frac{kB_o^{\,v}}{\omega}\; p_r \begin{Bmatrix} \cos \\ i \sin \end{Bmatrix} ra/2 \; e^{\,i\left[kz - \alpha t + \alpha_r (x - a)\right]} \qquad (48)$$

$$B_z^{\,v} = \sum_{r\neq 0} \frac{\alpha_r B_o^{\,v}}{\omega}\; p_r \begin{Bmatrix} \cos \\ i \sin \end{Bmatrix} ra/2 \; e^{\,i\left[kz - \omega t + \alpha_r (x - a)\right]}$$

and in the vacuum region $x \leq 0$

$$E_y^{\,v} = \sum_{r\neq 0} \frac{B_o^{\,v}}{c}\; p_r \begin{Bmatrix} \cos \\ -i \sin \end{Bmatrix} ra/2 \; e^{\,i(kz - \omega t - \alpha_r x)}$$

$$B_x^{\,v} = -\sum_{r\neq 0} \frac{k\,B_o^{\,v}}{\omega}\; p_r \begin{Bmatrix} \cos \\ -i \sin \end{Bmatrix} ra/2 \; e^{\,i(kz - \omega t - \alpha_r x)} \qquad (49)$$

$$B_z^{\,v} = -\sum_{r\neq 0} \frac{\alpha_r B_o^{\,v}}{\omega}\; p_r \begin{Bmatrix} \cos \\ -i \sin \end{Bmatrix} ra/2 \; e^{\,i(kz - \omega t - \alpha_r x)}$$

6. Modes in a Plasma Bounded by a Neutral Gas

We now assume that the regions $x \geq a$, $x \leq 0$ contain neutral gas with pressure and density equal to that of the plasma, so that we may write

$$p_o^p = p_o^v = p_o$$
$$\rho_o^p = \rho_o^v = \rho_o \qquad (50)$$

The acoustic velocity c_o is assumed to have the same value in the plasma and the neutral gas.

The unit normal to the static boundaries is

$$\underline{n}_o = (\pm 1,\ 0,\ 0) \text{ on } x = \begin{Bmatrix} 0 \\ a \end{Bmatrix}$$

and the zero order boundary condition (14) (or (15)) are satisfied by taking

$$\underline{B}_o^{\,v} = \underline{B}_o^{\,p} = (0,\ 0,\ B_o)$$

In this case the neutral gas particle pressure supports the plasma particle pressure.

Since $\underline{n}_o \cdot \underline{B}_o = 0$, first order boundary conditions (17) apply. By use of equations (19), (21), and (29) these may be reduced to

$$\begin{aligned} v_x^{\,p} &= v_x^{\,v} \\ E_z^{\,v} &= 0 \\ E_y^{\,v} &= \frac{B_o}{c}\, v_x^{\,p} \\ p^v + \frac{B_o B_z^{\,v}}{4\pi} &= p^p + \frac{B_o B_z^{\,p}}{4\pi} \end{aligned} \qquad (51)$$

on x = 0, 2 for variations of the form $e^{i(kz - \omega t)}$

The principal modes of equations (24) and the TE modes of equations (27) are to be matched to the electromagnetic and acoustic neutral gas modes given by (28), (29), and (31), under the requirement that the neutral gas modes are to be outgoing or damped. Therefore the sign of α is chosen as in the vacuum case, while the sign of β is chosen such that if $\omega^2/k^2 > c_o^{\,2}$, $\beta > 0$ while if $\omega^2/k^2 < c_o^{\,2}$, $i\beta < 0$. Under this sign convention the acoustic waves in the region $x \geq a$ vary as $e^{i(kz - \omega t + \beta x)}$, while those in the region $x \leq 0$ vary as $e^{i(kz - \omega t - \beta x)}$.

For the principal modes

$$v_x^{\,p} = p^p = B_z^{\,p} = 0$$

throughout the plasma. Hence from (51)

$$\left.\begin{aligned} v_x^{\,v} &= 0 \\ E_z^{\,v} &= 0 \\ E_y^{\,v} &= 0 \\ p^v + \frac{B_o B_z^{\,v}}{4\pi} &= 0 \end{aligned}\right\} \quad \text{on } x = 0,\ a$$

and from (28), (29), and (31) this implies that there are no waves in the neutral gas associated with the principal modes, which have the form given in equations (36).

Next we consider the TE modes. Since $E_z^v = 0$ on $x = 0, a$, we may set

$$E_x^v = E_z^v = B_y^v = 0$$

throughout the neutral gas.

For the remaining variables we consider first the region $x \geq a$. On $x = a$, from (27) and (51)

$$v_x^v = A_{o1}\, e^{-i\omega(t - z/V_A')} + \sum_{r\neq 0} (A_r\, e^{ira} + A_{-r}\, e^{-ira})\, e^{i(kz - \omega t)} \tag{52}$$

$$E_y^v = \frac{B_o}{c} A_{o1}\, e^{-i\omega(t - z/V_A')} + \sum_{r\neq 0} \frac{B_o}{c} (A_r\, e^{ira} + A_{-r}\, e^{-ira})\, e^{i(kz - \omega t)} \tag{53}$$

$$p^v + \frac{B_o B_z^v}{4\pi} = c_o \rho_o A_{o2}\, e^{-i\omega(t - z/c_o)} + \sum_{r\neq 0} \frac{r\rho_o}{\omega(\omega^2 - c_o^2 k^2)} \left[\omega^2(c_o^2 + V_A^2) - c_o^2 V_A^2 k^2\right] (A_r\, e^{ira} - A_{-r}\, e^{-ira})\, e^{i(kz - \omega t)} \tag{54}$$

From (29) and (31) the fields for $x \geq a$ may be written as

$$E_y^v = D_{o1}\, e^{-i\omega(t - z/V_A') + i\alpha_{o1}x} + D_{o2} e^{-i\omega(t - z/c_o) + i\alpha_{o2}x} + \sum_{r\neq 0} D_r\, e^{i(kz - \omega t + \alpha_r x)} \tag{55}$$

$$B_z^v = \frac{c\,\alpha_{o1}}{\omega} D_{o1} e^{-i\omega(t - z/V_A') + i\alpha_{o1}x} + \frac{c\,\alpha_{o2}}{\omega} D_{o2}\, e^{-i\omega(t - z/c_o) + i\alpha_{o2}x} + \sum_{r\neq 0} \frac{c\,\alpha_r}{\omega} D_r\, e^{i(kz - \omega t + \alpha_r x)} \tag{56}$$

and

$$p^v = d_{o1} e^{-i\omega(t - z/V_A') + i\beta_{o1} x} + d_{o2} e^{-i\omega(t - z/c_o)} + \sum_{r \neq 0} d_r e^{i(kz - \omega t + i\beta_r x)} \tag{57}$$

$$v_x^v = \frac{\beta_{o1}}{\omega \rho_o} d_{o1} e^{-i\omega(t - z/V_A') + i\beta_{o1} x} + \sum_{r \neq 0} \frac{\beta_r}{\omega \rho_o} d_r e^{i(kz - \omega t + \beta_r x)} \tag{58}$$

where α_r is defined as in Section 5,

$$\beta_{o1}^2 = \omega^2 \left(\frac{1}{c_o^2} - \frac{1}{V_A'^2} \right)$$

$$\beta_{o2}^2 = 0$$

$$\beta_r^2 = \omega^2 / c_o^2 - k^2$$

and ω and k satisfy (25) for each value of r. The constants D_r and d_r are to be determined. Comparison of coefficients of $e^{i(kz - \omega t)}$ on $x = a$ then gives, from (52) - (58)

$$\begin{aligned} D_{o1} &= 0 \\ D_{02} &= 0 \\ D_r &= \frac{B_o}{c} (A_r e^{ira} + A_{-r} e^{-ira}) e^{-i\alpha_r a} \end{aligned} \tag{59}$$

$$\begin{aligned} d_{o1} &= 0 \\ d_{02} &= c_o \rho_o A_{o2} \\ d_r &= \frac{\omega \rho_o}{\beta_r} (A_r e^{ira} + A_{-r} e^{-ira}) e^{-i\beta_r a} \end{aligned} \tag{60}$$

and for consistency

$$A_{o1} = 0$$

$$\frac{A_r e^{ira} + A_{-r} e^{-ira}}{A_r e^{ira} - A_{-r} e^{-ira}} = \frac{r \left[\omega^2 (c_o^2 + V_A^2) - c_o^2 V_A^2 k^2 \right]}{(\alpha_r V_A^2 + \omega^2 / \beta_r)(\omega^2 - c_o^2 k^2)} \tag{61}$$

This matching process must be repeated for the other boundary $x = 0$, as in the vacuum case. Note that the transverse wave for which $r = 0$, $\omega^2 = (V_A')^2 k^2$ cannot propagate in the plasma-neutral gas system, but that the acoustic wave for which $r = 0$, $\omega^2 = c_o^2 k^2$ is able to propagate in this system.

From the consistency conditions for the two boundaries we find

$$A_r^2 e^{ira} = A_{-r}^2 e^{-ira}$$

and we may therefore set

$$A_r = \frac{p_r}{2} e^{-ira/2}, \quad A_{-r} = \pm \frac{p_r}{2} e^{ira/2} \tag{62}$$

The consistency condition (61) then becomes

$$ir\begin{Bmatrix}\tan \\ -\cot\end{Bmatrix} ra/2 = \frac{(\alpha_r V_A^2 + \omega^2/\beta_r)(\omega^2 - c_o^2 k^2)}{\omega^2 (c_o^2 + V_A^2) - c_o^2 V_A^2 k^2} \tag{63}$$

which together with (26) determines the possible propagating modes. From (26) we see that for real ω and k, r is either real or pure imaginary and $r\begin{Bmatrix}\tan \\ -\cot\end{Bmatrix} ra/2$ is therefore always real. Consistent solutions of (26) and (63) are thus only possible if $(\alpha_r V_A^2 + \omega^2/\beta_r)$ is imaginary i.e. if

$$\omega^2/k^2 < c_o^2$$

The fields corresponding to these TE modes are as follows:

Within the plasma, for $0 \le x \le a$

$$v_x^p = \sum_{r\neq 0} p_r \begin{Bmatrix}\cos \\ i\sin\end{Bmatrix} r(x - a/2) \, e^{i(kz - \omega t)}$$

$$v_z^p = A_{o2} e^{i\omega(t - z/c_o)} + \sum_{r\neq 0} \frac{r k c_o^2}{\omega^2 - c_o^2 k^2} p_r \begin{Bmatrix}i\sin \\ \cos\end{Bmatrix} r(x - a/2) \, e^{i(kz - \omega t)}$$

$$B_x^p = -\sum_{r\neq 0} \frac{B_o k}{\omega} p_r \begin{Bmatrix}\cos \\ i\sin\end{Bmatrix} r(x - a/2) \, e^{i(kz - \omega t)}$$

$$B_z^p = \sum_{r\neq 0} \frac{r B_o}{\omega} p_r \begin{Bmatrix}i\sin \\ \cos\end{Bmatrix} r(x - a/2) \, e^{i(kz - \omega t)}$$

$$E_y^p = \sum_{r\neq 0} \frac{B_o}{c} p_r \begin{Bmatrix}\cos \\ i\sin\end{Bmatrix} r(x - a/2) \, e^{i(kz - \omega t)}$$

$$p^P = c_o \rho_o A_{o2} e^{-i\omega(t - z/c_o)} + \sum_{r=0} \frac{r\,\omega \rho_o c_o^2}{\omega^2 - c_o^2 k^2} p_r \begin{Bmatrix} i \sin \\ \cos \end{Bmatrix} r(x - a/2) e^{i(kz - \omega t)}$$

$$j_y^P = \sum_{r \neq 0} \frac{i B_o}{4\pi\omega c} \quad \omega^2 - c^2(k^2 + r^2) \quad p_r \begin{Bmatrix} \cos \\ i \sin \end{Bmatrix} r(x - a/2) e^{i(kz - \omega t)} \qquad \text{(64) cont.}$$

In the neutral gas region $x \geq a$

$$E_y^v = \sum_{r \neq 0} \frac{B_o}{c} \quad p_r \begin{Bmatrix} \cos \\ i \sin \end{Bmatrix} ra/2 \; e^{i[kz - \omega t + a_r (x - a)]}$$

$$B_x^v = -\sum_{r \neq 0} \frac{k B_o}{\omega} \quad p_r \begin{Bmatrix} \cos \\ i \sin \end{Bmatrix} ra/2 \; e^{i[kz - \omega t + a_r (x - a)]} \qquad (65)$$

$$B_z^v = \sum_{r \neq 0} \frac{a_r B_o}{\omega} \quad p_r \begin{Bmatrix} \cos \\ i \sin \end{Bmatrix} ra/2 \; e^{i[kz - \omega t + a_r (x - a)]}$$

and

$$p^v = c_o \rho_o A_{o2} \; e^{-i\omega(t - z/c_o)} + \sum_{r \neq 0} \frac{\omega \rho_o}{\beta_r} \quad p_r \begin{Bmatrix} \cos \\ i \sin \end{Bmatrix} ra/2 \; e^{i[kz - \omega t + \beta_r (x - a)]}$$

$$v_x^v = \sum_{r \neq 0} p_r \begin{Bmatrix} \cos \\ i \sin \end{Bmatrix} ra/2 \; e^{i[kz - \omega t + \beta_r (x - a)]} \qquad (66)$$

$$v_z^v = A_{o2} \; e^{-i\omega(t - z/c_o)} + \sum_{r=0} \frac{k}{\beta_r} \quad p_r \begin{Bmatrix} \cos \\ i \sin \end{Bmatrix} ra/2 \; e^{i[kz - \omega t + \beta_r (x - a)]}$$

and similarly for the neutral gas region $x \leq 0$.

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