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NASr-21(07)

MEMORANDUM
RM-4651-NASA
JULY 1965

N65-30837
(ACCESSION NUMBER) (THRU)
11 1
(PAGES) (CODE)
CR 64336 29
(NASA CR OR TMX OR AD NUMBER) (CATEGORY)
FACILITY FORM 602

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GPO PRICE $ ________

CSFTI PRICE(S) $ ________

Hard copy (HC) 1.00

Microfiche (MF) .50

ff 653 July 65

PREPARED FOR:
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

The RAND Corporation
SANTA MONICA • CALIFORNIA

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This research is sponsored by the National Aeronautics and Space Administration under Contract No. NASr-21. This report does not necessarily represent the views of the National Aeronautics and Space Administration.

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Published by The RAND Corporation

PREFACE

This Memorandum is part of RAND's continuing study of Satellite Meteorology for the National Aeronautics and Space Administration under contract NASr-21(07). The relations which are derived will lead to the increasingly effective use of computers in the numerical study of radiative transfer in planetary atmospheres. The Memorandum will be especially useful to atmospheric physicists, astrophysicists, and applied mathematicians.

SUMMARY

A commonly occurring situation in planetary and stellar physics is that parallel rays of radiation illuminate a slab in which both absorption and isotropic scattering take place. The ultimate fate of an incident photon is either to be diffusely reflected, transmitted, or absorbed by the slab. While much has been accomplished in the study of the probability of transmission and reflection, less attention has been paid to the probability of absorption as a function of the thickness of the slab and the angle of incidence.

The authors derive a differential-integral equation for the dissipation function. In addition they derive a conservation relationship connecting the reflection, transmission, and dissipation functions. These relations are useful for both analytic and computational purposes.

CONTENTS

I. INTRODUCTION

The physical situation to be considered in this Memorandum is as follows: parallel rays of radiation are incident on a finite homogeneous slab which absorbs and scatters radiation isotropically. These diffusely transmitted and reflected fields have been studied intensively. In earlier papers(1-3) we have shown the importance of the dissipation function in various analytical studies of transport in a rod. In this Memorandum we derive an equation for the dissipation function of a slab and write the conservation relation which relates the reflection, transmission, and dissipation functions.

II. DERIVATION OF INVARIANT IMBEDDING EQUATION FOR THE ABSORPTION FUNCTION

Consider a plane-parallel, homogeneous and isotropically scattering medium of finite optical thickness τ_1. Suppose that a parallel beam of radiation of constant net flux π per unit area normal to the incident direction is incident on the upper surface $\tau = 0$ at a fixed angle whose cosine is $\mu_0 (0 < \mu_0 \leq 1)$ with respect to the inward normal. We follow the standard nomenclature of Chandrasekhar.(4)

The intensity of radiation which is diffusely reflected from the slab with direction cosine μ is $S(\tau_1;\mu,\mu_0)/4\mu$, and the diffusely transmitted intensity with direction cosine μ is $T(\tau_1;\mu,\mu_0)/4\mu$. The directly transmitted intensity is $\pi \exp(-\tau_1/\mu_0)$ in the direction of incidence.

We define the absorption function L in the following fashion. Let

$\pi L(\tau_1,\mu_o)$ = the rate of production of truly absorbed particles in a cylinder of unit base area extending from $\tau = 0$ to $\tau = \tau_1$, the input having direction cosine μ_o and the net incident flux being π.

(It is clear that L is also the probability of ultimate absorption of a particle with direction cosine μ_o which is incident on a slab of thickness τ_1.) We add an infinitesimal layer of optical thickness Δ to the lower surface τ_1, and we consider its effect on the rate of production of absorbed particles. We obtain the equation

$$\pi L(\tau_1+\Delta,\mu_o)$$
$$= \pi L(\tau_1,\mu_o) + \left[\pi\mu_o\, e^{-\tau_1/\mu_o}\frac{\Delta}{\mu_o} + \int_0^1 \frac{T(\tau_1;\mu',\mu_o)}{4\,\mu'}\,\mu'\,\frac{\Delta}{\mu'}\,2\pi d\mu'\right] \tag{1}$$
$$\cdot\left[(1-\lambda) + \frac{\lambda}{4\pi}\int_0^1 \frac{L(\tau_1,\mu_o')}{\mu_o'}\,2\pi d\mu_o'\right] + o(\Delta),$$

where λ is the albedo for single scattering. The first term on the right-hand side of the equation accounts for the absorption of particles which never enter the thin slab. The second term accounts for those particles which interact in the thin slab and then are absorbed. The first bracketed expression represents the rate of production of interacting particles in the cylinder of unit base area extending from $\tau = \tau_1$ to $\tau = \tau_1 + \Delta$, and the second bracketed expression is the probability that an interacting particle is ultimately absorbed. All other processes have probabilities of order Δ^2 or greater and are accounted for in the term $o(\Delta)$. Letting $\Delta \to 0$, we obtain the partial differential integral equation

$$\frac{\partial L(\tau_1,\mu_o)}{\partial \tau_1} = \left[e^{-\tau_1/\mu_o} + \frac{1}{2}\int_0^1 T(\tau_1;\mu',\mu_o)\,\frac{d\mu'}{\mu'}\right]\cdot\left[1 - \lambda + \frac{\lambda}{2}\int_0^1 L(\tau_1,\mu_o')\,\frac{d\mu_o'}{\mu_o'}\right]. \tag{2}$$

The initial condition is

$$L(0,\mu_o) = 0. \tag{3}$$

III. CHECK FOR THE CASE OF NO REEMISSION

For the case of no reemission of interacting particles, the rate of production of absorbed particles is

$$\begin{aligned}\pi L(\tau_1,\mu_o) &= \int_0^{\tau_1} \pi\, e^{-\tau/\mu_o}\, d\tau \\ &= \pi\,\mu_o(1-e^{-\tau_1/\mu_o})\ .\end{aligned} \tag{4}$$

As a check, we put $\lambda = 0$ in Eq. (2), and the result agrees with Eq. (4).

IV. CONSERVATION RELATIONSHIP

The particles incident on a unit of horizontal area are either directly transmitted, truly absorbed, diffusely reflected, or diffusely transmitted. This leads to the conservation relationship

$$\pi\mu_o = \pi\mu_o\, e^{-\tau_1/\mu_o} + \pi L(\tau_1,\mu_o) + \int_0^1 \frac{S(\tau_1,\mu',\mu_o)}{4\mu'}\,\mu' 2\pi\, d\mu' + \int_0^1 \frac{T(\tau_1,\mu',\mu_o)}{4\mu'}\,\mu' 2\pi\, d\mu', \tag{5}$$

which reduces to

$$1 = e^{-\tau_1/\mu_o} + \frac{L(\tau_1,\mu_o)}{\mu_o} + \frac{1}{2\mu_o}\int_0^1 S(\tau_1,\mu',\mu_o)\,d\mu' + \frac{1}{2\mu_o}\int_0^1 T(\tau_1,\mu',\mu_o)\,d\mu'. \tag{6}$$

V. DISCUSSION

For reference, we present the complete set of integro-differential equations for the diffuse reflection, diffuse transmission and absorption functions. These equations are

$$\frac{\partial S(\tau_1;\mu,\mu_o)}{\partial \tau_1} + \left(\frac{1}{\mu} + \frac{1}{\mu_o}\right)S$$
$$= \lambda\left[1+\frac{1}{2}\int_0^1 S(\tau_1;\mu,\mu_o')\,\frac{d\mu_o'}{\mu_o'}\right]\cdot\left[1+\frac{1}{2}\int_0^1 S(\tau_1;\mu',\mu_o)\,\frac{d\mu'}{\mu'}\right], \tag{7}$$

$$S(0;\mu,\mu_o) = 0, \tag{7'}$$

$$\frac{\partial T(\tau_1;\mu,\mu_o)}{\partial \tau_1} + \frac{1}{\mu}\,T$$
$$= \lambda\left[1+\frac{1}{2}\int_0^1 S(\tau_1;\mu,\mu_o')\,\frac{d\mu_o'}{\mu_o'}\right]\cdot\left[e^{-\tau_1/\mu_o}+\frac{1}{2}\int_0^1 T(\tau_1;\mu',\mu_o)\,\frac{d\mu'}{\mu'}\right], \tag{8}$$

$$T(0;\mu,\mu_o) = 0, \tag{8'}$$

$$\frac{\partial L(\tau_1,\mu_o)}{\partial \tau_1}$$
$$= \left[e^{-\tau_1/\mu_o}+\frac{1}{2}\int_0^1 T(\tau_1;\mu',\mu_o)\,\frac{d\mu'}{\mu'}\right]\cdot\left[1-\lambda+\frac{\lambda}{2}\int_0^1 L(\tau_1,\mu_o')\,\frac{d\mu_o'}{\mu_o'}\right], \tag{9}$$

$$L(\tau_1,\mu_o) = 0. \tag{9'}$$

This set of equations can be approximated via the use of Gaussian quadrature formulas which leads to an effective computational scheme for determining S, T, and L.

The conservation equation, which can be derived analytically as well as physically, connects the solutions of Eqs. (7), (8), and (9).

It can serve as an automatic check on the accuracy of the numerical calculations of S, T, and L. For the inhomogeneous or anisotropic cases, the appropriate equations for the absorption functions are readily derived.

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