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ESTIMATION OF INTERNAL SOURCE DISTRIBUTIONS USING EXTERNAL FIELD MEASUREMENTS IN RADIATIVE TRANSFER

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PREFACE

This Memorandum was prepared as part of RAND's continuing study of Satellite Meteorology for the National Aeronautics and Space Administration under contract number NASr-21(07). It should be useful in estimating source distributions in stellar and planetary atmospheres based on observations of emergent intensity patterns.

SUMMARY

A finite homogeneous slab which absorbs radiation and scatters it isotropically possesses internal isotropic sources of radiation. The authors first show how to determine the intensity of the emergent radiation, making use of invariant imbedding techniques, and then show how to determine the distribution of the internal sources that best accounts for an observed emergent radiation pattern. This inverse problem is viewed as a nonlinear two-point boundary value problem which can be resolved numerically using quasilinearization.

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I. INTRODUCTION

A basic problem in science and engineering is inferring causes on the basis of observed effects. In the construction of model stellar and planetary atmospheres and in the study of planetary entry phenomena, this problem is particularly significant. In this Memorandum we shall discuss some aspects of this problem in relation to the ability of the modern computer to integrate systems of a thousand or more ordinary nonlinear differential equations.

The physical situation is as follows: A slab of finite optical thickness absorbs radiation and scatters it isotropically. It possesses internal sources of isotropic radiation. First, we discuss the direct problem of determining the intensity of the radiation emerging from the slab, with particular emphasis on its angular distribution. We use standard ideas from the theory of invariant imbedding; ^(1,2) the X and Y functions of Chandrasekhar play a fundamental role. ⁽³⁾ Then we shift to the inverse problem.

We assume that measurements of the emergent radiation field are made, and it is desired to determine the distribution of internal sources which best explains the measurements. This is viewed mathematically as a nonlinear two-point boundary value problem which is solved numerically using quasilinearization. ^(4,5) We conclude with a discussion of some of the extensions of the model which are possible.

The importance of such problems was brought out in conversations with Dr. Ralph Zirkind of the Advanced Research Projects Agency.

II. DERIVATION OF THE BASIC EQUATIONS

Consider a homogeneous slab bounded by two planes separated by an optical distance x_0 . The slab both absorbs radiation and scatters it isotropically, the albedo for single scattering is λ . Within the slab are isotropic sources of radiation. At each point y above the bottom surface, the strength of the source is $B(y)$ per unit volume per unit solid angle per unit time.

We first derive computationally useful equations for the intensity of the radiation emerging from the top surface. To do this, we consider the slab extending from the bottom to the optical altitude x , add a layer of thickness Δ , and note the changes that occur in the intensity of the radiation emerging from the top of the slab. Let

$T(v,x)dv$ = the rate of emission of energy per unit horizontal area through the upper surface of a slab of thickness x , having a direction cosine, with respect to the upward-directed normal, between v and $v + dv$. (1)

In addition let

$p(v,x)dv$ = the fraction of the energy which is isotropically emitted at the top of the slab of thickness x and which ultimately emerges from the top with a direction cosine between v and $v + dv$. (2)

Then we may write

$$T(v, x+\Delta)dv = \left(1 - \frac{\Delta}{v}\right)T(v, x)dv + \left[4\pi B(x)\Delta + \int_0^1 T(v', x)dv' \frac{\Delta}{v^2} \lambda \right] p(v, x)dv + o(\Delta). \quad (3)$$

The first term on the right-hand side accounts for the losses encountered in passing through the slab of thickness Δ . The first factor in the second term is the total rate of production of scattered radiation in a cylinder with unit base area and altitude Δ , located at the top of the slab of thickness x . The second factor is the fraction of such radiation which ultimately emerges from the top with a direction cosine v . The third term, $o(\Delta)$, accounts for all higher order processes and consists of terms in Δ of powers higher than the first. The limiting form, as Δ tends to zero, is the partial differential-integral equation

$$T_x(v, x) = -\frac{1}{v} T(v, x) + 4\pi p(v, x)B(x) + \lambda p(v, x) \int_0^1 T(v', x) \frac{dv'}{v^2}. \quad (4)$$

The slab of thickness 0 is easily analyzed. We assume the bottom surface is a perfect absorber which leads to the condition

$$T(v, 0) = 0. \quad (5)$$

Situations in which the bottom surface is an emitter and a reflector are readily treated, though we do not discuss them here.

In order to integrate Eq. (4) it is necessary to derive an equation for the function $p(v,x)$. Let us introduce the function

$q(v,x)dv$ = the fraction of the energy isotropically emitted at the top of the slab of thickness x which ultimately emerges from the bottom with a direction cosine, with respect to the downward-directed normal, between v and $v + dv$. (6)

Then we may write

$$p(v,x+\Delta) = p(v,x) + q(v,x) \lambda \int_0^1 q(v',x) dv' \frac{\Delta}{v} + o(\Delta) . \quad (7)$$

In Eq. (7) we have considered the slab of thickness $x + \Delta$ to be made by adding a slab of thickness Δ to the bottom of a slab of thickness x . In addition, we have

$$q(v,x+\Delta) = q(v,x) \left(1 - \frac{\Delta}{v}\right) + p(v,x) \lambda \int_0^1 q(v',x) dv' \frac{\Delta}{v} + o(\Delta) . \quad (8)$$

The limiting forms of these equations, as Δ tends to zero, are

$$p_x(v,x) = \lambda q(v,x) \int_0^1 q(v',x) \frac{dv'}{v} \quad (9)$$

and

$$q_x(v,x) = -\frac{1}{v} q(v,x) + \lambda p(v,x) \int_0^1 q(v',x) \frac{dv'}{v} . \quad (10)$$

For thin slabs of thickness Δ we have

$$p(v, \Delta) dv = \frac{2\pi}{4\pi} \frac{dv}{v} + 0(\Delta), \quad (11)$$

$$q(v, \Delta) dv = \frac{1}{2} dv + 0(\Delta). \quad (12)$$

Thus

$$p(v, 0) = \frac{1}{2}, \quad (13)$$

$$q(v, 0) = \frac{1}{2}. \quad (14)$$

Our basic equations are

$$p_x = \lambda q \int_0^1 q(v', x) \frac{dv'}{v'}, \quad p(v, 0) = \frac{1}{2}, \quad (15)$$

$$q_x = -\frac{1}{v} q + \lambda p \int_0^1 q(v', x) \frac{dv'}{v'}, \quad q(v, 0) = \frac{1}{2}, \quad (16)$$

$$T_x = -\frac{1}{v} T + 4\pi p B + \lambda p \int_0^1 T(v', x) \frac{dv'}{v'}, \quad T(v, 0) = 0. \quad (17)$$

Experience with many similar systems of nonlinear differential-integral equations leads us to believe that this system can be readily integrated numerically by approximating the integrals through the use of finite sums with Gaussian quadrature formulas.^(1,2)

Before we discuss this, we will make several substitutions to put the system of equations (15)-(17) into a more convenient form.

We write

$$p(v, x) = \frac{1}{2} X(v, x), \quad (18)$$

$$q(v, x) = \frac{1}{2} Y(v, x), \quad (19)$$

so that Eqs. (15) and (16) become

$$X_x = \frac{1}{2} \lambda Y \int_0^1 Y(v', x) \frac{dv'}{v'}, \quad X(v, 0) = 1, \quad (20)$$

$$Y_x = -\frac{1}{v} Y + \frac{\lambda}{2} X \int_0^1 Y(v', x) \frac{dv'}{v'}, \quad Y(v, 0) = 1. \quad (21)$$

Thus X and Y are the standard X and Y functions of radiative transfer. ^(3,6,7) The intensity of the radiation emerging from the upper surface of the slab of thickness x with direction cosine v is denoted $t(v, x)$ and is given by the formula

$$2\pi v t(v, x) = T(v, x). \quad (22)$$

With this substitution, Eq. (17) becomes

$$t_x = -\frac{1}{v} t + \frac{B}{v} X + \frac{\lambda X}{2v} \int_0^1 t(v', x) dv', \quad t(v, 0) = 0. \quad (23)$$

III. APPROXIMATE SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

Our basic system of equations contains integrals on the interval (0,1). We approximate such an integral of a function $g(v)$ using a Gaussian quadrature formula of order N ,

$$\int_0^1 g(v) dv \cong \sum_{i=1}^N w_i g(v_i), \quad (24)$$

where w_i , $i=1,2,\dots,N$, are the Christoffel numbers, and v_i , $i=1,2,\dots,N$, are the abscissas at which the integrand is to be evaluated. These are tabulated in Ref. 2. From experience in similar problems, ^(7,8) we expect that $N=7$ will yield high accuracy. Let us then introduce the functions $x_i(x)$, $y_i(x)$ and $t_i(x)$, $0 \leq x \leq x_0$, as solutions of the system of ordinary differential equations

$$\dot{x}_i = \frac{1}{2} \lambda y_i \sum_{j=1}^N y_j \frac{w_j}{v_j}, \quad x_i(0) = 1, \quad (25)$$

$$\dot{y}_i = -\frac{1}{v_i} y_i + \frac{1}{2} \lambda x_i \sum_{j=1}^N y_j \frac{w_j}{v_j}, \quad y_i(0) = 1, \quad (26)$$

$$\dot{t}_i = -\frac{1}{v_i} t_i + \frac{B}{v_i} x_i + \frac{\lambda}{2v_i} x_i \sum_{j=1}^N t_j w_j, \quad t_i(0) = 0, \quad (27)$$

$i=1,2,\dots,N.$

This is a nonlinear system of $3N$ ordinary differential equations with a complete system of initial conditions. Its integration on a modern computer is routine for $0 \leq x \leq x_0$. In this way we produce the values of the emergent intensity at the top, $t(v_i, x_0)$, $i=1,2,\dots,N$, for a given internal source distribution $B(y)$, $0 \leq y \leq x_0$.

IV. INVERSE PROBLEM AND QUASILINEARIZATION

Now let us turn to the problem of determining the source distribution function $B(y)$ which would best explain an observed emergent intensity pattern. First we will consider the case in which

$$B(y) = a + by + cy^2, \quad (28)$$

where a , b , and c are the three constants to be determined. We suppose that the observed intensity of the emergent radiation with direction cosine v_i is b_i ; i.e.,

$$t_i(x_0) \cong b_i, \quad i=1,2,\dots,N. \quad (29)$$

We wish to minimize the sum of the squares of the deviations,

$$S = \sum_{j=1}^N \left\{ t_j(x_0) - b_j \right\}^2,$$

through an appropriate choice of the constants a , b , c , which enter into Eqs. (25), (26), and (27) through B .

This problem is readily solved numerically using the quasilinearization technique.^(4,9) We now present the basic formalism. A system of M equations is written in vector form

$$\dot{x} = f(x, \alpha), \quad x(0) = c, \quad (30)$$

where α is an R -dimensional vector constant. We wish to determine the vector α so as to minimize the form

$$F = (x(T) - \beta, Q(x(T) - \beta)), \quad (31)$$

where Q is a positive definite square matrix of order M , and β is a given vector. We consider α to be a function of time satisfying the differential equation

$$\dot{\alpha} = 0. \quad (32)$$

Now the initial value of α is unknown. Thus, our original problem is equivalent to one in which

$$\dot{y} = g(y), \quad y(0) = w, \quad (33)$$

and some of the components of w , say the first L , are free, and the remaining ones are specified. We wish to minimize a quadratic form in $y(T)$, the final value of y . This can be done by starting with an initial approximation, $w = w^{(0)}$, and integrating the system, Eq. (33), to produce $y^{(0)}(t)$, $0 \leq t \leq T$. The next approximation is produced by finding $y^{(1)}(t)$ so that

$$\dot{y}^{(1)} = J(y^{(0)}) (y^{(1)} - y^{(0)}) + g(y^{(0)}), \quad (34)$$

where F is minimized; $J(y^{(0)})$ is the Jacobian matrix

$$J_{ij} = \frac{\partial g_i}{\partial y_j}. \quad (35)$$

This is done by numerically producing an appropriate system of L independent solutions of the homogeneous equation corresponding to Eq. (34) and also a particular solution. In an obvious notation, the possible solutions of Eq. (34) are

$$y^{(1)}(t) = p(t) + \sum_{i=1}^L c_i h_i(t), \quad (36)$$

where c_i , $i=1,2,\dots,L$, are arbitrary constants. The minimizing values of c_i , $i=1,2,\dots,L$, are determined by substituting $y^{(1)}(T)$ (as determined from Eq. (36)) into F (from Eq. (31)). A linear algebraic system of equations for the optimal choices of c_i , $i=1,2,\dots,L$, results from

$$\frac{\partial F}{\partial c_i} = 0, \quad i=1,2,\dots,L. \quad (37)$$

Many previous calculations attest to the efficacy of the procedure. However, at times, special devices are required to carry through all the steps.^(4,10) The procedure is then repeated to produce $y^{(2)}(t)$, and so on. When the initial estimate is sufficiently good, the process will converge quadratically; that is, the number of correct digits in each approximant doubles asymptotically.

The case of a layered slab is discussed in Ref. 8.

If $B(y)$ is known only as the solution of a differential equation involving some unknown constant vector β

$$\frac{dB}{dy} = g(y,\beta), \quad (38)$$

essentially the same procedure is used. Equation (38) is adjoined to the system of Eqs. (25)-(27), and the calculation proceeds as before.⁽¹¹⁾

V. DISCUSSION

In the previous paragraphs we have indicated that a judicious use of concepts from the theories of invariant imbedding and quasi-linearization, together with modern digital computers, could lead to the determination of internal source distributions using external field measurements. It is possible to examine the effects of errors in the measurements on the accuracy of the source distribution estimation. This might be of value in planning experiments. In addition, the results of the calculations might serve as a stimulus to the development of new theories concerning the sources.

It would probably be necessary to start a computational program with the simple physical model sketched. The basic invariant imbedding equations for various generalizations are available. These include spherical and cylindrical geometry, time dependence,⁽²⁾ anisotropy of the scattering law,⁽⁹⁾ inhomogeneity of the medium,⁽¹²⁾ and energy dependence.⁽¹³⁾

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