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To NASA

For Period Sept 15 - Dec 14, 1964

Contract No. NASw-986

BOUNDED PHASE

COORDINATE CONTROL

RESEARCH LABORATORY

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BOUNDED PHASE COORDINATE CONTROL

Contributors

E. B. Lee
J. Y. Luh

Approved by: E. R. Rang
E. R. Rang
Research Supervisor

Approved by: O. H. Schuck
O. H. Schuck
Director
MPG Research

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INTRODUCTION

This is the second quarterly progress report prepared for the National Aeronautics and Space Administration under Contract No. NASw-986. The body of this report will be primarily qualitative in nature. Preliminary versions of detailed technical reports will be given as appendices. The final report will contain a complete and detailed description and summary of the work carried on under the contract.

The objective of the research being carried out is the development of methods by means of which bounded phase coordinate controllers for large flexible launch boosters can be realized. The areas for investigation are the determination of the zero cost sets, the determination of extremal controls, and computational procedures for finding optimal controls.

SUMMARY OF ACCOMPLISHMENTS

The two efforts during the second quarter under Contract No. NASw-986 were devoted toward continuing the development of computational procedures for optimal controls. The first of these continued from the results reported in the first quarterly progress report. The second more fully developed the approximation procedure outlined on pages 3 to 5 of the first quarterly progress report.

Appendix C of the first progress report presented sufficient and uniqueness conditions for bounded phase coordinate control.

Necessary conditions and singular control were considered during this reporting period. Singular control (control while on the phase boundary) was investigated in an attempt to provide additional information to assist in using the sufficient conditions for finding optimal controls. Examples demonstrate that information on the number of arcs on the phase boundaries and distributions between phase boundaries would assist in using the sufficient conditions to find the optimal controls. The singular control investigation may provide these data.

The appendix of this report presents the theoretical development of a computational algorithm for approximation (by use of a penalty function) of bounded phase coordinate control. Computer simulations to determine its workability were initiated.

AN APPROXIMATION TO LINEAR BOUNDED PHASE COORDINATE CONTROL PROBLEMS*

E. B. Lee†

1. Introduction

In many control problems both restraints on the magnitudes of the control variables and various system variables may occur. Certain results [6] are available for the determination of optimal controllers for some classes of linear and nonlinear systems involving such restraints. These results take the from of necessary and sufficient conditions for optimal control, and are a satisfactory solution to the theoretical problem, but leave much to be desired in the way of a practical solution. To use the necessary and sufficient conditions for synthesizing an optimal controller it is necessary to solve a two-point boundary value problem in terms of a number of free parameters and multipliers where the number of parameters is not even known as well as certain jump conditions [4]. A backing out procedure is also available if one is interested in flooding the domain of controllability with responses and then keeping track (storing) of the corresponding control magnitude for each such point.

We here offer a procedure which has several advantages over the above schemes, but is only an approximate solution. Its main advantage is that no discontinuities will be encountered in the adjoint solution which determines the optimum controller and therefore the resulting two point boundary value problem may be more readily

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*Prepared under contract NASw-986 for the NASA.

†Research Consultant, Honeywell, Roseville, Minnesota, Associate Professor, University of Minnesota

solved. In fact, we offer an outline of a computer routine which should solve such problems. A study is now under way to determine the usefulness of this technique in actually solving such control problems.

2. <u>The approximate linear time optimal control process with bounded phase coordinates</u>. Consider a linear control process described by the differential system

$\mathcal{L})\ \dot{x} = A(t)x + B(t)u(t).$

The coefficient matrices A(t) and B(t) are composed of known continuous functions on the time interval $[t_0,t_1]$. The controller u(t) is to be chosen from a set $\Omega:|u^j| \leq 1; j = 1,2...m$ so as to steer the response, $x_u(t)$, of $\mathcal{L}$) from an initial point x_o at time t_o to a prescribed compact target set $\tilde{G} \subset R^n$ and it is required that $x_u(t)$ remain within a given constraint set, Λ, during its entire response. Here R^n is the n dimensional real number space.

The problem of time optimal control is to find that controller u(t) which steers $x_u(t)$ from x_o to $\tilde{G} \subset \Lambda$ in minimum time, that is, minimizes $C(u) = t_1 - t_o$ with $x_u(t_1) \epsilon \tilde{G}$ and $x_u(t) \epsilon \Lambda$, $t_o \leq t \leq t_1$. Later, in section 5, we discuss other optimum control cost functionals.

There are certain difficulties involved when one directly solves for this optimum controller. We shall therefore be content with solving the following apparently simpler problem: Find that controller u(t) with graph in Ω which steers $x_u(t)$ from x_o at t_o to $\tilde{G}$ at t_1 with $x_u^o(t_1) \leq \beta$ and $t_1 - t_o$ a minimum. $x_u^o(t)$ is defined below.

It is assumed that Λ is a closed convex set, (for convenience we could even let $\Lambda = \{x \mid x'H\,x \leq c\}$, where H is a positive semi-definite matrix and c = constant > 0.) Let F(x) be a convex continuous differentiable function which is such that†

$$\begin{aligned} F(x) &\neq 0 \quad \text{if } x \notin \Lambda \\ &= 0 \quad \text{if } x \in \Lambda \end{aligned}$$

Then define

$$x_u^\circ(t_1) = \int_{t_o}^{t_1} F(x_u(t))dt.$$

$x_u^\circ(t)$ essentially measures the excursions of the response $x_u(t)$ to a controller u(t) outside of the region Λ. By keeping $x_u^\circ(t_1)$ small the response $x_u(t)$ is restricted to stay close to or within Λ. The above minimum time optimal control problem is approximately solved by finding a controller which steers $\hat{x}_u(t) = (x_u^\circ(t), x_u(t))$ from $(0,x_o)$ to $G = \{x^\circ, x \mid x \in \tilde{G},\ 0 \leq x^\circ \leq \beta\}$ in the minimum time interval $t_1 - t_o$ if $\beta > 0$ is sufficiently small.

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†There is of course some question as to whether such a function F(x) exists for an arbritrary convex set Λ contained in R^n. We now cite an example which shows that there are such functions in a number of interesting cases. Suppose $\Lambda = \{x^1, x^2, \ldots x^n \mid |x^2| \leq 1\}$. Then pick

$$\begin{aligned} F(x) &= \tfrac{1}{2}(x^2 - 1)^2 \quad \text{if } x^2 > 1 \\ &= 0 \quad \text{if } |x^2| \leq 1 \\ &= \tfrac{1}{2}(x^2+1)^2 \quad \text{if } x^2 \leq -1 \end{aligned}$$

Thus if only one coordinate (or a linear combination) is restricted the problem is easily handled as in the example, where F(x) is continuous and has continuous partial derivatives. Other Λ's can be approximately handled as in the example.

In the next section we give necessary and sufficient conditions for this approximation problem and in section 4 we describe a computational technique for solving the remaining two point boundary value problem as found in section 3.

3. <u>The necessary and sufficient conditions for the approximate linear time optimal problems</u> We augment the system $\mathcal{L}$ by considering the equation system

$$\hat{\mathcal{L}}) \quad \dot{x}^{\circ} = F(x)$$
$$\dot{x} = A(t)x + B(t)\,u(t)$$

obtained from $\mathcal{L}$) by adding the equation for $\dot{x}^{\circ}$ with $x^{\circ}(t_o) = 0$. Here $A(t)$, $B(t)$ are bounded and continuous on $[t_o,t_1]$ and $F(x)$ is a convex function with $F(x) = 0$ for $x \in \Lambda$. $\frac{\partial F}{\partial x}(x)$ is assumed to exist and be continuous everywhere.

The set of attainability $\hat{K}(t_1) \subset R^{n+1}$ is the collection of end points $\hat{x}_u(t_1)$ of responses $\hat{x}_u(t) = (x^{\circ}_u(t), x_u(t))$ of $\hat{\mathcal{L}}$ which initiate at $(0,x_o)$ at time t_o corresponding to all measurable controllers $u(t)$ which are such that $|u^j(t)| \leq 1$ on $[t_o,t_1]$, for $j = 1,2...,m$. (Such controllers are referred to as admissible controllers.)

In the following theorems we establish various properties for $\hat{K}(t_1)$ and $\partial\hat{K}(t_1)$ as required in synthesizing optimal controllers.

<u>Theorem 1 Consider the above system $\hat{\mathcal{L}}$) with initial point $\hat{x}_o$, restraint set Ω and set of attainability $\hat{K}(t_1)$. Then:</u>

$\hat{K}(t_1)$ is a nonempty compact subset of R^{n+1} in variables (x^0, x) with convex lower surface (as defined below) for each $t_0 \leq t_1 < \infty$.

Proof $\hat{K}(t_1)$ is nonempty since any measurable controller $u(t) \subset \Omega$ gives rise to an end point $\hat{x}_u(t_1) \in \hat{K}(t_1)$. $\hat{K}(t_1)$ is compact because the system $\hat{\mathcal{S}}$) satisfies the hypothesis of the existence theorem 1 of reference 2.

The lower surface of $\hat{K}(t)$ is where exterior normal n+1 vectors $\hat{\eta}$ to $\hat{K}(t)$ at points of $\partial\hat{K}(t)$ have their first component $\eta_0 \leq 0$. We now show that if $\hat{x}_1$ and $\hat{x}_2$ are points of $\hat{K}(t_1)$ then the point $\hat{y} = \lambda\hat{x}_1 + (1-\lambda)\ \hat{x}_2 = (y^\circ, y)$, $0 \leq \lambda \leq 1$, is such that

$$y = x_{\bar{u}}(t_1)$$

and

$$y^\circ \geq x^0_{\bar{u}}(t_1),$$

where $\bar{u}(t) = \lambda\ u_1(t) + (1-\lambda)\ u_2(t)$ and $u_1(t)$ and $u_2(t)$ are such that $\hat{x}_{u_1}(t_1) = \hat{x}_1$ and $\hat{x}_{u_2}(t_1) = \hat{x}_2$. The convexity of the lower surface of $\hat{K}(t_1)$ then follows because in order for it to be nonconvex it is necessary that there exist two points $\hat{x}_1$, $\hat{x}_2$ on this lower boundary, with the property that the point $\lambda\ \hat{x}_1 + (1-\lambda)\ \hat{x}_2$ is below the set $\hat{K}(t_1)$ for some $0 < \lambda < 1$, which will then be impossible.

With $\bar{u}(t) = \lambda\ u_1(t) + (1-\lambda)\ u_2(t)$ we find that

$$x_{\bar{u}}(t_1) = \Phi(t_1)x_o + \int_{t_o}^{t_1} \Phi(t_1)\Phi^{-1}(s)B(s)\bar{u}(s)ds$$

$$= \lambda \left[\Phi(t_1)x_o + \int_{t_o}^{t_1} \Phi(t_1)\Phi^{-1}(s)B(s)u_1(s)ds\right]$$

$$+ (1-\lambda) \left[\Phi(t_1)x_o + \int_{t_o}^{t_1} \Phi(t_1)\Phi^{-1}(s)B(s)u_2(s)ds\right]$$

$$= \lambda\, x_{u_1}(t_1) + (1-\lambda)\, x_{u_2}(t_1) =$$

$$= \lambda\, x_1 + (1-\lambda)\, x_2 = y$$

where $\Phi(t)$ is the fundamental solution matrix of $\mathcal{L}$ with $\Phi(t_o) = I$. We also calculate

$$x^o_{\bar{u}}(t_1) = \int_{t_o}^{t_1} F(x_{\bar{u}}(t))dt$$

and $\lambda\, x^o_{u_1}(t_1) + (1-\lambda)\, x^o_{u_2}(t_1)$ for comparison. Since $F(x)$ is a convex function of x if follows that for $0 \leq \lambda \leq 1$,

$$F(x_{\bar{u}}(t)) = F(\lambda\, x_{u_1}(t) + (1-\lambda)x_{u_2}(t)) \leq \lambda\, F(x_{u_1}(t)) + (1-\lambda)\, F(x_{u_2}(t))$$

and so

$$x^o_{\bar{u}}(t_1) = \int_{t_o}^{t_1} F(x_{\bar{u}}(t))dt = \int_{t_o}^{t_1} F(\lambda\, x_{u_1}(t) + (1-\lambda)\, x_{u_2}(t))dt$$

$$\leq \lambda \int_{t_o}^{t_1} F(x_{u_1}(t))dt + \int_{t_o}^{t_1} (1-\lambda)F(x_{u_2}(t))dt = y^o.$$

QED.

We will now consider those controllers $u(t)$ on $[t_0,t_1]$ which steer $\hat{x}_u(t)$ from $\hat{x}_0$ at t_0 to points $\hat{x}_1$ contained in the lower boundary of $\hat{K}(t_1)$ (written $\partial K^-(t_1)$). Such controllers will be called extremal and they will play a significant part in the selection of optimal controllers.

Let $u(t) \in \Omega$ on $t_0 \leq t \leq t_1$ be a controller for the convex control process

$$\hat{\mathcal{S}}) \quad \dot{x}^0 = F(x)$$

$$\dot{x} = A(t)\,x + B(t)u(t)$$

with initial point $\hat{x}_0 = (0,x_0)$ at t_0. If the corresponding response $\hat{x}_u(t)$ has an end point $\hat{x}(t_1) \in \partial K^-(t_1)$, then $u(t)$ is called an <u>extremal control</u> and $\hat{x}_u(t)$ an <u>extremal response</u> on $[t_0,t_1]$.

The <u>adjoint response</u> $\hat{\eta}(t) = (\eta_0(t), \eta(t))$ corresponding to a controller $u(t)$ is a row n+1 vector satisfying the differential system

$$\dot{\eta} = -\eta\, A(t) - \eta_0 \frac{\partial F'}{\partial x}(x_u(t))$$

$$\eta_0 = \text{constant} \lessgtr 0.$$

where $x_u(t)$ is the response of $\mathcal{S}$) corresponding to the controller $u(t)$. In the following theorem 2 we characterize the extremal controllers, that is, those controllers which steer to the lower surface of $\hat{K}(t_1)$.

Theorem 2 Consider the convex control process†

$$\hat{\mathcal{S}}) \quad \dot{x}^0 = F(x)$$

$$\dot{x} = A(t)x + B(t)u(t)$$

with initial point $\hat{x}_0 = (0,x_0)$ at time t_0. An admissible controller $u(t) \subset \Omega$ on $[t_0,t_1]$ is extremal for $\hat{\mathcal{S}}$ if and only if there exists a nonvanishing adjoint response $\hat{\eta}(t)$ of

$$\dot{\eta} = -\eta A(t) - \eta_0 \frac{\partial F'}{\partial x}(x_u(t))$$

$$\eta_0 = \text{constant} \lessgtr 0$$

so that

$$\eta(t)B(t)u(t) = \max_{u \in \Omega} \{\eta(t)B(t)u\}$$

almost always on $[t_0,t_1]$.

Proof Assume that $u(t)$ on $[t_0,t_1]$ is extremal and so steers $\hat{x}_u(t)$ from $(0,x_0)$ at t_0 to $\hat{x}_1 \in \partial\hat{K}^-(t_1)$.

Since $\hat{K}(t_1)$ is closed with a convex lower surface there exists a support plane π to $\hat{K}(t_1)$ at $\hat{x}_1$. Let $\hat{\eta}(t_1) = (\eta_0,\eta(t_1))$

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† The necessary portion of this theorem follows from L. S. Pontryagin's Maximum Principle(4). For completeness the simple arguments to establish the necessary part are presented.

be a nonzero vector normal to π directed into the halfspace defined by π which does not meet $\hat{K}(t_1)$. Note $\eta^\circ \bar{<} 0$.

Let $\hat{\eta}(t)$ with $\hat{\eta}(t_1)$ as above be the response of the adjoint equation corresponding to the controller $u(t)$.

Consider the admissible controller $\bar{u}(t) = \operatorname{sgn}\{\eta(t)B(t)\}$ defined for $t \in [t_o, t_1]$. Note

$$\eta(t)B(t)\bar{u}(t) = \max_{u \in \Omega} \{\eta(t)B(t)u\}$$

on $[t_o, t_1]$.

Let τ_ϵ be an interval of total length $\epsilon > 0$ contained in $\mathcal{J} = [t_o, t_1]$ whereon

$$\delta + \eta(t)B(t)u(t) < \max_{u \in \Omega} \{\eta(t)B(t)u\}$$

For each $\delta > 0$ consider the modified controller

$$u_\epsilon(t) = u(t) \text{ on } \mathcal{J} - \tau_\epsilon$$

$$\bar{u}(t) \text{ on } \tau_\epsilon,$$

and calculate

$$\frac{d\hat{\eta}(t)\hat{x}_\epsilon}{dt} = \dot{\hat{\eta}}\hat{x}_\epsilon + \eta\dot{\hat{x}}_\epsilon$$

and

$$\frac{d\hat{\eta}(t)\hat{x}}{dt} = \dot{\hat{\eta}}\hat{x} + \hat{\eta}\dot{\hat{x}},$$ where $\hat{x}_\epsilon$ refers to a response of $\hat{\mathcal{X}}$ corresponding to the modified controller $u_\epsilon(t)$.

Integration from t_o to t_1 yields

$$\hat{\eta}((t)\ x_\epsilon(t_1) - \hat{\eta}(t)\hat{x}_\epsilon(t_o) = \int_{t_o}^{t_1} \left[-\eta\ A(t) + \frac{\partial F}{\partial x}(x(t))\right] x_\epsilon(t)$$

$$+ \int_{t_o}^{t_1} \eta(t)\left[A(t)x_{\epsilon(t)} + B(t)u(t)\right] - F(x_\epsilon(t))dt$$

and

$$\hat{\eta}(t_1)\hat{x}(t_1) - \hat{\eta}(T)\ \hat{x}(t_o) = \int_{t_o}^{t_1} \left\{\left[-\eta\ A(t) + \frac{\partial F}{\partial x}\ (x(t))\right] x(t)\right.$$

$$\left. + \eta(t)\left[A(t)x(t) + B(t)u(t)\right] - F(x(t))\right\}dt \text{ for } \eta_o = -1.$$

Combining terms and using the assumed continuity for F and $\frac{\partial F}{\partial x}$ we easily find that

$$\hat{\eta}(t_1)\hat{x}_\epsilon(t_1) - \hat{\eta}(t_1)\hat{x}_1(t_1) \geq \delta\epsilon + o(\epsilon) \text{ for } \epsilon \text{ sufficiently small}$$

where $o(\epsilon)$ corresponds to terms of higher than first order in ϵ.

and therefore for ϵ sufficiently small

$$\hat{\eta}(t)x_{\epsilon}(t_1) - \hat{\eta}(t)\hat{x}_1(t_1) > 0$$ contradicting the construction of $\hat{\eta}(t_1)$ as the outward normal to $K(t_1)$ at $\hat{x}_1$.

Hence there exists no such interval τ_{ϵ} so

$$\eta(t)B(t)u(t) = \underset{u\in\Omega}{\text{Max}}\ \eta(t)B(t)u \text{ almost everywhere on } \mathcal{J}.$$

Conversely, assume that u(t) and corresponding response $\eta(t)$ are such that

$$\eta(t)B(t)u(t) = \underset{u\in\Omega}{\text{Max}}\ \eta(t)Bu$$

a.e. on $\mathcal{J}$ with $\eta_o \lessgtr 0$. Let $\bar{u}(t)$ be any controller in Ω with corresponding response $x_{\bar{u}}(t)$. If we calculate

$$\frac{d\hat{\eta}\hat{x}_u}{dt} \text{ and } \frac{d\hat{\eta}\hat{x}_{\bar{u}}}{dt} \text{ as above,}$$

and then integrate from t_o to t_1 using the assumed convexity of F(x) we find that

$$\hat{\eta}(t_1)\ \hat{x}_u(t_1) \geq \hat{\eta}(t_1)\ \hat{x}_{\bar{u}}(t_1) = \hat{\eta}(t_1)\hat{w}$$

Where $\hat{w} = \hat{x}$ is any point of $\hat{K}(t_1)$. Since $|\eta(t_1)| \neq 0$, and $\eta_o \leq 0$, the above inequality implies that $\hat{x}_u(t_1)$ is contained in the lower boundary of the compact set $\hat{K}(t_1)$ with convex lower boundary and hence u(t) is extremal. QED.

Theorem 2 indicates that to stay at a lower boundary point we must continuously steer maximally in the direction of the vector $\hat{\eta}(t)$. This remark is summarized as a corollary.

Corollary 2.1 Let u(t) on $[t_o, t_1]$ be an extremal controller for $\hat{\mathcal{L}}$, with corresponding response $\hat{x}_u(t)$ and adjoint response $\hat{\eta}(t)$ so that,

$$\eta(t)B(t)u(t) = \underset{u\in\Omega}{\text{Max}}\ \eta(t)B(t)u$$

a.e. on $[t_o, t_1]$. Then on each subinterval $[t_o \tau], \subset [t_o, t_1]$, u(t) is also an extremal controller with $\hat{x}_u(\tau)\epsilon\partial\hat{K}(\tau)$. Moreover $\hat{\eta}(\tau)$ is an exterior normal to $\hat{K}(\tau)$ at $\hat{x}(\tau)$.

Proof Replace t_1 by τ in the proof of theorem 2 to obtain that

$$\hat{\eta}(\tau)\ \hat{x}_u(\tau) \geq \hat{\eta}(\tau)\ \hat{x}_{\bar{u}}(\tau) = \hat{\eta}(\tau)\ \hat{w}(\tau)$$

for all $\hat{w}(\tau)$ in $\hat{K}(\tau)$. From this inequality the conclusion of the corollary can be drawn.

We next show that the set of attainability $\hat{K}(t_1)$ depends continuously on the parameter t_1.

Define the distance between a point p and a compact set $G_1 \subset R^n$ to be

$$d(p,G_1) = \min_{g \in G_1} |p-g|$$

and define the distance between two compact sets G_1, $G_2 \subset R^n$ to be

$$d(G_2,G_2) = \mathrm{Max}\left\{\max_{p_1 \in G_1} d(p_1,G_2), \max_{p_2 \in G_2} d(p_2,G_1)\right\}.$$ Here $|p| = \sum_{i=1}^{n} |p^i|$.

The set $\hat{K}(t_2) \subset R^{n+1}$ varies continuously with t_2 if given an $\epsilon > 0$ there exists a $\delta > 0$ so that for $|t_2-t_1| < \delta$,

$$d(\hat{K}(t_1), \hat{K}(t_2)) < \epsilon$$

<u>Lemma 1: Consider the system $\hat{\mathcal{L}}$ as above with attainable set $\hat{K}(t_1)$.</u>

<u>Then:</u> <u>$\hat{K}(t_1)$ varies continuously with t_1</u>

<u>Proof</u> We need only show that each point $\hat{x}(t_1)$ of $\hat{K}(t_1))$ is close to some point $\hat{x}(t_2)$ of $\hat{K}(t_2)$ and conversely. That is, we need show that given $\epsilon > 0$ there exists a $\delta > 0$ so that when $|t_1 - t_2| < \delta$ there exists $\hat{x}(t_1) \in \hat{K}(t_1)$ such that $|x(t_1) - x(t_2)| < \epsilon$ for each $\hat{x}(t_2) \in \hat{K}(t_2)$ and conversely.

Let $u_1(t)$ be an admissible controller on $[t_o,t_1+1]$ and $\hat{x}_1(t)$ the corresponding response. For $t_1 \leq t_2 \leq t_1 + 1$ calculate

$$x_1^o(t_2) - x_1^o(t_1) = \int_{t_o}^{t_2} F(x_1(t))dt - \int_{t_o}^{t_1} F(x_1(t))dt$$

and

$$x_1(t_2) - x_1(t_1) = \Phi(t_2) \int_{t_o}^{t_2} \Phi(s)^{-1} B(s)u_1(s)ds$$

$$- \Phi(t_2) \int_{t_o}^{t_1} \Phi(s)^{-1}[B(s)u_1(s)]ds$$

$$+ [\Phi(t_2) - \Phi(t_1)]\left[\int_{t_o}^{t_1} \Phi(s)^{-1} B(s)u_1(s)ds\right].$$

So

$$x_1^o(t_2) - x_1^o(t_1) = \int_{t_1}^{t_2} F(x_1(t))dt$$

and

$$x_1(t_2) - x_1(t_1) = \Phi(t_2) \int_{t_1}^{t_2} \Phi(s)^{-1}u_1(s)ds$$

$$+\left[\Phi(t_2) - \Phi(t_1)\right]\left[\int_{t_o}^{t_1} \Phi(s)^{-1} B(s)u_1(s)ds\right]$$

Since $A(t)$ is bounded and continuous on $[t_o, t_1+1]$ so is $\Phi(t)$ and therefore there exists a constant C_1 so that

$$|\Phi(t)| < C_1$$

and

$$|\Phi(t)^{-1}| < C_1 \text{ on } [t_o, t_1+1].$$

Also since $B(s)$ has bounded continuous elements $b_j^i(t)$ and $u_1(t)$ is bounded and measurable there exists the constant C_2 so that

$$\left|\int_{t_o}^{t_1} \Phi(s)^{-1} B(s)\, u_1(s)ds\right| < C_2.$$ Integration is a continuous operation, therefore, given an $\epsilon > 0$ there exists a $\delta > 0$ so that

$$\left|\int_{t_1}^{t} F(x_1(t))dt\right| < \frac{\epsilon}{3},$$

$$\left|\int_{t_1}^{t} \Phi(s)^{-1} B(s)u_1(s)ds\right| < \frac{\epsilon}{3C_2}$$

for $|t-t_1| < \delta < 1$.

Hence

$$|\hat{x}_1(t_2) - \hat{x}_1(t_1)| < \frac{\epsilon}{3} + C_1 \frac{\epsilon}{3C_1} + \frac{\epsilon}{3C_2} C_2 = \epsilon$$

for $|t_2 - t_1| < \delta < 1$.

The other way we consider $u_1(t) = u(t)$ on $[t_o,t_1]$ where $u(t)$ steers to $\hat{x}(t_1)$ and extend it to $[t_o,t_1+1]$ by letting $u_1(t) = u(t_1)$ for $t \in [t_1,t_1+1]$. The above calculation is then repeated to find $|\hat{x}(t_2) - \hat{x}(t_1)| < \epsilon$ for $|t_2-t_1| < \delta < 1$ so $K(t_1)$ varies continuously with t_1.

Theorem 3 Consider the system $\hat{\mathcal{S}}$ as above with initial data $x_o = (0,x_o)$, compact restraint set Ω and set of attainability $\hat{K}(t_1)$. Let the target set $G = \{x^o,x \mid 0 \leq x^o \leq \beta,\ x = \tilde{G}\}$ where $\beta > 0$ is a constant and $\tilde{G}$ is a compact set of R^n. Suppose G meets the interior of $\hat{K}(t_1)$ then there is a $\delta > 0$ such that G meets $\hat{K}(t)$ for $|t - t_1| < \delta$.

Proof Since G meets the interior of $\hat{K}(t_1)$, and a ball neighborhood $N(\hat{p})$ of radius r contained in $\hat{K}(t_1)$. Consider the hyperplane $x^o = p^o - r/2$ and in this plane pick $n+1$ independent points $\hat{x}_1, \hat{x}_2 \ldots \hat{x}_n$, $\hat{x}_{n+1}$ of boundary of the ball $N(\hat{p})$, all equally spaced. Let $\hat{x}_1(t)$, $\hat{x}_2(t), \ldots \hat{x}_n(t)$, $\hat{x}_{n+1}(t)$ be responses of $\hat{\mathcal{S}}$ with initial data $x_o = (0,x_o)$ and corresponding to controllers $u_1(t), u_2(t), \ldots u_{n+1}(t)$ $t_o \leq t \leq t_1 + 1$ which are such that $\hat{x}_1(t_1) = \hat{x}_1, \ldots \hat{x}_{n+1}(t_1) = \hat{x}_{n+1}$. Pick $1 > \delta > 0$ so small that for $|t-t_1| \leq \delta$ the points $\hat{x}_1(t), \ldots \hat{x}_{n+1}(t)$ lie within spheres of radius $r/10$ of the points $\hat{x}_1 \ldots \hat{x}_{n+1}$. This being possible because of the previous lemma 1.

Consider the convex combination of controllers $u_\lambda(t) = \lambda_1 u_1(t) + \lambda_2 u_2(t) + \ldots \lambda_{n+1} u_{n+1}(t) \sum \lambda_i \geq 0$, (Note $|u_\lambda^i| \leq 1$) and the corresponding responses $\hat{x}_\lambda(t)$ of $\mathscr{L}$ with initial data $(0,x_o)$. For each fixed $t, |t-t_1) \leq \delta$ these response end points $x_\lambda(\hat{t})$ sweep out a surface section $\tilde{S}$ which lies below the plane $x^o = p$ by convexity, above the plane $x^o = 0$ because of the positive nature of F and intersect the line segment $\{0 \leq x^o \leq p^o, x = p\}$ (see proof of theorem 1). Hence G meets $\hat{K}(t)$ for $|t-t_1| \lesssim \delta < 1$.

We now consider the problem of existence of optimum controllers.

<u>Theorem 4 Consider the system $\hat{\mathscr{L}}$ as above with compact restraint set $\Omega = \{u| \; |u^i| \leq 1, i=1,2\ldots m\} \subset R^m$, initial point $(0,x_o) \epsilon R^n$ at time t_o and constant compact target set $G = \{x^o, x | 0 \leq x^o \leq \beta, x \epsilon \tilde{G}\}$ for $\beta > 0$. If there exists an admissible controller $u(t) \subset \Omega$ steering x_o to G on $t_o \leq t \leq t_1$ then there exists an optimum controller (also admissible) steering x_o to G in minimum time duration $t^* - t_o$.</u>

<u>Proof</u> If $(0,x_o) \epsilon$ G then $t^* = t_o$ and optimum control is not required. So assume $(0,x_o) \not\epsilon$ G and consider the set of attainability $\hat{K}(t_1)$ for $t_1 \geq t_o$. Since there is one controller which steers

$(0,x_o)$ to G the set $\hat{K}(t_1)$ meets G for some $t_1 > t_o$. Define t^* to be the greatest lower bound of all times t_1 such that $\hat{K}(t_1)$ meets G. By the continuous dependence of $\hat{K}(t_1)$ on t_1 the set of times for which $\hat{K}(t_1)$ meets G is a closed set in R^1. Hence t^* is the first time $\hat{K}(t_1)$ meets G and therefore pick as the optimum controller $u^*(t)$, $t_o \leq t \leq t^*$ a controller which steers to

$K(t^*) \cap G$.

The next theorem asserts that for optimum control we need only consider points of the lower boundary of the set of attainability and therefore by theorem 2 extremal controllers. A sufficiency condition is also included.

Theorem 5. Consider the system $\hat{\mathcal{S}}$ as above with compact rectangular restraint set Ω, initial point $(0,x_o)$ at t_o and compact convex target set $G = \{x^o_1 x \mid 0 \leq x^o \leq \beta;\ x \epsilon \tilde{G};\ \beta > 0\}$. Let $u^*(t)$ be a minimal time optimal controller steering $\hat{x}^*(t)$ from $\hat{x}_o$ to G. Then $u^*(t)$ is extremal, that is, there exists a nonvanishing adjoint response $\hat{\eta}(t) = (\eta_o, \eta(t))$ with $\eta_o \leq 0$ so that

$$\eta(t)B(t)u^*(t) = \underset{u \epsilon \Omega}{\text{Max}} \{\eta(t)B(t)u\}$$

almost always on $[t_o, t^*]$ with $\hat{\eta}(t^*)$ an outward normal of $\hat{K}(t^*)$ at $\hat{x}^*(t^*)$ on $\partial\hat{K}(t^*)$ and $\hat{\eta}(t^*)$ satisfies the transversality condition, namely, $\hat{\eta}(t^*)$ is normal to a supporting hyperplane π of G and the set of attainability $\hat{K}(t^*)$ which separates $\hat{K}(t^*)$ from G.

Moreover if for each point of $\bar{x} \epsilon G$ there exists a nonextremal controller $\bar{u}(t) \subset \Omega$ so that on $\bar{t} \leq t < \infty$ the response $x_{\bar{u}}(t)$ initiating at $\bar{x} = x_{\bar{u}}(\bar{t})$ is contained in G than when u(t) is an admissible extremal controller steering x_o to G by means of a response satisfying the transversality condition it is an optimum controller.

Proof By assumption there exists a controller steering $\hat{x}_o$ to G so G meets $\hat{K}(t^*)$. Suppose G meets the interior of $K(t^*)$. This is impossible because then G meets the interior of $\hat{K}(t)$ for $|t-t^*| < \delta, \delta > 0$, by theorem 3 and this contradicts the optimality of the controller. Hence ∂G meets $\partial\hat{K}(t^*)$ so that the optimum controller must steer to $\partial\hat{K}(t^*)$. We must show that it steers to a lower boundary point to conclude that it is extremal. This follows at once because $\hat{K}(t)$ always first makes contact with G at a lower boundary point as can be seen by considering how the compact set $\hat{K}(t_1)$ with convex lower surface moves with respect to the set G. Thus if $u^*(t)$ is optimal it is extremal and by theorem 2 there exists the nonvanishing adjoint response $\hat{\eta}(t)$ so that

$$\eta(t)B(t)u^*(t) = \max_{u \epsilon \Omega} \eta(t)B(t)u$$

where $\hat{\eta}(t^*)$ satisfies the transversality condition since G and the lower boundary of $\hat{K}(t^*)$ are convex they can be separated by a supporting hyperplane π and we choose $\hat{\eta}(t^*)$ to be normal to π and directed into the halfspace containing G.

When $u(t)$ is an admissible extremal controller steering $\hat{x}_o$ to G and satisfying the transversality condition it must be an optimum controller if G has the property that through each point $\bar{x}\epsilon G$ there passes a nonextremal response which remains forever in G. This follows because once G and $\hat{K}(t)$ come together the interior of $\hat{K}(t)$ has a nonempty intersection with G so that the transversality condition can only be satisfied once and therefore there is only one time, namely t*, for which an extremal controller can steer to G and satisfy the transversality condition. Thus any such extremal controller satisfying the transversality condition is an optimum controller.

Q.E.D.

We have therefore reduced the problem of finding an optimum controller for the approximation problem to that of finding a solution to the two point value problem as given by the 2n+2 equations

$$\dot{x}^o = F(x)$$
$$\dot{x} = A(t)x + B(t)\operatorname{sgn}\{\eta(t)B(t)\}$$
$$\dot{\eta} = -\eta A(t) - \eta_o \frac{\partial F'}{\partial x}(x)$$
$$\dot{\eta}_o = 0 \qquad (\eta_o \leq 0)$$

with boundary conditions $\hat{x}(t_o) = \hat{x}_o$, $\hat{x}(t^*)\epsilon\, \partial G$ with $\hat{\eta}(t^*)$ an interior normal to G at $\hat{x}(t^*)$.

Next we explore the possibility of solving this two point boundary value problem by calculating initial conditions for the adjoint response as required with a computer machine.

4) Discussion of synthesis methods for approximate bounded phase coordinate time optimal control

It is assumed for simplicity that the target set $\tilde{G}$ is just the origin and hence that G is just the line segment $(0 \leq x^o \leq \beta,\ x = 0)$. The two point boundary value problem then reduces to finding $\hat{\eta}(t_o)$ so that $\hat{x}(t_o) = (0, x_o)$ and $\hat{x}(t_1) \in G$ (that is, $x^o(t_1) \leq \beta$ and $x(t) = 0$) with $\eta^o = 0$ if $x^o(t) < \beta$.

Consider the 2n+2 system of equations

8)
$$\dot{x}^o = F(x)$$
$$\dot{x} = A(t)x + B(t)\mathrm{sgn}\{\eta\ B(t)\}$$
$$\dot{\eta}_o = 0$$
$$\dot{\eta} = -\eta\ A(t) - \eta_o \frac{\partial F'}{\partial x}(x)$$

with $\hat{x}(t_o) = (0, x_o)$ and $\hat{\eta}(t_o) = (c_o, \ldots c_n)$ with $c_o < 0$.

If we assume $\hat{c} = (c_o, \ldots c_n)$ as known then 8) can be solved to obtain the response pair

$x(t,c),\ \eta(t,c)$.

We shall seek the dependence of $\hat{c}$ on a parameter σ so that $\hat{c}(\sigma) \to \hat{c}^*$, as σ increases, where c^* is an initial condition for $\hat{\eta}(t)$ which solves the two point boundary value problem of the previous section.

Let $\hat{f}(t,\hat{c},\alpha) = \hat{\eta}(t,\hat{c})[\hat{x}(t,\hat{c}) - \alpha]$ where [Ref. 8]

$$\alpha = \begin{bmatrix} \alpha_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and } 0 \leq \alpha_1 \leq \beta$$

increasing t until $\hat{f}(t,\hat{c},\alpha) = 0$ for some $0 \leq \alpha_1 \leq \beta$, corresponds to carrying the hyperplane $\hat{\eta}(t,\hat{c})\ [\hat{x}(t,\hat{c})-\hat{x}]= 0$ along until $\hat{x} = \alpha$ is a point of it for some $0 \leq \alpha_1 \leq \beta$. We shall use this condition as a stopping condition for the integration to be described. To obtain such a root t it is of course necessary that we start with a good initial guess for $\hat{c}$ and that $f(t,\hat{c},\alpha)$ then moves to a zero as t increases. It is therefore assumed that $\hat{c}(0)$ is chosen so that† $\hat{f}(0,\hat{c}(0),\alpha) < 0$ for all $0 \leq \alpha_1 \leq \beta$. We next show that $\hat{f}(t,\hat{c},\alpha)$ is a nondecreasing function of t, in fact, if a condition similar to normality (in the sense of La Salle 7) is introduced $\hat{f}(t,\hat{c},\alpha)$ is strictly increasing with t.

Formally we calculate

$$\frac{d\ \hat{f}(t,\hat{c},\alpha)}{dt} = \eta_o \dot{x}^o(t,c) + \dot{\eta}(t,\hat{c})x(t,\hat{c}) + \hat{\eta}(t,\hat{c})\ \dot{x}(t,\hat{c})$$

$$= \eta_o\ F(x(t,\hat{c})) + \left[-\eta(t,\hat{c})A(t)-\eta_o \frac{\partial F'}{\partial x}\ (x(t,\hat{c}))\right] x(t,\hat{c})$$

$$+ \eta(t,\hat{c}) \left[A(t)x(t,\hat{c}) + B(t)\mathrm{sgn}[\eta(t,\hat{c})B(t)]\right]$$

$$= \eta_o\ F(x(t,\hat{c}))- \eta_o \frac{\partial F'}{\partial x}(x(t,\hat{c})\ x\ (t,\hat{c}) + |\eta(t,\hat{c})B(t)|$$

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†It is therefore necessary to consider only the case where $\alpha_1 = \beta$ if $c_o(\sigma)$ is restricted to be less than or equal to 0.

There are two cases:

1. If $x(t,\hat{c}) \in \Lambda$ then

$$\frac{d\hat{f}}{dt} = |\eta(t,\hat{c})B(t)| \geq 0$$

2. If $x(t,\hat{c}) \notin \Lambda$ then

$$\frac{d\hat{f}}{dt} = + \eta_o \left[F(x(t,\hat{c}) - \frac{\partial F'}{\partial x}(x(t,\hat{c}))x(t,\hat{c})\right] + \eta(t,\hat{c})B(t)|$$

which is also greater than or equal to zero for $\eta_o \leq 0$ since

$F(0) = 0$ and therefore $[F(x) - \frac{\partial F'}{\partial x}(x)x] \leq 0$ for the convex function $F(x)$.

Thus $\frac{d\hat{f}(t,\hat{c},\alpha)}{dt} \geq 0$ and is thus a nondecreasing function of t. In fact if $\eta(t,c)B(t)$ has no collection of zeros on $[t_o,t_1]$ then $\frac{d\hat{f}}{dt} > 0$ a.e. on $[t_o,t_1]$. Let T be the first $t > t_o$ for which $\hat{f}(t,\hat{c},\alpha) = 0$, for $\alpha_1 = \beta$. We note that $T = T(\hat{c})$. Clearly $T \leq t^*$, where t^* is the optimum time. This follows because $x(t,\hat{c})$ is on the boundary of $\hat{K}(t)$ where the vector $\eta'(t,\hat{c})$ is an exterior normal (see corollary) to theorem 2. Thus T assumes its maximum for $c = c^*$. T is the function which we will maximize to compute c^*.

Consider corrections to c by choosing the dependence of $\hat{c}$ on σ through the differential equation

$$\frac{d\hat{c}}{d\sigma} = k \frac{\partial T}{\partial \hat{c}}$$

where $k = \text{constant} > 0$.

Now $\frac{dT}{d\sigma} = \frac{\partial T}{\partial \hat{c}} \frac{\partial \hat{c}'}{\partial \sigma} = k \frac{\partial T}{\partial \hat{c}} \frac{\partial T'}{\partial \hat{c}} > 0.$

Hence T is increasing with σ, and hopefully T has no local maxima or minima.

The big problem remaining is the calculation of $\frac{\partial T}{\partial \hat{c}}$. To calculate this gradient one can compute

$$\frac{\partial T}{\partial \hat{c}} = \frac{\partial \hat{f}}{\partial \hat{c}} \Big/ \frac{\partial \hat{f}}{\partial T}$$

where

$$\frac{\partial \hat{f}}{\partial T} = |\eta(T,c)B(T)| \quad \text{if } x(T,\hat{c}) \in \Lambda$$

$$= -c_o[F(x(t,\hat{c}) - \frac{\partial F}{\partial x} x(t,\hat{c})] + |\eta(T,c)B(T)|$$

$$\text{if } x(T,c) \notin \Lambda \text{ and}$$

compute

$$\frac{\partial \hat{f}}{\partial \hat{c}} = \frac{\partial \hat{f}}{\partial \hat{\eta}} \frac{\partial \hat{\eta}'}{\partial \hat{c}} + \frac{\partial \hat{f}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial \hat{c}}$$

Thus the only difficult items to obtain are $\frac{\partial \hat{\eta}}{\partial \hat{c}}$, and $\frac{\partial \hat{x}}{\partial \hat{c}}$

To find these partial derivatives probably the best and most straight forward method is to calculate $(\hat{\eta}(t,\hat{c}), \hat{x}(t,\hat{c}))$ and then $(\hat{x}(t,\hat{c} + \delta_i\hat{c}), \hat{\eta}(t,\hat{c} + \delta_i\hat{c}))$, $i = 0,..,2,...n$, where $\delta_i\hat{c}$ is a small change in the ith component of $\hat{c}$. The various partial derivatives are then approximately known.

This scheme is now under study on a computer machine and the resuts will be published elsewhere.

5) Remarks on the approximate bounded phase coordinate problems with integral cost

As before consider the linear control process

$\mathcal{L}$) $\dot{x} = A(t)x + B(t)u(t)$

satisfying the conditions stated at the beginning of section 2. As a cost functional of control consider

$$C(u) = g(x(t)) + \int_{t_o}^{t_1} \{f^\circ(x,t) + h^\circ(u,t)\} dt$$

where t_1 = fixed time $> t_o$ and the real functions $f^\circ(x,t)$ and $h^\circ(u,t)$ are continuously differentiable and $f^\circ(x,t)$ is a convex function of x for each t.

The problem of optimal control is to pick an admissible controller $u(t)$ on $[t_o,T]$ so that the response $x_u(t)$ of $\mathcal{L}$ moves from x_o to a target set $\tilde{G} \subset R^n$ at t_1, ($\tilde{G}$ may be the whole space) and minimizes $C(u)$ with the entire response $x_u(t)$ contained is the closed convex restraint set Λ.

As before we introduce the convex differentiable function F(x) satisfying the conditions

$$F(x) > 0 \quad \text{if } x \notin \Lambda$$
$$= 0 \quad \text{if } x \in \Lambda$$

The approximation problem is obtained by adding F(x) to the integrand of the cost functional C(u) to obtain a new cost functional

$$C_\lambda(u) = g(x(t)) + \int_{t_o}^{T} \{f^\circ(x,t) + \lambda F(x) + h^\circ(u,t)\} dt$$
$$= \int_{t_o}^{T} \{f^\circ(x,t) + h^\circ(u,t)\} dt,$$

here $\lambda \geq 0$. If λ is sufficiently large then one would expect that the contribution from the term λ F(x) can be small only if the response stays near Λ or within it. The approximation problem is to find that controller u(t) which minimizes C(u) and steers to $\tilde{G} \subset R^n$.

We shall assume that $h^\circ(u,t)$ is convex in u for each t or that the controller is bounded and h is a positive function of u for each t. In either case the previous theory can be applied after slight modification by noting that $f^\circ(x,t) = f^\circ(x,t) + \lambda F(x)$ is a convex function of x for each t since both f° and F were convex functions and by noting the contribution to $x^\circ(T)$ made by the terms $h^\circ(u,t)$. That is, the problem has now been cast as one which is covered by the sufficiency results of reference 5 which are also necessary [reference 4] and can be obtained as a slight modification of the results of section 3.

REFERENCES

1. Russell, D. L. "Linear Programming and Bounded Phase Coordinate Control" MH MPG Report 1541-TR 8.

2. Lee, E. B., and Markus, L., "Optimal Control for Nonlinear Processes", Archive for Rational Mechanics and Analysis, Vol. 8, No. 1, 1961.

3. Harvey, C. A., and Lee, E. B., "On the Uniqueness of Time-Optimal Control", J. Math, Anal. and App. Vol. 5, 1962.

4. Pontryagin, L.S., Boltyanskii, V. G., Gamkrelidze, R. V., Mishehenko,E.F, "The Mathematical Theory of Optimal Processes", John Wiley and Sons, New York, 1962.

5. Lee, E. B., "A Sufficient Condition in the Theory of Optimal Control" SIAM Jour of Control, Vol 1, No 3, 1963.

6. Russell, D. "Time Optimal Bounded Phase Coordinate Control of Linear Systems Parts I, II, III, Appendix C of Honeywell MPG Report 12006-QR 1 (The First Quarterly Progress Report on Control NASw-986)", 30 September, 1964.

7. LaSalle, J.P., "The Time Optimal Control Problem" Am. Math. Studies No. 45, 1-24, 1960.

8. Neustadt, L. W., "Synthesizing Time Optimal Control Systems", J. Math. Anal. and Appl. Vol 1, 1960, pp 484-493.