NASA CONTRACTOR REPORT

NASA CR-285

NASA CR-289

TABLES OF ELLIPTIC INTEGRALS

by William Joel Nellis

Prepared under Grant No. NsG-293 *by*
IOWA STATE UNIVERSITY
Ames, Iowa

for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • AUGUST 1965

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I. INTRODUCTION

The tables present a method of evaluating elliptic integrals of the first and second kinds by means of the R function, a hypergeometric function of several variables (1). The general method is to reduce an integral of elliptic type to an R function by means of the integral formulas of Table 1. The formulas of Table 2 (for complete integrals) or Table 3 (for incomplete integrals) are then used to reduce the R function to a linear combination of two standard R functions and an algebraic function. The two standard functions, R_K and R_E (in the complete case) or R_F and R_G (in the incomplete case), are normal elliptic integrals of the first and second kinds (2). Table 4 gives numerical values of the normal integrals over a limited range; it is intended only to give an idea of the behavior of the functions and should not be used for interpolation near $x = 0$ or $y = 0$.

Table 1 can also be used to express elliptic integrals of the third kind (as well as many other integrals) in terms of the R function, but tables for reduction and numerical evaluation of integrals of the third kind are not included. Until such tables are developed, the reader is referred to conventional tables of elliptic integrals, for example (3, 4, 5), which deal with integrals of all three kinds. For integrals of the first two kinds, it is believed that the present scheme of reduction offers significant advantages in conciseness.

II. DEFINITION AND PROPERTIES OF THE R FUNCTION

A few basic properties (1) of the R function are essential for the use of these tables. We shall give first the definition of R as an infinite series and the most important integral representation. Let a function R of n complex variables $z_1, \ldots, z_n$ and n+1 complex parameters a, $b_1, \ldots, b_n$ be defined by the following power series if $|1-z_i| < 1 (i=1, \ldots, n)$ and by its analytic continuation if $|\arg z_i| < \pi$:

$$R(a;b_1, \ldots, b_n; z_1, \ldots, z_n)$$
$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b_1,m_1) \ldots (b_n,m_n)}{(b_1+\ldots+b_n,m_1+\ldots+m_n)m_1! \ldots m_n!}(1-z_1)^{m_1} \ldots (1-z_n)^{m_n}, \tag{2.1}$$

where $(a,m) = a(a+1) \ldots (a+m-1) = \Gamma(a+m)/\Gamma(a)$ and Γ is the gamma function. Let

$$c = b_1 + \ldots + b_n$$

and define a' by

$$c = a + a'.$$

We assume that c is not zero or a negative integer. Provided that a and a' have positive real parts, the R function can be represented by a single integral:

$$B(a,a')R(a;b_1, \ldots, b_n; z_1, \ldots, z_n) = \int_0^{\infty} t^{a'-1} \prod_{i=1}^{n} (t+z_i)^{-b_i} dt, \tag{2.2}$$

where $B(a,a') = \Gamma(a)\Gamma(a')/\Gamma(c)$ is the beta function.

The R function has two important properties, symmetry and homogeneity. Symmetry is apparent from either representation: $R(a;b_1, \ldots, b_n; z_1, \ldots, z_n)$ is invariant under permutation of the subscripts 1, . . ., n, that is, R does not change when the b's and z's are permuted together.

A scale factor common to all the z's can be removed by changing the variable of integration in 2.2. Thus, R is homogeneous of degree $-a$ in the variables $z_1, \ldots, z_n$:

$$R(a;b_1, \ldots, b_n; sz_1, \ldots, sz_n) = s^{-a} R(a;b_1, \ldots, b_n; z_1, \ldots, z_n). \tag{2.3}$$

This property is often used to make all the arguments of an R function less than or equal to unity.

The substitution $t = \tau^{-1}$ in 2.2 leads to the Euler transformation, which allows us to change the value of the a-parameter:

$$R(a;b_1, \ldots, b_n; z_1, \ldots, z_n) = \left(\prod_{i=1}^{n} z_i^{-b_i}\right) R(a';b_1, \ldots, b_n; z_1^{-1}, \ldots, z_n^{-1}). \tag{2.4}$$

For the case $n = 2$, there is another relation which can be used to change the value of the a-parameter:

$$z_2^{a} R(a;b_1,b_2;z_1,z_2) = z_2^{b_1} R(b_1;a,a';z_1,z_2) . \tag{2.5}$$

Equation 2.5 has no analogue for $n > 2$. Equations 2.3, 2.4, and 2.5 are actually valid without restriction on a or a′.

There are a few special cases of the parameters and variables which provide very useful relations. If one of the b-parameters in 2.1 is zero, say b_n, then

$$(b_n, m_i) = (0, m_i) = \delta_{m_i 0}$$

is a Kronecker delta, and

$$R(a; b_1, \ldots, b_{n-1}, 0; z_1, \ldots, z_n) = R(a; b_1, \ldots, b_{n-1}; z_1, \ldots, z_{n-1}). \tag{2.6}$$

If any two arguments are equal, say $z_1 = z_2$, then 2.2 shows that

$$R(a; b_1, \ldots, b_n; z_2, z_2, z_3, \ldots, z_n) = R(a; b_1 + b_2, b_3, \ldots, b_n; z_2, \ldots, z_n). \tag{2.7}$$

If any one of the arguments vanishes, say z_1, 2.2 again shows that

$$B(a, a')R(a; b_1, \ldots, b_n; 0, z_2, \ldots, z_n) = B(a, a'-b_1)R(a; b_2, \ldots, b_n; z_2, \ldots, z_n), \tag{2.8}$$

provided that $\mathrm{Re}(a'-b_1) > 0$.

Tables 2 and 3 were derived from recursion formulas for the R function. Let

$$R = R(a; b_1, \ldots, b_n; z_1, \ldots, z_n),$$

$$R(a \pm 1) = R(a \pm 1; b_1, \ldots, b_n; z_1, \ldots, z_n),$$

$$R(b_i \pm 1) = R(a; b_1, \ldots, b_i \pm 1, \ldots, b_n; z_1, \ldots, z_n).$$

For any n the basic recursion formulas are

$$(c-1)R(b_i-1) = (a'-1)R+az_iR(a+1) \quad , \ (i=1,. . .,n), \tag{2.9}$$

$$cR = \sum_{i=1}^{n} b_iR(b_i+1), \tag{2.10}$$

$$cR(a-1) = \sum_{i=1}^{n} b_iz_iR(b_i+1), \tag{2.11}$$

$$acR(a+1) = \sum_{i=1}^{n} b_iz_i^{-1} [cR-a'R(b_i+1)]. \tag{2.12}$$

For n = 2, these relations imply three further formulas that are particularly useful in practice:

$$b_i(z_j-z_i)R(b_i+1) = c[z_jR-R(a-1)], \tag{2.13}$$

$$b_1b_2(z_1-z_2)^2R(b_1+1,b_2+1) = c(c+1)[-z_1z_2R+(z_1+z_2)R(a-1)-R(a-2)], \tag{2.14}$$

$$az_1z_2R(a+1) = a'R(a-1) - \sum_{i=1}^{2} (b_i-a)z_iR. \tag{2.15}$$

For n = 3, 2.9-2.12 imply two formulas that are likewise quite useful in practice:

$$a'b_i(z_i-z_j)(z_i-z_k)R(b_i+1) = cR[z_1z_2z_3 \sum_{m=1}^{3} b_mz_m^{-1}-a'z_i(z_j+z_k)]$$

$$+ a'z_icR(a-1)-acz_1z_2z_3R(a+1), \tag{2.16}$$

$$a(a+1)z_1z_2z_3R(a+2) = az_1z_2z_3 \sum_{i=1}^{3} (1-a'+b_i)z_i^{-1}R(a+1)$$

$$+ (a'-1) \Sigma_{i=1}^{3} (a-b_i)z_i \; R(a) + a'(a'-1) \; R(a-1) \qquad (2.17)$$

An R function of four variables $(0 < x < y, z, w)$ with $a = a' = 1$ can be reduced to a linear combination of two R functions of three variables:

$$(1-\alpha)(y-x)^{\beta}(z-x)^{\gamma}(w-x)^{2-\alpha-\beta-\gamma}R(1;\alpha,\beta,\gamma,2-\alpha-\beta-\gamma;x,y,z,w)$$

$$= R(1-\alpha;\beta,\gamma,2-\alpha-\beta-\gamma;(y-x)^{-1},(z-x)^{-1},(w-x)^{-1})$$

$$- x^{1-\alpha}R(1-\alpha;\beta,\gamma,2-\alpha-\beta-\gamma;y(y-x)^{-1},z(z-x)^{-1},w(w-x)^{-1}). \qquad (2.18)$$

By applying an Euler transformation, we obtain the alternative form

$$(1-\alpha)R(1;\alpha,\beta,\gamma,2-\alpha-\beta-\gamma;x,y,z,w) = R(1;\beta,\gamma,2-\alpha-\beta-\gamma;y-x,z-x,w-x)$$

$$- x^{1-\alpha}y^{-\beta}z^{-\gamma}w^{\alpha+\beta+\gamma-2} \; R(1;\beta,\gamma,2-\alpha-\beta-\gamma; \; y^{-1}(y-x),z^{-1}(z-x),w^{-1}(w-x)).$$

$$(2.19)$$

An R function with two b-parameters and having complex conjugate arguments can be reduced (if elliptic) to a linear combination of R_K and R_E. The transformations of the complete normal integrals are then

$$R_K(z,\bar{z}) = R_K(|z|,u), \qquad (2.20)$$

$$R_E(z,\bar{z}) = 2R_E(|z|,u)-|z|R_K(|z|,u), \qquad (2.21)$$

where $2u = |z| + \mathrm{Re}\ z$ and $\bar{z}$ is the complex conjugate of z.

If an R function has three b-parameters, one real argument, and two complex conjugate arguments, it can be reduced (if elliptic) to a linear combination of R_F, R_G, and an algebraic function. The transformations of the incomplete normal integrals are then

$$R_F(x,z,\bar{z}) = R_F(u,v,w), \tag{2.22}$$

$$2R_G(x,z,\bar{z}) = 4R_G(u,v,w) - |z| R_F(u,v,w) - \sqrt{x}, \tag{2.23}$$

where x is real and

$$2u = |z| + \mathrm{Re}\ z,$$

$$2v = |z| + x + (|z|^2 - 2x\ \mathrm{Re}\ z + x^2)^{1/2},$$

$$2w = |z| + x - (|z|^2 - 2x\ \mathrm{Re}\ z + x^2)^{1/2}.$$

III. DESCRIPTION OF TABLE 1

The first section of Table 1 contains integrals such as

$$\int_{\lambda}^{\mu} (t-x)^{\alpha}(t-y)^{\beta}(t-z)^{\gamma}(t-w)^{\delta}\, dt,$$

where one of the limits, λ or μ, is a singularity of the integrand, that is, either λ or μ is equal to x, y, z, w or $\pm\infty$. If x is complex, t-x is to be given its principal value: $|\arg(t-x)|<\pi$. Integrals of the form

$$\int_{\lambda}^{\mu} t^{\alpha}(t^2-x)^{\beta}(t^2-y)^{\gamma}(t^2-z)^{\delta}\, dt$$

can be reduced to the above form (with $w = 0$) by substituting $t^2 = \tau$, and similarly for factors of type (t^3-x), etc.

In 6.3 and 6.4 the upper limit is assumed greater than the lower limit. In 6.3 the upper limit is a singularity of the integrand, and in 6.4, the lower limit. If the integral to be evaluated contains a factor $(z-t)^{\beta}$ instead of $(t-z)^{\beta}$, that is, if z lies to the right instead of to the left of the range of integration, one need only replace $(t-z)^{\beta}$ by $(z-t)^{\beta}$ on the left side of 6.3 or 6.4 and $(y-z)^{\beta}$ by $(z-y)^{\beta}$ in the coefficient on the right side (since t takes the value y at one point of the interval of integration). The arguments of the R function are unchanged; both x-z and y-z will now be negative, but their ratio is positive as before.

The second section of Table 1 contains some integrals with trigonometric integrands. Statements analogous to those about algebraic integrals can also be made about trigonometric ones. For example, in 6.7 if $(1-p\cos\phi)$ is negative, replace $(1-p\cos\theta)^{\delta}$ by $(p\cos\theta-1)^{\delta}$ on the left hand side and $(1-p)^{\delta}$ by $(p-1)^{\delta}$ in the coefficient of R on the right hand side. The arguments of the R function are unchanged; both $1-p\cos\phi$ and $1-p$ will now be negative, but their ratio is still positive.

IV. DESCRIPTION OF TABLES 2 AND 3

Table 2 (the complete case) expresses a number of R functions with half-integral a- and b-parameters as linear combinations of two R functions, R_K and R_E, which are taken as normal elliptic integrals of the first and second kinds, respectively:

$$R_K(x,y) = R(\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2};x,y),$$

$$R_E(x,y) = R(-\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2};x,y).$$

Table 3 (the incomplete case) expresses a number of R functions with half integral a- and b-parameters as linear combinations of three standard R functions, one of which is algebraic:

$$R_F(x,y,z) = R(\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};x,y,z),$$

$$R_G(x,y,z) = R(-\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};x,y,z),$$

$$(xyz)^{-1/2} = R(\tfrac{3}{2};\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};x,y,z).$$

Each complete integral is a multiple of an incomplete integral with one vanishing argument, as shown by 2.8:

$$2R_F(x,y,0) = \pi\ R_K(x,y), \tag{4.1}$$

$$4R_G(x,y,0) = \pi\ R_E(x,y). \tag{4.2}$$

We are concerned primarily with R functions in which all the parameters a, $b_1, \ldots, b_n$ are half-integral or integral. If the a-parameter is a negative integer, the series 2.1 terminates and the R function is a polynomial. If a' is a negative integer, the Euler transformation 2.4 shows that R is then algebraic. A number of such cases are included in Table 3(the coefficients multiplying the normal elliptic integrals being zero), but they cannot arise from single integrals of the type in Table 1, since the integral representation 2.2 converges only if a and a' have positive real parts.

Excluding such algebraic cases, we have the following rules of thumb:

(1) If $n = 2$ and all parameters but the c-parameter are half-integral, we have a complete elliptic integral of the first or second kind; e.g., $R(\frac{3}{2};\frac{3}{2},\frac{1}{2};x,y)$.

(2) If there are three b-parameters, all half-integral, and if one of the pair (a,a') is a half integer (the other being an integer), we have an incomplete elliptic integral of the first or second kind; e.g., $R(\frac{1}{2};\frac{3}{2},\frac{1}{2},\frac{1}{2};x,y,z)$.

(3) If there are three b-parameters, two half-integral and one integral, and if the a-parameter is half-integral, we have a complete elliptic integral of the third kind; e.g., $R(\frac{1}{2};\frac{1}{2},\frac{1}{2},1;x,y,z)$.

(4) If there are four b-parameters, three half-integral and one integral, we have an incomplete elliptic integral of the third kind; e.g., $R(\frac{1}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2},1;w,x,y,z)$.

(5) If we have four b-parameters, all half-integral, and if $a = a' = 1$, we have an incomplete integral of the first or second kind; e.g., $R(1;\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2};w,x,y,z)$.

There are many R functions which fall into one of these classes only after being subjected to some (perhaps quadratic or cubic) transformation. For example, $R(\frac{1}{4};\frac{3}{4},\frac{1}{4};x,y)$ is a complete elliptic integral of the first kind.

Table 2 gives some reduction formulas for class (1) in which the c-parameter is positive and the largest b-parameter is 3/2. Table 3 gives some reduction formulas for class (2). Equation 2.18 reduces R functions of class (5) to two R functions of class (1) or (2). If an R function is not found in Table 2 or 3 but falls into class (1) or (2), one can extend these tables by use of 2.9-2.17, especially by 2.13-2.15 in the complete case and by 2.16 and 2.17 in the incomplete case.

V. DESCRIPTION OF TABLE 4

Table 4 gives numerical values of R_F and R_G. Homogeneity can always be used to make the arguments of R less than or equal to unity. Thus, the table contains values of $R_F(x,y,1)$ and $R_G(x,y,1)$ only in the domain $0 \leq x,y \leq 1$. The table was calculated with the IBM 7074 computer of the Iowa State University Computation Center. Two distinct methods were employed. One program was based on the power series expansion of R given by 2.1. The other program utilized descending Gauss or ascending Landen transformations. The algorithms for the latter method are described in detail in (6). The power series expansion converges very slowly for values of the arguments close to zero. The program based on it yielded results only in the region $.20 \leq x,y \leq 1$. The other program computed the normal integrals throughout the whole domain. Both programs yielded identical results in the region where both computed the functions.

The table is intended to give an idea of the behavior of R_F and R_G; it should not be used for interpolation. If numerical values not contained in the table are desired, the algorithms in (6) or, if a computer is available, the Fortran subroutine in the Appendix may be employed.

Complete integrals are contained in the table as the case in which one of the arguments is zero. For example, by 4.1

$$R_K(3/4,1) = (2/\pi)R_F(0,3/4,1).$$

The table is redundant in that both R_F and R_G are symmetric in x and y, that is, $R_F(x,y,1) = R_F(y,x,1)$, and similarly for R_G. Note also that R_G increases and R_F decreases as either x or y increases.

VI. EXAMPLES

1. Reduce the following integral to an R function:

$$I = \int_{-2}^{4} (t+2)^{-1/2}(t+4)^{-3/2}(6-t)^{1/2}(t+8)^{-1/2}dt.$$

Rewrite the integral to obtain

$$I = \int_{-2}^{4} [t-(-2)]^{-1/2}[t-(-4)]^{-3/2}(6-t)^{1/2}[t-(-8)]^{-1/2}dt.$$

The lower limit is a singularity and is less than the upper limit. In 6.4, on the left side let

$$(t-v)^{\delta} \rightarrow (v-t)^{\delta} = (6-t)^{1/2},$$

and on the right side let

$$(y-v)^{\delta} \rightarrow (v-y)^{\delta} = (6+2)^{1/2}.$$

Make the correspondence

$$x = 4 \qquad z = -8 \qquad v = 6 \qquad \alpha = -\frac{1}{2} \qquad \gamma = -\frac{3}{2}$$

$$y = -2 \qquad w = -4 \qquad a = \frac{1}{2} \qquad \beta = -\frac{1}{2} \qquad \delta = \frac{1}{2} \quad .$$

Therefore,

$$I = 2R(\tfrac{1}{2};\tfrac{1}{2},\tfrac{3}{2},-\tfrac{1}{2},0;2,4,\tfrac{1}{4},1).$$

By 2.6,

$$I = 2R(\tfrac{1}{2};\tfrac{1}{2},\tfrac{3}{2},-\tfrac{1}{2};2,4,\tfrac{1}{4}).$$

2. Reduce the following integral to an R function:

$$I = \int_g^h (t+d)^{1/2}(t-f)^{1/2}(h-t)^{-1/2}dt, \qquad -d < f < g < h.$$

Equation 6.3 is appropriate. We have the choice of setting either β, γ, or δ equal to zero. Make the correspondence

$$\begin{array}{llll} y = h & w = -d & \alpha = -\frac{1}{2} & \gamma = \frac{1}{2} \\ x = g & v = f & \beta = 0 & \delta = \frac{1}{2} \end{array} .$$

Therefore,

$$I = 2(h-g)^{1/2}(h+d)^{1/2}(h-f)^{1/2}R(\tfrac{1}{2};0,-\tfrac{1}{2},\tfrac{1}{2},\tfrac{5}{2};\tfrac{z-g}{z-h},\tfrac{d+g}{d+h},\tfrac{g-f}{h-f}, 1).$$

Using symmetry and 2.6 we obtain

$$I = 2(h-g)^{1/2}(h+d)^{1/2}(h-f)^{1/2}\ R(\tfrac{1}{2};\tfrac{5}{2},-\tfrac{1}{2},-\tfrac{1}{2};\ 1,\tfrac{d+g}{d+h},\ \tfrac{g-f}{h-f}).$$

3. Express Legendre's incomplete elliptic integral of the first kind as an R function.

$$I = \int_0^x (1-t^2)^{-1/2}(1-k^2t^2)^{-1/2}dt, \ 0 < k,\ x < 1.$$

Substitute $t^2 = \tau$ to obtain

$$2I = \int_0^{x^2} \tau^{-1/2}(1-\tau)^{-1/2}(1-k^2\tau)^{-1/2}d\tau,$$

and

$$2kI = \int_0^{x^2} t^{-1/2}(1-t)^{-1/2}(1/k^2-t)^{-1/2}dt.$$

In 6.4 choose $\beta = 0$. On the left hand side of 6.4 let

$$(t-w)^{\gamma} \rightarrow (w-t)^{\gamma} = (1-t)^{-1/2},$$

$$(t-v)^{\delta} \rightarrow (v-t)^{\delta} = (1/k^2-t)^{-1/2}.$$

On the right hand side let

$$(y-w)^{\gamma} \rightarrow (w-y)^{\gamma} = 1,$$

$$(y-v)^{\delta} \rightarrow (v-y)^{\delta} = k.$$

Make the correspondence

$$y = 0 \qquad w = 1 \qquad \alpha = -\frac{1}{2} \qquad \gamma = -\frac{1}{2}$$

$$x \rightarrow x^2 \qquad v = \frac{1}{k^2} \qquad \beta = 0 \qquad \delta = -\frac{1}{2}.$$

The integral becomes

$$I = xR(\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};1-x^2,1-k^2x^2,1),$$

$$I = xR_F(1-x^2,1-k^2x^2,1).$$

When x is equal to unity the integral is Legendre's complete integral of the first kind, and is given by

$$I = R_F(0,1-k^2,1), \quad x = 1.$$

4. Reduce the following integral to a standard R function:

$$I = \int_0^1 t^{-1/2}(t^2+1)^{-1/2}dt.$$

The substitution $t^2 = \tau$ yields

$$I = 2R(\tfrac{1}{4};\tfrac{3}{4},\tfrac{1}{2};1,2).$$

We can obtain an R function whose a- and b-parameters are all half-integral by reducing the integral to an R with one real argument and two complex conjugate arguments, and then using the transformation 2.22.

Rewrite the integral as

$$I = \int_0^1 t^{-1/2}(t+i)^{-1/2}(t-i)^{-1/2}dt.$$

In 6.4 make the correspondence

$$\begin{array}{llll} x = 1 & w = -i & \alpha = -\frac{1}{2} & \gamma = -\frac{1}{2} \\ y = 0 & v = i & \beta = 0 & \delta = -\frac{1}{2} \end{array} .$$

The integral becomes

$$I = 2R_F(1+i,1-i,1).$$

By 2.22,

$$I = 2R_F(u,v,w),$$

where

$$2u = \sqrt{2} + 1$$

$$2v = \sqrt{2} + 2$$

$$2w = \sqrt{2}.$$

5. Reduce the following integral to an R function whose a-parameter is half-integral:

$$I = \int_0^\infty t^{1/2}(t+f)^{-3/2}(t+g)^{-1/2}(t+h)^{-1/2}\,dt, \quad f,g,h > 0.$$

In 2.2 make the correspondence

$$b_1 = \tfrac{3}{2} \qquad b_3 = \tfrac{1}{2} \qquad a' = \tfrac{3}{2}$$

$$b_2 = \tfrac{1}{2} \qquad c = \tfrac{5}{2} \qquad a = 1.$$

The integral becomes

$$3I = 2R(1;\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2};\ f,g,h).$$

Using the Euler transformation 2.4, we obtain the result,

$$3f(fgh)^{1/2}I = 2R(\tfrac{3}{2};\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2};f^{-1},\ g^{-1},\ h^{-1}).$$

6. Express the following R function as another R whose arguments lie between zero and unity, and then evaluate numerically:

$$R = R(\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};1,2,4).$$

Use homogeneity 2.3 to obtain

$$2R = R_F(0.25,\ 0.50,\ 1.00).$$

Table 4 yields

$$R = 0.3764.$$

7. Evaluate numerically the following definite integral:

$$I = \int_2^3 (t+1)^{-3/2}(t-1)^{1/2}(3-t)^{-1/2}dt.$$

Using 6.3, 2.6, and symmetry, the integral becomes

$$I = 2^{-3/2}R(\tfrac{1}{2};\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2};\tfrac{3}{4},1,\tfrac{1}{2}).$$

By the reduction formula of Table 3 and the symmetry of R_F and R_G, we obtain

$$I = \sqrt{2}[R_F(\tfrac{1}{2},\tfrac{3}{4},\ 1) - 2R_G(\tfrac{1}{2},\tfrac{3}{4},\ 1)+(2/3)^{1/2}].$$

Table 4 gives the numerical result:

$$I = \sqrt{2}[1.1682 - 2(0.8628) + 0.8165]$$

$$I = 0.3664.$$

8. Express the following integral as an R function:

$$I = \int_{\phi}^{\pi/2} (k^2\sin^2\theta - 1)^{-1/2}\, d\theta, \quad k\sin\phi > 1.$$

In 6.6 make the correspondence

$$\alpha = 0 \qquad \gamma = 0 \qquad p = -1 \qquad r = k^2$$

$$\beta = 0 \qquad \delta = -\frac{1}{2} \qquad q = 0\,.$$

The integral becomes

$$I = (k^2-1)^{-1/2}\cos\phi\ R_F(\sin^2\phi, \frac{k^2\sin^2\phi - 1}{k^2-1}, 1).$$

9. Express $R(\frac{1}{2};\frac{5}{2},\frac{1}{2};x,y)$ as a linear combination of R_K and R_E. The purpose of this example is to illustrate how the reduction formula tables may be extended.

Use 2.13 to increase by one a b-parameter of $R(\frac{1}{2};\frac{3}{2},\frac{1}{2};x,y)$. For this R function,

$$a = \frac{1}{2}\ ,\ a' = \frac{3}{2}\ ,\ c = 2.$$

By 2.13,

$$3(y-x)R(\tfrac{1}{2};\tfrac{5}{2},\tfrac{1}{2};x,y) = 4[yR(\tfrac{1}{2};\tfrac{3}{2},\tfrac{1}{2};x,y) - R(-\tfrac{1}{2};\tfrac{3}{2},\tfrac{1}{2};x,y].$$

Using the reduction formulas of Table 2 for the two R functions on the right, we obtain

$$9(y-x)^2R(\tfrac{1}{2};\tfrac{5}{2},\tfrac{1}{2};x,y) = 12y[2yR_K(x,y)-2R_E(x,y]$$

$$-4[2xyR_K(x,y)-2(2x-y)R_E(x,y)].$$

Simplifying, we obtain the desired result,

$$9(y-x)^2 R(\tfrac{1}{2};\tfrac{5}{2},\tfrac{1}{2};x,y) = 8y(3y-x)R_K(x,y)+16(x-2y)R_E(x,y).$$

The identical result may be obtained by setting $z = 0$ in the reduction formula for $R(\frac{1}{2};\frac{5}{2},\frac{1}{2},\frac{1}{2};x,y,z)$ given in Table 3, and using 2.8.

VII. TABLES

Table 1. Integrals with algebraic or trigonometric integrands

In 6.1 and 6.2,

$$a = -(\alpha+\beta+\gamma+\delta+1), \ \mathrm{Re}\ a > 0.$$

(6.1)
$$\int_{-\infty}^{x} (z-t)^{\alpha}(w-t)^{\beta}(v-t)^{\gamma}(s-t)^{\delta}\, dt$$

$$= a^{-1}R(a;-\alpha,-\beta,-\gamma,-\delta;\ z-x,w-x,v-x,s-x),$$

$$(x \leq z,\ w,\ v,\ s),\ (\text{substitute } t = x-u).$$

(6.2)
$$\int_{x}^{\infty} (t-z)^{\alpha}(t-w)^{\beta}(t-v)^{\gamma}(t-s)^{\delta}\, dt$$

$$= a^{-1}R(a;-\alpha,-\beta,-\gamma,-\delta;\ x-z,x-w,x-v,x-s),$$

$$(z,w,v,s \leq x),\ (\text{substitute } t = x+u).$$

Equations 6.3 and 6.4 remain valid if both sides are multiplied by $(-1)^{\beta}$, that is, if $(t-z)^{\beta}$ is replaced by $(z-t)^{\beta}$ and $(y-z)^{\beta}$ by $(z-y)^{\beta}$. The same is true for $(-1)^{\gamma}$, $(-1)^{\delta}$, or any combination of the three, such as $(-1)^{\beta+\gamma}$ (see Section III). In 6.3 and 6.4,

$$a = \alpha+1, \ \mathrm{Re}\ \alpha > -1.$$

Table 1. (Continued)

(6.3) $$\int_x^y (y-t)^{\alpha}(t-z)^{\beta}(t-w)^{\gamma}(t-v)^{\delta}\, dt$$

$$= a^{-1}(y-x)^{\alpha+1}(y-z)^{\beta}(y-w)^{\gamma}(y-v)^{\delta}$$

$$\cdot R(a;-\beta,-\gamma,-\delta,\alpha+\beta+\gamma+\delta+2;\ \frac{x-z}{y-z},\ \frac{x-w}{y-w},\ \frac{x-v}{y-v},\ 1),$$

(substitute $t = (uy+x)(u+1)^{-1}$).

(6.4) $$\int_y^x (t-y)^{\alpha}(t-z)^{\beta}(t-w)^{\gamma}(t-v)^{\delta} dt$$

$$= a^{-1}(x-y)^{\alpha+1}(y-z)^{\beta}(y-w)^{\gamma}(y-v)^{\delta}$$

$$\cdot R(a;-\beta,\ -\gamma,\ -\delta,\ \alpha+\beta+\gamma+\delta+2\ ;\ \frac{x-z}{y-z},\ \frac{x-w}{y-w},\ \frac{x-v}{y-v},\ 1),$$

(substitute $t = (uy+x)(u+1)^{-1}$).

(6.5) $$\int_0^{\phi} (\sin\theta)^{\alpha}(\sin^2\phi-\sin^2\theta)^{\beta}(\cos\theta)^{\gamma}(p+q\cos^2\theta+r\sin^2\theta)^{\delta}\, d\theta$$

$$= \frac{1}{2}\ (p+q)^{\delta}(\sin\phi)^{\alpha+2\beta+1}B(\frac{\alpha+1}{2}\ ,\ \beta+1)$$

$$\cdot R(\frac{\alpha+1}{2};\ \frac{1-\gamma}{2},\ -\delta,\ \frac{\alpha+2\beta+\gamma+2\delta+2}{2}\ ;\ \cos^2\phi,\frac{p+q\cos^2\phi+r\sin^2\phi}{p+q},\ 1),$$

($0 \le \phi \le \frac{\pi}{2}$; Re α, Re $\beta > -1$), (substitute $\sin\theta = (1+t)^{-1/2}\sin\phi$).

Table 1. (Continued)

(6.6) $$\int_{\phi}^{\pi/2} (\cos\theta)^{\alpha}(\cos^2\phi-\cos^2\theta)^{\beta}(\sin\theta)^{\gamma}(p+q\cos^2\theta+r\sin^2\theta)^{\delta}d\theta$$

$$= \tfrac{1}{2}(p+r)^{\delta}(\cos\phi)^{\alpha+2\beta+1}B(\tfrac{\alpha+1}{2},\beta+1)$$

$$\cdot R(\tfrac{\alpha+1}{2};\tfrac{1-\gamma}{2},-\delta,\tfrac{\alpha+2\beta+\gamma+2\delta+2}{2};\sin^2\phi,\tfrac{p+q\cos^2\phi+r\sin^2\phi}{p+r},\ 1),$$

$(0 \le \phi \le \frac{\pi}{2}; \mathrm{Re}\ \alpha,\ \mathrm{Re}\ \beta > -1)$, (substitute $\cos\theta = (1+t)^{-1/2}\cos\phi$).

(6.7) $$\int_{0}^{\phi} (\sin\theta)^{\alpha}(\cos\theta-\cos\phi)^{\beta}(\cos\theta)^{\gamma}(1-p\cos\theta)^{\delta}d\theta$$

$$= 2^{\alpha+\beta}(1-p)^{\delta}(\sin\tfrac{\phi}{2})^{\alpha+2\beta+1}B(\tfrac{\alpha+1}{2},\beta+1)$$

$$\cdot R(\tfrac{\alpha+1}{2};-\gamma,\tfrac{1-\alpha}{2},-\delta,\alpha+\beta+\gamma+\delta+1;\cos\phi,\tfrac{1+\cos\phi}{2},\tfrac{1-p\cos\phi}{1-p},1),$$

$(0 \le \phi \le \frac{\pi}{2}; \mathrm{Re}\ \alpha,\ \mathrm{Re}\ \beta > -1)$, (substitute $\cos\theta = (t+\cos\phi)(t+1)^{-1}$).

(6.8) $$\int_{\phi}^{\pi/2} (\cos\theta)^{\alpha}(\sin\theta-\sin\phi)^{\beta}(\sin\theta)^{\gamma}(1-p\sin\theta)^{\delta}d\theta$$

$$= 2^{\alpha+\beta}(1-p)^{\delta}[\sin(\tfrac{\pi}{4}-\tfrac{\phi}{2})]^{\alpha+2\beta+1}B(\tfrac{\alpha+1}{2},\ \beta+1)$$

$$\cdot R(\tfrac{\alpha+1}{2};-\gamma,\tfrac{1-\alpha}{2},-\delta,\ \alpha+\beta+\gamma+\delta+1;\sin\phi,\tfrac{1+\sin\phi}{2},\tfrac{1-p\sin\phi}{1-p},1),$$

$(0 \le \phi \le \frac{\pi}{2}; \mathrm{Re}\ \alpha,\ \mathrm{Re}\ \beta > -1)$, (substitute $\sin\theta = (t+\sin\phi)(t+1)^{-1}$).

Table 2. Reduction formulas for complete integrals of the first and second kinds

These formulas are of the form

$$AR(a;b_1,b_2;x,y) = A_K R_K + A_E R_E,$$

where

$$R_K = R(\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2};\ x,y),$$

$$R_E = R(-\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2};x,y).$$

The symbol $\underline{b}$ is defined as

$$\underline{b} = (b_1,b_2).$$

a	a'	$\underline{b}$	A	A_K	A_E
$\frac{5}{2}$	$\frac{1}{2}$		$3xy(x-y)^2$	$-16xy$	$8(x+y)$
$\frac{3}{2}$	$\frac{3}{2}$		$(x-y)^2$	$8(x+y)$	-16
$\frac{1}{2}$	$\frac{5}{2}$		$3(x-y)^2$	$-16xy$	$8(x+y)$
$-\frac{1}{2}$	$\frac{7}{2}$	$(\frac{3}{2}, \frac{3}{2})$	$15(x-y)^2$	$-8xy(x+y)$	$16(x^2+y^2-xy)$

Table 2. (Continued)

a	a′	b	A	A_K	A_E
$-\frac{3}{2}$	$\frac{9}{2}$	$(\frac{3}{2},\frac{3}{2})$	$105(x-y)^2$	$16xy(-2x^2-2y^2+xy)$	$64(x^3+y^3)-40xy(x+y)$
$-\frac{5}{2}$	$\frac{11}{2}$		$315(x-y)^2$	$24x^2y^2(x+y)-64xy(x^3+y^3)$	$-8xy(8x^2+8y^2+6xy)$ $+128(x^4+y^4)$
$\frac{5}{2}$	$-\frac{1}{2}$	$(\frac{3}{2},\frac{1}{2})$	$3x^2y(y-x)$	$-2xy$	$-2x+4y$
$\frac{3}{2}$	$\frac{1}{2}$		$x(y-x)$	$-2x$	2
$\frac{1}{2}$	$\frac{3}{2}$		$y-x$	$2y$	-2
$-\frac{1}{2}$	$\frac{5}{2}$		$3(y-x)$	$2xy$	$-4x+2y$
$-\frac{3}{2}$	$\frac{7}{2}$		$15(y-x)$	$2xy(4x-y)$	$-16x^2+4y^2+6xy$
$-\frac{5}{2}$	$\frac{9}{2}$		$105(y-x)$	$xy(48x^2-8y^2-10xy)$	$2xy(16x+9y)$ $+16(y^3-6x^3)$

Table 2. (Continued)

a	a′	<u>b</u>	A	A_K	A_E
$\frac{5}{2}$	$-\frac{3}{2}$	$(\frac{3}{2}, -\frac{1}{2})$	$3x^3y$	$-4xy$	$-x+8y$
$\frac{3}{2}$	$-\frac{1}{2}$		x^2	$-x$	2
$\frac{1}{2}$	$\frac{1}{2}$		x	0	1
$-\frac{1}{2}$	$\frac{3}{2}$		1	$-y$	2
$-\frac{3}{2}$	$\frac{5}{2}$		3	$-4xy$	$8x-y$
$-\frac{5}{2}$	$\frac{7}{2}$		15	$xy(-24x+y)$	$48x^2-2y^2-8xy$
$\frac{5}{2}$	$-\frac{3}{2}$	$(\frac{1}{2},\frac{1}{2})$	$3x^2y^2$	$-xy$	$2(x+y)$
$\frac{3}{2}$	$-\frac{1}{2}$		xy	0	1

Table 2. (Continued)

a	a′	b	A	A_K	A_E
$\frac{1}{2}$	$\frac{1}{2}$	$(\frac{1}{2},\frac{1}{2})$	1	1	0
$-\frac{1}{2}$	$\frac{3}{2}$		1	0	1
$-\frac{3}{2}$	$\frac{5}{2}$		3	$-xy$	$2(x+y)$
$-\frac{5}{2}$	$\frac{7}{2}$		15	$-4xy(x+y)$	$8(x+y)^2-9xy$

Table 3. Reduction formulas for incomplete integrals of the first and second kinds

These formulas are of the form

$$CR(a;b_1,b_2,b_3;x,y,z) = C_F R_F + C_G R_G + C_A (xyz)^{-1/2},$$

where

$$R_F = R(\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};x,y,z),$$

$$R_G = R(-\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};x,y,z),$$

$$(xyz)^{-1/2} = R(\tfrac{3}{2};\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};x,y,z).$$

The symbols s,p, and $\underline{b}$ are defined as

$$s = xy+xz+yz$$

$$p = xyz$$

$$\underline{b} = (b_1,b_2,b_3)$$

Table 3. (Continued)

a	a′	b	C	C_F	C_G	C_A
$\frac{3}{2}$	2	$(\frac{5}{2},\frac{1}{2},\frac{1}{2})$	$x(x-y)^2(x-z)^2$	$5x^2(3yz-s)$	$5x(x^2-4yz+s)$	$-5p(x^2-yz)$
$\frac{1}{2}$	3		$8(x-y)^2(x-z)^2$	$15(2yz-s)^2$ $-5x^2s$	$20x(x^2+5yz-2s)$	$-5p(x^2+8yz-3s)$
$\frac{3}{2}$	1	$(\frac{5}{2},\frac{1}{2},\frac{-1}{2})$	$x(x-y)^2(x-z)$	$x(s-3xy)$	$2x(x+y-2z)$	$-yz(2x^2+yz-3xz)$
$\frac{1}{2}$	2		$2(x-y)^2(x-z)$	$-(x-3y)s-6y^2z$	$4x(x-2y)$ $-2z(x-3y)$	$-yz(x^2+2yz-3xy)$
$\frac{3}{2}$	0	$(\frac{5}{2},\frac{-1}{2},\frac{-1}{2})$	x^2	0	0	yz
$\frac{1}{2}$	1		$3x(x-y)(x-z)$	$x(3yz-s)$	$2x(2x-y-z)$	$-yz(x^2-yz)$

Table 3. (Continued)

a	a′	b	C	C_F	C_G	C_A
$\frac{1}{2}$	3	$(\frac{3}{2},\frac{3}{2},\frac{1}{2})$	$8(x-y)^2(y-z)(x-z)$	$15z^2(x-y)^2$ $+15xy(s-3xy)$	$30x^2(y-z)$ $+30y^2(x-z)$	$15p(s-3xy)$
$\frac{3}{2}$	0	$(\frac{3}{2},\frac{3}{2},-\frac{3}{2})$	xy	0	0	z^2
$\frac{3}{2}$	1		$(x-y)(x-z)$	$3x$	-6	$3yz$
$\frac{1}{2}$	2	$(\frac{3}{2},\frac{1}{2},\frac{1}{2})$	$2(x-y)(x-z)$	$3(2yz-s)$	$6x$	$-3p$
$-\frac{1}{2}$	3		$8(x-y)(x-z)$	$3x(2yz-s)$	$6(x-y)(x-z)+6x^2$	$-3xp$
$-\frac{3}{2}$	4		$16(x-y)(x-z)$	$-s[(x-y)(x-z)+3x^2]$ $+6xp$	$(x-y)(x-z)(10x+4y+4z)$ $+6x^3$	$-[(x-y)(x-z)+3x^2]p$
$\frac{3}{2}$	0		x	0	0	z
$\frac{1}{2}$	1		$x-y$	$-y$	2	$-yz$
		$(\frac{3}{2},\frac{1}{2},-\frac{1}{2})$				

Table 3. (Continued)

a	a'	$\underline{b}$	C	C_F	C_G	C_A
$-\frac{1}{2}$	2	$(\frac{3}{2},\frac{1}{2},-\frac{1}{2})$	$2(x-y)$	$2yz-s$	$4x-2y$	$-p$
$-\frac{3}{2}$	3		$8(x-y)$	$-z(4x+y)(x-y)$ $-xy(4x-y)$	$2(x-y)(5x+2y-z)$ $+6x^2$	$-(4x-y)p$
$\frac{3}{2}$	-1	$(\frac{3}{2},\frac{1}{2},-\frac{3}{2})$	x^2y	0	0	$3yz(z-x)+xz^2$
$\frac{1}{2}$	0		x	0	0	z^2
$\frac{3}{2}$	-1	$(\frac{3}{2},-\frac{1}{2},-\frac{1}{2})$	x^2	0	0	$4yz-s$
$\frac{1}{2}$	0		x	0	0	yz
$-\frac{1}{2}$	1		1	$-y-z$	4	$-yz$
$-\frac{3}{2}$	2		1	$3yz-2s$	$8x-y-z$	$-2p$
$\frac{3}{2}$	0	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	1	0	0	1

Table 3. (Continued)

a	a′	b	C	C_F	C_G	C_A
$\frac{1}{2}$	1	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	1	1	0	0
$-\frac{1}{2}$	2		1	0	1	0
$-\frac{3}{2}$	3		8	-s	4(x+y+z)	-p
$-\frac{5}{2}$	4		24	3p-2(x+y+z)s	$8(x+y+z)^2-9s$	-2(x+y+z)p
$\frac{5}{2}$	-2	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	3xyp	0	0	$[s(4xz-s)-4yz(x^2-yz)]$
$\frac{3}{2}$	-1		xy	0	0	s-2xy
$\frac{1}{2}$	0		1	0	0	z
$-\frac{1}{2}$	1		1	-z	2	0
$-\frac{3}{2}$	2		2	-s	4(x+y)-2z	-p

Table 3. (Continued)

a	a′	b	C	C_F	C_G	C_A
$-\frac{5}{2}$	3	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	8	$3p+(-3x-3y+2z)s$	$4(3x^2+5xy+3y^2-2z^2)-5s$	$(-3x-3y+2z)p$
$\frac{3}{2}$	-2	$(\frac{1}{2},\frac{1}{2},-\frac{3}{2})$	x^2y^2	0	0	$4zp-3(s-2xy)^2$
$\frac{1}{2}$	-1		xy	0	0	$z(4xy-s)$
$-\frac{1}{2}$	0		1	0	0	z^2
$-\frac{3}{2}$	1		1	$s-3z^2$	$-4x-4y+8z$	p
$\frac{3}{2}$	-2	$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	xp	0	0	$s^2-4yz(x^2+yz)$
$\frac{1}{2}$	-1		x	0	0	$s-2yz$
$-\frac{1}{2}$	0		1	0	0	yz
$-\frac{3}{2}$	1		1	$s-3yz$	$-4x+2y+2z$	p

Table 4a. Numerical table of $R_F(x,y,1)$

(X,Y)	0.00	0.05	0.10	0.15	0.20	0.25	0.30
0.00	∞	2.9083	2.5781	2.3890	2.2572	2.1565	2.0754
0.05	2.9083	2.2349	2.0613	1.9502	1.8679	1.8024	1.7480
0.10	2.5781	2.0613	1.9168	1.8223	1.7513	1.6942	1.6465
0.15	2.3890	1.9502	1.8223	1.7375	1.6733	1.6215	1.5779
0.20	2.2572	1.8679	1.7513	1.6733	1.6140	1.5659	1.5254
0.25	2.1565	1.8024	1.6942	1.6215	1.5659	1.5207	1.4825
0.30	2.0754	1.7480	1.6465	1.5779	1.5254	1.4825	1.4461
0.35	2.0076	1.7015	1.6055	1.5404	1.4903	1.4493	1.4146
0.40	1.9496	1.6609	1.5695	1.5073	1.4594	1.4201	1.3867
0.45	1.8989	1.6249	1.5375	1.4778	1.4317	1.3939	1.3617
0.50	1.8541	1.5927	1.5087	1.4512	1.4067	1.3702	1.3391
0.55	1.8139	1.5634	1.4825	1.4270	1.3839	1.3485	1.3183
0.60	1.7775	1.5367	1.4585	1.4047	1.3629	1.3286	1.2993
0.65	1.7444	1.5121	1.4363	1.3841	1.3435	1.3101	1.2816
0.70	1.7139	1.4894	1.4158	1.3650	1.3255	1.2929	1.2651
0.75	1.6858	1.4682	1.3966	1.3471	1.3086	1.2768	1.2497
0.80	1.6596	1.4484	1.3787	1.3304	1.2928	1.2617	1.2352
0.85	1.6353	1.4298	1.3618	1.3147	1.2779	1.2475	1.2215
0.90	1.6124	1.4124	1.3459	1.2998	1.2638	1.2341	1.2086
0.95	1.5910	1.3959	1.3309	1.2858	1.2505	1.2213	1.1963
1.00	1.5708	1.3802	1.3166	1.2724	1.2378	1.2092	1.1847

Table 4a. (Continued)

(X,Y)	0.35	0.40	0.45	0.50	0.55	0.60	0.65
0.00	2.0076	1.9496	1.8989	1.8541	1.8139	1.7775	1.7444
0.05	1.7015	1.6609	1.6249	1.5927	1.5634	1.5367	1.5121
0.10	1.6055	1.5695	1.5375	1.5087	1.4825	1.4585	1.4363
0.15	1.5404	1.5073	1.4778	1.4512	1.4270	1.4047	1.3841
0.20	1.4903	1.4594	1.4317	1.4067	1.3839	1.3629	1.3435
0.25	1.4493	1.4201	1.3939	1.3702	1.3485	1.3286	1.3101
0.30	1.4146	1.3867	1.3617	1.3391	1.3183	1.2993	1.2816
0.35	1.3844	1.3577	1.3337	1.3119	1.2920	1.2736	1.2566
0.40	1.3577	1.3319	1.3088	1.2878	1.2686	1.2509	1.2344
0.45	1.3337	1.3088	1.2865	1.2662	1.2475	1.2304	1.2144
0.50	1.3119	1.2878	1.2662	1.2465	1.2284	1.2117	1.1962
0.55	1.2920	1.2686	1.2475	1.2284	1.2108	1.1946	1.1795
0.60	1.2736	1.2509	1.2304	1.2117	1.1946	1.1787	1.1640
0.65	1.2566	1.2344	1.2144	1.1962	1.1795	1.1640	1.1496
0.70	1.2407	1.2191	1.1995	1.1817	1.1654	1.1503	1.1362
0.75	1.2259	1.2047	1.1856	1.1682	1.1522	1.1374	1.1236
0.80	1.2119	1.1912	1.1725	1.1554	1.1397	1.1252	1.1117
0.85	1.1987	1.1784	1.1601	1.1433	1.1280	1.1137	1.1005
0.90	1.1862	1.1663	1.1483	1.1319	1.1168	1.1028	1.0898
0.95	1.1744	1.1548	1.1372	1.1211	1.1062	1.0925	1.0797
1.00	1.1631	1.1439	1.1266	1.1107	1.0961	1.0826	1.0701

Table 4a. (Continued)

(X,Y)	0.70	0.75	0.80	0.85	0.90	0.95	1.00
0.00	1.7139	1.6858	1.6596	1.6353	1.6124	1.5910	1.5708
0.05	1.4894	1.4682	1.4484	1.4298	1.4124	1.3959	1.3802
0.10	1.4158	1.3966	1.3787	1.3618	1.3459	1.3309	1.3166
0.15	1.3650	1.3471	1.3304	1.3147	1.2998	1.2858	1.2724
0.20	1.3255	1.3086	1.2928	1.2779	1.2638	1.2505	1.2378
0.25	1.2929	1.2768	1.2617	1.2475	1.2341	1.2213	1.2092
0.30	1.2651	1.2497	1.2352	1.2215	1.2086	1.1963	1.1847
0.35	1.2407	1.2259	1.2119	1.1987	1.1862	1.1744	1.1631
0.40	1.2191	1.2047	1.1912	1.1784	1.1663	1.1548	1.1439
0.45	1.1995	1.1856	1.1725	1.1601	1.1483	1.1372	1.1266
0.50	1.1817	1.1682	1.1554	1.1433	1.1319	1.1211	1.1107
0.55	1.1654	1.1522	1.1397	1.1280	1.1168	1.1062	1.0961
0.60	1.1503	1.1374	1.1252	1.1137	1.1028	1.0925	1.0826
0.65	1.1362	1.1236	1.1117	1.1005	1.0898	1.0797	1.0701
0.70	1.1231	1.1107	1.0991	1.0881	1.0776	1.0677	1.0583
0.75	1.1107	1.0986	1.0872	1.0764	1.0662	1.0565	1.0472
0.80	1.0991	1.0872	1.0760	1.0654	1.0554	1.0459	1.0367
0.85	1.0881	1.0764	1.0654	1.0550	1.0452	1.0358	1.0269
0.90	1.0776	1.0662	1.0554	1.0452	1.0355	1.0263	1.0175
0.95	1.0677	1.0565	1.0459	1.0358	1.0263	1.0172	1.0085
1.00	1.0583	1.0472	1.0367	1.0269	1.0175	1.0085	1.0000

Table 4b. Numerical table of $R_G(x,y,1)$

(X,Y)	0.00	0.05	0.10	0.15	0.20	0.25	0.30
0.00	0.5000	0.5302	0.5524	0.5717	0.5892	0.6055	0.6208
0.05	0.5302	0.5559	0.5763	0.5945	0.6112	0.6267	0.6415
0.10	0.5524	0.5763	0.5958	0.6133	0.6295	0.6446	0.6589
0.15	0.5717	0.5945	0.6133	0.6303	0.6460	0.6608	0.6748
0.20	0.5892	0.6112	0.6295	0.6460	0.6614	0.6759	0.6896
0.25	0.6055	0.6267	0.6446	0.6608	0.6759	0.6901	0.7036
0.30	0.6208	0.6415	0.6589	0.6748	0.6896	0.7036	0.7169
0.35	0.6354	0.6555	0.6726	0.6882	0.7028	0.7166	0.7297
0.40	0.6492	0.6689	0.6857	0.7011	0.7154	0.7290	0.7420
0.45	0.6625	0.6818	0.6983	0.7135	0.7276	0.7410	0.7538
0.50	0.6753	0.6942	0.7105	0.7254	0.7394	0.7527	0.7653
0.55	0.6877	0.7063	0.7223	0.7371	0.7509	0.7640	0.7765
0.60	0.6997	0.7180	0.7338	0.7484	0.7620	0.7750	0.7874
0.65	0.7113	0.7294	0.7450	0.7594	0.7729	0.7857	0.7980
0.70	0.7227	0.7404	0.7559	0.7701	0.7835	0.7962	0.8083
0.75	0.7337	0.7513	0.7665	0.7806	0.7938	0.8064	0.8185
0.80	0.7445	0.7618	0.7769	0.7908	0.8039	0.8164	0.8284
0.85	0.7551	0.7721	0.7871	0.8009	0.8139	0.8262	0.8381
0.90	0.7654	0.7823	0.7970	0.8107	0.8236	0.8359	0.8476
0.95	0.7755	0.7922	0.8068	0.8204	0.8331	0.8453	0.8570
1.00	0.7854	0.8019	0.8164	0.8299	0.8425	0.8546	0.8662

Table 4b. (Continued)

(X,Y)	0.35	0.40	0.45	0.50	0.55	0.60	0.65
0.00	0.6354	0.6492	0.6625	0.6753	0.6877	0.6997	0.7113
0.05	0.6555	0.6689	0.6818	0.6942	0.7063	0.7180	0.7294
0.10	0.6726	0.6857	0.6983	0.7105	0.7223	0.7338	0.7450
0.15	0.6882	0.7011	0.7135	0.7254	0.7371	0.7484	0.7594
0.20	0.7028	0.7154	0.7276	0.7394	0.7509	0.7620	0.7729
0.25	0.7166	0.7290	0.7410	0.7527	0.7640	0.7750	0.7857
0.30	0.7297	0.7420	0.7538	0.7653	0.7765	0.7874	0.7980
0.35	0.7423	0.7544	0.7661	0.7775	0.7885	0.7993	0.8098
0.40	0.7544	0.7664	0.7780	0.7892	0.8002	0.8108	0.8212
0.45	0.7661	0.7780	0.7895	0.8006	0.8114	0.8220	0.8323
0.50	0.7775	0.7892	0.8006	0.8116	0.8223	0.8328	0.8430
0.55	0.7885	0.8002	0.8114	0.8223	0.8330	0.8433	0.8535
0.60	0.7993	0.8108	0.8220	0.8328	0.8433	0.8536	0.8637
0.65	0.8098	0.8212	0.8323	0.8430	0.8535	0.8637	0.8736
0.70	0.8200	0.8314	0.8423	0.8530	0.8634	0.8735	0.8834
0.75	0.8301	0.8413	0.8522	0.8628	0.8731	0.8831	0.8929
0.80	0.8399	0.8510	0.8618	0.8723	0.8826	0.8925	0.9023
0.85	0.8495	0.8606	0.8713	0.8817	0.8919	0.9018	0.9115
0.90	0.8590	0.8699	0.8806	0.8909	0.9010	0.9109	0.9205
0.95	0.8682	0.8791	0.8897	0.9000	0.9100	0.9198	0.9294
1.00	0.8774	0.8882	0.8987	0.9089	0.9189	0.9286	0.9381

Table 4b. (Continued)

(X,Y)	0.70	0.75	0.80	0.85	0.90	0.95	1.00
0.00	0.7227	0.7337	0.7445	0.7551	0.7654	0.7755	0.7854
0.05	0.7404	0.7513	0.7618	0.7721	0.7823	0.7922	0.8019
0.10	0.7559	0.7665	0.7769	0.7871	0.7970	0.8068	0.8164
0.15	0.7701	0.7806	0.7908	0.8009	0.8107	0.8204	0.8299
0.20	0.7835	0.7938	0.8039	0.8139	0.8236	0.8331	0.8425
0.25	0.7962	0.8064	0.8164	0.8262	0.8359	0.8453	0.8546
0.30	0.8083	0.8185	0.8284	0.8381	0.8476	0.8570	0.8662
0.35	0.8200	0.8301	0.8399	0.8495	0.8590	0.8682	0.8774
0.40	0.8314	0.8413	0.8510	0.8606	0.8699	0.8791	0.8882
0.45	0.8423	0.8522	0.8618	0.8713	0.8806	0.8897	0.8987
0.50	0.8530	0.8628	0.8723	0.8817	0.8909	0.9000	0.9089
0.55	0.8634	0.8731	0.8826	0.8919	0.9010	0.9100	0.9189
0.60	0.8735	0.8831	0.8925	0.9018	0.9109	0.9198	0.9286
0.65	0.8834	0.8929	0.9023	0.9115	0.9205	0.9294	0.9381
0.70	0.8931	0.9026	0.9119	0.9210	0.9300	0.9388	0.9475
0.75	0.9026	0.9120	0.9212	0.9303	0.9392	0.9480	0.9566
0.80	0.9119	0.9212	0.9304	0.9394	0.9483	0.9570	0.9656
0.85	0.9210	0.9303	0.9394	0.9484	0.9572	0.9659	0.9744
0.90	0.9300	0.9392	0.9483	0.9572	0.9660	0.9746	0.9831
0.95	0.9388	0.9480	0.9570	0.9659	0.9746	0.9832	0.9916
1.00	0.9475	0.9566	0.9656	0.9744	0.9831	0.9916	1.0000

VIII. LITERATURE CITED

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IX. ACKNOWLEDGEMENTS

The author wishes to express his sincere gratitude to Dr. B. C. Carlson for both his professional and personal advice and encouragement. The author also wishes to thank two members of the Ames Laboratory Computer Service Group, Mr. James Delany and Mr. Frank Carlsen. Mr. Delany wrote the Fortran subroutine program to calculate R_F and R_G using quadratic transformations. Mr. Carlsen provided valuable advice for the computation of R_F and R_G directly from their power series expansions.

X. APPENDIX: FORTRAN SUBROUTINE PROGRAM FOR COMPUTING R_F AND R_G BY MEANS OF DESCENDING GAUSS OR ASCENDING LANDEN TRANSFORMATIONS

This program has been written under the assumption that $0 \leq x \leq y \leq z$. Particular values worthy of note are $R_F(0,0,1) = \infty$ and $R_G(0,0,1) = 1/2$.

```
      SUBROUTINE ELINT(X,Y,Z,RF,RG,INF)
      DIMENSION V(3),W(3)

 5    IF(X) 7,8,8
 8    SQRTZ=SQRTF(Z)
      IF(Y) 7,10,11
10    INF=1
      RG=.5*SQRTZ
      GOTO 7
11    INF=0
 9    IF(Z-X) 13,12,13
12    RF=1./SQRTZ
      RG=SQRTZ
      GOTO 7
13    D=Y+Y-X-Z
      A=SQRTF(Z-X)
      IF(D) 15,14,14
14    T=SQRTZ
      H=SQRTF(X)
      CS=Z-Y
      GOTO 21
15    T=SQRTF(X)
      H=SQRTZ
      CS=Y-X
21    T=.5*(T+SQRTF(Y))
      N=0
80    A=.5*(A+SQRTF(A*A-CS))
      CC=(.25*CS)/A
      CS=CC*CC
      N=N+1
      V(N)=A*A-CS
      W(N)=H
      IF(N-1) 81,82,81
```

```
82    E=CC/A
      IF(E-7.46E-3) 18,18,17
17    NMAX=3
      GOTO 81
18    IF(E-1.39E-5) 20,20,19
19    NMAX=2
      GOTO 81
20    NMAX=1
81    IF(N-NMAX) 23,27,23
23    IF(D) 25,24,24
24    Q=SQRTF(T*T-CS)
      GOTO 26
25    Q=SQRTF(T*T+CS)
26    H=(H*T)/Q
      T=.5*(T+Q)
      GOTO 80
27    IF(D) 29,28,28
28    RF=(ASINF(A/T))/A
      GOTO 30
29    AT=A/T
      RF=(LOGF(AT+SQRTF(1.+AT*AT)))/A
30    B=V(1)
      C=W(1)
      IF(NMAX-2) 31,32,33
33    B=B+4.*(V(3)-V(2))
      C=C+4.*(W(3)-W(2))
32    V21=V(2)-V(1)
      W21=W(2)-W(1)
      B=B+V21+V21
      C=C+W21+W21
31    IF(D) 34,35,35
34    U=Z-B
      GOTO 36
35    U=X+B
36    RG=.5*(U*RF+C)
 7    RETURN
      END
```


NASA-Langley, 1965 CR-289

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