THE DYADIC GREEN'S FUNCTION FOR A MOVING ISOTROPIC MEDIUM

C. T. Tai
The Radiation Laboratory
The University of Michigan

When the velocity of a moving isotropic medium is small compared to the velocity of light, the Maxwell-Minkowski equations have a relatively simple form.[1] The dyadic Green's function pertaining to these simplified wave equations can be found either by the methods of Fourier transform or by a more direct method.[2] Alternatively, the method of potentials can also be used to solve these equations.[3]

In this communication we shall present a derivation of the dyadic Green's function with no restriction upon the order of magnitude of the velocity. A compact result is obtained by transforming the wave equation into a conventional form and then solving it with the operational method originally due to Levine and Schwinger.[4]

The Maxwell's equations for a moving medium have the same form as for a stationary medium. For harmonically oscillative fields with a time convention $e^{j\omega t}$, they are:

$$\nabla \times \bar{E} = -j\omega\bar{B} \tag{1}$$

$$\nabla \times \bar{H} = \bar{J} + j\omega\bar{D} \tag{2}$$

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The constitutive relations between the field vectors for a uniformly moving isotropic medium were found by Minkowski[5] based upon the special theory of relativity. They are:

$$\bar{D} + \frac{1}{c^2}\,\bar{v} \times \bar{H} = \epsilon(\bar{E} + \bar{v} \times \bar{B}) \tag{3}$$

$$\bar{B} - \frac{1}{c^2}\,\bar{v} \times \bar{E} = \mu(\bar{H} - \bar{v} \times \bar{D}) \tag{4}$$

where ϵ and μ denote, respectively, the permittivity and permeability of the medium at rest which is assumed to be lossless. $\bar{v}$ and c denote, respectively, the velocity of the moving medium and the speed of light in vacuum. To simplify the derivation we assume

$$\bar{v} = v\hat{z}\ . \tag{5}$$

The above condition is not much of a restriction since a coordinate transformation of the result can easily take care of the general case.

By solving $\bar{D}$ and $\bar{B}$ from (3 - 4) in terms of $\bar{E}$ and $\bar{H}$ with $\bar{v}$ given by (5), we obtain the following relations:

$$\bar{D} = \epsilon\,\bar{\bar{\alpha}} \cdot \bar{E} + \bar{\Omega} \times \bar{H} \tag{6}$$

$$\bar{B} = \mu\,\bar{\bar{\alpha}} \cdot \bar{H} - \bar{\Omega} \times \bar{E} \tag{7}$$

where

$$\bar{\Omega} = \frac{(n^2 - 1)\beta}{(1 - n^2\beta^2)c}\,\hat{z} \tag{8}$$

$$\beta = v/c \tag{9}$$

$$n = \left(\frac{\mu\epsilon}{\mu_o\epsilon_o}\right)^{\frac{1}{2}} \tag{10}$$

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$$\bar{\bar{\alpha}} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad a = \frac{1-\beta^2}{1-n^2\beta^2} \tag{11}$$

Substitution of (6 - 7) into (1 - 2) yields the Maxwell-Minkowski equations for a moving isotropic medium. They are

$$(\nabla - j\omega\bar{\Omega}) \times \bar{E} = -j\omega\mu\bar{\bar{\alpha}} \cdot \bar{H} \tag{12}$$

$$(\nabla - j\omega\bar{\Omega}) \times \bar{H} = \bar{J} + j\omega\epsilon\bar{\bar{\alpha}} \cdot \bar{E} \quad . \tag{13}$$

We obtain the wave equation (14) for $\bar{E}$ by eliminating $\bar{H}$ between (12) and (13).

$$(\nabla - j\omega\bar{\Omega}) \times \left[\bar{\bar{\alpha}}^{-1} \cdot (\nabla - j\omega\bar{\Omega}) \times \bar{E}\right] - k^2\bar{\bar{\alpha}} \cdot \bar{E} = -j\omega\mu\bar{J} \tag{14}$$

where $\bar{\bar{\alpha}}^{-1}$ denotes the reciprocal of $\bar{\bar{\alpha}}$ defined by (11), i.e.,

$$\bar{\bar{\alpha}}^{-1} = \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{15}$$

To integrate (14) in an infinite region, we introduce the dyadic Green's function $\bar{\bar{G}}$ such that

$$\bar{\bar{\alpha}} \cdot \bar{E} = -j\omega\mu \iiint \bar{\bar{G}} \cdot \bar{J}(\bar{R}')\,dv' \quad . \tag{16}$$

Substituting (16) into (14) and making use of the identify

$$\iiint \bar{J}(\bar{R}')\,\delta(\bar{R}/\bar{R}')\,dv' = \bar{J}(\bar{R}) \tag{17}$$

where $\delta(\bar{R}/\bar{R}')$ denotes the three-dimensional delta function, one finds that $\bar{\bar{G}}$ must satisfy the following equation:

$$(\nabla - j\omega\bar{\Omega}) \times \left\{\bar{\bar{\alpha}}^{-1} \cdot \left[(\nabla - j\omega\bar{\Omega}) \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{G}})\right]\right\} - k^2\bar{\bar{G}} = \bar{\bar{I}}\,\delta(\bar{R}/\bar{R}') \tag{18}$$

where $\bar{\bar{I}}$ denotes the idem factor. Equation (18) can be reduced to a simpler form if we introduce a function $\bar{\bar{g}}$ such that

$$\bar{\bar{G}} = e^{j\omega\Omega z}\,\bar{\bar{g}} . \tag{19}$$

Then,

$$\nabla \times \left[\bar{\bar{\alpha}}^{-1} \cdot \nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{g}})\right] - k^2 \bar{\bar{g}} = e^{-j\omega\Omega z'}\,\bar{\bar{I}}\,\delta(\bar{R}/\bar{R}') . \tag{20}$$

The first term of (20) can be decomposed into two terms, namely,

$$\nabla \times \left[\bar{\bar{\alpha}}^{-1} \cdot \nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{g}})\right] = \frac{1}{a}\left(-\nabla \cdot \nabla_a \bar{\bar{g}} + \nabla_a \nabla \cdot \bar{\bar{g}}\right) \tag{21}$$

where ∇_a is defined as follows:

$$\nabla_a = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{a\partial z} = \frac{1}{a}\,\bar{\bar{\alpha}} \cdot \nabla . \tag{22}$$

Following the operational method originally due to Levine and Schwinger,[4] we take the divergence of (20) giving

$$-k^2 \nabla \cdot \bar{\bar{g}} = e^{-j\omega\Omega z'}\,\nabla\delta(\bar{R}/\bar{R}') . \tag{23}$$

As a result of (21) and (23), (20) can be written in the form

$$\nabla \cdot \nabla_a \bar{\bar{g}} + k^2 a \bar{\bar{g}} = -a e^{-j\omega\Omega z'}\left(\bar{\bar{I}} + \frac{1}{k^2 a}\nabla_a \nabla\right)\delta(\bar{R}/\bar{R}') . \tag{24}$$

Thus, $\bar{\bar{g}}$ can be determined if we can find a scalar function g_o such that

$$\bar{\bar{g}} = a e^{-j\omega\Omega z'}\left(\bar{\bar{I}} + \frac{1}{k^2 a}\nabla_a \nabla\right) g_o \tag{25}$$

with g_o satisfying

$$\nabla \cdot \nabla_a g_o + k^2 a g_o = -\delta(\bar{R}/\bar{R}') . \tag{26}$$

The solution for g_o in an infinite region is obviously given by

$$g_o = \frac{e^{-jka^{1/2}\left[(x-x')^2 + (y-y')^2 + a(z-z')^2\right]^{1/2}}}{4\pi\left[(x-x')^2 + (y-y')^2 + a(z-z')^2\right]^{1/2}} \tag{27}$$

That completes the derivation.

To summarize the result, a recapitulation of the successive steps is given below with some simplification and rearrangement of the terms. The numbering of the equations is the same as the one originally labelled.

$$\bar{E} = -j\omega\mu \iiint \bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{G}} \cdot \bar{J}(\bar{R}')\, dv' \quad (16)$$

$$= -j\omega\mu \iiint \bar{\bar{\alpha}}^{-1} \cdot e^{j\omega\Omega z}\, \bar{\bar{g}} \cdot \bar{J}(\bar{R}')\, dv' \quad (19)$$

$$= -j\omega\mu a \iiint e^{j\omega\Omega(z-z')} \bar{\bar{\alpha}}^{-1} \cdot (\bar{\bar{I}} + \frac{1}{k^2 a^2} \bar{\bar{\alpha}} \cdot \nabla \nabla)\, g_o \cdot \bar{J}(\bar{R}')\, dv' \quad (25)$$

$$= -j\omega\mu a\, e^{j\omega\Omega z} (\bar{\bar{\alpha}}^{-1} + \frac{1}{k^2 a^2} \nabla \nabla) \cdot \iiint e^{-j\omega\Omega z'} g_o \bar{J}(\bar{R}')\, dv'$$

The corresponding expression for $\bar{H}$ is given by

$$\bar{H} = \frac{1}{-j\omega\mu} \bar{\bar{\alpha}}^{-1} \cdot (\nabla - j\omega\bar{\Omega}) \times \bar{E} \quad (28)$$

$$= a e^{j\omega\Omega z} \bar{\bar{\alpha}}^{-1} \cdot \nabla \times \left(\bar{\bar{\alpha}}^{-1} \cdot \iiint e^{-j\omega\Omega z'} g_o \bar{J}(\bar{R}')\, dv'\right).$$

Once the dyadic Green's function for an open region is known, numerous problems involving a radiating system can be investigated.

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