5/ \$ 1.500 HC

THE DYADIC GREEN'S FUNCTION FOR N 65 32104

A MOVING ISOTROPIC MEDIUM

C. T. Tai
The Radiation Laboratory
The University of Michigan

MASA OS 93/6 Cat, 23

When the velocity of a moving isotropic medium is small compared to the velocity of light, the Maxwell-Minkowski equations have a relatively simple form. The dyadic Green's function pertaining to these simplified wave equations can be found either by the methods of Fourier transform or by a more direct method. Alternatively, the method of potentials can also be used to solve these equations.

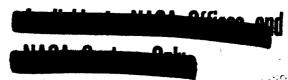
In this communication we shall present a derivation of the dyadic Green's function with no restriction upon the order of magnitude of the velocity. A compact result is obtained by transforming the wave equation into a conventional form and then solving it with the operational method originally due to Levine and Schwinger.

The Maxwell's equations for a moving medium have the same form as for a stationary medium. For harmonically oscillative fields with a time convention $e^{j\omega t}$, they are:

$$\nabla \times \overline{E} = -j \omega \overline{B}$$
 (1)

$$\nabla \mathbf{x} \ \mathbf{\bar{H}} = \mathbf{\bar{J}} + \mathbf{j} \, \omega \mathbf{\bar{D}} \tag{2}$$

⁺The research reported in this paper was sponsored by National Aeronautics and Space Administration Grant NsG444.



The constitutive relations between the field vectors for a uniformly moving isotropic medium were found by Minkowski⁵ based upon the special theory of relativity. They are:

$$\bar{\mathbf{D}} + \frac{1}{\mathbf{c}^2} \bar{\mathbf{v}} \times \bar{\mathbf{H}} = \epsilon \left(\bar{\mathbf{E}} + \bar{\mathbf{v}} \times \bar{\mathbf{B}} \right) \tag{3}$$

$$\bar{\mathbf{B}} - \frac{1}{c^2} \bar{\mathbf{v}} \times \bar{\mathbf{E}} = \mu (\bar{\mathbf{H}} - \bar{\mathbf{v}} \times \bar{\mathbf{D}})$$
 (4)

where ϵ and μ denote, respectively, the permittivity and permeability of the medium at rest which is assumed to be lossless. $\bar{\mathbf{v}}$ and \mathbf{c} denote, respectively, the velocity of the moving medium and the speed of light in vacuum. To simplify the derivation we assume

$$\bar{\mathbf{v}} = \mathbf{v} \, \hat{\mathbf{z}} . \tag{5}$$

The above condition is not much of a restriction since a coordinate transformation of the result can easily take care of the general case.

By solving \bar{D} and \bar{B} from (3 - 4) in terms of \bar{E} and \bar{H} with \bar{v} given by (5), we obtain the following relations:

$$\overline{D} = \epsilon \, \overline{\alpha} \cdot \overline{E} + \overline{\Omega} \times \overline{H}$$
 (6)

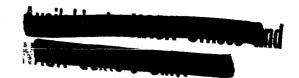
$$\bar{\mathbf{B}} = \mu \, \bar{\alpha} \cdot \bar{\mathbf{H}} - \bar{\Omega} \times \bar{\mathbf{E}} \tag{7}$$

where

$$\bar{\Omega} = \frac{(n^2 - 1)\beta}{(1 - n^2 \beta^2)c} \hat{z}$$
 (8)

$$\beta = \mathbf{v/c} \tag{9}$$

$$n = \left(\frac{\mu \epsilon}{\mu c}\right)^{\frac{1}{2}} \tag{10}$$



$$\bar{\bar{\alpha}} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} , \qquad a = \frac{1 - \beta^2}{1 - n^2 \beta^2}$$
 (11)

Substitution of (6-7) into (1-2) yields the Maxwell-Minkowski equations for a moving isotropic medium. They are

$$(\nabla - j\omega \bar{\Omega}) \times \bar{E} = -j\omega \mu \bar{\alpha} \cdot \bar{H}$$
 (12)

$$(\nabla - j\omega\bar{\Omega}) \times \bar{\mathbf{H}} = \bar{\mathbf{J}} + j\omega\epsilon\bar{\bar{\alpha}}\cdot\bar{\mathbf{E}} \qquad (13)$$

We obtain the wave equation (14) for \bar{E} by eliminating \bar{H} between (12) and (13).

$$(\nabla - j\omega \hat{\Omega}) \times \left[\bar{\alpha}^{-1} \cdot (\nabla - j\omega \hat{\Omega}) \times \bar{E}\right] - k^2 \bar{\alpha} \cdot \bar{E} = -j\omega \mu \bar{J}$$
(14)

where $\bar{\alpha}^{-1}$ denotes the reciprocal of $\bar{\alpha}$ defined by (11), i.e.,

$$\frac{\mathbf{a}-1}{\alpha} = \begin{pmatrix} \mathbf{a}^{-1} & 0 & 0 \\ 0 & \mathbf{a}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{15}$$

To integrate (14) in an infinite region, we introduce the dyadic Green's function = G such that

$$\bar{\bar{\alpha}} \cdot \bar{E} = -j\omega \mu \iiint \bar{\bar{G}} \cdot \bar{J} (\bar{R}') dv' . \qquad (16)$$

Substituting (16) into (14) and making use of the identify

$$\iiint \bar{J}(\bar{R}') \, \delta(\bar{R}/\bar{R}') \, dv' = \bar{J}(\bar{R}) \qquad (17)$$

where $\delta(\bar{R}/\bar{R}')$ denotes the three-dimensional delta function, one finds that \bar{G} must satisfy the following equation:

$$(\nabla - j\omega\bar{\Omega}) \times \left\{ \bar{\alpha}^{-1} \cdot \left[(\nabla - j\omega\bar{\Omega}) \times (\bar{\alpha}^{-1} \cdot \bar{G}) \right] \right\} - k^2 \bar{G} = \bar{I} \delta (\bar{R}/\bar{R}')$$
(18)

where I denotes the idem factor. Equation (18) can be reduced to a simpler

form if we introduce a function \bar{g} such that

$$\bar{\mathbf{G}} = \mathbf{e}^{\mathbf{j}\,\omega\,\Omega\,\mathbf{z}}\,\bar{\mathbf{g}}\,.\tag{19}$$

Then,

$$\nabla \mathbf{x} \left[\bar{\bar{\alpha}}^{-1} \cdot \nabla \mathbf{x} \left(\bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{\mathbf{g}}} \right) \right] - \mathbf{k}^2 \bar{\bar{\mathbf{g}}} = \mathbf{e}^{-\mathbf{j} \,\omega \Omega \, \mathbf{z}'} \, \bar{\bar{\mathbf{I}}} \, \delta(\bar{\mathbf{R}}/\bar{\mathbf{R}}') \, . \tag{20}$$

The first term of (20) can be decomposed into two terms, namely,

$$\nabla_{\mathbf{X}} \left[\bar{\bar{\alpha}}^{-1} \cdot \nabla_{\mathbf{X}} \left(\bar{\bar{\alpha}}^{-1} \cdot \bar{\bar{\mathbf{g}}} \right) \right] = \frac{1}{\mathbf{a}} \left(-\nabla \cdot \nabla_{\mathbf{a}} \bar{\bar{\mathbf{g}}} + \nabla_{\mathbf{a}} \nabla \cdot \bar{\bar{\mathbf{g}}} \right)$$
 (21)

where $\nabla_{\mathbf{a}}$ is defined as follows:

$$\nabla_{\mathbf{a}} = \hat{\mathbf{x}} \frac{\partial}{\partial \mathbf{x}} + \hat{\mathbf{y}} \frac{\partial}{\partial \mathbf{y}} + \hat{\mathbf{z}} \frac{\partial}{\partial \mathbf{z}} = \frac{1}{\mathbf{a}} \bar{\alpha} \cdot \nabla . \tag{22}$$

Following the operational method originally due to Levine and Schwinger, we take the divergence of (20) giving

$$-k^{2} \nabla \cdot \bar{g} = e^{-j\omega\Omega z'} \nabla_{\delta}(\bar{R}/\bar{R}') . \qquad (23)$$

As a result of (21) and (23), (20) can be written in the form

$$\nabla \cdot \nabla_{\mathbf{a}} \bar{\bar{\mathbf{g}}} + k^{2} a \bar{\bar{\mathbf{g}}} = -a e^{-j \omega \Omega z'} (\bar{\bar{\mathbf{I}}} + \frac{1}{k^{2} a} \nabla_{\mathbf{a}} \nabla) \delta(\bar{\mathbf{R}}/\bar{\mathbf{R}}').$$
 (24)

Thus, \bar{g} can be determined if we can find a scalar function g_0 such that

$$\bar{g} = a e^{-j \omega \Omega z'} (\bar{I} + \frac{1}{k^2 a} \nabla_a \nabla) g_0$$
 (25)

with g satisfying

$$\nabla \cdot \nabla_{\mathbf{a}} \mathbf{g}_{0} + \mathbf{k}^{2} \mathbf{a} \mathbf{g}_{0} = -\delta(\mathbf{\bar{R}}/\mathbf{\bar{R'}}) \qquad (26)$$

The solution for g_0 in an infinite region is obviously given by

$$g_{O} = \frac{e^{-jka^{\frac{1}{2}} \left[(x-x')^{2} + (y-y')^{2} + a(z-z')^{2} \right]^{\frac{1}{2}}}}{4\pi \left[(x-x')^{2} + (y-y')^{2} + a(z-z')^{2} \right]^{\frac{1}{2}}}$$
(27)

That completes the derivation.

To summarize the result, a recapitulation of the successive steps is given below with some simplification and rearrangement of the terms. The numbering of the equations is the same as the one originally labelled.

$$\vec{E} = -j\omega\mu \iiint \vec{\bar{\alpha}}^{-1} \cdot \vec{\bar{G}} \cdot \vec{J}(\vec{R}') dv'$$
(16)

$$= -j\omega\mu \iiint \bar{\bar{\alpha}}^{-1} \cdot e^{j\omega\Omega z} \, \bar{\bar{g}} \cdot \bar{J}(\bar{R}') \, dv'$$
 (19)

$$=-j\omega\mu a\iiint e^{j\omega\Omega(z-z')}\bar{\bar{\alpha}}^{-1}\cdot(\bar{\bar{I}}+\frac{1}{k^2a^2}\bar{\bar{\alpha}}\cdot\nabla\nabla)g_{0}\cdot\bar{\bar{J}}(\bar{R}')dv'$$
(25)

$$= -j\omega\mu \, a \, e^{j\omega\Omega\,z} (\bar{\bar{\alpha}}^{-1} + \frac{1}{k^2 a^2} \, \nabla \, \nabla) \cdot \iiint e^{-j\omega\Omega\,z'} \, g_0 \, \bar{J}(\bar{R}') \, dv'$$

The corresponding expression for \bar{H} is given by

$$\bar{\mathbf{H}} = \frac{1}{-j\,\omega\mu}\,\bar{\bar{\alpha}}^{-1}\cdot(\nabla - j\,\omega\bar{\Omega})\,\mathbf{x}\,\bar{\mathbf{E}} \tag{28}$$

$$= a e^{j\omega\Omega z} \bar{\bar{\alpha}}^{-1} \cdot \nabla x (\bar{\bar{\alpha}}^{-1} \cdot \iiint e^{-j\omega\Omega z'} g_{0} \bar{J}(\bar{R}') dv').$$

Once the dyadic Green's function for an open region is known, numerous problems involving a radiating system can be investigated.

REFERENCES

- 1. C. T. Tai, "A Study of Electrodynamics of Moving Media," Proc. IEEE, <u>52</u>, pp. 685-690, (June, 1964).
- 2. R. T. Compton, Jr. and C. T. Tai, "The Dyadic Green's Function for an Infinite Moving Medium," Tech Report 1691-3, Antenna Laboratory, Ohio State University, (January, 1964).
- 3. J. R. Collier and C. T. Tai, "Guided Waves in Moving Media," an oral paper presented at the URSI Meeting in Washington, (March, 1964).
- 4. H. Levine and J. Schwinger, 'On the Theory of Electromagnetic Waves Diffraction by an Aperture in an Infinite Plane Conducting Screen, 'A Symposium on the Theory of Electromagnetic Waves, Interscience Publishers, Inc. (1951).
- 5. A. Sommerfeld, Electrodynamics, Academic Press, Inc., New York, N.Y. (1952).