PROGRESS REPORT TO THE

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

(Supplement on Project No. 8)

Research Grant No. NsG 239-62

August, 1965

Foundations of Linear Inelastic Shell Theory

Texas A&M University

College Station, Texas

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P R E F A C E

The material presented in this report forms the basis of an introductory course in linear thin shell analysis as taught at Texas A&M University. Its primary purpose is to acquaint the student with the assumptions and limitations of linear shell analysis as based on the Kirchoff hypotheses. As a consequence, there are no example problems or, for that matter, any special treatments such as shallow shell analysis or symmetrically loaded shells of revolution.

In developing the course material, a choice in approach had to be made. Tensor analysis could have been used and the results presented so as to show their generality. Even more appealing would have been the freedom of choice of coordinate system that tensors would have allowed. In spite of these advantages, the vector approach was thought to be the more feasible one. The chief consideration leading to this conculsion was that the students were more familiar with vectors than tensors and hence be able to better cope with the presented material. Since principal curvalinear coordinates were to be used exclusively, the resulting equations when developed by vector methods would not be particularly complex and many of the important concepts could be readily grasped. If the course material were such as to inspire further studies in shell analysis, then the tensor approach could be found in the various books and articles dealing with this topic.

In developing the course, a great deal of stress has been placed on differential geometry. It was felt that a great deal of confusion with regard to strains, curvature changes, twist and compatibility could be eliminated by considering a surface and its deformation. But even more so, by first developing the general equations for a surface, a means would be available for deriving the corresponding non-linear expressions.

Much of the material has been typed directly from class notes and as a consequence the English tends to be stilted. However, it is hoped that the material is sufficiently clear in exposition so as to be readable. One comment on the presentation: a great deal of the analysis depends on the orthnormal triad of vectors of the undeformed surface and their derivatives. The derivatives of these vectors is given at the end in the Addendum rather than in Chapter II as would normally be expected.

There is no claim for the originality of the work. Much of the material can be found in one form or another in texts dealing with shell analysis. However, those works which most directly influenced the present compilation are A. V. Pogorelov, "Differential Geometry", V. V. Novozhilov, "The Theory of Thin Shells", Delft, 1959, "Proceedings of the Symposium on the Theory of Thin Elastic Shells", P. M. Naghdi, "Progress in Solid Mechanics", Volume IV.

CHAPTER I

INTRODUCTION

1.1 Definition of a Thin Shell

A thin shell is a body bounded by two curved surfaces, the distance between the surfaces being small in comparison with the radii of curvature of the surfaces.

Smallness in this instance must be defined. It will be tacitly assumed that quantities of order of magnitude (δ/R) in comparison with unity may be neglected. (The reason for this assumption will be brought out later when studying the Kirchoff hypothesis.) Since a maximum error of 5% is normally admisable in shell analysis, the above approximation is equivalent to stating that

$$\max\ (\delta/R) \leq 1/20$$

Thus the definition of a thin shell is now quantitatively evaluated.

There are other assumptions which will be implied in the resulting development. These assumptions are stated as follows.

a) Shell is thin $(\delta/R \leq 1/20)$

b) Material is homogeneous and isotropic

c) Material remains elastic throughout the stressed range and obeys Hooke's Law.

d) Deflections are sufficiently small so that linear theory is applicable. This is equivalent to stating that products of displacements and their derivatives may be neglected in the analysis

e) Edge of the shells are plane curves and cuts are made perpendicular to the middle surface

1.2. Method of Solution

Basically, a shell is nothing more than a three dimensional elastic body subjected to external loads. As a consequence, the equations derivable from the theory of elasticity are applicable to such a body. Thus there are two basic methods by which shell problems may be solved. The first is to express the equilibrium equations in terms of stresses, formulate the compatibility equations in terms of stresses and combine together. The resulting equations are called the <u>Beltrami-Michell Equations</u> and are given below.

$$\left.\begin{aligned}
&\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + X = 0 \\
&\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + Y = 0 \\
&\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + Z = 0
\end{aligned}\right] \quad \text{Equilibrium}$$

$$\left.\begin{aligned}
&\nabla^2 \sigma_{xx} + \frac{1}{(1+\nu)} \frac{\partial^2 \Phi}{\partial x^2} = -\frac{\nu}{(1-\nu)} \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right) - 2 \frac{\partial X}{\partial x} \\
&\nabla^2 \sigma_{yy} + \frac{1}{(1+\nu)} \frac{\partial^2 \Phi}{\partial y^2} = -\frac{\nu}{(1-\nu)} \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right) - 2 \frac{\partial Y}{\partial y} \\
&\nabla^2 \sigma_{zz} + \frac{1}{(1+\nu)} \frac{\partial^2 \Phi}{\partial z^2} = -\frac{\nu}{(1-\nu)} \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right) - 2 \frac{\partial Z}{\partial z} \\
&\nabla^2 \sigma_{xy} + \frac{1}{(1+\nu)} \frac{\partial^2 \Phi}{\partial x \partial y} = -\left(\frac{\partial X}{\partial y} + \frac{\partial Y}{\partial x} \right) \\
&\nabla^2 \sigma_{yz} + \frac{1}{(1+\nu)} \frac{\partial^2 \Phi}{\partial y \partial z} = -\left(\frac{\partial Y}{\partial z} + \frac{\partial Z}{\partial y} \right) \\
&\nabla^2 \sigma_{xz} + \frac{1}{(1+\nu)} \frac{\partial^2 \Phi}{\partial z \partial x} = -\left(\frac{\partial Z}{\partial x} + \frac{\partial X}{\partial z} \right)
\end{aligned}\right] \quad \text{compatibility}$$

Strictly speaking, the Beltrami-Michell equations refer to the transformation of the compatibility equations, which are stated in terms of strains, to a set of equations in terms of stress. Note that X, Y, Z refer to body forces and not surface forces and furthermore that

$$\Phi = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

The second method of postulating the elasticity problem is to express the equilibrium equations in terms of displacement functions. In this manner, the need of the compatibility equations is circumvented since these equations when expressed in displacement form are identically satisfied. Another advantage in the displacement formulation is that the total number of equations is reduced to only three, but note that the order of derivative is increased. The equations of elasticity, when stated in displacement form are termed the <u>Navier Equations</u> and are given as follows.

$$\mu \nabla^2 u + (\lambda + \mu)\frac{\partial \Delta}{\partial x} + X = 0$$

$$\mu \nabla^2 v + (\lambda + \mu)\frac{\partial \Delta}{\partial y} + Y = 0$$

$$\mu \nabla^2 w + (\lambda + \mu)\frac{\partial \Delta}{\partial z} + Z = 0$$

where in the above, X, Y, Z are again body forces and;

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad \text{(volume dilatation)}$$

$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$; $\mu = \frac{E}{2(1+\nu)}$ Lamé constants of elaticity

Thus any solution to an elasticity problem and any exact solution to the shell problem must satisfy one or the other of the above system of equations.

Various attempts have been made to solve three dimensional elasticity problems but except for a limited class of problems, this area is in general unexplored. In the application of the elasticity equations to shells, certain simplifications can be made due to the thinness of the material. Thus attempts have been made to expand the various functions such as stress in power series in parameters of (z/R) where z is the distance measured normal to the shell. Some success has been attained using this approach but the results did not warrant the effort.

The classical method of shell analysis is based on the Kirchoff hypotheses first formulated in the study of elastic plates. These hypotheses are three in number and are given below.

i) Lines initially normal to a shell surface remain so after deformation

ii) Line segments oriented normal to the shell surface suffer no extensions or contractions

iii) Normal stresses acting on planes tangent to the shell surface may be neglected in comparison with other stresses.

These three assumptions have already been encountered in plates and are generalizations of the "plane sections remaining plane" assumption in simple beam bending theory. Their application to shells is first attributed to G. Aron and further exploited by A. E. H. Love. In fact, the Kirchoff assumptions together with the shell development as presented by Love is still the standard reference work though modifications have taken place. Love's presentation is frequently called "first order shell theory approximation".

A number of things should be mentioned about the Kirchoff hypothesis. To begin with, it is in general an approximation and hence introduces an inherent error into the analysis. Various investigators have looked into the resulting error and found that it is in general of order (δ/R), or one that falls within the scope of the thin shell approximation. However the magnitude of this error is dependent on the loading condition and where rapidly varying loads are present, recent papers have shown that the error is sizeably increased over that normally expected. Secondly, note that the condition of undeformed normal implies the lack of transverse shear stresses, a situation virtually never encountered. Thus the Kirchoff hypothesis is basically an erroneous one and as a consequence, it must be concluded that any shell theory based on such a hypothesis cannot be improved in accuracy. The retention of terms smaller than order (δ/R) is superfulous since it does nothing for the basic accuracy of the theory.

Another thing to note about the Kirchoff hypothesis is that it introduces a contradiction. One of its assumptions is that the normal deformation and hence normal strain in a direction perpendicular to the shell surface is zero. In essence, this is analogous to the condition of plane strain. However, the third of its assumptions is that the normal stress in a direction perpendicular to the shell middle surface is negligible which is analogous to the condition of plane stress. For a plane stress and plane strain condition to exist simultaneouly a necessary condition is that the remaining two normal stress be dependent. Again this condition is seldom realized.

The Kirchoff hypothesis does not restrict the method of solution. Thus either a displacement or stress formulation of the resulting equations is possible. If a stress formulation is to be utilized, than rather then use the Beltrami-Michell equations as stated, the stresses are reduced to stress resultants, or forces per unit length of some reference surface, usually the shell middle surface. This procedure is analogous to that used in deriving the plate equations. The equilibrium and compatibility equations may then be stated in terms of stress resultants.

In the case of a displacement formulation, the Navier equations together with the consequences of the Kirchoff hypothesis are utilized. The resulting equations are then operated on so as to yield a set of equations in terms of the components of the displacement of some reference surface, again usually the middle surface.

From the standpoint of historical development and predominant use, the stress resultant formulation is most frequently encountered. This is an odd situation when one considers that the Kirchoff hypotheses are conditions placed more on displacement than on stresses. The reason for the dominant use of stress resultants are obscure, but perhaps the greatest reason is one of historic development. Thus the use of the Kirchoff hypothesis in plates preceded its use in shells and since the success in plate solution came from stating the equilibrium equations in terms of stress resultants, it would be reasonably expected that the first attempts at a shell equation formulation would closely parallel those of the plate. Aron and Love used the stress resultant formulation and others that followed built on their historic developments. Another reason for the use of the stress resultant formulation is that unlike the displacement formulation, it is insensitive to the discrepancy between a plane strain and plane stress formulation. In the displacement formulation, the assumption of non-extensibility of normals to the shell middle surface must be discarded if a resultant plane stress formulation of the shell equations is desired.

Work has been done on the displacement formulation of the shell equations. This work is chiefly attributed to V. Z. Vlasov. However some fundamental questions have as yet to be answered.

1.3. Consequences of the Kirchoff hypotheses

In elementary beam bending theory, the assumption of plane sections remaining plane led to a simple formulation of the stress equation and the displacement equation.

$$\sigma = \frac{My}{I} \; ; \quad EI\frac{d^2y}{dx^2} = M$$

Now if the deflection of the elastic curve were known, then the stresses, strains and displacements could all be calculated geometrically. Thus the plane sections assumptions reduced the problem to one of finding the deflection of the elastic curve. Now in the case of plates, the Kirchoff hypothesis allowed an assumption of linear variation of the displacements and hence strains through the plate thickness. Thus again, if the deflection of the plate, (actually the plate middle surface) were known, then stresses, strains and displacements could be found by simple geometric means. Hence the solution of the plate problem was reduced to the solution of the deflection of a plate surface.

$$\nabla^4 w = q/D$$

When the Kirchoff hypothesis is used in shells, the conclusions are the same as previously encountered. Namely, the Kirchoff hypothesis allows one to assume a linear variation of the displacements through the shell thickness. Hence the displacements, stresses may all be calculated in terms of the deflections of some reference surface (again the middle surface). Thus in shells, the solutions to the shell problem reduces itself to predicting the deformation of some surface. However, unlike the case of the plate or beam, the resultant equations are stated in terms of stress resultants which ultimately are dependent on the deflections of a surface.

It may then be seen, that the most important consequence of the Kirchoff hypothesis is that it reduces the analysis of a three dimensional problem to the study of a single surface. Since the study of surfaces is so important to shell analysis, the next chapter will be devoted to a review of analytic and differential geometry.

CHAPTER II

Surface Study

The study of thin elastic shells is such as to finally reduce the various expressions to functions acting on the shell middle surface. This is no more than a generalization and explicit statement of what has occurred in the study of simple beams and flat plates. Thus in the former case, the elastic line and its deformation was all important and in the latter, the planform shape, or the plane was essential in the formulation of the plate equations.

Since the study of the middle surface will become so important in the study of shells, it will be advantageous to briefly review and survey some results of the geometry of surfaces.

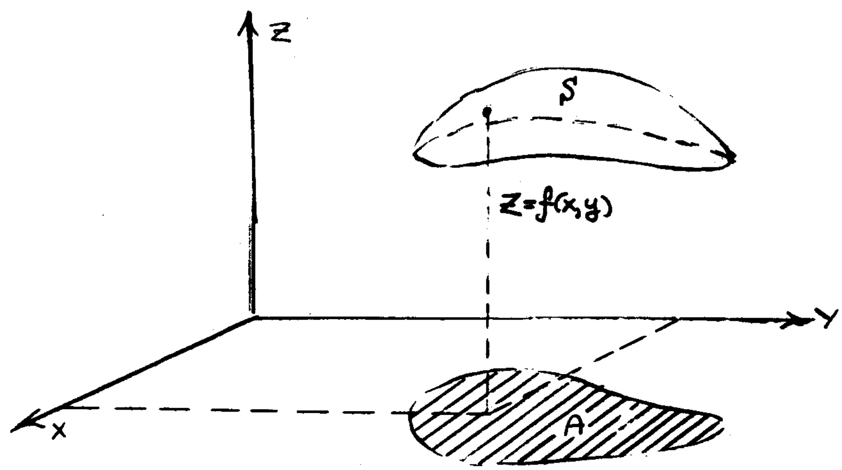
2.1. Specification Of A Surface And Its Properties In The Large

A surface may be defined as a configuration of points having a two dimensional character; that is, a point moving on the surface, but otherwise unrestricted, has two degrees of freedom. Thus to completely specify a surface, two independent coordinates will always be necessary.

Assuming a Cartesian coordinate system, an explicit or implicit equation may be used to describe the surface. An example of an explicit representation is the following equation;

$$z = f(x,y)$$

In this representation, x and y are independent variables and z is assumed to a single valued function of these variables. This equation can also be looked upon as a mapping of points from one set, those in the x-y plane, to those in space defining the surface. However, the boundary of the points in the x-y plane is not rectangular but in shape the same as the projection of the surface on the x-y plane. This situation is shown in the sketch below.



An implicit functional representation defining a surface is given by an equation of the form

$F(x,y,z) = 0$

In this instance the choice of independent variables is purely discretionary. However, it might be noted that frequently the implicit representation is used when the variables cannot be conveniently solved for an explicit relation.

Some examples of surface equations are as follows:

a) Right circular cylinder

$x^2 + y^2 = a^2$

b) Sphere

$x^2 + y^2 + z^2 = a^2$

c) Cone

$x^2 + y^2 = k^2 z^2$

d) Body of revolution

$x^2 + y^2 = f(z)$

e) Plane

$$Ax + By + Cz = D$$

Note that all of the above examples are defined in implicit form and with the exception of the plane, are all bodies of revolution about the z axis.

Analytically, there is a yet more convenient way to express the equation of a surface than either by explicit or implicit method. The basis for this method lies in the fact that only two independent coordinates are necessary to define a surface. Consider now two independent variables α and β defined in an $\alpha - \beta$ plane and defined in a rectangular region such that

$$0 \leq \alpha \leq \alpha_0 \quad ; \quad 0 \leq \beta \leq \beta_0$$

Then relative to the Cartesian coordinate system, the points (x,y,z) of the surface may be written as;

$$x = x(\alpha,\beta)$$

$$y = y(\alpha,\beta)$$

$$z = z(\alpha,\beta)$$

A representation of a surface in such a manner is called a parametric representation. In a mathematical sense, what is being done is a rectangular region on the $\alpha - \beta$ plane is being mapped on to a spacial surface, the mapping transformations being the functional relations that exist between α, β and x,y,z. Note that the explicit equation form of a surface may be called a parametric representation. Thus letting $\alpha = x$ and $\beta = y$, the explicit form given as

$$z = f(x,y)$$

may now be written as

$$z = f(\alpha, \beta)$$
$$x = \alpha$$
$$y = \beta$$

However note the difference between this representation and a true parametric representation. Here, α and β are defined in a definite region in general non-rectangular. Thus, the region of definition of α and β itself depends on the shape of the surface, a situation which is not true in a true parametirazation.

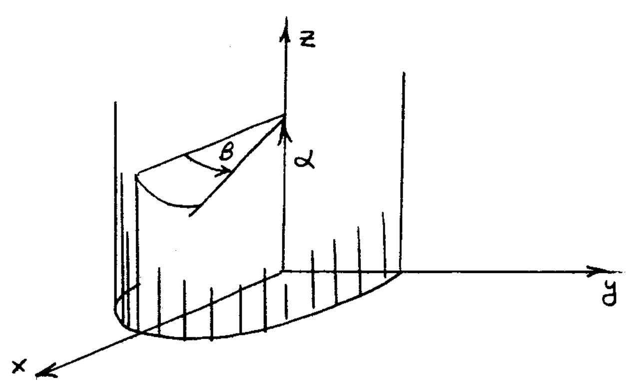
Consider now the paremetric representations of the surfaces previously considered.

a) Right circular cylinder

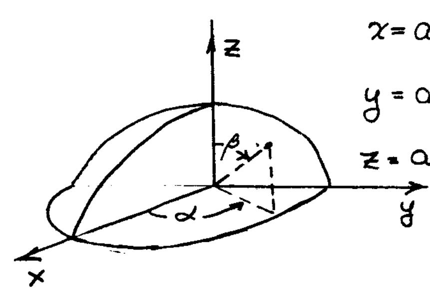
$x = a \text{ Cos}\,\beta \qquad 0 \leq \beta \leq 2\pi$

$y = a \text{ Sin}\,\beta \qquad 0 < \alpha$

$z = \alpha$



b) Sphere



$x = a\,\text{Sin}\beta\,\text{Sin}\alpha \qquad 0 \leq \alpha \leq 2\pi$

$y = a\,\text{Sin}\beta\,\text{Cos}\alpha \qquad 0 \leq \beta \leq \pi$

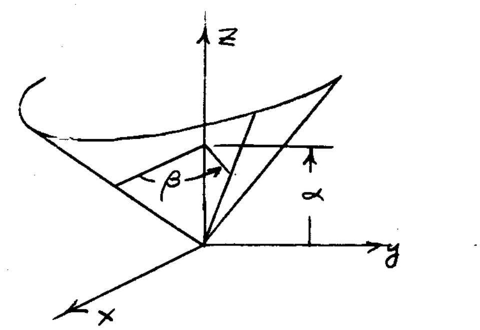
$z = a\,\text{Cos}\beta$

c) Cone

$x = k\alpha \cos\beta \; ; \; 0 \le \beta \le 2\pi$

$y = k\alpha \sin\beta \; ; \; 0 < \alpha$

$z = \alpha$

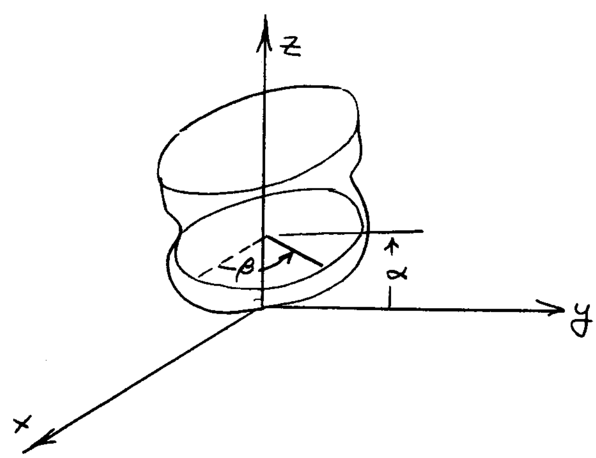


Body of Revolution

$x = f(\alpha) \cos\beta \; ; \; 0 \le \beta \le 2\pi$

$y = f(\alpha) \sin\beta \; ; 0 < \alpha$

$z = \alpha$



Now in the examples cited, note that α and β have direct gepmetrical interpretation in the Cartesian coordinate system. However, note that their definition is such that they continuously and arbitrarily vary in some rectangular region of the ($\alpha - \beta$) plane. The fact that we draw an angular measurement by means of circular segments is just an aid in visualization. Note further that the parameters α and β are not unique. The ones that were used in the examples were the most nautral and convenient ones to use. However, any other set of parameters would have equally described the surfaces.

Summary:

1. All surfaces must be described by the use of two independent coordinates.
2. There are three ways of describing a surface.
   a) Explicit relation:
   $$z = f(x,y)$$
   b) Implicit relation:
   $$F(x,y,z) = 0$$
   c) Parametric:
   $$x = x(\alpha,\beta)$$
   $$y = y(\alpha,\beta)$$
   $$z = z(\alpha,\beta)$$
3. The principle value of curvalinear coordinates is that their domain of definition is a rectangular plane area and hence independent of the shape of the surface.
4. The parametric representation of a surface is not unique.

2.2. Surface Properties In The Small

Since the purpose of the present surface study is to facilitate the development of a set of differential shell equations, it might reasonably be expected that the properties in the small would be more important than those in the large.

2.3. Concept Of a Tangent Plane And Normal To A Surface

Consider for a moment the equation of a surface given as

$$z = f(x,y)$$

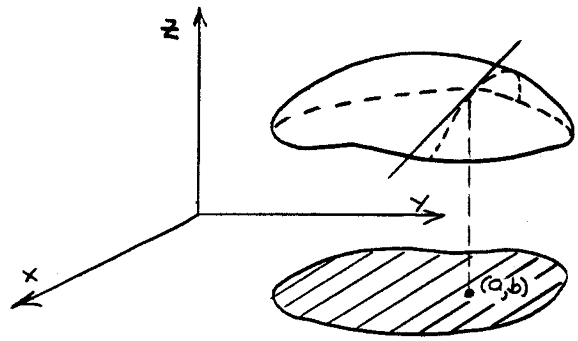
and consider a tangent plane to this surface. Now the general equation of a plane is given as:

$$Ax + By + Cz = D$$

where A, B and C are defined as the direction numbers of the normal to the plane. If now the point of tangency to the surface is at the point (a,b,c), then since this must also be a point on the surface, the equation of the plane may be written as:

$$A(x-a) + B(y-b) + C(z-c) = 0$$

Now consider the partial derivative $\left.\partial f/\partial x\right|_{(a,b)}$;



this partial derivative represents the slope of the line of intersection of the surfaces

$f(x,y)$; $y = b$

at the point (a,b). Hence, points lying on this tangent line are given by the equations:

$$(z-c) = (x-a)\frac{\partial f}{\partial x}(a,b) \quad ; \quad y = b$$

In a completely analagous manner, the partial derivative $\frac{\partial f}{\partial y}(a,b)$ represents the slope of the line of intersection of the surfaces $f(x,y)$ and $x = a$ at the point (a,b). Thus, points lying along this line are given by the equation;

$$(z-c) = (y-b)\frac{\partial f}{\partial y}(a,b) \quad ; \quad x = a$$

If a tangent plane is being sought to the surface at the point (a,b), then this tangent plane must contain the two tangent lines to the surface and hence the points lying on that line. Applying this principle to the equation of the tangent plane previously stated, it is now possible to calculate the coefficients A, B and C. Substituting, the result becomes:

$$\frac{\partial f}{\partial x}(a,b)\,(x-a) + \frac{\partial f}{\partial y}(a,b)\,(y-b) - (z-c) = 0$$

Thus, the direction numbers of the normal to the tangent plane and hence to the surface are:

$$\frac{\partial f}{\partial x}\ ;\ \frac{\partial f}{\partial y}\ ;\ -1$$

Suppose now that equation of the surface is stated in implicit form, that is,

$F(x,y,z) = 0$

Then by the implicit function chain rule;

$$\frac{\partial z}{\partial x} = -\frac{\left(\partial F/\partial x\right)}{\left(\partial F/\partial z\right)}\ ; \qquad \frac{\partial z}{\partial y} = -\frac{\left(\partial F/\partial y\right)}{\left(\partial F/\partial z\right)}$$

and hence the direction numbers of the surface normal become;

$$\frac{\partial F}{\partial x}\ ;\ \frac{\partial F}{\partial y}\ ;\ \frac{\partial F}{\partial z}$$

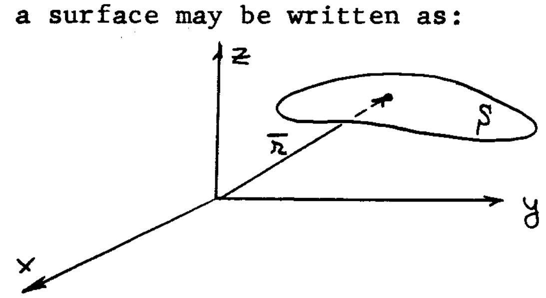
Consider now the parametric definition of a surface, namely;

$$x = x(\alpha,\beta)\ ;\quad y = y(\alpha,\beta)\ ;\quad z = z(\alpha,\beta)$$

In this particular instance, it may be more advantageous to develop the equation of the tangent plane and hence in this manner determine the direction numbers of the normal. Furthermore, in dealing with the surface, it is easier to deal with the vector equation of a surface.

In vector form, the equation of a surface may be written as:

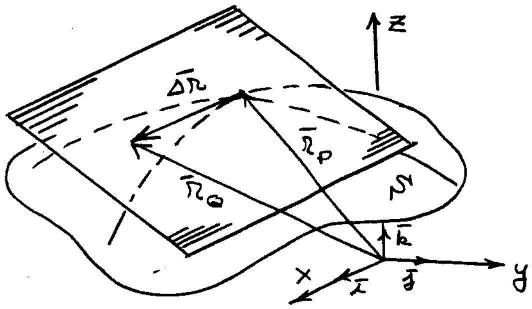
$$\bar{r} = \bar{r}(\alpha,\beta)$$



In Cartesian coordinates, the vector $\bar{r}$ becomes:

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

where $\bar{i}$, $\bar{j}$ and $\bar{k}$ are the orthornormal triad corresponding to the x, y and axis. Again let the point P, (a,b,c) be the one at which the tangent plane is desired. Let (x,y,z) be some arbitrary point Q on the plane. Let the vector to the point (a,b,c) be designated as $\bar{r}_p$ and that to Q as $\bar{r}_q$



Now the vector $\Delta\bar{r} = \bar{r}_q - \bar{r}_p$ lies in the tangent plane. Consider now the derivative of the surface vector $\bar{r}$ with respect to each of the coordinates. For this purpose, consider first $\partial\bar{r}/\partial\alpha$. Now $\bar{r}(\alpha,\beta)$ is a vector to some point on the surface. Letting α increase defines a new vector, $\bar{r}(\alpha+\Delta\alpha, \beta)$ which again is a vector to some new point on the surface. Hence the vector; $[\bar{r}(\alpha+\Delta\alpha,\beta) - \bar{r}(\alpha,\beta)]$ corresponds to a secant vector on the surface and letting $\Delta\alpha \rightarrow 0$ would imply that this vector becomes tangent to the surface. Hence the vectors $\partial\bar{r}/\partial\alpha$ and $\partial\bar{r}/\partial\beta$ evaluated at the point P represent vectors lying in the tangent plane to the surface. Thus the triple product

$$\left(\frac{\partial\bar{r}}{\partial\alpha} \times \frac{\partial\bar{r}}{\partial\beta}\right) \cdot \Delta\bar{r} = 0$$

Now:

$$\frac{\partial \bar{r}}{\partial \alpha} = \frac{\partial x}{\partial \alpha}\bar{i} + \frac{\partial y}{\partial \alpha}\bar{j} + \frac{\partial z}{\partial \alpha}\bar{k}$$

$$\frac{\partial \bar{r}}{\partial \beta} = \frac{\partial x}{\partial \beta}\bar{i} + \frac{\partial y}{\partial \beta}\bar{j} + \frac{\partial z}{\partial \beta}\bar{k}$$

$$\Delta \bar{r} = (x-a)\,\bar{i} + (y-b)\,\bar{j} + (z-c)\,\bar{k}$$

Then:

$$\left(\frac{\partial \bar{r}}{\partial \alpha} \times \frac{\partial \bar{r}}{\partial \beta}\right) = \left(\frac{\partial y}{\partial \alpha}\frac{\partial z}{\partial \beta} - \frac{\partial y}{\partial \beta}\frac{\partial z}{\partial \alpha}\right)\bar{i} + \left(\frac{\partial z}{\partial \alpha}\frac{\partial x}{\partial \beta} - \frac{\partial z}{\partial \beta}\frac{\partial x}{\partial \alpha}\right)\bar{j} + \left(\frac{\partial x}{\partial \alpha}\frac{\partial y}{\partial \beta} - \frac{\partial x}{\partial \beta}\frac{\partial y}{\partial \alpha}\right)\bar{k}$$

Hence, forming the inner product, the result becomes;

$$\left(\frac{\partial y}{\partial \alpha}\frac{\partial z}{\partial \beta} - \frac{\partial y}{\partial \beta}\frac{\partial z}{\partial \alpha}\right)(x-a) + \left(\frac{\partial z}{\partial \alpha}\frac{\partial x}{\partial \beta} - \frac{\partial z}{\partial \beta}\frac{\partial x}{\partial \alpha}\right)(y-b) + \left(\frac{\partial x}{\partial \alpha}\frac{\partial y}{\partial \beta} - \frac{\partial x}{\partial \beta}\frac{\partial y}{\partial \alpha}\right)(z-c) = 0$$

where all the derivatives are evaluated at the point P(a,b,c). Thus, the direction numbers of the normal to the surface defined in parametric form are given as:

$$\left(\frac{\partial y}{\partial \alpha}\frac{\partial z}{\partial \beta} - \frac{\partial y}{\partial \beta}\frac{\partial z}{\partial \alpha}\right);\left(\frac{\partial x}{\partial \beta}\frac{\partial z}{\partial \alpha} - \frac{\partial x}{\partial \alpha}\frac{\partial z}{\partial \beta}\right);\left(\frac{\partial x}{\partial \alpha}\frac{\partial y}{\partial \beta} - \frac{\partial x}{\partial \beta}\frac{\partial y}{\partial \alpha}\right)$$

Or in Jacobian Form;

$$\frac{\partial(y,z)}{\partial(\alpha,\beta)}\,;\quad \frac{\partial(z,x)}{\partial(\alpha,\beta)}\,;\quad \frac{\partial(x,y)}{\partial(\alpha,\beta)}$$

<u>Summary of Results</u>

The direction numbers of the normal for a surface are given as the following:

a) Explicit Form:

$$z = f(x,y)$$

$$\frac{\partial f}{\partial x};\ \frac{\partial f}{\partial y};\ -1$$

b) Implicit Form:

$$F(x,y,z) = 0$$

$$\frac{\partial F}{\partial x};\ \frac{\partial F}{\partial y};\ \frac{\partial F}{\partial z}$$

c) Parametric Form:

$$x = x(\alpha,\beta);\ y = y(\alpha,\beta);\ z = z(\alpha,\beta)$$

$$\left(\frac{\partial y}{\partial \alpha}\frac{\partial z}{\partial \beta} - \frac{\partial y}{\partial \beta}\frac{\partial z}{\partial \alpha}\right);\ \left(\frac{\partial x}{\partial \beta}\frac{\partial z}{\partial \alpha} - \frac{\partial x}{\partial \alpha}\frac{\partial z}{\partial \beta}\right);\ \left(\frac{\partial x}{\partial \alpha}\frac{\partial y}{\partial \beta} - \frac{\partial x}{\partial \beta}\frac{\partial y}{\partial \alpha}\right)$$

2.4. Definition Of A Curve And Its Representation

A curve may be defined as an ordered continuous configuration of points possessing a one dimensional character. An arc is defined as a curve which does not intersect itself and has two distinct and finite ends. A closed curve with no self intersections is termed a simple or Jordon type of curve. A rectifiable curve is one whose length may be approxiated by the length of secants.

Curves are frequently represented as the intersection of two surfaces. Thus given two surfaces, $F(x,y,z) = 0$ and $G(x,y,z) = 0$, the equation of the curve formed by their intersection would be

$$F(x,y,z) = 0$$

$$G(x,y,z) = 0$$

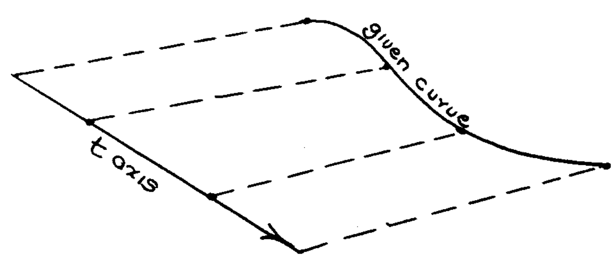
There is a more appealing manner of specifying curves, and that is parametrically. Since a curve is a one dimensional configuration of points, it should be possible to find but one parameter, say t, such that the x, y, z coordinates of every point on the curve would be given as :

$$x = x(t)$$

$$y = y(t)$$

$$z = (t)$$

The parameter t varies continuously between a and b. In a sense, the functional representation represents a transformation of a straight line segment, the t axis, to the given curve. It is assumed that every point on the t axis has its image on the given curve.



With the parametric representation, there is associated the vector representation. That is, given a radius vector from some origin to some point on the curve, the equation of the radius vector may be written as:

$$\bar{r} = \bar{r}(t)$$

The vector representation has the convenience of not being tied down to a particular coordinate system.

2.5. Length Of A Curve

Consider now a rectifiable curve given in parametric form. Then by the Pathagerion Theorem;

$$ds^2 = dx^2 + dy^2 + dz^2$$

If the given curve is stated in parametric form, the length may be finally expressed as:

$$\Delta s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

In vector form, the equation is given as:

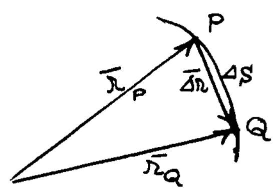
$$\Delta s = \int_a^b \sqrt{\frac{d\bar{r}}{dt} \cdot \frac{d\bar{r}}{dt}}\, dt$$

2.6. Tangent To A Curve

Consider again a space curve given by the equation

$\bar{r} = \bar{r}(t)$

Let $\bar{T}$ be a unit vector in the direction of the curve and tangent to it and consider the differential $\Delta \bar{r}$.



$$\Delta \bar{r} = \bar{r}_Q - \bar{r}_P$$

Hence, the vector $\Delta \bar{r}$ is the secant vector for the point P + Q of the curve. Since the curve is rectifiable, then as $Q \to P$, $\Delta \bar{r}$ approaches the tangent to the curve. Now;

$$|\Delta \bar{r}| \cong \Delta s$$

Thus, the unit tangent vector to the curve becomes:

$$\bar{T} = \frac{d\bar{r}}{ds}$$

Or in terms of the parameter t;

$$\bar{T} = \frac{1}{\left(\frac{ds}{dt}\right)} \frac{d\bar{r}}{dt}$$

2.7. Principle Normal To A Curve

Consider again the curve $\bar{r} = \bar{r}(t)$. Consider now the derivative $d\bar{T}/ds$.

Thus;

$$\frac{d\bar{T}}{dS} = -\frac{1}{\left(\frac{dS}{dt}\right)^2}\frac{d^2S}{dt^2}\frac{d\bar{R}}{dt} + \frac{1}{\left(\frac{dS}{dt}\right)}\frac{d^2\bar{R}}{dt^2}$$

Consider first determing the direction of this vector with respect to the vector $\bar{T}$. Forming the inner product;

$$\bar{T}\cdot\frac{d\bar{T}}{dS} = -\frac{1}{\left(\frac{dS}{dt}\right)^3}\frac{d^2S}{dt^2}\left(\frac{d\bar{R}}{dt}\cdot\frac{d\bar{R}}{dt}\right) + \frac{1}{\left(\frac{dS}{dt}\right)^2}\left(\frac{d\bar{R}}{dt}\cdot\frac{d^2\bar{R}}{dt^2}\right)$$

Now;

$$\frac{d\bar{R}}{dt}\cdot\frac{d\bar{R}}{dt} = \left(\frac{dS}{dt}\right)^2$$

and;

$$\frac{d\bar{R}}{dt}\cdot\frac{d^2\bar{R}}{dt^2} = \frac{d}{dt}\left(\frac{d\bar{R}}{dt}\cdot\frac{d\bar{R}}{dt}\right) - \frac{d^2\bar{R}}{dt^2}\cdot\frac{d\bar{R}}{dt}$$

But since

$$\frac{d\bar{R}}{dt}\cdot\frac{d\bar{R}}{dt} = \left(\frac{dS}{dt}\right)^2 \qquad \text{and} \qquad \frac{d}{dt}\left(\frac{dS}{dt}\right)^2 = 2\frac{dS}{dt}\frac{d^2S}{dt^2},$$

the result becomes;

$$2\frac{d\bar{R}}{dt}\cdot\frac{d^2\bar{R}}{dt^2} = 2\frac{dS}{dt}\frac{d^2S}{dt^2}$$

Substituting;

$$\bar{T}\cdot\frac{d\bar{T}}{dS} = -\frac{1}{\left(dS/dt\right)}\frac{d^2S}{dt^2} + \frac{1}{\left(dS/dt\right)}\frac{d^2S}{dt^2} \equiv 0$$

Thus the vectors $\bar{T}$ and $\frac{d\bar{T}}{dS}$ are orthogonal to each other.

Let the unit vector in the direction of $d\bar{T}/dS$ be designated as $\bar{N}$, which is defined as the direction of the principal normal. Let the magnitude of the vector be designated as k. The quantity k is called the curvature of the curve. Thus;

$$\frac{d\bar{T}}{dS} = k\bar{N}$$

Note that the magnitude of k^2 is given as;

$$k^2 = \frac{d^2\bar{r}}{dS^2} \cdot \frac{d^2\bar{r}}{dS^2}$$

Or more conveniently;

$$k^2 = \frac{1}{\left(\frac{ds}{dt}\right)^4}\left(\frac{d^2s}{dt^2}\right)\left(\frac{d\bar{r}}{dt}\cdot\frac{d\bar{r}}{dt}\right) + \frac{1}{\left(\frac{ds}{dt}\right)^2}\left(\frac{d^2\bar{r}}{dt^2}\cdot\frac{d^2\bar{r}}{dt^2}\right)$$
$$- \frac{2}{\left(\frac{ds}{dt}\right)^3}\left(\frac{d^2s}{dt^2}\right)\left(\frac{d\bar{r}}{dt}\cdot\frac{d^2\bar{r}}{dt^2}\right)$$

Simplifying by noting that;

$$\frac{d\bar{r}}{dt}\cdot\frac{d\bar{r}}{dt} = \left(\frac{ds}{dt}\right)^2 \quad ; \quad \frac{d\bar{r}}{dt}\cdot\frac{d^2\bar{r}}{dt^2} = \frac{ds}{dt}\frac{d^2s}{dt^2}$$

then;

$$k^2 = -\frac{1}{\left(\frac{ds}{dt}\right)^2}\left(\frac{d^2s}{dt^2}\right)^2 + \frac{1}{\left(\frac{ds}{dt}\right)^2}\left(\frac{d^2\bar{r}}{dt^2}\cdot\frac{d^2\bar{r}}{dt^2}\right)$$

Note that any further simplification leads to an identity.

2.8. Binormal Torsion

The vector $d\bar{T}/dS$ has been shown to be perpendicular to the tangent vector, and its direction was called the direction of the principal normal. However, it is obvious that other normals to the tangent vector may exist, and in fact, there are an infinite number of such normals.

Consider now forming the cross product and defining the vector $\bar{B}$. Thus;

$$\bar{B} = \bar{T} \times \bar{N}$$

The vector $\bar{B}$ is defined as the binormal of the curve. Now the plane of $\bar{T}$ and $\bar{N}$ is defined as the osculating plane. Note that every curve which has a tangent and a normal will contain the binormal. Now consider forming the derivative; $\frac{d\bar{B}}{dS}$. The magnitude of this vector will be called the torsion of the curve and designated as τ.

Forming the derivative;

$$\frac{d\bar{B}}{dS} = \bar{T} \times \frac{d\bar{N}}{dS} + \frac{d\bar{T}}{dS} \times \bar{N}$$

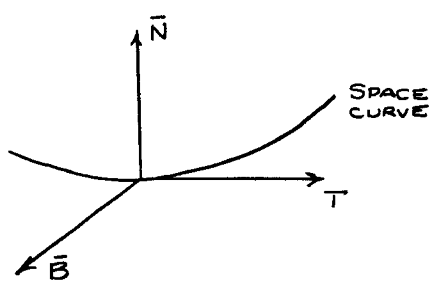
However the direction of $d\bar{T}/dS$ is by definition the direction of $\bar{N}$ and hence;

$$\frac{d\bar{B}}{dS} = \bar{T} \times \frac{d\bar{N}}{dS}$$

Now

$$0 = \frac{d}{dS}(\bar{B}.\bar{B}) = \bar{B}.\frac{d\bar{B}}{dS} + \frac{d\bar{B}}{dS}.\bar{B} = 2\bar{B}.\frac{d\bar{B}}{dS}$$

Hence, the vector $d\bar{B}/dS$ is perpendicular to the vector $\bar{B}$. But $\bar{B}$ is perpendicular to $\bar{T}$ and $\bar{N}$. The situation is shown below;



Thus $d\bar{B}/dS$ must lie in the osculating plane. However, $d\bar{B}/dS$ is also perpendicular to the plane of $\bar{T}$ and $d\bar{N}/dS$. Thus it must also be perpendicular to the vector $\bar{T}$. Thus it must be concluded that the vector $d\bar{B}/dS$ must be in the direction of $\bar{N}$. Defining τ as;

$$\frac{d\bar{B}}{dS} = -\tau\,\bar{N}$$

The quantity τ may now be determined knowing $\bar{T}$ + $\bar{N}$. Note that the torsion is an indicator of the deviation of the curve from a plane curve and hence, an indication of its twist. For a plane curve; $\tau = 0$.

Summary:

The equation of a curve is specified in vector parametric form as;

$$\bar{r} = \bar{r}(t)$$

The length of a line is given as;

$$dS^2 = \left(\frac{d\bar{r}}{dt} \cdot \frac{d\bar{r}}{dt}\right) dt^2$$

The tangent vector is given as;

$$\bar{T} = \frac{1}{\left(\frac{dS}{dt}\right)} \frac{d\bar{r}}{dt}$$

The principal normal and curvature are given as;

$$k\bar{N} = \frac{d^2\bar{r}}{dS^2} = -\frac{1}{\left(\frac{dS}{dt}\right)^2} \frac{d^2S}{dt^2} \frac{d\bar{r}}{dt} + \frac{1}{\left(\frac{dS}{dt}\right)} \frac{d^2\bar{r}}{dt^2}$$

The binormal is given as;

$$\bar{B} = \bar{T} \times N$$

The torsion is given as;

$$-\tau\bar{N} = \frac{d\bar{B}}{dS}$$

In Pure vector form, it can be shown that;

$$T = \frac{\bar{r}'}{|r'|} \qquad \text{where } \bar{r}' = \frac{d\bar{r}}{dt}$$

$$\kappa = \frac{|\bar{r}' \times \bar{r}''|}{|\bar{r}'|^3}$$

$$\tau = \frac{(\bar{r}' \times \bar{r}'') \cdot \bar{r}'''}{|\bar{r}' \times \bar{r}''|^2}$$

2.9. Vector Representation Of A Surface

In dealing with the parametric form of a surface, it was stated that the surface coordinates might be represented in terms of two coordinates, α and β such that;

$$x = x(\alpha,\beta); \; y = y(\alpha,\beta); \; z = z(\alpha,\beta)$$

If now on $\bar{i}$, $\bar{j}$, $\bar{k}$ unit vector system is used, then a vector $\bar{r}$ may be defined such that

$$\bar{r} = x(\alpha,\beta)\bar{i} + y(\alpha,\beta)\bar{j} + z(\alpha,\beta)\bar{k}$$

Thus, the vector $\bar{r}$ uniquely defines the surface. Now the advantage of using $\bar{r}$ rather than x, y and z is that the representation of the surface is freed of a specific coordinate system. Hence, the vector equation of a surface is given as;

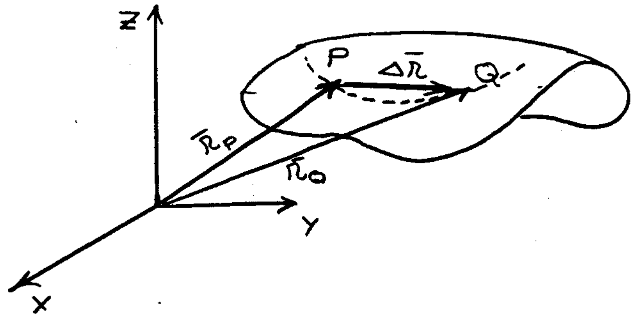
$$\bar{r} = \bar{r}(\alpha,\beta)$$

2.10. Length Of A Curve On A Surface (First Quadratic Form)

Consider now some surface whose equation is $\bar{r}(\alpha,\beta)$, and consider a point P on the surface and another point, say Q, close to this surface. Let the values of α and β corresponding to point P be (α_P, β_P) and those corresponding to Q be (α_Q, β_Q). Now if Q is close to P, then it is reasonable to expect that the corresponding points in the (α-β) plane will also be close to each other so that;

$$\alpha_Q = \alpha_P + \Delta\alpha \; ; \; \beta_Q = \beta_P + \Delta\beta$$

Let $\Delta\bar{r} = \bar{r}_Q - \bar{r}_P$. The situation is shown on the following sketch.



Now there exists some curve whose points are P and Q and for which the vector $\Delta\bar{r}$ is a secant. If $\Delta\bar{r}$ remains the secant for this curve as $Q \rightarrow P$, then it is obvious that the curve must be on the given surface and hence, in the limit, the magnitude of the vector $\Delta\bar{r}$ becomes equal to the length of the curve.

Consider now finding $\Delta\bar{r}$. By a Taylor expansion about the point P;

$$\Delta\bar{r} = \frac{\partial\bar{r}}{\partial\alpha}\Delta\alpha + \frac{\partial\bar{r}}{\partial\beta}\Delta\beta + \cdots$$

Then the square of the scalor length of $\Delta\bar{r}$, which in the limit is given as dS, becomes;

$$ds^2 = \left(\frac{\partial\bar{r}}{\partial\alpha}\cdot\frac{\partial\bar{r}}{\partial\alpha}\right)(d\alpha)^2 + 2\left(\frac{\partial\bar{r}}{\partial\alpha}\cdot\frac{\partial\bar{r}}{\partial\beta}\right)(d\alpha)(d\beta) + \left(\frac{\partial\bar{r}}{\partial\beta}\cdot\frac{\partial\bar{r}}{\partial\beta}\right)(d\beta)^2$$

the higher order terms dropping out.

The above expression for a differential line length on a surface is called the first quadratic form of a surface.

Consider now the above expression in Cartesian coordinates when

$$\bar{r} = x(\alpha,\beta)\bar{i} + y(\alpha,\beta)\bar{j} + z(\alpha,\beta)\bar{k}$$

Substituting, the result becomes;

$$ds^2 = \left[\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 + \left(\frac{\partial z}{\partial \alpha}\right)^2\right](d\alpha)^2 + 2\left[\left(\frac{\partial x}{\partial \alpha}\right)\left(\frac{\partial x}{\partial \beta}\right) + \left(\frac{\partial y}{\partial \alpha}\right)\left(\frac{\partial y}{\partial \beta}\right) +\right.$$

$$\left. + \left(\frac{\partial z}{\partial \alpha}\right)\left(\frac{\partial z}{\partial \beta}\right)\right](d\alpha)(d\beta) + \left[\left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2 + \left(\frac{\partial z}{\partial \beta}\right)^2\right](d\beta)^2$$

For convenience in writing the above expression, let

$$E = \left(\frac{\partial \bar{r}}{\partial \alpha} \cdot \frac{\partial \bar{r}}{\partial \alpha}\right); \quad F = \left(\frac{\partial \bar{r}}{\partial \alpha} \cdot \frac{\partial \bar{r}}{\partial \beta}\right); \quad G = \left(\frac{\partial \bar{r}}{\partial \beta} \cdot \frac{\partial \bar{r}}{\partial \beta}\right)$$

Hence, the expression for dS^2 may also be written as;

$$ds^2 = E(d\alpha)^2 + 2F(d\alpha)(d\beta) + G(d\beta)^2$$

To digress, note that the expression for dS^2 defines a metric on the surface. To illustrate this, let $x(\alpha,\beta) = x_1$; $y(\alpha,\beta) = x_2$; $z(\alpha,\beta) = x_3$ hence, the expression for the differential line element may be written as;

$$ds^2 = \sum_{j=1;k=1}^{2} \sum_{i=1}^{3} \frac{\partial x_i}{\partial \alpha_j} \frac{\partial x_i}{\partial \beta_k} d\alpha_j d\beta_k$$

Letting

$$g_{jk} = \sum_{i=1}^{3} \frac{\partial x_i}{\partial \alpha_j} \frac{\partial x_i}{\partial \beta_k} \quad ; \quad \textit{then}$$

$$ds^2 = \sum_{j=1;k=1}^{2} g_{jk}\, d\alpha_j d\beta_k$$

The quantity g_{jk} is termed the fundamental metric tensor of the surface.

2.11. Angle Between Curves On A Surface

Consider now a curve on the surface $\bar{r}(\alpha,\beta)$. As pointed previously, the equation of a curve is expressed parametrically in terms of one parameter, say t. Thus the general equation of a curve is $\bar{r}(t)$. Now if the curve lies on the given surface, then points on the curve must be coincident with points on the surface. Thus for the points on the curve, there exists a separate parametrization such that $\alpha = \alpha(t)$ and $\beta = \beta(t)$. Thus the equation of a curve on a surface is;

$$\bar{r} = \bar{r}\left[\alpha(t), \beta(t)\right]$$

The direction of the line at any position is given by the direction of its tangent vector $\bar{T}$. Now

$$\bar{T} = \frac{1}{dS} d\bar{r}$$

where dS is the differential segment of length and has been shown to be

$$dS = \sqrt{E(d\alpha)^2 + 2F(d\alpha)(d\beta) + G(d\beta)^2}$$

Note now that E, F and G are surface and not line properties. The only quantities which depend on the curve length are the quantities $d\alpha$ and $d\beta$. Note now that

$$d\bar{r} = \frac{\partial \bar{r}}{\partial \alpha} d\alpha + \frac{\partial \bar{r}}{\partial \beta} d\beta$$

Consider now two curves on the surface, let one of the curves be designated as $\bar{r}_1$, and the other by $\bar{r}_2$. Thus;

$$\bar{r}_1 = \bar{r}_1\left[\alpha(t_1); \beta(t_1)\right]$$
$$\bar{r}_2 = \bar{r}_2\left[\alpha(t_2); \beta(t_2)\right]$$

Assume now that the two curves intersect at the point P on the surface. Let the value of α and β corresponding to this point be designated as (α_P, β_P). Then the tangent vectors to the two curves at this point become;

$$\bar{T}_1 = \frac{1}{dS_1} d\bar{r}_1$$

$$\bar{T}_2 = \frac{1}{dS_2} d\bar{r}_2$$

The angle between the two tangent vectors and hence, the curves become;

$$\cos\theta = \bar{T}_1 \cdot \bar{T}_2 = \frac{1}{dS_1 dS_2}(d\bar{r}_1 \cdot d\bar{r}_2)$$

The derivative $d\bar{r}$ is given as;

$$d\bar{r} = \frac{\partial\bar{r}}{\partial\alpha} d\alpha + \frac{\partial\bar{r}}{\partial\beta} d\beta$$

where now the derivatives are evaluated at point P. Note now that although there are two vectors, $\bar{r}_1$ and $\bar{r}_2$, both vectors are the surface vector. Hence, the derivative $\partial\bar{r}/\partial\alpha$ is the same for the two curves. The same obviously holds true for $\partial\bar{r}/\partial\beta$. However, $d\alpha$ and $d\beta$ represent an incremental change along each of the curves and hence, are indicators of the directions of the two curves. Thus;

$$d\bar{r}_1 = \frac{\partial\bar{r}}{\partial\alpha} d\alpha_1 + \frac{\partial\bar{r}}{\partial\beta} d\beta_1$$

$$d\bar{r}_2 = \frac{\partial\bar{r}}{\partial\alpha} d\alpha_2 + \frac{\partial\bar{r}}{\partial\beta} d\beta_2$$

Hence, the expression for the angle θ becomes;

$$\cos\theta = \frac{1}{dS_1 dS_2}\left[\left(\frac{\partial\bar{r}}{\partial\alpha}\cdot\frac{\partial\bar{r}}{\partial\alpha}\right) d\alpha_1 d\alpha_2 + \left(\frac{\partial\bar{r}}{\partial\alpha}\cdot\frac{\partial\bar{r}}{\partial\beta}\right) d\alpha_1 d\beta_2 \right.$$

$$\left. + \left(\frac{\partial\bar{r}}{\partial\alpha}\cdot\frac{\partial\bar{r}}{\partial\beta}\right) d\alpha_2 d\beta_1 + \left(\frac{\partial\bar{r}}{\partial\beta}\cdot\frac{\partial\bar{r}}{\partial\beta}\right) d\beta_1 d\beta_2\right]$$

Substituting the defined expressions of E, F and G;

$$\cos\theta = \frac{1}{dS_1 dS_2}\left[E\, d\alpha_1 d\alpha_2 + F d\alpha_1 d\beta_2 + F d\alpha_2 d\beta_1 + G d\beta_1 d\beta_2\right]$$

Examples:

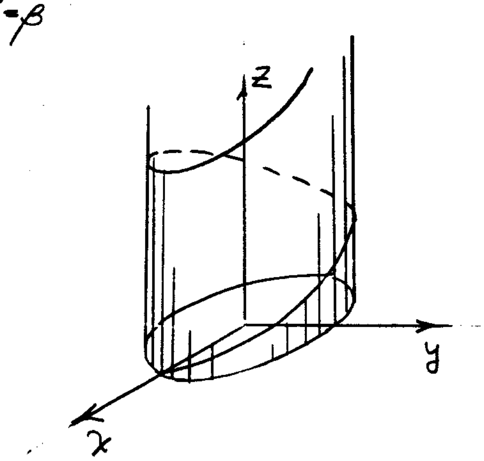
Consider now a right circular cylinder

$x^2 + y^2 = a^2$

Its parametrized form is given as

$x = a \cos\alpha ; y = a \sin\alpha ; z = \beta$

$0 \leq \alpha \leq 2\pi ; \beta > 0$



For this cylinder, the vector equation is given as;

$$\bar{r} = a \cos\alpha \, \bar{i} + a \sin\alpha \, \bar{j} + \beta \bar{k}$$

Consider now a curve on the cylinder. For this curve, let

$$\alpha = k_1 t \quad ; \quad \beta = k_1 t \quad ; 0 < t$$

Hence, the equation of the line becomes;

$$x = a \cos k_1 t \quad ; \quad y = a \sin k_1 t \quad ; \quad z = k_1 t \quad ; 0 < t$$

The resulting curve is a helix drawn on the cylinder. Consider now another helix given by the equation;

$$x = a \cos k_2 t \, ; \quad y = a \sin k_2 t \quad ; z = k_2 t \quad ; 0 < t$$

and consider now bending the angle between the two helices at the point $(\alpha = 0 ; \beta = 0)$.

Now;

$$d\alpha_1 = k_1 dt \qquad\qquad d\beta_1 = k_1 dt$$

$$d\alpha_2 = k_2 dt \qquad\qquad d\beta_2 = k_2 dt$$

Furthermore, the vector equation for the two curves is given as;

$$\bar{r}_1 = a\,\mathrm{Cos}\,k_1 t\,\bar{i} + a\,\mathrm{Sin}\,k_1 t\,\bar{j} + k_1 t\,\bar{k}$$

$$\bar{r}_2 = a\,\mathrm{Cos}\,k_2 t\,\bar{i} + a\,\mathrm{Sin}\,k_2 t\,\bar{j} + k_2 t\,\bar{k}$$

Consider now calculating the quantities E, F, G for the surface.

$$E = \frac{\partial \bar{r}}{\partial \alpha} \cdot \frac{\partial \bar{r}}{\partial \alpha} = a^2 \mathrm{Sin}^2\alpha + a^2 \mathrm{Cos}^2\alpha = a^2$$

$$F = \frac{\partial \bar{r}}{\partial \alpha} \cdot \frac{\partial \bar{r}}{\partial \beta} = 0$$

$$G = \frac{\partial \bar{r}}{\partial \beta} \cdot \frac{\partial \bar{r}}{\partial \beta} = 1$$

Hence, E, F, G are constants for a circular cylinder. Now the differential lengths for each of the curves are given as;

$$ds_1^2 = a^2 d\alpha_1^2 + d\beta_1^2 \;;\quad ds_2^2 = a^2 d\alpha_2^2 + d\beta_2^2$$

Substituting for $d\,\alpha_1$ and $d\,\alpha_2$, the result becomes;

$$ds_1^2 = a^2 k_1^2 dt^2 + k_1^2 dt^2 \;;\quad ds_2^2 = a^2 k_2^2 dt^2 + k_2^2 dt^2$$

Substituting into the expression for Cos θ;

$$\mathrm{Cos}\,\theta = \frac{1}{k_1 k_2 (a^2+1) dt^2}\left(a^2 k_1 k_1 + k_1 k_2\right) dt^2$$

Or;

$$\mathrm{Cos}\,\theta = 1 \quad ; \quad \theta = 0$$

Hence, the two curves coincide. This is not a surprising conclusion for consider the point at which the curve intersects the (z-y) plane. For curve 1;

$$k_1 t = \frac{\pi}{2}$$

and for curve 2;

$$k_2 t = \pi/2$$

Hence, the heights of the two curves are the same above the x-y plane and therefore the two curves are identical. The only thing that k_1 and k_2 do is to speed up the drawing of the curve.

2.12. Curvature Of A Surface And Second Quadratic Form Of A Surface

Second Quadratic Form Of A Surface:

In the previous section, the first quadratic form of a surface had been introduced. To recapitulate, this form, designated usually by the symbol I, had been derived on the basis of a length of curve. That is;

$$I = d\bar{r} \cdot d\bar{r} = ds^2 = E\,d\alpha^2 + 2F\,d\alpha\,d\beta + G\,d\beta^2$$

Now associated with surfaces, there is a quantity called the second quadratic form and is defined as $(-d\bar{r} \cdot d\bar{n})$ where $\bar{n}$ is the unit normal to the surface. The second quadratic form is designated by the symbol II. Thus;

$$II = -d\bar{r} \cdot d\bar{n}$$

Consider now expressing the values of $d\bar{r}$ + $d\bar{n}$. As found for $d\bar{r}$

$$d\bar{r} = \frac{\partial \bar{r}}{\partial \alpha} d\alpha + \frac{\partial \bar{r}}{\partial \beta} d\beta$$

The expression for $\bar{n}$ has as yet not been developed but from its definition, it obviously is perpendicular to the tangent plane to the surface. Now as has been pointed out, $\partial\bar{r}/\partial\alpha$ and $\partial\bar{r}/\partial\beta$ lay in the tangent plane. Hence;

$$\bar{n} = \frac{\left(\partial\bar{r}/\partial\alpha\right) \times \left(\partial\bar{r}/\partial\beta\right)}{\left|\left(\partial\bar{r}/\partial\alpha\right) \times \left(\partial\bar{r}/\partial\beta\right)\right|}$$

Now it can be shown that

$$\left|\frac{\partial\bar{r}}{\partial\alpha} \times \frac{\partial\bar{r}}{\partial\beta}\right|^2 = \left(\partial\bar{r}/\partial\alpha \cdot \partial\bar{r}/\partial\alpha\right)\left(\partial\bar{r}/\partial\beta \cdot \partial\bar{r}/\partial\beta\right) - \left(\partial\bar{r}/\partial\alpha \cdot \partial\bar{r}/\partial\beta\right)^2$$

or $\quad EG-F^2$

Note that the proof of this statement can be gotten by going back to the definition of the direction numbers for a normal to the surface and expanding the results. Thus the equation for the normal $\bar{n}$ becomes.

$$\bar{n} = \frac{1}{\sqrt{EG-F^2}}\left(\frac{\partial \bar{r}}{\partial \alpha} \times \frac{\partial \bar{r}}{\partial \beta}\right)$$

Consider then the expression for $d\bar{n}$. Now $\bar{n}$ is a function of α & β.

Hence;

$$d\bar{n} = \partial \bar{n}/\partial \alpha \, d\alpha + \partial \bar{n}/\partial \beta \, d\beta$$

Substituting into the expression for II,

$$II = -\left(\frac{\partial \bar{r}}{\partial \alpha} d\alpha + \frac{\partial \bar{r}}{\partial \beta} d\beta\right) \cdot \left(\frac{\partial \bar{n}}{\partial \alpha} d\alpha + \frac{\partial \bar{n}}{\partial \beta} d\beta\right)$$

Expanding:

$$II = -\left(\frac{\partial \bar{r}}{\partial \alpha} \cdot \frac{\partial \bar{n}}{\partial \alpha}\right)(d\alpha)^2 - \left(\frac{\partial \bar{r}}{\partial \alpha} \cdot \frac{\partial \bar{n}}{\partial \beta} + \frac{\partial \bar{r}}{\partial \beta} \cdot \frac{\partial \bar{n}}{\partial \alpha}\right)(d\alpha d\beta) - \left(\frac{\partial \bar{r}}{\partial \beta} \cdot \frac{\partial \bar{n}}{\partial \beta}\right)(d\beta)^2$$

Define the quantities L, M and N as follows;

$$L = -\partial \bar{r}/\partial \alpha \cdot \partial \bar{n}/\partial \alpha$$

$$M = -\tfrac{1}{2}\left(\partial \bar{r}/\partial \alpha \cdot \partial \bar{n}/\partial \beta + \partial \bar{r}/\partial \beta \cdot \partial \bar{n}/\partial \alpha\right)$$

$$N = -\partial \bar{r}/\partial \beta \cdot \partial \bar{n}/\partial \beta$$

Hence the second quadratic form of the surface becomes;

$$II = L\,(d\alpha)^2 + 2M(d\alpha\, d\beta) + N\,(d\beta)^2$$

Consider now more convenient ways of evaluating the quantities L, M and N.

Again from the definition of the normal $\bar{n}$;

$$d\bar{r} \cdot \bar{n} = \left(\frac{\partial \bar{r}}{\partial \alpha} \cdot \bar{n}\right) d\alpha + \left(\frac{\partial \bar{r}}{\partial \beta} \cdot \bar{n}\right) d\beta = 0$$

Since $\partial \bar{r}/\partial \alpha$ and $\partial \bar{r}/\partial \beta$ lie in the tangent plane to the surface. Thus, the above equation is satisfied for all α and β. Forming now second differential

$$d(d\bar{r} \cdot \bar{n}) = d^2\bar{r} \cdot \bar{n} + d\bar{r} \cdot d\bar{n} = 0$$

Or;

$$II = -d\bar{r} \cdot d\bar{n} = d^2\bar{r} \cdot \bar{n}$$

Consider then evaluating $d^2\bar{r}$. Now;

$$d^2\bar{r} = \frac{\partial^2\bar{r}}{\partial\alpha^2}(d\alpha)^2 + 2\frac{\partial^2\bar{r}}{\partial\alpha\partial\beta}(d\alpha d\beta) + \frac{\partial^2\bar{r}}{\partial\beta^2}(d\beta)^2$$

Thus;

$$d^2\bar{r}\cdot\bar{n} = \frac{1}{\sqrt{EG-F^2}}\left[\frac{\partial^2\bar{r}}{\partial\alpha^2}\cdot\left(\frac{\partial\bar{r}}{\partial\alpha}\times\frac{\partial\bar{r}}{\partial\beta}\right)(d\alpha)^2 + 2\frac{\partial^2\bar{r}}{\partial\alpha\partial\beta}\cdot\left(\frac{\partial\bar{r}}{\partial\alpha}\times\frac{\partial\bar{r}}{\partial\beta}\right)(d\alpha d\beta)\right.$$
$$\left. + \frac{\partial^2\bar{r}}{\partial\beta^2}\cdot\left(\frac{\partial\bar{r}}{\partial\alpha}\times\frac{\partial\bar{r}}{\partial\beta}\right)(d\beta)^2\right]$$

Thus by analogy it follows that;

$$L = \frac{\frac{\partial^2\bar{r}}{\partial\alpha^2}\cdot\left(\frac{\partial\bar{r}}{\partial\alpha}\times\frac{\partial\bar{r}}{\partial\beta}\right)}{\sqrt{EG-F^2}} \qquad M = \frac{\frac{\partial^2\bar{r}}{\partial\alpha\partial\beta}\cdot\left(\frac{\partial\bar{r}}{\partial\alpha}\times\frac{\partial\bar{r}}{\partial\beta}\right)}{\sqrt{EG-F^2}}$$

$$N = \frac{\frac{\partial^2\bar{r}}{\partial\beta^2}\cdot\left(\frac{\partial\bar{r}}{\partial\alpha}\times\frac{\partial\bar{r}}{\partial\beta}\right)}{\sqrt{EG-F^2}}$$

Note now the reasons for the second form of the second quadratic form of a surface. All the quantities in the expressions may be readily evaluated and furthermore, the quantity;

$$\frac{\left(\partial\bar{r}/\partial\alpha \times \partial\bar{r}/\partial\beta\right)}{\sqrt{EG-F^2}}$$

represents nothing more than the unit normal to the surface and hence is involved with the direction cosines.

Consider now some meaning and distinction between the first and second quadratic forms of a surface. The first quadratic form of a surface is basically a measuring form since it defines a length on the surface. Furthermore, it can be shown that first quadratic form uniquely defines the surface area since it can be shown that

$$S = \int_{\sigma} \sqrt{EG - F^2}\, dS$$

The second quadratic form of a surface tends to give some idea of its shape. Now $d\bar{r}$ lies in the tangent plane to the surface. The vector $d\bar{n}$ may or may not and furthermore, may or may not lie in the same direction as $d\bar{r}$. Thus, their dot product gives some idea of the curvature of the surface encountered. Note that according to the definition, $d\bar{r} \cdot d\bar{n}$ may be zero even if $d\bar{n}$ and $d\bar{r}$ both lie in the tangent plane since they may be perpendicular toward each other.

2.13.Curvature Of A Surface

Consider now a surface whose equation is $\bar{r}(\alpha, \beta)$ and consider now a curve lying on the surface whose equation is $\bar{r}\,[\alpha(t), \beta(t)]$. Consider now the curvature of the curve at the point P on the surface.

The principle normal and curvature of the curve are defined as;

$$K \bar{N} = \frac{d\bar{T}}{dS}$$

where K is the curvature, $\bar{N}$ is the unit normal and defined as the principle normal and $\bar{T}$ is the tangent vector to the curve. Now as has been shown,

$$\bar{T} = \frac{d\bar{r}}{dS}$$

Hence; $$\frac{d\bar{T}}{dS} = \frac{d^2\bar{r}}{dS^2}$$

But $\bar{r}$ is also the radius vector to the surface and hence;

$$\frac{d\bar{r}}{dS} = \frac{\partial \bar{r}}{\partial \alpha}\frac{d\alpha}{dS} + \frac{\partial \bar{r}}{\partial \beta}\frac{d\beta}{dS}$$

Forming the second derivative;

$$\frac{d^2\bar{r}}{ds^2} = \left(\frac{\partial^2\bar{r}}{\partial\alpha^2}\frac{d\alpha}{ds} + \frac{\partial^2\bar{r}}{\partial\alpha\partial\beta}\frac{d\beta}{ds}\right)\frac{d\alpha}{ds} + \frac{\partial\bar{r}}{\partial\alpha}\frac{d^2\alpha}{ds^2} + \left(\frac{\partial^2\bar{r}}{\partial\alpha\partial\beta}\frac{d\alpha}{ds} + \frac{\partial^2\bar{r}}{\partial\beta^2}\frac{d\beta}{ds}\right)\frac{d\beta}{ds}$$
$$+ \frac{\partial\bar{r}}{\partial\beta}\frac{d^2\beta}{ds^2}$$

Consider now forming the inner product of the curvature of the curve and normal to the surface $\bar{n}$. Thus;

$$\kappa\bar{N}\cdot\bar{n} = \kappa\cos\vartheta$$

where ϑ is the angle between the two normals. Thus;

$$\kappa\cos\vartheta = \left(\frac{\partial^2\bar{r}}{\partial\alpha^2}\cdot\bar{n}\right)\left(\frac{d\alpha}{ds}\right)^2 + 2\left(\frac{\partial^2\bar{r}}{\partial\alpha\partial\beta}\cdot\bar{n}\right)\left(\frac{d\alpha}{ds}\right)\left(\frac{d\beta}{ds}\right) + \left(\frac{\partial^2\bar{r}}{\partial\beta^2}\cdot\bar{n}\right)\left(\frac{d\beta}{ds}\right)^2$$
$$+ \left(\frac{\partial\bar{r}}{\partial\alpha}\cdot\bar{n}\right)\left(\frac{d^2\alpha}{ds^2}\right) + \left(\frac{\partial\bar{r}}{\partial\beta}\cdot\bar{n}\right)\left(\frac{d^2\beta}{ds^2}\right)$$

But $\left(\partial\bar{r}/\partial\alpha\right)\cdot\bar{n} = 0$ and $\left(\partial\bar{r}/\partial\beta\right)\cdot\bar{n} = 0$ since the derivatives lie in the tangent plane. Hence the result becomes;

$$\kappa\cos\vartheta = \frac{1}{ds^2}\left[\left(\frac{\partial^2\bar{r}}{\partial\alpha^2}\cdot\bar{n}\right)(d\alpha)^2 + 2\left(\frac{\partial^2\bar{r}}{\partial\alpha\partial\beta}\cdot\bar{n}\right)(d\alpha d\beta) + \left(\frac{\partial^2\bar{r}}{\partial\beta^2}\cdot\bar{n}\right)(d\beta)^2\right]$$

But inspection of the bracketed form reveals that this is the expression for the second quadratic form of the surface. Furthermore, the numerator, ds^2) is nothing more than the first quadratic form of the surface. Thus;

$$\kappa\cos\vartheta = \frac{L\,d\alpha^2 + 2M\,d\alpha d\beta + N\,d\beta^2}{E\,d\alpha^2 + 2F\,d\alpha d\beta + G\,d\beta^2} = \frac{II}{I}$$

Consider now the meaning of the above expression. The quantities L, M, N, E, F, G are all surface properties defined at each point. However, the quantities $d\alpha$ and $d\beta$ do belong to the curve drawn on the surface since $\alpha = \alpha(t)$ and $\beta = \beta(t)$. Now the direction of $\bar{T}$ is the same as that of $d\bar{r}$. But

$$d\bar{r} = \frac{\partial \bar{r}}{\partial \alpha} d\alpha + \frac{\partial \bar{r}}{\partial \beta} d\beta$$

However, a given point on the surface, $\partial \bar{r}/\partial \alpha$ and $\partial \bar{r}/\partial \beta$ are surface properties and independent of the curve passing through the point. Thus the direction of $\bar{T}$ is determined by the quantities $d\alpha$, $d\beta$. Thus if there be a series of surface curves passing through a given point on the surface and all have the same tangent vector at the point, the quantities $d\alpha$ and $d\beta$ will be the same for all curves. It must be concluded, then, that the expression $\kappa \cos \vartheta$ is independent of the type surface curve passing through the point P but be solely dependent on the direction of the curve. This can be shown as follows; Rewriting II/I as;

$$\kappa \cos \vartheta = \frac{L\left(\frac{d\alpha}{ds}\right)^2 + 2M\left(\frac{d\alpha}{ds}\right)\left(\frac{d\beta}{ds}\right) + N\left(\frac{d\beta}{ds}\right)^2}{E\left(\frac{d\alpha}{ds}\right)^2 + 2F\left(\frac{d\alpha}{ds}\right)\left(\frac{d\beta}{ds}\right) + G\left(\frac{d\beta}{ds}\right)^2}$$

Now;

$$\bar{T}_1 = \frac{d\bar{r}}{ds_1} = \frac{\partial \bar{r}}{\partial \alpha}\left(\frac{d\alpha_1}{ds_1}\right) + \frac{\partial \bar{r}}{\partial \beta}\left(\frac{d\beta_1}{ds_1}\right)$$

where for a surface curve, the partial derivatives are fixed at a point. Hence, for two tangent vectors, $\bar{T}_1$ and $\bar{T}_2$ to be equal;

$$\left(\frac{d\alpha_1}{ds_1}\right) = \left(\frac{d\alpha_2}{ds_2}\right) \quad ; \quad \left(\frac{d\beta_1}{ds_1}\right) = \left(\frac{d\beta_2}{ds_2}\right)$$

and thus the expression for $\kappa \cos \vartheta$ is invariant with the curve but is dependent only on the direction of its tangent.

The normal curvature of a surface, κ_0, is defined as

$$\kappa_0 = \kappa \cos \vartheta$$

where κ_0 is the curvature of the surface in the direction $d\alpha : d\beta$

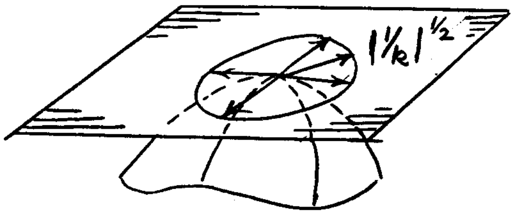
This is known as Meusiniers' Theorem.

It is obvious that a given point on the surface will have infinite values of curvature corresponding to the infinite possible directions on the tangent plane to the surface. Note that the normal surface curvature is directed in the same direction as the normal to the surface and will have the same sense of direction as the principle curvature of the line.

2.13. Surface Curvatures And The Indicatrix Of Curvature

In the previous section, the curvature of a surface at a point was defined and it was shown the value of the curvature was directionally dependent. Thus at a given point on a surface there are an infinite number of values possible for the curvature. These values may be classified by means of the indicatrix of curvature.

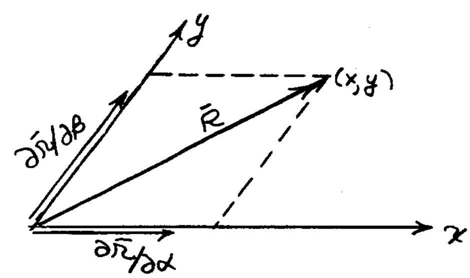
Consider now a point P on a given surface and at that point construct a tangent plane to the surface. Consider calculating all the possible values of the curvatures at the point P corresponding to differently oriented line segments passing through the point. Let k be the curvature of the surface. Now in the tangent plane lay off values of $|1/k|^{1/2}$ in the directions from which the curvatures were calculated. The situation appears as follows



The locus of the end of the segments drawn is a plane curve lying in the tangent plane and is defined as the indicatrix of curvature at the point P.

Certain facts should be noted about the resulting curve. The first and most important is that the curve is symmetric with respect to the origin.

Consider now determining the equation of the indicatrix. Toward this end, introduce an x and y axis. However, rather than being orthogonal, let x lie in the direction $\partial\bar{r}/\partial\alpha$ and y in the direction $\partial\bar{r}/\partial\beta$. Thus x and y lie in the directions of the tangent vectors to the coordinate curves. If the coordinate curves are orthogonal, then x and y will be orthogonal. Let $\bar{R}$ be the radius vector in a particular direction in the tangent plane whose length will be $|(1/k)|^{1/2}$. The situation on the tangent plane is as shown.



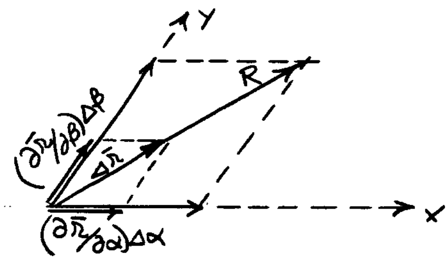
Consider now expressing $\bar{R}$. Remembering that $\bar{R}$ is on the same direction as the curve from which it has been calculated and the direction of the curve is described by $d\alpha$ and $d\beta$, then one expression for $\bar{R}$ is;

$$\bar{R} = |(1/k)|^{1/2} \left[\frac{(\partial\bar{r}/\partial\alpha)d\alpha + (\partial\bar{r}/\partial\beta)d\beta}{|(\partial\bar{r}/\partial\alpha)d\alpha + (\partial\bar{r}/\partial\beta)d\beta|} \right]$$

However, $\bar{R}$ may also be calculated in terms of its components along the x and y axis. Letting the tip of the vector $\bar{R}$ have the coordinates x and y, an alternate form for $\bar{R}$ may be derived as follows:

$$\Delta\bar{r} = \frac{\partial\bar{r}}{\partial\alpha}\Delta\alpha + \frac{\partial\bar{r}}{\partial\beta}\Delta\beta$$

Now $\partial \bar{r}/\partial \alpha$ and $\partial \bar{r}/\partial \beta$ are constant vectors. Hence, $(\partial \bar{r}/\partial \alpha)\Delta \alpha$ and $(\partial \bar{r}/\partial \beta)\Delta \beta$ represent the components of the vector $\Delta \bar{r}$ along the x and y axis as shown.



The quantities $\Delta \alpha$ and $\Delta \beta$ are scale factors which multiply the assumed base vectors $\partial \bar{r}/\partial \alpha$ and $\partial \bar{r}/\partial \beta$. Now consider the vector $\bar{R}$ and the coordinate axis x and y. The length of measure along the x and y axis will not be the same but rather will be modified by the ratio

$$\frac{\Delta x}{\Delta y} = \frac{\left|\left(\partial \bar{r}/\partial \alpha\right)\right|}{\left|\left(\partial \bar{r}/\partial \beta\right)\right|}$$

where x and y are equal increments of measure. Thus the x and y components of the vector $\bar{R}$ will be, respectively

$$\left(\frac{\partial \bar{r}}{\partial \alpha}\right)x \qquad \text{and} \qquad \left(\frac{\partial \bar{r}}{\partial \beta}\right)y$$

Thus $\bar{R}$ may now be written as;

$$\bar{R} = \left(\frac{\partial \bar{r}}{\partial \alpha}\right)x + \left(\frac{\partial \bar{r}}{\partial \beta}\right)y$$

Equating the two expressions;

$$\left(\frac{\partial \bar{r}}{\partial \alpha}\right)x + \left(\frac{\partial \bar{r}}{\partial \beta}\right)y = \left|\left(\frac{1}{k}\right)\right|^{1/2} \left[\frac{\left(\partial \bar{r}/\partial \alpha\right)d\alpha + \left(\partial \bar{r}/\partial \beta\right)d\beta}{\left|\left(\partial \bar{r}/\partial \alpha\right)d\alpha + \left(\partial \bar{r}/\partial \beta\right)d\beta\right|}\right]$$

Dot multiplying the vectors by themselves, i.e., $(\bar{R} \cdot \bar{R})$;

$$\left(\frac{\partial \bar{r}}{\partial \alpha}\cdot\frac{\partial \bar{r}}{\partial \alpha}\right)x^2 + 2\left(\frac{\partial \bar{r}}{\partial \alpha}\cdot\frac{\partial \bar{r}}{\partial \beta}\right)xy + \left(\frac{\partial \bar{r}}{\partial \beta}\cdot\frac{\partial \bar{r}}{\partial \beta}\right)y^2 = \left|\left(\frac{1}{k}\right)\right| \cdot \frac{1}{|ds|^2}\left[\left(\frac{\partial \bar{r}}{\partial \alpha}\cdot\frac{\partial \bar{r}}{\partial \alpha}\right)d\alpha^2 + 2\left(\frac{\partial \bar{r}}{\partial \alpha}\cdot\frac{\partial \bar{r}}{\partial \beta}\right)d\alpha\, d\beta + \left(\frac{\partial \bar{r}}{\partial \beta}\cdot\frac{\partial \bar{r}}{\partial \beta}\right)d\beta^2\right]$$

But $kdS^2 = II = Ld\alpha^2 + 2Md\alpha\, d\beta + Nd\beta^2$ by definition. Hence, the above expression becomes;

$$Ex^2 + 2Fxy + Gy^2 = \frac{Ed\alpha^2 + 2F\,d\alpha\, d\beta + G\,d\beta^2}{|L\,d\alpha^2 + 2M\,d\alpha\, d\beta + N\,d\beta^2|}$$

However, from the nature of definition of x and y;

$$\frac{d\alpha}{d\beta} = \frac{x}{y} = \text{constant}$$

Thus;

$$Ex^2 + 2Fxy + Gy^2 = \frac{Ex^2 + 2Fxy + Gy^2}{|Lx^2 + 2Mxy + Ny^2|}$$

Thus, if the equality is to hold;

$$|Lx^2 + 2Mxy + Ny^2| = 1$$

This is the equation of the indicatrix of curvature.

Note now that this equation may be plotted in conventional cortesian coordinates. In doing so, a number of forms result. Thus;

a) $(LN-M^2) > 0$, illipse (illiptical point)

b) $(LN-M^2) < 0$, a pair of conjugate hyperbolas (hyperbolie point)

c) $(LN-M^2) = 0$, pair of parallel straight lines (parabolic point)

2.14. Principle Directions On A Surface

The equation of the indicatrix of curvature of a surface is a quadratic in x and y coordinates and which furthermore possesses radial symmetry. Now the resulting values of the curvatures will take on extreme values and it can be shown that the extreme values correspond with the directions of the axis of symmetry for the indicatrix of curvature, and furthermore, these axes are orthogonal to each other.

The principal curvatures of a surface at a point are defined as the values of the extreme value of the curvatures and furthermore, the corresponding directions are defined as the principal directions.

One of the consequences of choosing principal directions is contained in <u>Rodriques' Theorem,</u> as follows:

If the direction (d) is a principal direction on a surface, then

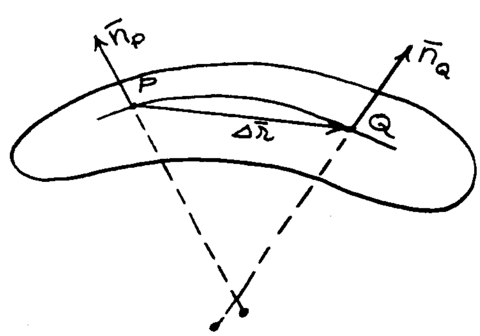
$$d\bar{n} = -k d\bar{r}$$

where k is the normal curvature in this direction. Conversely, if it can be shown that

$$d\bar{n} = \lambda\, d\bar{r}$$

where λ is some constant, then the direction (d) is a principal direction and $\lambda = -k.$

The implications of Rodriques' Theorem is extremely important especially where elements of line length of the surface are contained. To illustrate, consider a portion of a surface and two points P and Q through which some space curve passes.

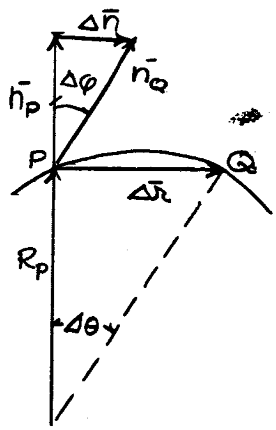


Now from the sketch, it is obvious that without any restrictions on $\Delta\ \bar{r}$ and $\bar{n}$, there will not be any assurance that the two normal vectors to P and Q will intersect. In fact in general, this is not the case. Rodriques' Theorem states that if the line direction is a principal curvature direction, then in fact, the intersection of the two normal vectors is assured. This can be shown as follows;

Consider looking into the plane of the vectors $\Delta \bar{r}$ and $\bar{n}_p$. Define a quantity termed a radius of curvature of the surface such that

$$1/R = k$$

The sketch appears as follows:



Note that as $\Delta \bar{r} \longrightarrow 0$, then $\Delta \bar{n}$ and $\Delta \bar{r}$ approach perpendicularity to the vector $\bar{n}$. Now from the figure it is obvious that

$$\Delta\theta = \tan\Delta\theta \approx \frac{|\Delta\bar{r}|}{R_P}$$

Hence; $|\Delta\bar{r}| = \Delta S = R_P \Delta\theta$. Thus, this expression is true no matter what the orientation of $\Delta \bar{r}$ is to $\Delta \bar{n}$. However $\Delta\theta$ is indeterminate in that it is the angle between R_p and a line drawn from the center of curvature of point P and point Q.

Consider now the situation when $\Delta \bar{r}$ and $\Delta \bar{n}$ are parallel. To begin with, the two triangles shown in the figure are all in the same plane. Furthermore;

$$\tan\Delta\varphi = \frac{|\Delta\bar{n}|}{|\bar{n}|}; \qquad \tan\Delta\theta = k_P |\Delta\bar{r}|$$

But $|\bar{n}| = 1$ and $\Delta\bar{n} = k_p \Delta\bar{r}$. Hence;

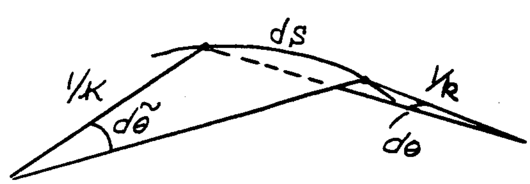
$$\tan\Delta\varphi = k_P |\Delta\bar{r}|$$

and thus it is concluded that

$$\Delta\varphi = \Delta\Theta$$

The condition on the angles implies that $\bar{n}_Q$ and the line connecting Q to the center of curvature of P are parallel and thus this latter line is in the direction of the normal to Q. Now for sufficiently small values of $\Delta\bar{r}$, the curvature from P to Q changes by a second order magnitude. Thus, the curvature at Q may be considered to be the same as that of P. The resultant conclusion is that $\Delta\Theta$ measures the angular deviation between the two radii of curvature between P and Q.

A curious and unique condition then exists on lines drawn on a surface in so far as measuring differential lengths are concerned. Given a line and its curvature $\mathcal{K}$, the length dS of the line may be written as; $dS = \mathcal{K}\, d\tilde{\Theta}$ where $d\tilde{\Theta}$ is the subtended angle between the two curvatures of the line. However, if the line is in a principal direction on the surface, this same differential length may be written as $dS = k\,d\Theta$ where k is the curvature of the surface and $d\Theta$ is the subtended angle between the two principal surface curvatures. This situation is shown on the sketch below.



2.15. Principal Curvalinear Coordinates

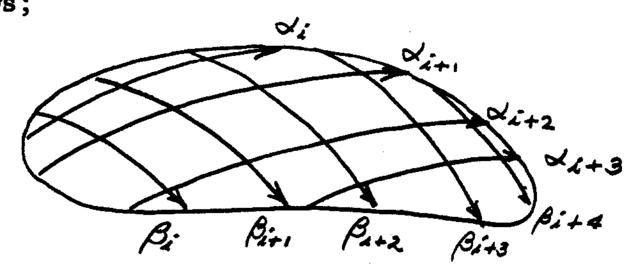
Consider again a coordinization of a surface

$$\bar{\mathcal{R}} = \bar{\mathcal{R}}(\alpha,\beta)$$

and consider now a system of curves on the surface corresponding to a variation of each of the surface parameters. That is, a system of curves obeying the equations;

$$\left.\begin{array}{l}\bar{r}_1 = \bar{r}_1(\alpha, \beta_j) \\ \bar{r}_2 = \bar{r}_2(\alpha_j, \beta)\end{array}\right\} ; \; j = 1, 2, \cdots\cdots$$

Since each value of and determine a point on the surface, the resulting curves $\bar{r}_1$ and $\bar{r}_2$ form an intersecting mesh. The system is shown as follows;



Thus, along a curve for which β is a constant, α varies continuously. Such a curve will be termed an "α curvalinear coordinate curve." The converse of the argument will suffice for a "β curvalinear coordinate curve." Now assume further that the α and β parametrization had been so chosen that the resulting curves coincide with the principal directions on the surface. The resulting system of curves are then termed "principal curvalinear coordinate curves." It is this system of coordinate curves which will be assumed to exist on the surface.

Consider now some of the previously derived expressions when applied to principal curvalinear coordinate curves.

For coordinate curves, it had been shown that the angle between the tangents is given as;

$$\cos\theta = \frac{F}{\sqrt{EG}}$$

However, principal directions are orthogonal to each other, and hence, $\cos\theta = 0$ which implies that

$F = 0$

Now the expression for the differential line element becomes;

$$I = ds^2 = E(d\alpha)^2 + G(d\beta)^2$$

Consider now the expression for the second quadratic form of the surface. By Rodriques' theorem;

$d\bar{n} = -k d\bar{r}$

where the direction chosen is along a principal curvalinear coordinate line and k is the curvature of the surface corresponding to that direction. Hence, it must be concluded that

$$\frac{\partial \bar{n}}{\partial \alpha} = -k_\alpha \frac{\partial \bar{r}}{\partial \alpha}$$

where k_α is the curvature in the direction of the α coordinate line. Similarily;

$$\frac{\partial \bar{n}}{\partial \beta} = -k_\beta \frac{\partial \bar{r}}{\partial \beta}$$

Since the coordinate lines are orthogonal, it becomes obvious that

$$\frac{\partial \bar{r}}{\partial \alpha} \cdot \frac{\partial \bar{n}}{\partial \beta} = 0 \quad ; \quad \frac{\partial \bar{r}}{\partial \beta} \cdot \frac{\partial \bar{n}}{\partial \alpha} = 0$$

and hence, the expression for M in the second quadratic form becomes;

$M = 0$

Consider now the expressions for L and N. Now;

$$L = -\frac{\partial \bar{r}}{\partial \alpha} \cdot \frac{\partial \bar{n}}{\partial \alpha} = k_\alpha \left(\frac{\partial \bar{r}}{\partial \alpha} \cdot \frac{\partial \bar{r}}{\partial \alpha} \right)$$

Hence, $L = k_\alpha E$

By an analogous argument;

$$N = k_\beta G$$

Thus the second quadratic form may be written as;

$$II = k_\alpha E(d\alpha)^2 + k_\beta G(d\beta)^2$$

And the resulting expression for the curvature in an arbitrary direction becomes;

$$k = \frac{k_\alpha E(d\alpha)^2 + k_\beta G(d\beta)^2}{E(d\alpha)^2 + G(d\beta)^2}$$

2.16. General Comments And Summary Of Relations For Principal Curvalinear Coordinates

1. Principal curvalinear coordinates are orthogonal to each other and are oriented in the direction of the principal curvature of a surface.
2. The first quadratic form of a surface, the length of a line, is given as;

$$ds^2 = E(d\alpha)^2 + G(d\beta)^2$$

3. The normals for two points on a principal curvalinear coordinate line intersect and subtend on angle $d\theta$ such that

   $dS = d\theta/k$

   where k is a principal curvature.
4. The second quadratic form of a surface becomes;

$$II = k_\alpha E(d\alpha)^2 + k_\beta G(d\beta)^2$$

5. The vector $d\bar{n}$ and the vector $d\bar{r}$ are parallel and related by the expression

   $d\bar{n} = -k d\bar{r}$
6. The principal curvatures are dependent on the direction of the surface normal and the magnitude and direction of the principal curvature of the principal curvalinear coordinate curve. Let $\mathcal{K}$ be the curvature of the line and ϑ the angle subtended between the surface normal and the principal normal to the line. Then, the curvature of the

surface is defined as;

$$k = K \cos \vartheta$$

7. The theorem of Bonnet states that if the first and second quadratic forms of a surface are known and if the coefficients satisfy the Gauss-Peterson-Codazzi conditions, then a surface, unique to within its position in space, is completely defined. Now for principal curvalinear coordinates, it has been shown that

$$I = E(d\alpha)^2 + G(d\beta)^2$$
$$II = k_\alpha E(d\alpha)^2 + k_\beta G(d\beta)^2$$

The conditions of Codazzi and Gauss will be derived in subsequent chapters. The important conclusion is the following:

If a surface, parametrized by principal curvalinear coordinates exists, knowing the coefficients of the first quadratic form and the principal curvatures is sufficient for a complete description of the surface.

CHAPTER III

DEFORMED SURFACES. LINEAR THEORY

For the undeformed surface, it had been shown that for principal curvalinear coordinates and a surface equation of the type $\bar{R}=\bar{R}(\alpha,\beta)$, the first and second quadratic forms become;

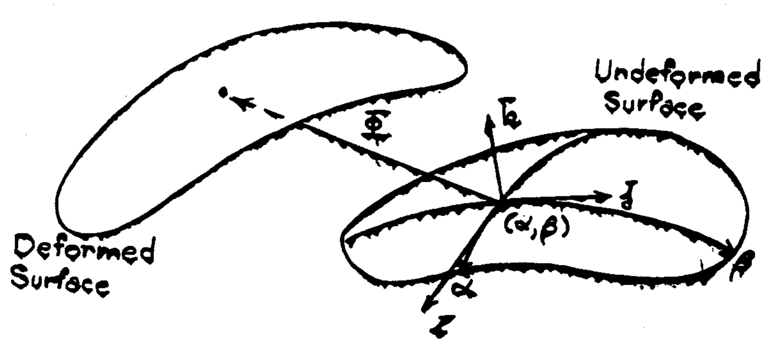
$$I = A^2(d\alpha)^2 + B^2(d\beta)^2$$
$$II = k_\alpha A^2(d\alpha)^2 + k_\beta B^2(d\beta)^2$$

where k_α and k_β are the principal curvatures of the surface.

Consider now the deformed surface $\bar{R}'$ and assume that it may be derived from the undeformed surface in the following manner. With each point (α,β) on the undeformed surface assume there exists a vector function $\bar{\Phi}$ such that relative to the orthonormal triad $(\bar{I},\bar{J},\bar{k})$ on the undeformed surface;

$$\bar{\Phi} = u\bar{I} + v\bar{J} + w\bar{k}$$

The situation is shown on the accompanying sketch.



Thus it is obvious that $\bar{\Phi} = \bar{\Phi}(\alpha,\beta)$. Now the equation of the deformed surface may be written as;

$$\bar{R}' = \bar{R} + \bar{\Phi}$$

Since $\bar{r} = \bar{r}(\alpha, \beta)$ and $\bar{\Phi} = \bar{\Phi}(\alpha, \beta)$, it is obvious that α and β will also be parameterization of the deformed surface and hence α and β coordinate lines will exist on this surface.

Consider now the general expressions for the first and second quadratic forms of any surface.

$$I = E(d\alpha)^2 + 2F(d\alpha)(d\beta) + G(d\beta)^2$$

$$II = L(d\alpha)^2 + 2M(d\alpha)(d\beta) + N(d\beta)^2$$

where,

$$E = \frac{\partial \bar{r}}{\partial \alpha} \cdot \frac{\partial \bar{r}}{\partial \alpha} \qquad F = \frac{\partial \bar{r}}{\partial \alpha} \cdot \frac{\partial \bar{r}}{\partial \beta} \qquad G = \frac{\partial \bar{r}}{\partial \beta} \cdot \frac{\partial \bar{r}}{\partial \beta}$$

$$L = -\frac{\partial \bar{r}}{\partial \alpha} \cdot \frac{\partial \bar{n}}{\partial \alpha} \qquad M = -\left(\frac{\partial \bar{r}}{\partial \alpha} \cdot \frac{\partial \bar{n}}{\partial \beta} + \frac{\partial \bar{r}}{\partial \beta} \cdot \frac{\partial \bar{n}}{\partial \alpha}\right) \qquad N = -\frac{\partial \bar{r}}{\partial \beta} \cdot \frac{\partial \bar{n}}{\partial \beta}$$

Or equivalently;

$$L = \frac{\partial^2 \bar{r}}{\partial \alpha^2} \cdot \bar{n} \qquad M = \frac{\partial^2 \bar{r}}{\partial \alpha \partial \beta} \cdot \bar{n} \qquad N = \frac{\partial^2 \bar{r}}{\partial \beta^2} \cdot \bar{n}$$

3.1. First Quadratic Form

Consider now evaluating the coefficients E, F, G. The equation of the deformed surface is given as;

$$\bar{r}' = \bar{r} + \bar{\Phi}$$

Forming the various derivatives;

$$\frac{\partial \bar{r}'}{\partial \alpha} = \frac{\partial \bar{r}}{\partial \alpha} + \frac{\partial \bar{\Phi}}{\partial \alpha} \quad ; \quad \frac{\partial \bar{r}'}{\partial \beta} = \frac{\partial \bar{r}}{\partial \beta} + \frac{\partial \bar{\Phi}}{\partial \beta}$$

Evaluating the coefficients;

a) E

$$\frac{\partial \bar{r}'}{\partial \alpha} \cdot \frac{\partial \bar{r}'}{\partial \alpha} = \left(\frac{\partial \bar{r}}{\partial \alpha} \cdot \frac{\partial \bar{r}}{\partial \alpha}\right) + 2\left(\frac{\partial \bar{r}}{\partial \alpha} \cdot \frac{\partial \bar{\Phi}}{\partial \alpha}\right) + \left(\frac{\partial \bar{\Phi}}{\partial \alpha} \cdot \frac{\partial \bar{\Phi}}{\partial \alpha}\right)$$

However as has been found for the undeformed shell;

$$\frac{\partial \bar{r}}{\partial \alpha} \cdot \frac{\partial \bar{r}}{\partial \alpha} = A^2 \quad \text{and} \quad \frac{\partial \bar{r}}{\partial \alpha} = A\bar{i} \; ; \; \frac{\partial \bar{r}}{\partial \beta} = B\bar{j}$$

and for the deformed shell;

$$\bar{\Phi} = u\bar{i} + v\bar{j} + w\bar{k}$$

Consider forming the various derivatives of I;

$$\frac{\partial \bar{\Phi}}{\partial \alpha} = \frac{\partial u}{\partial \alpha}\bar{i} + u\frac{\partial \bar{i}}{\partial \alpha} + \frac{\partial v}{\partial \alpha}\bar{j} + v\frac{\partial \bar{j}}{\partial \alpha} + \frac{\partial w}{\partial \alpha}\bar{k} + w\frac{\partial \bar{k}}{\partial \alpha}$$

Substituting; and combining;

$$\frac{\partial \bar{\Phi}}{\partial \alpha} = \left[\frac{\partial u}{\partial \alpha} + \frac{1}{B}\frac{\partial A}{\partial \beta}v + k_\alpha A w\right]\bar{i} + \left[\frac{\partial v}{\partial \alpha} - \frac{1}{B}\frac{\partial A}{\partial \beta}u\right]\bar{j} + \left[\frac{\partial w}{\partial \alpha} - k_\alpha A u\right]\bar{k}$$

And similarly;

$$\frac{\partial \bar{\Phi}}{\partial \beta} = \left[\frac{\partial u}{\partial \beta} - \frac{1}{A}\frac{\partial B}{\partial \alpha}v\right]\bar{i} + \left[\frac{\partial v}{\partial \beta} + \frac{1}{A}\frac{\partial B}{\partial \alpha}u + k_\beta B w\right]\bar{j} + \left[\frac{\partial w}{\partial \beta} - k_\beta B v\right]\bar{k}$$

Hence, forming the expression for **E**;

$$E = A^2 + 2A\left[\frac{\partial u}{\partial \alpha} + \frac{1}{B}\frac{\partial A}{\partial \beta}v + k_\alpha A w\right] + \left[\frac{\partial u}{\partial \alpha} + \frac{1}{B}\frac{\partial A}{\partial \beta}v + A k_\alpha w\right]^2 + \left[\frac{\partial v}{\partial \alpha} - \frac{1}{B}\frac{\partial A}{\partial \beta}u\right]^2 + \left[\frac{\partial w}{\partial \alpha} - k_\alpha A u\right]^2$$

Consider now introducing linearity into the problem by arbitrarily stating that products of displacement functions and their derivatives will be neglected. Thus the linear expression for **E** becomes;

$$E = A^2\left[1 + 2\left(\frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v + k_\alpha w\right)\right]$$

b) F

Now;

$$F = \frac{\partial \bar{r}'}{\partial \alpha}\cdot\frac{\partial \bar{r}'}{\partial \beta} = \left(\frac{\partial \bar{r}}{\partial \alpha} + \frac{\partial \bar{\Phi}}{\partial \alpha}\right)\cdot\left(\frac{\partial \bar{r}}{\partial \beta} + \frac{\partial \bar{\Phi}}{\partial \beta}\right)$$

Expanding;

$$\left(\frac{\partial \bar{r}}{\partial \alpha}\cdot\frac{\partial \bar{r}}{\partial \beta}\right) + \left(\frac{\partial \bar{r}}{\partial \alpha}\cdot\frac{\partial \bar{\Phi}}{\partial \beta}\right) + \left(\frac{\partial \bar{r}}{\partial \beta}\cdot\frac{\partial \bar{\Phi}}{\partial \alpha}\right) + \left(\frac{\partial \bar{\Phi}}{\partial \alpha}\cdot\frac{\partial \bar{\Phi}}{\partial \beta}\right)$$

But on the undeformed surface, $(\partial \bar{r}/\partial \alpha \cdot \partial \bar{r}/\partial \beta) = 0$ and hence;

$$\frac{\partial \bar{r}}{\partial \alpha}\cdot\frac{\partial \bar{\Phi}}{\partial \beta} = \left[\frac{\partial u}{\partial \beta} - \frac{1}{A}\frac{\partial B}{\partial \alpha}v\right]A$$

$$\frac{\partial \bar{r}}{\partial \beta}\cdot\frac{\partial \bar{\Phi}}{\partial \alpha} = \left[\frac{\partial v}{\partial \alpha} - \frac{1}{B}\frac{\partial A}{\partial \beta}u\right]B$$

$$\frac{\partial \bar{\Phi}}{\partial \alpha}\cdot\frac{\partial \bar{\Phi}}{\partial \beta} = \left(\frac{\partial u}{\partial \alpha} + \frac{1}{B}\frac{\partial A}{\partial \beta}v + k_\alpha A w\right)\left(\frac{\partial u}{\partial \beta} - \frac{1}{A}\frac{\partial B}{\partial \alpha}v\right) + \left(\frac{\partial v}{\partial \alpha} - \frac{1}{B}\frac{\partial A}{\partial \beta}u\right)\left(\frac{\partial v}{\partial \beta} + \frac{1}{B}\frac{\partial B}{\partial \alpha}u + k_\beta B w\right) + \left(\frac{\partial w}{\partial \alpha} - k_\alpha A u\right)\left(\frac{\partial w}{\partial \beta} - k_\beta B w\right)$$

Note that the last expression leads to non linear terms in the displacements and hence will be discarded. Then the expression for F becomes;

$$F = A\frac{\partial u}{\partial \beta} - \frac{\partial B}{\partial \alpha} v + B\frac{\partial v}{\partial \alpha} - \frac{\partial A}{\partial \beta} u$$

and which may also be written as;

$$F = AB\left[\frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha} v + \frac{1}{A}\frac{\partial v}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta} u\right]$$

which is equivalent to the final form

$$F = AB\left[\frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right)\right]$$

c) G

Now;

$$G = \frac{\partial \bar{r}'}{\partial \beta} \cdot \frac{\partial \bar{r}'}{\partial \beta} = \left(\frac{\partial \bar{r}}{\partial \beta} + \frac{\partial \bar{\Phi}}{\partial \beta}\right) \cdot \left(\frac{\partial \bar{r}}{\partial \beta} + \frac{\partial \bar{\Phi}}{\partial \beta}\right)$$

Expanding;

$$G = \left(\frac{\partial \bar{r}}{\partial \beta} \cdot \frac{\partial \bar{r}}{\partial \beta}\right) + 2\left(\frac{\partial \bar{r}}{\partial \beta} \cdot \frac{\partial \bar{\Phi}}{\partial \beta}\right) + \left(\frac{\partial \bar{\Phi}}{\partial \beta} \cdot \frac{\partial \bar{\Phi}}{\partial \beta}\right)$$

Substituting;

$$\left(\frac{\partial \bar{r}}{\partial \beta} \cdot \frac{\partial \bar{r}}{\partial \beta}\right) = B^2$$

$$\left(\frac{\partial \bar{r}}{\partial \beta} \cdot \frac{\partial \bar{\Phi}}{\partial \beta}\right) = B\left(\frac{\partial v}{\partial \beta} + \frac{1}{A}\frac{\partial B}{\partial \alpha} u + k_\beta B w\right)$$

$$\left(\frac{\partial \bar{\Phi}}{\partial \beta} \cdot \frac{\partial \bar{\Phi}}{\partial \beta}\right) = \text{terms non linear in the displacements.}$$

Hence the linear expression for G becomes;

$$G = B^2 + 2B^2\left(\frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha} u + k_\beta w\right)$$

The resulting expression for the first quadratic form becomes;

$$I = A^2\left[1 + 2\left(\frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta} v + k_\alpha w\right)\right](d\alpha)^2 + 2AB\left[\frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right)\right](d\alpha)(d\beta) + B^2\left[1 + 2\left(\frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha} u + k_\beta w\right)\right](d\beta)^2$$

Define;

$$e_{\alpha\alpha} = 1/A\ \partial u/\partial\alpha + 1/AB\ \partial A/\partial\beta\ v + k_\alpha w$$

$$e_{\alpha\beta} = A/B\ \partial/\partial\beta(u/A) + B/A\ \partial/\partial\alpha(v/B)$$

$$e_{\beta\beta} = 1/B\ \partial v/\partial\beta + 1/AB\ \partial B/\partial\alpha\ u + k_\beta w$$

Then the first quadratic form may be written as;

$$I = A^2(1+2e_{\alpha\alpha})(d\alpha)^2 + 2ABe_{\alpha\beta}(d\alpha)(d\beta) + B^2(1+2e_{\beta\beta})(d\beta)^2$$

3. 2. Second Quadratic Form of a Surface

The constants L, M, N have been previously defined. Now consider the expression for $\bar{n}$;

$$\bar{n} = \frac{1}{\sqrt{EG-F^2}}\left(\frac{\partial\bar{r}'}{\partial\alpha}\times\frac{\partial\bar{r}'}{\partial\beta}\right)$$

However;

$$\frac{\partial\bar{r}'}{\partial\alpha} = \frac{\partial\bar{r}}{\partial\alpha} + \frac{\partial\bar{\Phi}}{\partial\alpha}\ ;\quad \frac{\partial\bar{r}'}{\partial\beta} = \frac{\partial\bar{r}}{\partial\beta} + \frac{\partial\bar{\Phi}}{\partial\beta}$$

Hence;

$$\frac{\partial\bar{r}'}{\partial\alpha}\times\frac{\partial\bar{r}'}{\partial\beta} = \left(\frac{\partial\bar{r}}{\partial\alpha}\times\frac{\partial\bar{r}}{\partial\beta}\right) + \left(\frac{\partial\bar{r}}{\partial\alpha}\times\frac{\partial\bar{\Phi}}{\partial\beta}\right) + \left(\frac{\partial\bar{\Phi}}{\partial\alpha}\times\frac{\partial\bar{r}}{\partial\beta}\right) + \left(\frac{\partial\bar{\Phi}}{\partial\alpha}\times\frac{\partial\bar{\Phi}}{\partial\beta}\right)$$

To recapitulate;

$$\frac{\partial\bar{r}}{\partial\alpha} = A\bar{i}\ ;\quad \frac{\partial\bar{r}}{\partial\beta} = B\bar{j}\ ;\quad \frac{\partial\bar{\Phi}}{\partial\alpha} = Ae_{\alpha\alpha}\bar{i} + \left(\frac{\partial v}{\partial\alpha} - \frac{1}{B}\frac{\partial A}{\partial\beta}u\right)\bar{j} + \left(\frac{\partial w}{\partial\alpha} - k_\alpha A w\right)\bar{k}$$

$$\frac{\partial\bar{\Phi}}{\partial\beta} = \left(\frac{\partial u}{\partial\beta} - \frac{1}{A}\frac{\partial B}{\partial\alpha}v\right)\bar{i} + Be_{\beta\beta}\bar{j} + \left(\frac{\partial w}{\partial\beta} - k_\beta B v\right)\bar{k}$$

Consider now forming the various cross products

$$\frac{\partial\bar{r}}{\partial\alpha}\times\frac{\partial\bar{r}}{\partial\beta} = AB\bar{k}$$

$$\frac{\partial\bar{r}}{\partial\alpha}\times\frac{\partial\bar{\Phi}}{\partial\beta} = -A\left(\frac{\partial w}{\partial\beta} - k_\beta B v\right)\bar{j} + ABe_{\beta\beta}\bar{k}$$

$$\frac{\partial\bar{\Phi}}{\partial\alpha}\times\frac{\partial\bar{r}}{\partial\beta} = -B\left(\frac{\partial w}{\partial\alpha} - k_\alpha A u\right)\bar{i} + ABe_{\alpha\alpha}\bar{k}$$

$$\frac{\partial\bar{\Phi}}{\partial\alpha}\times\frac{\partial\bar{\Phi}}{\partial\beta}$$ = non linear terms in the displacements.

Hence the normal $\bar{n}$ may be written as;

$$\bar{n} = \frac{1}{\sqrt{EG-F^2}}\left[\left(k_\alpha AB u - B\frac{\partial w}{\partial \alpha}\right)\bar{i} + \left(k_\beta AB v - A\frac{\partial w}{\partial \beta}\right)\bar{j} + AB\left(1+e_{\alpha\alpha}+e_{\beta\beta}\right)\bar{k}\right]$$

Consider now the second derivatives of $\bar{r}'$.

a) $\dfrac{\partial^2 \bar{r}'}{\partial \alpha^2}$

Now;

$$\frac{\partial^2 \bar{r}'}{\partial \alpha^2} = \frac{\partial^2 \bar{r}}{\partial \alpha^2} + \frac{\partial^2 \bar{\Phi}}{\partial \alpha^2}$$

Substituting;

$$\frac{\partial^2 \bar{r}}{\partial \alpha^2} = \frac{\partial}{\partial \alpha}(A\bar{i}) = A\frac{\partial \bar{i}}{\partial \alpha} + \frac{\partial A}{\partial \alpha}\bar{i}$$

Hence;

$$\frac{\partial^2 \bar{r}}{\partial \alpha^2} = \frac{\partial A}{\partial \alpha}\bar{i} - \frac{A}{B}\frac{\partial A}{\partial \beta}\bar{j} - k_\alpha A^2\bar{k}$$

And;

$$\frac{\partial^2 \bar{\Phi}}{\partial \alpha^2} = \frac{\partial}{\partial \alpha}(Ae_{\alpha\alpha})\bar{i} + \left[\frac{\partial^2 v}{\partial \alpha^2} - \frac{\partial}{\partial \alpha}\left(\frac{1}{B}\frac{\partial A}{\partial \beta}u\right)\right]\bar{j} + \left[\frac{\partial^2 w}{\partial \alpha^2} - \frac{\partial}{\partial \alpha}\left(k_\alpha A u\right)\right]\bar{k}$$

$$+ Ae_{\alpha\alpha}\frac{\partial \bar{i}}{\partial \alpha} + \left(\frac{\partial v}{\partial \alpha} - \frac{1}{B}\frac{\partial A}{\partial \beta}u\right)\frac{\partial \bar{j}}{\partial \alpha} + \left(\frac{\partial w}{\partial \alpha} - k_\alpha A u\right)\frac{\partial \bar{k}}{\partial \alpha}$$

Substituting for the derivatives;

$$\frac{\partial^2 \bar{\Phi}}{\partial \alpha^2} = \left[\frac{\partial}{\partial \alpha}(Ae_{\alpha\alpha}) + \frac{1}{B}\frac{\partial A}{\partial \beta}\frac{\partial v}{\partial \alpha} - \frac{1}{B^2}\left(\frac{\partial A}{\partial \beta}\right)^2 u + k_\alpha A\frac{\partial w}{\partial \alpha} - k_\alpha^2 A^2 u\right]\bar{i} + \left[\frac{\partial^2 v}{\partial \alpha^2}\right.$$

$$\left. - \frac{\partial}{\partial \alpha}\left(\frac{1}{B}\frac{\partial A}{\partial \beta}u\right) - \frac{A}{B}\frac{\partial A}{\partial \beta}e_{\alpha\alpha}\right]\bar{j} + \left[\frac{\partial^2 w}{\partial \alpha^2} - \frac{\partial}{\partial \alpha}\left(k_\alpha A u\right) - k_\alpha A^2 e_{\alpha\alpha}\right]\bar{k}$$

Hence;

$$\frac{\partial^2 \bar{r}'}{\partial \alpha^2} = \left[\frac{\partial A}{\partial \alpha} + \frac{\partial}{\partial \alpha}(Ae_{\alpha\alpha}) + \frac{1}{B}\frac{\partial A}{\partial \beta}\frac{\partial v}{\partial \alpha} - \frac{1}{B^2}\left(\frac{\partial A}{\partial \beta}\right)^2 u + k_\alpha A\frac{\partial w}{\partial \alpha} - k_\alpha^2 A^2 u\right]\bar{i}$$

$$+ \left[-\frac{A}{B}\frac{\partial A}{\partial \beta} + \frac{\partial^2 v}{\partial \alpha^2} - \frac{\partial}{\partial \alpha}\left(\frac{1}{B}\frac{\partial A}{\partial \beta}u\right) - \frac{A}{B}\frac{\partial A}{\partial \beta}e_{\alpha\alpha}\right]\bar{j} + \left[-k_\alpha A^2 + \frac{\partial^2 w}{\partial \alpha^2}\right.$$

$$\left. - \frac{\partial}{\partial \alpha}\left(k_\alpha A u\right) - k_\alpha A^2 e_{\alpha\alpha}\right]\bar{k}$$

b) $\dfrac{\partial^2 \bar{r}'}{\partial \beta^2}$

Now

$$\frac{\partial^2 \bar{r}'}{\partial \beta^2} = \frac{\partial^2 \bar{r}}{\partial \beta^2} + \frac{\partial^2 \bar{\Phi}}{\partial \beta^2}$$

Substituting;

$$\frac{\partial^2 \bar{r}}{\partial \beta^2} = \frac{\partial}{\partial \beta}(B\bar{j}) = -\frac{B}{A}\frac{\partial B}{\partial \alpha}\bar{i} + \frac{\partial B}{\partial \beta}\bar{j} - k_\beta B^2 \bar{k}$$

Consider now the second function;

$$\frac{\partial^2 \bar{\Phi}}{\partial \beta^2} = \left[\frac{\partial^2 u}{\partial \beta^2} - \frac{\partial}{\partial \beta}\left(\frac{1}{A}\frac{\partial B}{\partial \alpha} v\right)\right]\bar{i} + \frac{\partial}{\partial \beta}(B e_{\beta\beta})\bar{j} + \left[\frac{\partial^2 w}{\partial \beta^2} - \frac{\partial}{\partial \beta}(k_\beta B v)\right]\bar{k}$$

$$+ \left[\frac{\partial u}{\partial \beta} - \frac{1}{A}\frac{\partial B}{\partial \alpha} v\right]\frac{\partial \bar{i}}{\partial \beta} + B e_{\beta\beta}\frac{\partial \bar{j}}{\partial \beta} + \left[\frac{\partial w}{\partial \beta} - k_\beta B v\right]\frac{\partial \bar{k}}{\partial \beta}$$

Substituting;

$$\frac{\partial^2 \bar{\Phi}}{\partial \beta^2} = \left[\frac{\partial^2 u}{\partial \beta^2} - \frac{\partial}{\partial \beta}\left(\frac{1}{A}\frac{\partial B}{\partial \alpha} v\right) - \frac{B}{A}\frac{\partial B}{\partial \alpha} e_{\beta\beta}\right]\bar{i} + \left[\frac{\partial}{\partial \beta}(B e_{\beta\beta}) + \frac{1}{A}\frac{\partial B}{\partial \alpha}\frac{\partial u}{\partial \beta} - \right.$$

$$\left. \frac{1}{A^2}\left(\frac{\partial B}{\partial \alpha}\right)^2 v + k_\beta B \frac{\partial w}{\partial \beta} - k_\beta^2 B^2 v\right]\bar{j} + \left[\frac{\partial^2 w}{\partial \beta^2} - \frac{\partial}{\partial \beta}(k_\beta B v) - k_\beta B^2 e_{\beta\beta}\right]\bar{k}$$

And hence;

$$\frac{\partial^2 \bar{r}'}{\partial \beta^2} = \left[-\frac{B}{A}\frac{\partial B}{\partial \alpha} + \frac{\partial^2 u}{\partial \beta^2} - \frac{\partial}{\partial \beta}\left(\frac{1}{A}\frac{\partial B}{\partial \alpha} v\right) - \frac{B}{A}\frac{\partial B}{\partial \alpha} e_{\beta\beta}\right]\bar{i} + \left[\frac{\partial B}{\partial \beta} + \frac{\partial}{\partial \beta}(B e_{\beta\beta})\right.$$

$$\left. + \frac{1}{A}\frac{\partial B}{\partial \alpha}\frac{\partial u}{\partial \beta} - \frac{1}{A^2}\left(\frac{\partial B}{\partial \alpha}\right)^2 v + k_\beta B \frac{\partial w}{\partial \beta} - k_\beta^2 B^2 v\right]\bar{j} + \left[-k_\beta B^2\right.$$

$$\left. + \frac{\partial^2 w}{\partial \beta^2} - \frac{\partial}{\partial \beta}(k_\beta B v) - k_\beta B^2 e_{\beta\beta}\right]\bar{k}$$

c) $\dfrac{\partial^2 \bar{r}'}{\partial \alpha \partial \beta} = \dfrac{\partial}{\partial \alpha}\left(\dfrac{\partial \bar{r}'}{\partial \beta}\right)$

Now;

$$\frac{\partial}{\partial \alpha}\left(\frac{\partial \bar{r}'}{\partial \beta}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial \bar{r}}{\partial \beta}\right) + \frac{\partial}{\partial \alpha}\left(\frac{\partial \bar{\Phi}}{\partial \beta}\right)$$

But

$$\frac{\partial}{\partial \alpha}\left(\frac{\partial \bar{r}}{\partial \beta}\right) = \frac{\partial A}{\partial \beta}\bar{i} + \frac{\partial B}{\partial \alpha}\bar{j}$$

And;

$$\frac{\partial}{\partial \alpha}\left(\frac{\partial \bar{\Phi}}{\partial \beta}\right) = \left[\frac{\partial^2 u}{\partial \alpha \partial \beta} - \frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial B}{\partial \alpha} v\right)\right]\bar{i} + \frac{\partial}{\partial \alpha}(B e_{\beta\beta})\bar{j} + \left[\frac{\partial^2 w}{\partial \alpha \partial \beta} - \frac{\partial}{\partial \alpha}(k_\beta B v)\right]\bar{k}$$

$$+ \left[-\frac{1}{B}\frac{\partial A}{\partial \beta}\frac{\partial u}{\partial \beta} + \frac{1}{AB}\frac{\partial A}{\partial \beta}\frac{\partial B}{\partial \alpha} v\right]\bar{j} + \left[-k_\alpha A \frac{\partial u}{\partial \beta} + k_\alpha \frac{\partial B}{\partial \alpha} v\right]\bar{k}$$

$$+ \frac{\partial A}{\partial \beta} e_{\beta\beta}\bar{i} + \left[k_\alpha A \frac{\partial w}{\partial \beta} - k_\alpha k_\beta A B v\right]\bar{i}$$

Combining;

$$\frac{\partial}{\partial\alpha}\left(\frac{\partial\bar{\Phi}}{\partial\beta}\right)=\left[\frac{\partial^2 u}{\partial\alpha\partial\beta}-\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}v\right)+k_\alpha A\frac{\partial w}{\partial\beta}-k_\alpha k_\beta ABv+\frac{\partial A}{\partial\beta}e_{\beta\beta}\right]\bar{i}$$
$$+\left[\frac{\partial}{\partial\alpha}(Be_{\beta\beta})-\frac{1}{B}\frac{\partial A}{\partial\beta}\frac{\partial u}{\partial\beta}+\frac{1}{AB}\frac{\partial A}{\partial\beta}\frac{\partial B}{\partial\alpha}v\right]\bar{j}+\left[\frac{\partial^2 w}{\partial\alpha\partial\beta}-\frac{\partial}{\partial\alpha}(k_\beta Bv)\right.$$
$$\left.-k_\alpha A\frac{\partial u}{\partial\beta}+k_\alpha\frac{\partial B}{\partial\alpha}v\right]\bar{k}$$

Hence;

$$\frac{\partial^2\bar{r}'}{\partial\alpha\partial\beta}=\left[\frac{\partial A}{\partial\beta}+\frac{\partial^2 u}{\partial\alpha\partial\beta}-\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}v\right)+k_\alpha A\frac{\partial w}{\partial\beta}-k_\alpha k_\beta ABv+\frac{\partial A}{\partial\beta}e_{\beta\beta}\right]\bar{i}$$
$$+\left[\frac{\partial B}{\partial\alpha}+\frac{\partial}{\partial\alpha}(Be_{\beta\beta})-\frac{1}{B}\frac{\partial A}{\partial\beta}\frac{\partial u}{\partial\beta}+\frac{1}{AB}\frac{\partial A}{\partial\beta}\frac{\partial B}{\partial\alpha}v\right]\bar{j}+\left[\frac{\partial^2 w}{\partial\alpha\partial\beta}-\frac{\partial}{\partial\alpha}(k_\beta Bv)\right.$$
$$\left.-k_\alpha A\frac{\partial u}{\partial\beta}+k_\alpha\frac{\partial B}{\partial\alpha}v\right]\bar{k}$$

Now for continuity

$$\frac{\partial^2\bar{r}'}{\partial\alpha\partial\beta}=\frac{\partial^2\bar{r}'}{\partial\beta\partial\alpha}$$

Consider then evaluating the second combination.

$$\frac{\partial}{\partial\beta}\left(\frac{\partial\bar{r}'}{\partial\alpha}\right)=\frac{\partial}{\partial\beta}\left(\frac{\partial\bar{r}}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{\partial\bar{\Phi}}{\partial\alpha}\right)$$

Now;

$$\frac{\partial}{\partial\beta}\left(\frac{\partial\bar{r}}{\partial\alpha}\right)=\frac{\partial A}{\partial\beta}\bar{i}+\frac{\partial B}{\partial\alpha}\bar{j}$$

And;

$$\frac{\partial}{\partial\beta}\left(\frac{\partial\bar{\Phi}}{\partial\alpha}\right)=\frac{\partial}{\partial\beta}(Ae_{\alpha\alpha})\bar{i}+\left[\frac{\partial^2 v}{\partial\alpha\partial\beta}-\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}u\right)\right]\bar{j}+\left[\frac{\partial^2 w}{\partial\alpha\partial\beta}-\frac{\partial}{\partial\beta}(k_\alpha Au)\right]\bar{k}$$
$$+\frac{\partial B}{\partial\alpha}e_{\alpha\alpha}\bar{j}+\left[-\frac{1}{A}\frac{\partial B}{\partial\alpha}\frac{\partial v}{\partial\alpha}+\frac{1}{AB}\frac{\partial A}{\partial\beta}\frac{\partial B}{\partial\alpha}u\right]\bar{i}+\left[-k_\beta B\frac{\partial v}{\partial\alpha}+\right.$$
$$\left.k_\beta\frac{\partial A}{\partial\beta}u\right]\bar{k}+\left[k_\beta B\frac{\partial w}{\partial\alpha}-k_\alpha k_\beta ABu\right]\bar{j}$$

Combining;

$$\frac{\partial}{\partial\beta}\left(\frac{\partial\bar{r}'}{\partial\alpha}\right)=\left[\frac{\partial A}{\partial\beta}+\frac{\partial}{\partial\beta}(Ae_{\alpha\alpha})-\frac{1}{A}\frac{\partial B}{\partial\alpha}\frac{\partial v}{\partial\alpha}+\frac{1}{AB}\frac{\partial A}{\partial\beta}\frac{\partial B}{\partial\alpha}u\right]\bar{i}+\left[\frac{\partial B}{\partial\alpha}+\frac{\partial B}{\partial\alpha}e_{\alpha\alpha}\right.$$
$$\left.+\frac{\partial^2 v}{\partial\alpha\partial\beta}-\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}u\right)+k_\beta B\frac{\partial w}{\partial\alpha}-k_\alpha k_\beta ABu\right]\bar{j}+\left[\frac{\partial^2 w}{\partial\alpha\partial\beta}-\frac{\partial}{\partial\beta}(k_\alpha Au)\right.$$
$$\left.-k_\beta B\frac{\partial v}{\partial\alpha}+k_\beta\frac{\partial A}{\partial\beta}u\right]\bar{k}$$

Inspection of the two expressions for $\left(\partial^2 \bar{r}'/\partial\alpha\partial\beta\right)$ indicates that one may be derived from the other by an interchange of letters. However, as might be expected, the resultant expression should be symmetric.

Hence letting;

$$\frac{\partial^2 \bar{r}'}{\partial\alpha\partial\beta} = \frac{1}{2}\left[\frac{\partial}{\partial\alpha}\left(\frac{\partial \bar{r}'}{\partial\beta}\right) + \frac{\partial}{\partial\beta}\left(\frac{\partial \bar{r}'}{\partial\alpha}\right)\right]$$

Then the final form for the mixed derivative becomes;

$$2\frac{\partial^2 \bar{r}'}{\partial\alpha\partial\beta} = \left[2\frac{\partial A}{\partial\beta} + \frac{\partial^2 u}{\partial\alpha\partial\beta} - \frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}v\right) + k_\alpha A\frac{\partial w}{\partial\beta} - k_\alpha k_\beta ABv + \frac{\partial A}{\partial\beta}e_{\beta\beta} + \frac{\partial}{\partial\beta}(Ae_{\alpha\alpha})\right.$$

$$\left. - \frac{1}{A}\frac{\partial B}{\partial\alpha}\frac{\partial v}{\partial\alpha} + \frac{1}{AB}\frac{\partial A}{\partial\beta}\frac{\partial B}{\partial\alpha}u\right]\bar{i} + \left[2\frac{\partial B}{\partial\alpha} + \frac{\partial}{\partial\alpha}(Be_{\beta\beta}) - \frac{1}{B}\frac{\partial A}{\partial\beta}\frac{\partial u}{\partial\beta} + \frac{1}{AB}\frac{\partial A}{\partial\beta}\frac{\partial B}{\partial\alpha}v\right.$$

$$\left. + \frac{\partial B}{\partial\alpha}e_{\alpha\alpha} + \frac{\partial^2 v}{\partial\alpha\partial\beta} - \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}u\right) + k_\beta B\frac{\partial w}{\partial\alpha} - k_\alpha k_\beta ABu\right]\bar{j} + \left[2\frac{\partial^2 w}{\partial\alpha\partial\beta}\right.$$

$$\left. - \frac{\partial}{\partial\alpha}(k_\beta Bv) - k_\alpha A\frac{\partial u}{\partial\beta} + k_\alpha\frac{\partial B}{\partial\alpha}v - \frac{\partial}{\partial\beta}(k_\alpha Au) - k_\beta B\frac{\partial v}{\partial\alpha} + k_\beta\frac{\partial A}{\partial\beta}u\right]\bar{k}$$

Consider finding the coefficients L, M and N. In doing so, only terms linear in the displacements will be retained.

a) L

$$L = \frac{\partial^2 \bar{r}'}{\partial\alpha^2}\cdot\bar{n}'$$

Substituting;

$$L = \frac{1}{\sqrt{EG-F^2}}\left[k_\alpha AB\frac{\partial A}{\partial\alpha}u - B\frac{\partial A}{\partial\alpha}\frac{\partial w}{\partial\alpha} - k_\beta A^2\frac{\partial A}{\partial\beta}v + \frac{A^2}{B}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta} - k_\alpha A^3B\right.$$

$$\left. + AB\frac{\partial^2 w}{\partial\alpha^2} - AB\frac{\partial}{\partial\alpha}(k_\alpha Au) - 2k_\alpha A^3Be_{\alpha\alpha} - k_\alpha A^3Be_{\beta\beta}\right]$$

Consider now evaluating $\sqrt{EG-F^2}$. As found on the section on the first quadratic form;

$$E = A^2(1+2e_{\alpha\alpha})\ ;\quad F = ABe_{\alpha\beta}\ ;\quad G = B^2(1+2e_{\beta\beta})$$

Forming the products and linearizing in the displacements;

$$\sqrt{EG-F^2} = AB\left[1+2\,(e_{\alpha\alpha}+e_{\beta\beta})\right]^{1/2}$$

and hence by the binomial expression;

$$\frac{1}{\sqrt{EG-F^2}} = \frac{1}{AB}\left[1-(e_{\alpha\alpha}+e_{\beta\beta})\right]$$

Thus the linearized expression for L becomes;

$$L = -k_\alpha A^2 + \frac{\partial^2 w}{\partial \alpha^2} - \frac{1}{A}\frac{\partial A}{\partial \alpha}\frac{\partial w}{\partial \alpha} + \frac{A}{B^2}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \beta} - k_\beta \frac{A}{B}\frac{\partial A}{\partial \beta} v - A\frac{\partial}{\partial \alpha}(k_\alpha u)$$
$$- k_\alpha A^2 e_{\alpha\alpha}$$

b) M

$$M = \frac{\partial^2 \bar{x}'}{\partial \alpha \partial \beta}\cdot \bar{n}'$$

Substituting;

$$M = \frac{1}{\sqrt{EG-F^2}}\Big[k_\alpha AB\frac{\partial A}{\partial \beta}u - B\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \alpha} + k_\beta AB\frac{\partial B}{\partial \alpha}v - A\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \beta}$$
$$+ AB\frac{\partial^2 w}{\partial \alpha \partial \beta} - \frac{AB}{2}\frac{\partial}{\partial \alpha}(k_\beta B v) - \frac{k_\alpha A^2 B}{2}\frac{\partial u}{\partial \beta} + \frac{k_\alpha AB}{2}\frac{\partial B}{\partial \alpha}v - \frac{AB}{2}\frac{\partial}{\partial \beta}(k_\alpha A u)$$
$$- \frac{k_\beta AB^2}{2}\frac{\partial v}{\partial \alpha} + \frac{k_\beta AB}{2}\frac{\partial A}{\partial \beta}u\Big]$$

Or substituting for $\sqrt{EF-G^2}$;

$$M = \frac{\partial^2 w}{\partial \alpha \partial \beta} - \frac{1}{B}\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \beta} - \frac{1}{A}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \alpha} + k_\alpha\frac{\partial A}{\partial \beta}u + k_\beta\frac{\partial B}{\partial \alpha}v - \frac{1}{2}\frac{\partial}{\partial \alpha}(k_\beta B v)$$
$$- \frac{1}{2}\frac{\partial}{\partial \beta}(k_\alpha A u) + \frac{k_\alpha}{2}\frac{\partial B}{\partial \alpha}v + \frac{k_\beta}{2}\frac{\partial A}{\partial \beta}u - \frac{k_\alpha A}{2}\frac{\partial u}{\partial \beta} - \frac{k_\beta B}{2}\frac{\partial v}{\partial \alpha}$$

The terms involving the tangential displacements u & v may be considerably simplified. Thus, expanding and using Codazzi conditions;

$$k_\alpha\frac{\partial A}{\partial \beta}u + k_\beta\frac{\partial B}{\partial \alpha}v - \frac{k_\alpha}{2}\frac{\partial B}{\partial \alpha}v - \frac{k_\beta B}{2}\frac{\partial v}{\partial \alpha} - \frac{k_\beta}{2}\frac{\partial A}{\partial \beta}u - \frac{k_\alpha A}{2}\frac{\partial u}{\partial \beta} + \frac{k_\alpha}{2}\frac{\partial B}{\partial \alpha}v$$
$$+ \frac{k_\beta}{2}\frac{\partial A}{\partial \beta}u - \frac{k_\alpha A}{2}\frac{\partial u}{\partial \beta} - \frac{k_\beta B}{2}\frac{\partial v}{\partial \alpha}$$

Combining;

$$k_\alpha\frac{\partial A}{\partial \beta}u + k_\beta\frac{\partial B}{\partial \alpha}v - k_\beta B\frac{\partial v}{\partial \alpha} - k_\alpha A\frac{\partial u}{\partial \beta}$$

Hence the expression for M becomes;

$$M = \frac{\partial^2 w}{\partial\alpha\,\partial\beta} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} + k_\alpha\left(\frac{\partial A}{\partial\beta}u - A\frac{\partial u}{\partial\beta}\right) + k_\beta\left(\frac{\partial B}{\partial\alpha}v - B\frac{\partial v}{\partial\alpha}\right)$$

N

$$N = \frac{\partial^2 \bar{R}'}{\partial\beta^2} \cdot \bar{n}'$$

Substituting;

$$N = \frac{1}{\sqrt{EG-F^2}}\Big[-k_\alpha B^2\frac{\partial B}{\partial\alpha}u + \frac{B^2}{A}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha} + k_\beta AB\frac{\partial B}{\partial\beta}v - A\frac{\partial B}{\partial\beta}\frac{\partial w}{\partial\beta} - k_\beta AB^3 + AB\frac{\partial^2 w}{\partial\beta^2} - AB\frac{\partial}{\partial\beta}(k_\beta Bv) - k_\beta AB^3 e_{\beta\beta} - k_\beta AB^3 e_{\alpha\alpha} - k_\beta AB^3 e_{\beta\beta}\Big]$$

Substituting in the value of $\sqrt{EG-F^2}$

$$N = -k_\beta B^2 + \frac{\partial^2 w}{\partial\beta^2} + \frac{B}{A^2}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha} - \frac{1}{B}\frac{\partial B}{\partial\beta}\frac{\partial w}{\partial\beta} + k_\beta\frac{\partial B}{\partial\beta}v - k_\alpha\frac{B}{A}\frac{\partial B}{\partial\alpha}u - \frac{\partial}{\partial\beta}(k_\beta Bv) - k_\beta AB^2 e_{\beta\beta}$$

Or recombining the tangential displacement terms

$$N = -k_\beta B^2 + \frac{\partial^2 w}{\partial\beta^2} + \frac{B}{A^2}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha} - \frac{1}{B}\frac{\partial B}{\partial\beta}\frac{\partial w}{\partial\beta} - k_\alpha\frac{B}{A}\frac{\partial B}{\partial\alpha}u - B\frac{\partial}{\partial\beta}(k_\beta v) - k_\beta\frac{B^2}{A}e_{\beta\beta}$$

Define the following quantities;

$$\mathcal{K}_\alpha = -\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} + \frac{1}{A^3}\frac{\partial A}{\partial\alpha}\frac{\partial w}{\partial\alpha} - \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta} + \frac{1}{A}\frac{\partial}{\partial\alpha}(k_\alpha u) + \frac{k_\beta}{AB}\frac{\partial A}{\partial\beta}v$$

$$\mathcal{K}_\beta = -\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2} + \frac{1}{B^3}\frac{\partial B}{\partial\beta}\frac{\partial w}{\partial\beta} - \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha} + \frac{1}{B}\frac{\partial}{\partial\beta}(k_\beta v) + \frac{k_\alpha}{AB}\frac{\partial B}{\partial\alpha}u$$

$$\Upsilon = -\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta} + \frac{1}{A^2B}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} + \frac{1}{AB^2}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta} + k_\alpha\left(\frac{1}{B}\frac{\partial u}{\partial\beta} - \frac{1}{AB}\frac{\partial A}{\partial\beta}u\right) + k_\beta\left(\frac{1}{A}\frac{\partial v}{\partial\alpha} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}v\right)$$

The coefficients of the second quadratic form may then be written as;

$$L = -A^2\left[k_\alpha(1+e_{\alpha\alpha}) + \mathcal{K}_\alpha\right]$$

$$M = -AB\Upsilon$$

$$N = -B^2\left[k_\beta(1+e_{\beta\beta}) + \mathcal{K}_\beta\right]$$

Hence the second quadratic form becomes;

$$II = -A^2\left[k_\alpha(1+e_{\alpha\alpha})+K_\alpha\right](d\alpha)^2 - 2AB\gamma(d\alpha)(d\beta) - B^2\left[k_\beta(1+e_{\beta\beta})+K_\beta\right](d\beta)^2$$

<u>Summary;</u>

Defining;

$$\begin{cases}
e_{\alpha\alpha} = \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v + k_\alpha w \\
e_{\alpha\beta} = \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right) \\
e_{\beta\beta} = \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}u + k_\beta w \\
K_\alpha = -\frac{1}{A}\frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial w}{\partial \alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \beta} + \frac{1}{A}\frac{\partial}{\partial \alpha}(k_\alpha u) + \frac{k_\beta}{AB}\frac{\partial A}{\partial \beta}v \\
K_\beta = -\frac{1}{B}\frac{\partial}{\partial \beta}\left(\frac{1}{B}\frac{\partial w}{\partial \beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \alpha} + \frac{1}{B}\frac{\partial}{\partial \beta}(k_\beta v) + \frac{k_\alpha}{AB}\frac{\partial B}{\partial \alpha}u \\
\gamma = -\frac{1}{AB}\frac{\partial^2 w}{\partial \alpha \partial \beta} + \frac{1}{A^2B}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \alpha} + \frac{1}{AB^2}\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \beta} + k_\alpha\frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) + k_\beta\frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right)
\end{cases}$$

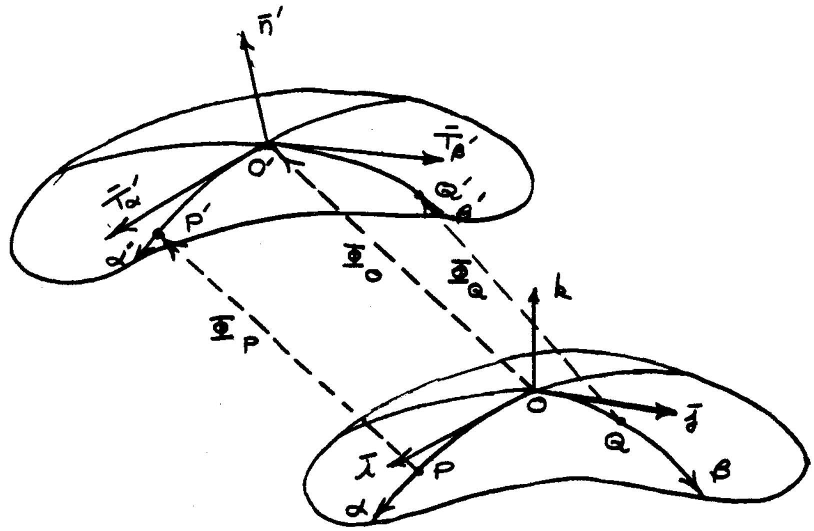
The first and second quadratic forms become

$$\begin{cases}
I = A^2(1+2e_{\alpha\alpha})(d\alpha)^2 + 2AB\,e_{\alpha\beta}(d\alpha)(d\beta) + B^2(1+2e_{\beta\beta})(d\beta)^2 \\
II = -A^2\left[k_\alpha(1+e_{\alpha\alpha})+K_\alpha\right](d\alpha)^2 - 2AB\gamma(d\alpha)(d\beta) - B^2\left[k_\beta(1+e_{\beta\beta})+K_\beta\right](d\beta)^2
\end{cases}$$

3. 3. Middle Surface Deformations

Consider again the deformed and undeformed surfaces of the previous section. On both of these surfaces there are α and β coordinate lines. In particular, this set of curves are the principal curvalinear coordinates of the undeformed surface and hence are orthogonal to each other. Further, the tangent vectors and the normal, $\bar{T}_\alpha$, $\bar{T}_\beta$, $\bar{n}$, of the undeformed surface formed an orthonormal triad of vectors, , , .

The case of the deformed surface is different. For one thing, there still exists a set of curvalinear coordinate lines on its surface which correspond to the α and β parametrization of undeformed surface. However, inspection of the first quadratic form of the deformed surface shows that these coordinate lines are not orthogonal to each other and hence are not principal curvalinear coordinates of the deformed surface. Thus the tangent vectors to the curvalinear coordinate lines of the deformed surface are not orthogonal, and, strictly speaking, an orthonormal triad cannot be constructed on the deformed surface using tangent vectors. The situation is shown on the accompanying sketch. Three points on the undeformed surface are shown as O, P, Q and these points on the deformed surface are noted as O', P', Q'. Note that each point on the deformed surface has a corresponding image point on the undeformed surface.



3. 4. Normal Strain in the α and β directions.

If the points O & P lie on the α coordinate line for the undeformed surface, then they will also lie on the α coordinate line of the deformed surface.

Define the normal strain in α direction;

$$\frac{\Delta S_\alpha' - \Delta S_\alpha}{\Delta S_\alpha} \ \text{Limit}_{\Delta S_\alpha \to 0}$$

Now from the first quadratic form of the undeformed surface;

$$dS_\alpha = A\, d\alpha$$

And the length from the first quadratic form of the deformed surface;

$$dS_\alpha' = A\sqrt{(1+2e_{\alpha\alpha})}\, d\alpha$$

Expanding by the binomial theorem and linearizing;

$$dS_\alpha' = A(1+e_{\alpha\alpha})\, d\alpha$$

Hence;

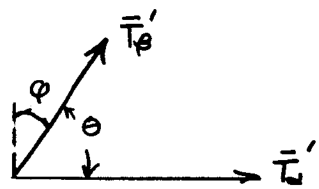
$$\frac{\Delta S_\alpha' - \Delta S_\alpha}{\Delta S_\alpha} \ \text{limit}_{\Delta S_\alpha \to 0} = e_{\alpha\alpha}$$

It is obvious that the normal strain in the β direction will be given as;

$$e_{\beta\beta} = \frac{\Delta S_\beta' - \Delta S_\beta}{\Delta S_\beta} \quad \lim_{\Delta S_\beta \to 0}$$

3. 5. Shear strain in the α-β directions.

As pointed, the tangent vectors on the undeformed surface, the $\bar{i}$ and $\bar{j}$ vectors, are initially orthogonal. As the surface deforms, this condition is no longer realized. The shear strain is defined as the tangent of the angular change between two initially orthogenal lines. Consider then the plane of the $\bar{T}_\alpha'$ and $\bar{T}_\beta'$ vectors.



Hence;

shear strain $= \tan\varphi$

But if the angle is small enough as is assumed in the present theory, then;

$$\tan\varphi \approx \varphi \approx \sin\varphi$$

But

$$\sin\varphi = \cos\theta$$

Thus;

$$\cos\theta \approx \text{shear strain}$$

Since $\bar{T}_\alpha'$ and $\bar{T}_\beta'$ are unit vectors, then;

$$\cos\theta = \bar{T}_\alpha' \cdot \bar{T}_\beta'$$

As has been shown in a previous section, for curvalinear coordinates

$$\cos\theta = \frac{F}{\sqrt{EG}}$$

Substituting;

$$EG = A^2B^2(1+2e_{\alpha\alpha})(1+2e_{\beta\beta})$$

$$F = AB\,e_{\alpha\beta}$$

Expanding;

$$\frac{F}{\sqrt{EG}} = \frac{e_{\alpha\beta}}{\sqrt{1+2(e_{\alpha\alpha}+e_{\beta\beta})}}$$

Linearizing by the binomial theorem;

$$\cos\theta = e_{\alpha\beta} = \text{Shear strain}$$

3. 6. Curvature Change and Torsion Expressions

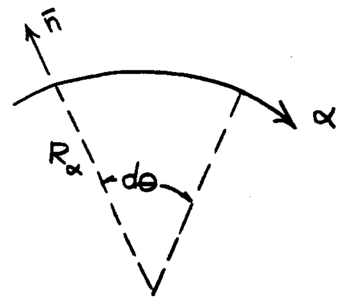
The components of deformation, $e_{\alpha\alpha}$, $e_{\alpha\beta}$, $e_{\beta\beta}$ measure the change in length dimensions but do nothing in describing the altered shell geometry. These latter quantities are measured by curvatures and a quantity called twist which as yet has to be defined. Consider now evaluating these parameters.

3. 7. Curvature Change

The curvature of a surface in a particular direction has been shown to be;

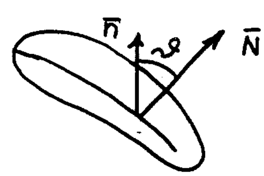
$$k = \frac{II}{I} = \frac{L(d\alpha)^2 + 2M(d\alpha)(d\beta) + N(d\beta)^2}{E(d\alpha)^2 + 2F(d\alpha)(d\beta) + G(d\beta)^2}$$

Before the above expression is applied, one important thing should be noted. The radius of curvature of say the α coordinate line was shown as;



Now according to the derivatives of the unit vectors R_α was chosen positive as shown, that is, the curvature was opposite to the positive $\bar{n}$ direction. Now in the definition of the curvature of a surface, the direction of the normal to the surface $\bar{n}$ and the principal normal $\bar{N}$ to a surface curve were used. Letting k_s be the curvature of the surface and k_c the curvature of the curve,

$$k_s = k_c \; (\bar{N} \cdot \bar{n})$$



Now in the shell assumptions used, the principal normal to the line and the normal to the surface made an obtuse angle with respect to each other so that $90^\circ \leq \vartheta \leq 180^\circ$. But if the above definition were used, then k_s would be negative, whereas it actually was chosen as positive. Thus; for the sign convention;

$$k = -\frac{1}{R}$$

3. 8. Curvature Change in the α Coordinate Direction.

The curvature of the deformed surface, $k_\alpha' = \frac{1}{R_\alpha'}$, in the α coordinate direction is defined as ($d\beta = 0$)

$$-k_\alpha' = \frac{-A^2[k_\alpha(1+e_{\alpha\alpha})+\kappa_\alpha](d\alpha)^2}{A^2(1+2e_{\alpha\alpha})(d\alpha)^2}$$

Simplifying and expanding the denominator by means of a binomial series;

$$k_\alpha' = [k_\alpha(1+e_{\alpha\alpha})+\kappa_\alpha](1-2e_{\alpha\alpha})$$

Expanding;

$$k_\alpha' = k_\alpha + k_\alpha e_{\alpha\alpha} + \kappa_\alpha - 2k_\alpha e_{\alpha\alpha}$$

Hence;

$$\frac{1}{R_\alpha'} - \frac{1}{R_\alpha} = k_\alpha' - k_\alpha = \kappa_\alpha - k_\alpha e_{\alpha\alpha}$$

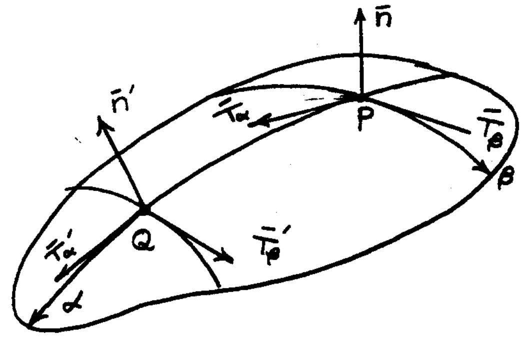
By direct analogy, the curvature change in the β direction is given as;

$$\frac{1}{R_\beta'} - \frac{1}{R_\beta} = \kappa_\beta - k_\beta e_{\beta\beta}$$

3. 9. Torsion or Twist

The first and the third coefficients of the second quadratic form have been explained and it was pointed out that these quantities represent the curvature changes of the surface from the deformed to the undeformed state. The problem now concerns the second or "M" term and its physical explaination. Note now that if principal curvalinear coordinates are not utilized then the first and second quadratic forms of a surface contain this middle term. Hence this term is not just limited to deformed surfaces but is connected to non-principal coordinization of surfaces.

Consider now a surface on which exist α and β coordinate curves. Assume that these curves are not in general principal curvalinear coordinate curves. The situation appears as shown.



The points P and Q are spaced an infinitesimal distance apart. Consider now the angular displacement of the vector $\bar{T}_\beta'$ in comparison with the vector $\bar{T}_\beta$. Since $\bar{T}_\beta$ and $\bar{T}_\beta'$ are both of unit length, then

$$\sin\theta = |\bar{T}_\beta' \times \bar{T}_\beta|$$

Since the vector T_β varies continously from point to point on the shell surface, it might be expected that in general there is an angular displacement. If now the points P and Q are chosen infinitesimally close to each other, then $\sin\theta \to 0$ and hence for small displacements, $\sin\theta \simeq \theta$.

If principal curvalinear coordinates were chosen for the and α & β coordinate lines, then $\bar{T}_\alpha \perp \bar{T}_\beta$ and further $\Delta\bar{n} \propto \Delta\lambda$. Effectively, this would mean that the only infinitesimal rotations allowable for $\bar{T}_\alpha'$, $\bar{T}_\beta'$ and $\bar{n}'$ vectors in going from P to Q would be rotations about the $\bar{n}$ axis. Rotations about the $\bar{T}_\alpha$ and $\bar{T}_\beta$ axis would then have to be associated with non-principal curvalinear coordinates for the α and β curves. It is in fact, this rotation that is now desired. In order to preclude the vanishing of the angle as Q→P, the result will be divided by the distance between P and Q namely dS_α .

Consider then the component of rotation along the α coordinate axis (in the $\bar{T}_\alpha$ direction). This quantity is given as;

$$(\bar{T}_\beta' \times \bar{T}_\beta) \cdot \frac{\bar{T}_\alpha}{dS_\alpha}$$

But if $\bar{T}_\beta'$ is continuous, then;

$$\bar{T}_\beta' = T_\beta + \frac{\partial \bar{T}_\beta}{\partial \alpha} d\alpha + \cdots$$

where now only first order differentials will be used.

Substituting;

$$\left[\left(\bar{T}_\beta + \frac{\partial \bar{T}_\beta}{\partial \alpha} d\alpha\right) \times \bar{T}_\beta\right] \cdot \frac{\bar{T}_\alpha}{dS_\alpha}$$

and expanding and further noting that $\bar{T}_\beta \times \bar{T}_\beta = 0$, the result becomes;

$$\left(\frac{d\alpha}{dS_\alpha}\right)\left(\frac{\partial \bar{T}_\beta}{\partial \alpha} \times \bar{T}_\beta\right) \cdot \bar{T}_\alpha$$

But from vectors, it is found that

$$(\bar{A} \times \bar{B}) \cdot \bar{C} = \bar{A} \cdot (\bar{B} \times \bar{C}) = \bar{B} \cdot (\bar{C} \times \bar{A})$$

Hence the above expression may be written as;

$$\frac{d\theta}{dS_\alpha} = \left(\frac{d\alpha}{dS_\alpha}\right)\left[\frac{\partial \bar{T}_\beta}{\partial \alpha} \cdot (\bar{T}_\beta \times \bar{T}_\alpha)\right]$$

But

$$\bar{T}_\alpha \times \bar{T}_\beta = \frac{d\alpha\, d\beta}{dS_\alpha\, dS_\beta}\left(\frac{\partial \bar{r}}{\partial \alpha} \times \frac{\partial \bar{r}}{\partial \beta}\right) = \sqrt{EG-F^2}\,\frac{(d\alpha\, d\beta)}{(dS_\alpha\, dS_\beta)}\,\bar{n}$$

Substituting;

$$\frac{d\theta}{dS_\alpha} = -\frac{(d\alpha)^2(d\beta)}{(dS_\alpha)^2(dS_\beta)}\sqrt{EG-F^2}\left(\frac{\partial \bar{T}_\beta}{\partial \alpha} \cdot \bar{n}\right)$$

However;

$$\frac{\partial \bar{T}_\beta}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left(\frac{\partial \bar{r}}{\partial S_\beta}\right) = \frac{d\beta}{dS_\beta}\,\frac{\partial^2 \bar{r}}{\partial \alpha\, \partial \beta}$$

and hence;

$$\frac{d\theta}{dS_\alpha} = -\frac{(d\alpha)^2(d\beta)^2}{(dS_\alpha)^2(dS_\beta)^2}\sqrt{EG-F^2}\left(\frac{\partial^2\bar{r}}{\partial\alpha\,\partial\beta}\cdot\bar{n}\right)$$

But $\left(\frac{\partial^2\bar{r}}{\partial\alpha\,\partial\beta}\cdot\bar{n}\right) = \bar{M}$; $dS_\alpha = \sqrt{E}\,d\alpha$; $dS_\beta = \sqrt{G}\,d\beta$ and thus

$$\frac{d\theta}{dS_\alpha} = -\frac{\sqrt{EG-F^2}}{EG}M$$

Note now that the above expression can be directly related to the non orthogenality of the coordinate curves. Letting the angle between $\bar{T}_\alpha$ and $\bar{T}_\beta$ be ψ , then as previously found;

$$\cos\psi = F/\sqrt{EG}$$

and hence the above expression may be written as;

$$\frac{d\theta}{dS_\alpha} = -\left(\frac{1}{\sqrt{EG}}\sin\psi\right)M$$

Consider now evaluating the above expression for the deformed surface; now;

$$E = A^2(1+2e_{\alpha\alpha}) \;;\qquad G = B^2(1+2e_{\beta\beta})$$

$$\sin\psi = \sqrt{1-e_{\alpha\beta}^2} = 1 - \tfrac{1}{2}e_{\alpha\beta}^2 + \cdots \approx 1$$

Hence;

$$\frac{1}{\sqrt{EG}} = \frac{1}{AB}\frac{1}{\sqrt{1+2(e_{\alpha\alpha}+e_{\beta\beta})}} = \frac{1}{AB}\left[1-(e_{\alpha\alpha}+e_{\beta\beta})\right]$$

Then;

$$\frac{d\theta}{dS_\alpha} = -\frac{1}{AB}\left[1-(e_{\alpha\alpha}+e_{\beta\beta})\right]M$$

But M is involved with displacements and their derivatives and so are the expressions for $e_{\alpha\alpha}$ and $e_{\beta\beta}$. Hence for linear results,

$$\frac{d\theta}{dS_\alpha} = -\frac{1}{AB}M$$

Substituting for M;

$$\frac{d\theta}{dS_\alpha} = \Upsilon$$

Note now the symmetry. Since γ is symmetric in A, B, α and β, then;

$$\frac{d\theta}{dS_\beta} = \gamma$$

Hence the same angular rotation of the vector $\bar{T}_\alpha$ about the $\bar{T}_\beta$ axis is experienced as for the vector $\bar{T}_\beta$ about the $\bar{T}_\alpha$ axis.

3. 10. Peterson-Codazzi-Gauss Equations (Compatibility).

When dealing with the undeformed surface, or more properly, a surface defined in principal curvalinear coordinates, the first and second quadratic forms of the surface took the form.

$$I = A^2(d\alpha)^2 + B^2(d\beta)^2$$

$$II = -k_\alpha A^2(d\alpha)^2 - k_\beta B^2(d\beta)^2$$

where in the second quadratic form, the sign of k_α and k_β has been chosen in accordance with the derivations subsequently used (i.e., $k_S = -k_q(\bar{N}\cdot\bar{n})$). Now the condition on the unit vectors was that;

$$\frac{\partial^2 \bar{i}}{\partial\alpha\partial\beta} = \frac{\partial^2 \bar{i}}{\partial\beta\partial\alpha} \; ; \; \frac{\partial^2 \bar{j}}{\partial\alpha\partial\beta} = \frac{\partial^2 \bar{j}}{\partial\beta\partial\alpha} \; ; \; \frac{\partial^2 \bar{k}}{\partial\alpha\partial\beta} = \frac{\partial^2 \bar{k}}{\partial\beta\partial\alpha}$$

and the result was the three equations;

$$\frac{\partial}{\partial\beta}(k_\alpha A) = k_\beta \frac{\partial A}{\partial\beta}$$

$$\frac{\partial}{\partial\alpha}(k_\beta B) = k_\alpha \frac{\partial B}{\partial\alpha}$$

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) = -k_\alpha k_\beta AB$$

The first two of the equations were called the conditions of Codazzi while the latter was defined as the condition of Gauss. Now these conditions may be looked upon as differentiability conditions, but note that they are involved with the coefficients of the second and quadratic forms of the surface. Thus these conditions yield the relations between the coefficients of the first and second quadratic form of the surface.

Consider now a surface parameterized by some α and β which lead to arbitrary α and β coordinate curves. It is now desired to find the relations between the coefficients of its first and second quadratic forms. To do this, the conditions on $\bar{i}, \bar{j}$ and $\bar{k}$ vectors cannot be utilized since in general, the curvalinear coordinates are non-orthogenal. However, remembering that

$$\bar{i} = \frac{1}{A}\frac{\partial \bar{r}}{\partial \alpha} \; ; \quad \bar{j} = \frac{1}{B}\frac{\partial \bar{r}}{\partial \beta} \; ; \quad \bar{k} = \bar{i} \times \bar{j}$$

then it is evident that cross derivatives of the unit vectors are really conditions on the third derivatives of $\bar{r}$. Thus for a general shell, the compatibility conditions take the form.

$$\frac{\partial}{\partial \beta}\left(\frac{\partial^2 \bar{r}}{\partial \alpha^2}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial^2 \bar{r}}{\partial \alpha \partial \beta}\right)$$

$$\frac{\partial}{\partial \alpha}\left(\frac{\partial^2 \bar{r}}{\partial \beta^2}\right) = \frac{\partial}{\partial \beta}\left(\frac{\partial^2 \bar{r}}{\partial \alpha \partial \beta}\right)$$

$$\frac{\partial}{\partial \beta}\left(\frac{\partial \bar{n}}{\partial \alpha}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial \bar{n}}{\partial \beta}\right)$$

where again;

$$\bar{n} = \frac{1}{\sqrt{EG-F^2}}\left(\frac{\partial \bar{r}}{\partial \alpha} \times \frac{\partial \bar{r}}{\partial \beta}\right)$$

Now by applying the above conditions, it can be shown that nine equations result, three of which become distinctive. These equations are given as;

$$(EG-2FF+GE)(L_\beta - M_\alpha) - (EN-2FM+GL)(E_\beta - F_\alpha) + \begin{vmatrix} E & E_\alpha & L \\ F & F_\alpha & M \\ G & G_\alpha & N \end{vmatrix} = 0$$

$$(EG-2FF+GE)(M_\beta - N_\alpha) - (EN-2FM+GL)(F_\beta - G_\alpha) + \begin{vmatrix} E & E_\beta & L \\ F & F_\beta & M \\ G & G_\beta & N \end{vmatrix} = 0$$

$$
\left\{ \begin{vmatrix} \left(-\tfrac{1}{2}G_{\alpha\alpha}+F_{\alpha\beta}-\tfrac{1}{2}E_{\beta\beta}\right) & \tfrac{1}{2}E_\alpha & \left(F_\alpha-\tfrac{1}{2}E_\beta\right) \\ \left(F_\beta-\tfrac{1}{2}G_\alpha\right) & E & F \\ \tfrac{1}{2}G_\beta & F & G \end{vmatrix} - \begin{vmatrix} 0 & \tfrac{1}{2}E_\beta & \tfrac{1}{2}G_\alpha \\ \tfrac{1}{2}E_\beta & E & F \\ \tfrac{1}{2}G_\alpha & F & G \end{vmatrix} \right\}
$$

$$
= \left(EG-F^2\right)^2 k_\alpha k_\beta
$$

In the above expressions, the subscript of a letter means differentiation with respect to that coordinate. The above equations are termed the Gauss, Peterson-Codazzi equations and do reduce to those previously defined for a surface parameterized with principal curvalinear coordinate curves.

Substituting;

$$
E = A^2(1+2e_{\alpha\alpha}) \; ; \quad F = AB\,e_{\alpha\beta} \; ; \quad G = B^2(1+2e_{\beta\beta})
$$

$$
L = -A^2\left[k_\alpha(1+e_{\alpha\alpha})+K_\alpha\right] ; \quad M = -AB\gamma \; ; \quad N = -B^2\left[k_\beta(1+e_{\beta\beta})+K_\beta\right]
$$

and linearizing the equations in the deformations, only three independent equations result. These equations are termed the compatibility equations of the deformed surface and are given as;

$$
A\frac{\partial K_\alpha}{\partial\beta} + \frac{\partial A}{\partial\beta}(K_\alpha - K_\beta) - B\frac{\partial\gamma}{\partial\alpha} - 2\frac{\partial B}{\partial\alpha}\gamma + k_\alpha\frac{\partial B}{\partial\alpha}e_{\alpha\beta} + k_\beta\Big[B\frac{\partial e_{\alpha\beta}}{\partial\alpha} + \frac{\partial B}{\partial\alpha}e_{\alpha\beta}
$$

$$
- A\frac{\partial e_{\alpha\alpha}}{\partial\beta} - \frac{\partial A}{\partial\beta}(e_{\alpha\alpha} - e_{\beta\beta})\Big] = 0
$$

$$
B\frac{\partial K_\beta}{\partial\alpha} + \frac{\partial B}{\partial\alpha}(K_\beta - K_\alpha) - A\frac{\partial\gamma}{\partial\beta} - 2\frac{\partial A}{\partial\beta}\gamma + k_\beta\frac{\partial A}{\partial\beta}e_{\alpha\beta} + k_\beta\Big[A\frac{\partial e_{\alpha\beta}}{\partial\beta} + \frac{\partial A}{\partial\beta}e_{\alpha\beta}
$$

$$
- B\frac{\partial e_{\beta\beta}}{\partial\alpha} - \frac{\partial B}{\partial\alpha}(e_{\beta\beta} - e_{\alpha\alpha})\Big] = 0
$$

$$
k_\beta K_\alpha + k_\alpha K_\beta + \frac{1}{AB}\Big\{\frac{\partial}{\partial\alpha}\cdot\frac{1}{A}\Big[B\frac{\partial e_{\beta\beta}}{\partial\alpha} + \frac{\partial B}{\partial\alpha}(e_{\beta\beta} - e_{\alpha\alpha}) - \frac{A}{2}\frac{\partial e_{\alpha\beta}}{\partial\beta} - \frac{\partial A}{\partial\beta}e_{\alpha\beta}\Big]
$$

$$
+ \frac{\partial}{\partial\beta}\cdot\frac{1}{B}\Big[A\frac{\partial e_{\alpha\alpha}}{\partial\beta} + \frac{\partial A}{\partial\beta}(e_{\alpha\alpha} - e_{\beta\beta}) - \frac{B}{2}\frac{\partial e_{\alpha\beta}}{\partial\alpha} - \frac{\partial B}{\partial\alpha}e_{\alpha\beta}\Big]\Big\} = 0
$$

3. 11. General Summary and Conclusions

1. Undeformed Surface

   a. First and second quadratic forms

$$\begin{cases} I = A^2(d\alpha)^2 + B^2(d\beta)^2 \\ II = -k_\alpha A^2(d\alpha)^2 - k_\beta B^2(d\beta)^2 \end{cases}$$

   b. Compatibility conditions

$$\left.\begin{bmatrix} \dfrac{\partial}{\partial\alpha}(k_\beta B) = k_\alpha \dfrac{\partial B}{\partial\alpha} \\[2ex] \dfrac{\partial}{\partial\beta}(k_\alpha A) = k_\beta \dfrac{\partial A}{\partial\beta} \end{bmatrix}\right\}$$

Conditions of Codazzi-Peterson

$$\left[\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) = -k_\alpha k_\beta AB\right.$$

Conditions of Gauss

2. Deformed Surface

   a. First and second quadratic forms

$$\begin{cases} I = A^2(1+2e_{\alpha\alpha})(d\alpha)^2 + 2AB\,e_{\alpha\beta}(d\alpha)(d\beta) + B^2(1+2e_{\beta\beta})(d\beta)^2 \\ II = -A^2\left[k_\alpha(1+e_{\alpha\alpha}) + K_\alpha\right](d\alpha)^2 - 2AB\tau(d\alpha)(d\beta) - B^2\left[k_\beta(1+e_{\beta\beta}) + K_\beta\right](d\beta)^2 \end{cases}$$

   b. Compatibility conditions

$$\begin{cases} B\dfrac{\partial K_\beta}{\partial\alpha} + \dfrac{\partial B}{\partial\alpha}(K_\beta - K_\alpha) - A\dfrac{\partial\tau}{\partial\beta} - 2\dfrac{\partial A}{\partial\beta}\tau + k_\beta\dfrac{\partial A}{\partial\beta}e_{\alpha\beta} + k_\alpha\Big[A\dfrac{\partial e_{\alpha\beta}}{\partial\beta} \\[2ex] \quad + \dfrac{\partial A}{\partial\beta}e_{\alpha\beta} - B\dfrac{\partial e_{\beta\beta}}{\partial\alpha} - \dfrac{\partial B}{\partial\alpha}(e_{\beta\beta} - e_{\alpha\alpha})\Big] = 0 \\[2ex] A\dfrac{\partial K_\alpha}{\partial\beta} + \dfrac{\partial A}{\partial\beta}(K_\alpha - K_\beta) - B\dfrac{\partial\tau}{\partial\alpha} - 2\dfrac{\partial B}{\partial\alpha}\tau + k_\alpha\dfrac{\partial B}{\partial\alpha}e_{\alpha\beta} + k_\beta\Big[B\dfrac{\partial e_{\alpha\beta}}{\partial\alpha} \\[2ex] \quad + \dfrac{\partial B}{\partial\alpha}e_{\alpha\beta} - A\dfrac{\partial e_{\alpha\alpha}}{\partial\beta} - \dfrac{\partial A}{\partial\beta}(e_{\alpha\alpha} - e_{\beta\beta})\Big] = 0 \\[2ex] k_\beta K_\alpha + k_\alpha K_\beta + \dfrac{1}{AB}\Big\{\dfrac{\partial}{\partial\alpha}\cdot\dfrac{1}{A}\Big[B\dfrac{\partial e_{\beta\beta}}{\partial\alpha} + \dfrac{\partial B}{\partial\alpha}(e_{\beta\beta} - e_{\alpha\alpha}) - \dfrac{A}{2}\dfrac{\partial e_{\alpha\beta}}{\partial\beta} \\[2ex] \quad - \dfrac{\partial A}{\partial\beta}e_{\alpha\beta}\Big] + \dfrac{\partial}{\partial\beta}\cdot\dfrac{1}{B}\Big[A\dfrac{\partial e_{\alpha\alpha}}{\partial\beta} + \dfrac{\partial A}{\partial\beta}(e_{\alpha\alpha} - e_{\beta\beta}) - \dfrac{B}{2}\dfrac{\partial e_{\alpha\beta}}{\partial\alpha} - \dfrac{\partial B}{\partial\alpha}e_{\alpha\beta}\Big]\Big\} = 0 \end{cases}$$

c. Displacement-strain and bending relations

$$
\begin{cases}
e_{\alpha\alpha} = \dfrac{1}{A}\dfrac{\partial u}{\partial \alpha} + \dfrac{1}{AB}\dfrac{\partial A}{\partial \beta} v + k_\alpha w \\[2ex]
e_{\beta\beta} = \dfrac{1}{B}\dfrac{\partial v}{\partial \beta} + \dfrac{1}{AB}\dfrac{\partial B}{\partial \alpha} u + k_\beta w \\[2ex]
e_{\alpha\beta} = \dfrac{A}{B}\dfrac{\partial}{\partial \beta}\left(\dfrac{u}{A}\right) + \dfrac{B}{A}\dfrac{\partial}{\partial \alpha}\left(\dfrac{v}{B}\right) \\[2ex]
K_\alpha = -\dfrac{1}{A}\dfrac{\partial}{\partial \alpha}\left(\dfrac{1}{A}\dfrac{\partial w}{\partial \alpha}\right) - \dfrac{1}{AB^2}\dfrac{\partial A}{\partial \beta}\dfrac{\partial w}{\partial \beta} + \dfrac{1}{A}\dfrac{\partial}{\partial \alpha}(k_\alpha u) + \dfrac{k_\beta}{AB}\dfrac{\partial A}{\partial \beta} v \\[2ex]
K_\beta = -\dfrac{1}{B}\dfrac{\partial}{\partial \beta}\left(\dfrac{1}{B}\dfrac{\partial w}{\partial \beta}\right) - \dfrac{1}{A^2 B}\dfrac{\partial B}{\partial \alpha}\dfrac{\partial w}{\partial \alpha} + \dfrac{1}{B}\dfrac{\partial}{\partial \beta}(k_\beta v) + \dfrac{k_\alpha}{AB}\dfrac{\partial B}{\partial \alpha} u \\[2ex]
\tau = -\dfrac{1}{AB}\dfrac{\partial^2 w}{\partial \alpha \partial \beta} + \dfrac{1}{A^2 B}\dfrac{\partial A}{\partial \beta}\dfrac{\partial w}{\partial \alpha} + \dfrac{1}{AB^2}\dfrac{\partial B}{\partial \alpha}\dfrac{\partial w}{\partial \beta} + \dfrac{k_\alpha A}{B}\dfrac{\partial}{\partial \beta}\left(\dfrac{u}{A}\right) + \dfrac{k_\beta B}{A}\dfrac{\partial}{\partial \alpha}\left(\dfrac{v}{B}\right)
\end{cases}
$$

d. Curvature change and twist of the middle surface.

Let k_α, k_β refer to the curvatures of the undeformed surface and these same quantities primed refer to the deformed surface.

$$
\begin{cases}
k_\alpha' - k_\alpha = K_\alpha - k_\alpha e_{\alpha\alpha} \\[2ex]
k_\beta' - k_\beta = K_\beta - k_\beta e_{\beta\beta}
\end{cases}
$$

Let θ represent the angular change in orientation of two lines on an opposite side of an element and let this angle be projection of the resultant angular change along either of the two coordinate axis. Then;

$$
\left[\dfrac{d\theta}{ds} = \tau \right.
$$

Conclusions

When the first and second quadratic forms of a surface are known and when the Gauss-Peterson-Codazzi conditions are specified, then the theorem of Bonnet states that a surface is uniquely specified up to its location in space.

In dealing with a deformed surface, it is tacitly assumed that the undeformed surface is completely known. It has been shown that for linear theory at least, the first and second quadratic forms of the deformed surface may be completely specified in terms of either the three displacement functions u, v and w, or the six deformation functions $e_{\alpha\alpha}$, $e_{\beta\beta}$, $e_{\alpha\beta}$, K_α, K_β, Υ. If the latter set are used to specify the deformed surface, then the Gauss-Peterson-Codazzi conditions yield three differential equation interrelating these quantities. Thus it is seem that whichever method of deformed surface specification is used, that surface suffers a third degree of indeterminateness in that three functions may always be chosen arbitrarily to define its configuration.

A given thin shell structure when subjected to a loading system will deform in a prescribed manner. Since the shell may be considered as being made up of an infinite number of laminaes' or surfaces, then each surface will deform in a prescribed manner. The problem is to find the deformed configuration of the shell and hence each of its laminaes. Surface study alone does not have the complete key to the problem since it has been shown that such a study leads to a third degree indeterminateness. If now three additional equations could be found relating the deformation functions or displacement functions, then the surface configurations would be completely specified.

Since it is a loading condition which causes the deformations, it would seem logical to presume that the additional equations will be found when the load condition is related to the condition of state of the shell. If the shell is statically loaded, the condition of equilibrium may be used or if the shell is vibrating, the dynamic equations may be utilized. However, no matter which state the shell is in, it is the shell and not the surface that will be considered. Hence in finding the additional equations to predict the deformed configuration of a shell surface, the thickness of the shell must be considered.

Remembering that a shell deformed surface was of third degree of indeterminateness, the equations of the condition of state of a shell must be reducible to three additional equations for a surface. However, the conditions of state will yield either the displacements or deformation functions for each point within the shell thickness and hence what is needed is to find the variation of these quantities with respect to some arbitrarily chosen reference surface within the shell.

Consider now the situation. In order to define completely the configuration of the deformed surface and hence shell, three additional equations are required. However if these equations are to be derived from the condition of state, then the variation of the deformation or displacement functions must be assumed. If the assumption on the variation is by chance one that is true for the particular shell geometry, edge conditions and loading, the equations governing the shell will all be satisfied. In practice however, this type of a guess is extremely rare.

The assumption that is normally used in shell analysis is the Kirchoff hypotheses, on extension of the "plane sections remaining plane" assumption used in simple beam bending theory. The condition prescribes the variation of displacements through the shell thickness. Physically, the assumption is an appealing one for thin shells. However, it is an approximation and its use introduces an error into shell analysis. The magnitude of this error will be subsequently discussed.

CHAPTER IV

4. 1. General Discussion

In the previous section, it has been shown that the study of the deformed surface ultimately leads to a third degree of indeterminacy in uniquely defining this surface. That is, three parameters, either the three displacement components, u, v, w, or three of the six deformations, $e_{\alpha\alpha}$, $e_{\beta\beta}$, $e_{\alpha\beta}$, K_α, K_β, τ, may be chosen arbitrarily without violating any of the precepts of the differential geometry. However, a thin elastic shell subjected to an external loading, surface or edge in nature, will deform in a unique manner. Thus a shell structure does not possess any indeterminacy.

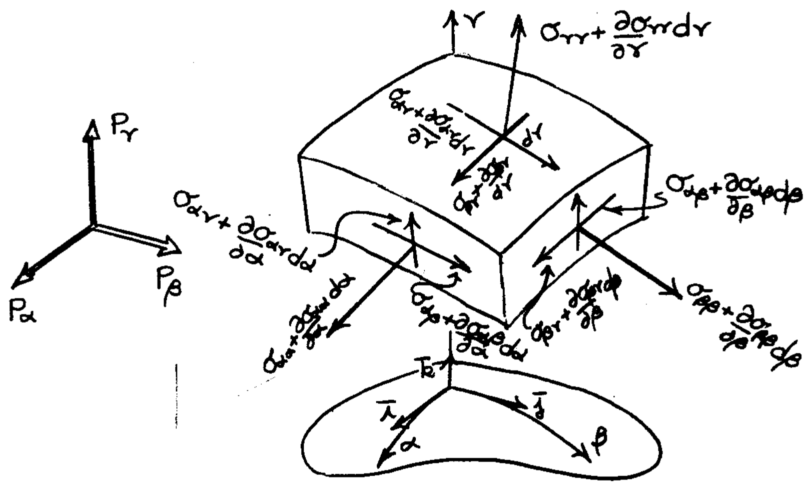
If a surface can represent a thin shell structure, then additional equations must be available whereby the deformed surface can be uniquely specified. From the viewpoint of uniquely defining the deformed surface, the number of additional equations must be three. The problem then is in finding the scource of these equations.

As has been mentioned, when a shell structure is subjected to a loading, it assumes a unique configuration. Hence the laminaes or surfaces which may be considered as making up the shell also assume unique configurations. Thus load and deformed configuration of a surface are intimately related. Now the indeterminacy of the deformed surface has been concluded on the basis of differential geometry. Nowhere has there been any mention of a shell structure let alone an external loading. Thus it must be concluded that the additional sought equations which will ultimately define the deformed surface uniquely must be related with loading on the shell structure and hence with the stress state of the shell.

4. 2. Stress State Within A Shell

Consider now a thin shell structure subjected to some external loading. Within the shell, each point will be subjected to a general stress state consisting of the six components of the stress tensor. In order to picture this stressed state and the assumed positive directions of the stresses, assume the following. Let the middle surface of the shell be the reference surface and let this surface be parameterized by some α , β coordinates. Assume that the coordinization is such that the α and β coordinate curves are principal curvalinear coordinate curves of the middle surface. Assume further, that the deformations are sufficiently small so that the stressed state geometry of an element of the shell may be approximated by the geometry of the unstressed state.

Let γ be the distance measured normal to the middle surface such that the α, β, γ coordinate axis form a right handed system. It is obvious that γ is collinear with and in the same sense as the $\bar{k}$ vector of the orthonormal triad of $\bar{i}$, $\bar{j}$, $\bar{k}$ vectors of the middle surface. Consider now forming an element within the shell by means of three intersecting surfaces. Let the first surface be parallel to the middle surface such that the distance γ between the two surfaces remains a constant. Let the second surface be normal to the middle surface and pass through the β curvalinear coordinate curve. The third surface also will be normal to the middle surface but will pass through the α curvalinear coordinate curve. The resulting element and the corresponding stressed state are shown in the sketch on the following page.



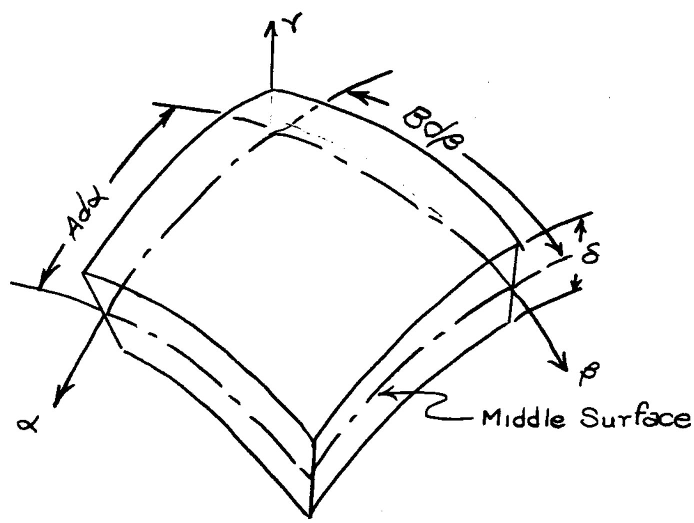
The body forces, p_α, p_β, p_γ, loads per unit volume, are shown beside the sketch in order to minimize the clutter.

On the basis of the differential element shown, and assuming a static equilibrium, a set of relations between the stress components could be derived. However such a set of relations would not prove immediately fruitful. Calling to mind the discussion at the beginning of the present chapter, it was pointed out that additional relations had to be found in order that the deformed middle surface would be uniquely defined. Thus the sought equations must be surface type equations. The relations that would be developed by considering the equilibrium of the differential element described would be volume type equations and in order to prove useful in defining the deformed middle surface, would have to be transformed to surface type equations.

4. 3. Stress Resultants

Rather than transform the equilibrium equations of an infinitesimal volume element into surface type equations, it is a simpler act to start with a surface element and consider its static equilibrium state. However to do so requires that the stresses which exist at a point in a three dimensional medium be transformed to some equivalent force state acting on a lamina or surface. Toward this end, the concept of a stress resultant will be introduced.

Consider now a differential element of a shell but one whose thickness is equal to the shell thickness δ . Assume that the element has been formed by intersecting cutting surfaces such that these surfaces are normal to the middle surface, pass through the principal curvalinear coordinates of the middle surface, and further, are such that a straight line segment normal to the middle surface at any point on the principal curvalinear coordinate curve would be contained in this cutting surface. Such a generated element with its dimensions is shown on the accompanying sketch.



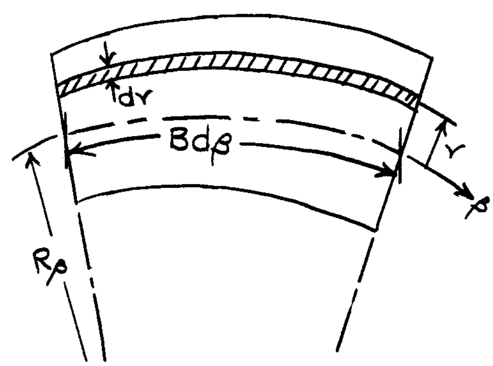
Note that the dimensions of the element are measured on the middle surface and the boundaries out from the shell are normal to the middle surface and consist of straight line segments. Thus knowing the middle surface dimensions of the element and the geometry of the middle surface is sufficient to completely describe the element. Now every point on each of the four lateral sides of the element is subjected to a stress condition of the type previously mentioned while the outer and inner surfaces of the element (which coincide with the outer and inner sides of the shell) are subjected to assumed known load conditions. The stress distribution and the external surface loadings will cause the element to be subjected to a force condition and if the shell is in static equilibrium, an element of the shell must also be in static equilibrium. Thus the resultant force and couple on the element must be zero.

Consider now the resultant forces due to the stress condition on the lateral sides of the element. Since the lateral dimensions are of infinitesimal length, then within first order approximation, it may be assumed that the stress variation in the α or β directions on any of the lateral sides may be neglected. However this does not preclude the stress variation in the α or β directions between any two parallel lateral sides. Since the element thickness is finite, a stress variation in the γ direction is assumed on any of the lateral sides.

Define the stress resultants as follows

$$
\left[
\begin{array}{ll}
T_{\alpha\alpha} = \int_{-\delta/2}^{\delta/2} \sigma_{\alpha\alpha}(1+k_\beta\gamma)\,d\gamma & T_{\beta\gamma} = \int_{-\delta/2}^{\delta/2} \sigma_{\beta\gamma}(1+k_\alpha\gamma)\,d\gamma \\
T_{\alpha\beta} = \int_{-\delta/2}^{\delta/2} \sigma_{\alpha\beta}(1+k_\beta\gamma)\,d\gamma & M_{\alpha\alpha} = \int_{-\delta/2}^{\delta/2} \sigma_{\alpha\alpha}\gamma(1+k_\beta\gamma)\,d\gamma \\
T_{\alpha\gamma} = \int_{-\delta/2}^{\delta/2} \sigma_{\alpha\gamma}(1+k_\beta\gamma)\,d\gamma & M_{\alpha\beta} = \int_{-\delta/2}^{\delta/2} \sigma_{\alpha\beta}\gamma(1+k_\beta\gamma)\,d\gamma \\
T_{\beta\alpha} = \int_{-\delta/2}^{\delta/2} \sigma_{\alpha\beta}(1+k_\alpha\gamma)\,d\gamma & M_{\beta\beta} = \int_{-\delta/2}^{\delta/2} \sigma_{\beta\beta}\gamma(1+k_\alpha\gamma)\,d\gamma \\
T_{\beta\beta} = \int_{-\delta/2}^{\delta/2} \sigma_{\beta\beta}(1+k_\alpha\gamma)\,d\gamma & M_{\beta\alpha} = \int_{-\delta/2}^{\delta/2} \sigma_{\alpha\beta}\gamma(1+k_\alpha\gamma)\,d\gamma
\end{array}
\right.
$$

Note now that the stress resultants T_{ij} have the units of force per unit length while the stress resultants M_{ij} have the units of couple per unit length. The physical meaning of these stress resultants can be shown as follows. Consider for a moment finding the resultant force normal to the lateral side for which the tangent to the α coordinate axis is a normal. On that side, the only stress which can yield a force normal to the area is the normal stress $\sigma_{\alpha\alpha}$. Hence the problem of finding the resultant force is reduced to finding the force resultant of the normal stress distribution $\sigma_{\alpha\alpha}$. A sketch of the lateral side is shown below.



The length of the middle surface is $Bd\beta$ and the curvature of the middle surface in the β direction is assumed to be k_β. Since the α and β coordinate lines are assumed to be principal curvalinear coordinate curves, then there is also associated with the surface a radius curvature R_β which for the infinitesimal element shown may be assumed to be constant. Now the element has been formed such that the sides shown are straight lines and normal to the middle surface and hence for β being a principal curvalinear coordinate curve, these sides, if extended, will intersect at the center of curvature for the surface. Because of these facts, it is now possible to easily express the length of any curve on this lateral face which is parallel to the β coordinate curve. Thus for a curve located a distance γ above the β coordinate curve, its length is given as;

$$ds_\beta(\gamma) = (1 + k_\beta \gamma) B d\beta$$

The force acting on the shaded area due to the normal stress $\sigma_{\alpha\alpha}$ is given as;

$$dF_{\alpha\alpha} = \sigma_{\alpha\alpha}\, ds_\beta(\gamma)\, d\gamma$$

Hence the force acting over the lateral side is the integral of this quantity over the lateral face. But notice that the stress variation $\sigma_{\alpha\alpha}$ is assumed to be a constant is the β direction for this surface since its dimension in that direction is infinitesimal. The same will be true for the quantity B and the curvature k_β . Thus in intergrating the force expression the only variation that need be taken into account is in the γ direction. Substituting for $ds_\beta(\gamma)$;

$$F_{\alpha\alpha} = B d\beta \int_{-\delta/2}^{\delta/2} \sigma_{\alpha\alpha}(1 + k_\beta \gamma)\, d\gamma$$

Now the force per unit length of the middle surface will be given as;

$$\frac{F_{\alpha\alpha}}{B d\beta} = \int_{-\delta/2}^{\delta/2} \sigma_{\alpha\alpha}(1 + k_\beta \gamma)\, d\gamma$$

But this is precisely the stress resultant $T_{\alpha\alpha}$. Hence it may be concluded that the force stress resultants, T_{ij}, represent the resultant forces per unit length of the middle surface on the lateral sides of the element.

Considering again the lateral side pictured, if a moment summation were taken about a tangent to the β coordinate curve of the middle surface, the result to a first order of approximation would be;

$$B d\beta \int_{-\delta/2}^{\delta/2} \sigma_{\alpha\alpha} \gamma (1 + k_{\beta} \gamma) \, d\gamma$$

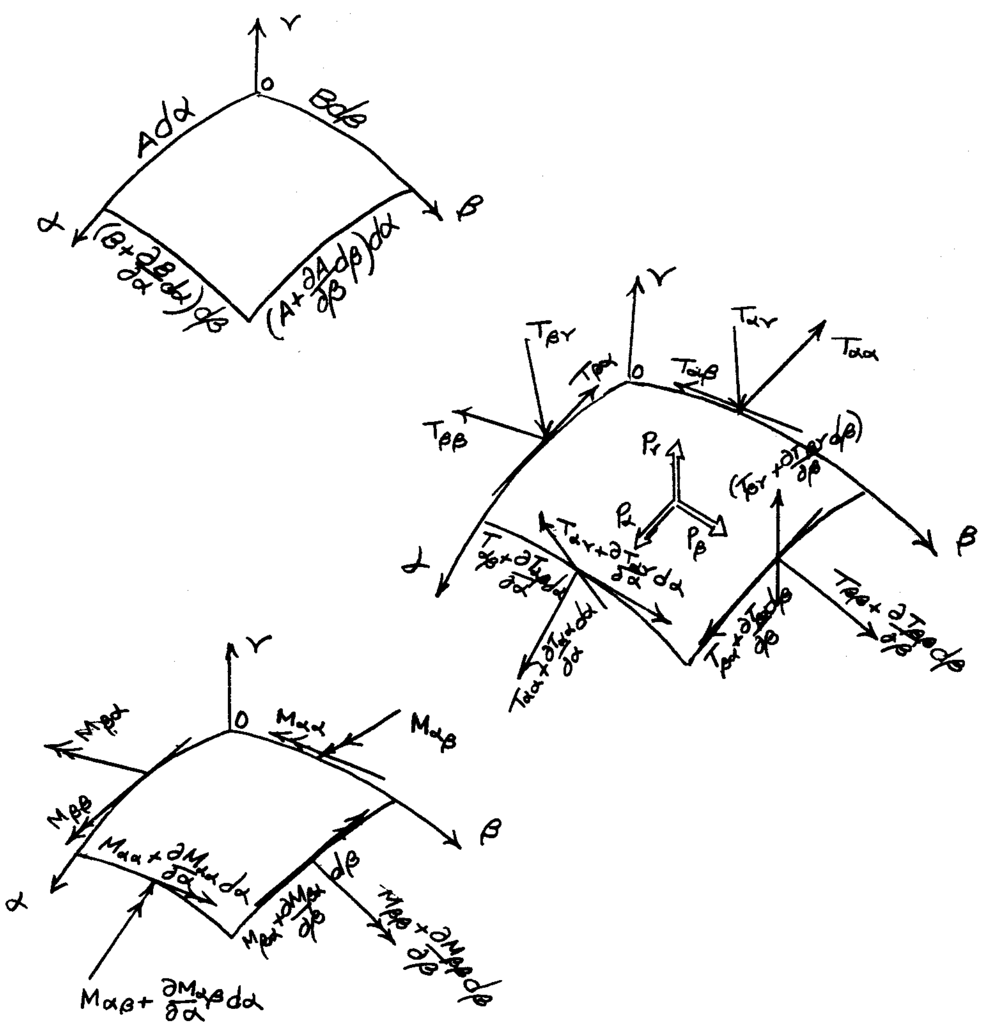
Thus the moment per unit length of the middle surface would be the stress resultant $M_{\alpha\alpha}$. By analogy, the moment per unit length of the middle surface about the α coordinate curve of the adjacent lateral face would be the stress resultant $M_{\beta\beta}$. The possible variation of the shear stress $\sigma_{\alpha\beta}$ in the γ direction causes a moment about a normal to the lateral face. If this normal is placed on the coordinate curve, then the twisting moment per unit length of the middle surface is the stress resultant $M_{\alpha\beta}$. Note that there is no twisting moment associated with the sheer stress $\sigma_{\alpha\gamma}$ since the only assumed variation of stress on a lateral face of the element occurs in the γ direction.

The inner and outer surfaces of the element are free surfaces. Hence only loading or stress condition on these surfaces is assumed to be prescribed. Assume that the resultant of the surface loading will consist of three components of load intensity, q_{α}, q_{β}, q_{γ}. These intensities are presumed to have dimensions of force per unit area of the middle surface.

The use of stress resultants on the element of finite thickness δ is statically equivalent to the stress distribution acting on that element. Further, the direction of the stress resultants will be dictated by the tangents to the α and β coordinate lines and the normal to the middle surface. Since these stress resultants are calculated on the basis of middle surface dimensions, then so far as equilibrium equations are concerned, the three dimensional element may be replaced by a surface element of the middle surface. Thus the state of equilibrium for a shell structure is finally reduced to a surface problem.

4. 4. Equilibrium Equations

Consider now an element of the middle surface subjected to stress resultants. In order to minimize the complexity of the diagram three sketches will be used. The first will be a sketch showing the dimensions of the element, the second will show the force stress resultants and the third the moment stress resultants.



In writing the equilibrium equations for the element, a number of things should be borne in mind. First, that the surface parameters A & B as well as the stress resultants are defined at the point 0. Secondly, that the dimensions of the element change between opposite sides. Further, that the magnitudes of the stress resultants change and finally that the directions of the stress resultants change. Now the force summations will be taken in the $\bar{i}$, $\bar{j}$, and $\bar{k}$ directions and the moment summations will be taken about the axis coinciding with these unit vectors. Positive moment and force will be said to exist if they are in the same directions as the unit vectors $\bar{i}$, $\bar{j}$, and $\bar{k}$.

To account for the changes in directions of the stress resultant, consider using a unit vector $\bar{\eta}_{ij}$ where $\bar{\eta}_{ij}$ will be a unit vector located at same position and in the same direction as a corresponding stress resultant. If the components of the vectors $\bar{\eta}_{ij}$ are known in the directions of the triad of vectors $\bar{i}$, $\bar{j}$, $\bar{k}$, then so also will be the components of the stress resultants. To account between the back faces and the front faces of an element, the subscript 1 and 2 will be used. Thus the unit vector acting collinearly with the stress resultant ($T_{\alpha\alpha} + \frac{\partial T_{\alpha\alpha}}{\partial \alpha} d\alpha$) will be designated as $\bar{\eta}_{\alpha\alpha_2}$ while the unit vector acting collinearly with the stress resultant $T_{\alpha\alpha}$ will be designated as $\bar{\eta}_{\alpha\alpha_1}$.

The components of the unit vector can be easily derived from the derivatives of the unit vectors $\bar{i}$, $\bar{j}$, and $\bar{k}$. As an example, consider the components of the unit vector $\bar{\eta}_{\alpha\alpha_2}$. Thus;

$$\bar{\eta}_{\alpha\alpha_2} = \bar{i} + \frac{\partial \bar{i}}{\partial \alpha} d\alpha + \frac{\partial \bar{i}}{\partial \beta} \frac{d\beta}{2}$$

Substituting for the derivatives of the unit vectors,

$$\bar{\eta}_{\alpha\alpha_2} = \bar{i} + \left(\frac{1}{A}\frac{\partial B}{\partial \alpha}\frac{d\beta}{2} - \frac{1}{B}\frac{\partial A}{\partial \beta} d\alpha\right)\bar{j} - k_\alpha A\, d\alpha\, \bar{k}$$

Table of the Unit Vectors

	$\bar{i}$	$\bar{j}$	$\bar{k}$
$\eta_{\alpha\alpha_1}$	1	$\frac{1}{A}\frac{\partial B}{\partial \alpha}\frac{d\beta}{2}$	—
$\eta_{\alpha\beta_1}$	$-\frac{1}{A}\frac{\partial B}{\partial \alpha}\frac{d\beta}{2}$	1	$-k_\beta B \frac{d\beta}{2}$
$\eta_{\alpha\gamma_1}$	—	$k_\beta B \frac{d\beta}{2}$	1
$\eta_{\beta\alpha_1}$	1	$-\frac{1}{B}\frac{\partial A}{\partial \beta}\frac{d\alpha}{2}$	$-k_\alpha A \frac{d\alpha}{2}$
$\eta_{\beta\beta_1}$	$\frac{1}{B}\frac{\partial A}{\partial \beta}\frac{d\alpha}{2}$	1	—
$\eta_{\beta\gamma_1}$	$k_\alpha A \frac{d\alpha}{2}$	—	1
$\eta_{\gamma\alpha_1}$	1	$-\frac{1}{A}\frac{\partial B}{\partial \alpha}\frac{d\beta}{2} + \frac{1}{B}\frac{\partial A}{\partial \beta}\frac{d\alpha}{2}$	$k_\alpha A \frac{d\alpha}{2}$
$\eta_{\gamma\beta_1}$	$\frac{1}{B}\frac{\partial A}{\partial \beta}\frac{d\alpha}{2} - \frac{1}{A}\frac{\partial B}{\partial \alpha}\frac{d\beta}{2}$	1	$-k_\beta B \frac{d\beta}{2}$
$\eta_{\gamma\gamma_1}$	$k_\alpha A \frac{d\alpha}{2}$	$k_\beta B \frac{d\beta}{2}$	1
$\eta_{\alpha\alpha_2}$	1	$\frac{1}{A}\frac{\partial B}{\partial \alpha}\frac{d\beta}{2} - \frac{1}{B}\frac{\partial A}{\partial \beta} d\alpha$	$-k_\alpha A\, d\alpha$
$\eta_{\alpha\beta_2}$	$\frac{1}{B}\frac{\partial A}{\partial \beta} d\alpha - \frac{1}{A}\frac{\partial B}{\partial \alpha}\frac{d\beta}{2}$	1	$-k_\beta B \frac{d\beta}{2}$
$\eta_{\alpha\gamma_2}$	$k_\alpha A\, d\alpha$	$k_\beta B \frac{d\beta}{2}$	1
$\eta_{\beta\alpha_2}$	1	$\frac{1}{A}\frac{\partial B}{\partial \alpha} d\beta - \frac{1}{B}\frac{\partial A}{\partial \beta}\frac{d\alpha}{2}$	$-k_\alpha A \frac{d\alpha}{2}$
$\eta_{\beta\beta_2}$	$\frac{1}{B}\frac{\partial A}{\partial \beta}\frac{d\alpha}{2} - \frac{1}{A}\frac{\partial B}{\partial \alpha} d\beta$	1	$-k_\beta B\, d\beta$
$\eta_{\beta\gamma_2}$	$k_\alpha A \frac{d\alpha}{2}$	$k_\beta B\, d\beta$	1

Summing forces in $\bar{i}$ direction;

$$\left[\left(T_{\alpha\alpha} + \frac{\partial T_{\alpha\alpha}}{\partial \alpha} d\alpha\right)\left(B + \frac{\partial B}{\partial \alpha} d\alpha\right) d\beta\right] + \left[\left(T_{\alpha\beta} + \frac{\partial T_{\alpha\beta}}{\partial \alpha} d\alpha\right)\left(B + \frac{\partial B}{\partial \alpha} d\alpha\right) d\beta\right]\left(\frac{1}{B}\frac{\partial A}{\partial \beta} d\alpha - \frac{1}{A}\frac{\partial B}{\partial \alpha}\frac{d\beta}{2}\right) + \left[\left(T_{\alpha\gamma} + \frac{\partial T_{\alpha\gamma}}{\partial \alpha} d\alpha\right) \times\right.$$

$$\left. \times \left(B + \frac{\partial B}{\partial \alpha} d\alpha\right) d\beta\right]\left(k_\alpha A\, d\alpha\right) + \left[\left(T_{\beta\alpha} + \frac{\partial T_{\beta\alpha}}{\partial \beta} d\beta\right)\left(A + \frac{\partial A}{\partial \beta} d\beta\right) d\alpha\right] + \left[\left(T_{\beta\beta} + \frac{\partial T_{\beta\beta}}{\partial \beta} d\beta\right)\left(A + \frac{\partial A}{\partial \beta} d\beta\right) d\alpha\right] \times$$

$$\times \left(\frac{1}{B}\frac{\partial A}{\partial \beta}\frac{d\alpha}{2} - \frac{1}{A}\frac{\partial B}{\partial \alpha} d\alpha\right) + \left[\left(T_{\beta\gamma} + \frac{\partial T_{\beta\gamma}}{\partial \beta} d\beta\right)\left(A + \frac{\partial A}{\partial \beta} d\beta\right) d\alpha\right]\left(k_\alpha A \frac{d\alpha}{2}\right) - \left[\left(T_{\alpha\alpha}\right)\left(A\, d\alpha\right)\right] - \left[\left(T_{\beta\beta}\right)\left(A\, d\alpha\right)\right] \times$$

$$\times \left(\frac{1}{B}\frac{\partial A}{\partial \beta}\frac{d\alpha}{2}\right) - \left[\left(T_{\beta\gamma}\right)\left(A\, d\alpha\right)\right]\left(k_\alpha A \frac{d\alpha}{2}\right) - \left[\left(T_{\alpha\alpha}\right) B\, d\beta\right] + \left[\left(T_{\alpha\beta}\right)\left(B\, d\beta\right)\right]\left(\frac{1}{A}\frac{\partial B}{\partial \alpha}\frac{d\beta}{2}\right) +$$

$$+ \left[\left(q_\alpha\right)\left(A\, d\alpha\right)\left(B\, d\beta\right)\right] + \left[\left(q_\beta\right)\left(A\, d\alpha\right)\left(B\, d\beta\right)\right]\left(\frac{1}{B}\frac{\partial A}{\partial \beta}\frac{d\alpha}{2} - \frac{1}{A}\frac{\partial B}{\partial \alpha}\frac{d\beta}{2}\right) + \left[\left(q_\gamma\right)\left(A\, d\alpha\right)\left(B\, d\beta\right)\right] \times$$

$$\left(k_\alpha A \frac{d\alpha}{2}\right) = 0$$

Expanding, simplifying, and dividing through by $A\,d\alpha\,d\beta$ and passing to the limit as $d\alpha \to 0$ and $d\beta \to 0$, the result becomes

$$\frac{1}{AB}\frac{\partial}{\partial\alpha}(BT_{\alpha\alpha}) + \frac{1}{AB}\frac{\partial}{\partial\beta}(AT_{\alpha\beta}) + \frac{1}{B}\frac{\partial A}{\partial\beta}T_{\alpha\beta} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}T_{\beta\beta}$$

$$+ k_\alpha T_{\alpha\gamma} + q_\alpha = 0$$

The remaining equilibrium equation can be found in an analagous manner. The resulting equations are given in the following summary.

4. 5. Summary of Equilibrium Equations

$$\begin{cases}
\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BT_{\alpha\alpha}) + \frac{\partial}{\partial\beta}(AT_{\beta\alpha}) + \frac{\partial A}{\partial\beta}T_{\alpha\beta} - \frac{\partial B}{\partial\alpha}T_{\beta\beta}\right] + k_\alpha T_{\alpha\gamma} + q_\alpha = 0 \\
\frac{1}{AB}\left[\frac{\partial}{\partial\beta}(AT_{\beta\beta}) + \frac{\partial}{\partial\alpha}(BT_{\alpha\beta}) + \frac{\partial B}{\partial\alpha}T_{\beta\alpha} - \frac{\partial A}{\partial\beta}T_{\alpha\alpha}\right] + k_\beta T_{\beta\gamma} + q_\beta = 0 \\
\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BT_{\alpha\gamma}) + \frac{\partial}{\partial\beta}(AT_{\beta\gamma})\right] - k_\alpha T_{\alpha\alpha} - k_\beta T_{\beta\beta} + q_\gamma = 0 \\
\frac{1}{AB}\left[\frac{\partial}{\partial\beta}(AM_{\beta\beta}) + \frac{\partial}{\partial\alpha}(BM_{\alpha\beta}) - \frac{\partial A}{\partial\beta}M_{\alpha\alpha} + \frac{\partial B}{\partial\alpha}M_{\beta\alpha}\right] - T_{\beta\gamma} = 0 \\
\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BM_{\alpha\alpha}) + \frac{\partial}{\partial\beta}(AM_{\beta\alpha}) - \frac{\partial B}{\partial\alpha}M_{\beta\beta} + \frac{\partial A}{\partial\beta}M_{\alpha\beta}\right] - T_{\alpha\gamma} = 0 \\
T_{\alpha\beta} - T_{\beta\alpha} + k_\alpha M_{\alpha\beta} - k_\beta M_{\beta\alpha} = 0
\end{cases}$$

4. 6. Commentary

The six equilibrium equations are involved with the undeformed surface parameters, A and B, and with ten stress resultants. Note that unlike stresses, neither the force stress resultants, $T_{\alpha\beta}$ and $T_{\beta\alpha}$, or the moment stress resultants, $M_{\alpha\beta}$ and $M_{\beta\alpha}$, are equal to each other unless the curvatures of the middle surface in the α and β directions are equal.

Initially, the idea behind the introduction of the equilibrium equations was to bring about the dependence of the deformed middle surface on the external loading. However, as matters presently stand, the equilibrium equations further complicate the problem in that they introduce ten new functions, the stress resultants, but are only six in number. If the equilibrium equations plus the deformed surface study uniquely define the deformed middle surface, then that surface, on the basis of the derived results, is analytically indeterminate to the seventh degree; three degrees from the surface study and four degrees from the equilibrium equation study.

Inspection of the defining equations for the stress resultants show that these quantities are dependent in integral form on the stresses within the shell. Since the material is assumed to be elastic, then the stresses may be converted to strains and hence the stress resultants become dependent on the strain variation within the shell. If now the strain variation with the depth of the shell, Υ, can be found, a solvable system of equations will result which will completely define the deformed surface. That this is so can be ascertained by an inspection of the equilibrium equations. Consider assuming that the strain, at any height Υ above or below the middle surface can be found as a function of the strains of the middle surface and the distance Υ. The stress resultant expression could then be integrated and the stress resultants solved as functions of the middle surface deformations. Hence there would be a total of nine equations and nine unknowns. (the strains $e_{\alpha\Upsilon}$, $e_{\beta\Upsilon}$, $e_{\Upsilon\Upsilon}$ would enter into the system from the equilibrium equations).

The problem of the strain variation within a shell can be and has been considered the major problem in shell analysis. It is as yet unresolved in that exact expressions which are convenient to problem solution have not been obtained. However, approximations to the true variation abound in the literature. The most important and most frequently encountered approximation is also historically the oldest. It was first used by Aaron and then Love in their shell theory development and is an extension of plate and simple beam bending theory where it is assumed that lines or planes originally normal to the neutral surface remain so after deformation. This hypothesis is known as the Kirchoff hypothesis and will be introduced in the following chapter.

CHAPTER V

5. 1. Kirchoff Hypothesis and Displacement Variations.

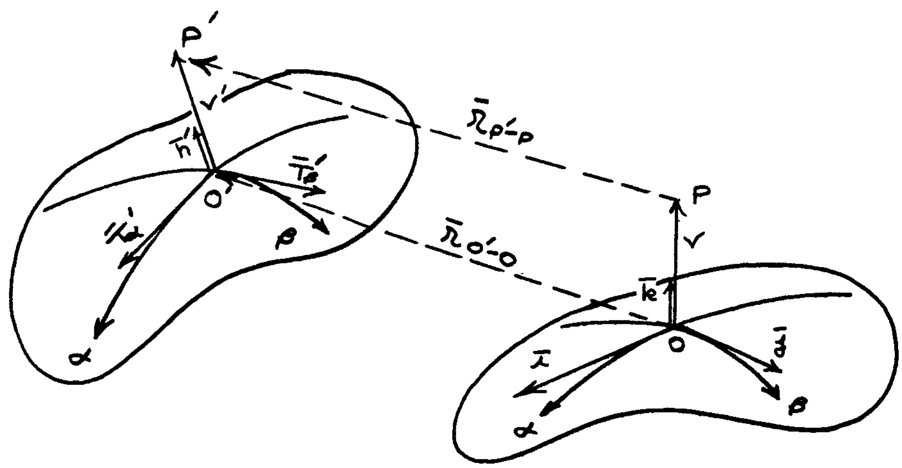
The Kirchoff hypotheses are three in number. Stated briefly, they are as follows.

i) Line segments initially normal to the shell middle surface remain so after deformation.

ii) Line segments initially normal to shell middle surface do not suffer any extensions or contractions.

iii) Normal stresses oriented in a direction normal to the shell middle surface are small in comparison with other stresses.

The consequences of the Kirchoff assumptions will be discussed later. For the moment, note that ii) and iii) are contradictory for the general shell problem in that the two assumptions state that a condition of plane strain and plane stress simultaneously exist. Now the Kirchoff hypothesis (more properly, the Kirchoff-Love hypotheses) are a direct extension of plate theory where it has been found that the plate problem like the beam problem is in approximation a plane stress problem. Hence as might be expected, the thin shell problem should also be a plane stress problem. To maintain this assumption, and to obviate the contradiction of simultaneously assuming plane strain and plane stress, consider relaxing assumption iii). Consider now an element of the shell middle surface before and after deformation.



Now by assumption, $\bar{r}_{o'-o} = u\bar{i}+v\bar{j}+w\bar{k}$. The problem now consisits of finding $\bar{r}_{p'-p}$.

Consider now the vector equation;

$$\bar{r}_{p'-p} = \bar{r}_{o'-o} + \gamma'\bar{n}' - \gamma\bar{k}$$

where $\bar{n}'$ is the normal to the deformed surface. Hence;

$$\bar{r}_{p'-p} = u\bar{i} + v\bar{j} + (w-\gamma)\bar{k} + \gamma' \frac{(\bar{T}_\alpha' \times \bar{T}_\beta')}{|(\bar{T}_\alpha' \times \bar{T}_\beta')|}$$

where $\bar{T}_\alpha'$ and $\bar{T}_\beta'$ are the tangents to the curvalinear coordinate curves of the middle surface. But as found in section dealing with surfaces,

$$\frac{(\bar{T}_\alpha' \times \bar{T}_\beta')}{|(\bar{T}_\alpha' \times \bar{T}_\beta')|} = \left(k_\alpha u - \frac{1}{A}\frac{\partial w}{\partial \alpha}\right)\bar{i} + \left(k_\beta v - \frac{1}{B}\frac{\partial w}{\partial \beta}\right)\bar{j} + \bar{k}$$

Thus the displacement vector of point P becomes;

$$\bar{r}_{p'-p} = \left[u + \left(k_\alpha u - \frac{1}{A}\frac{\partial w}{\partial \alpha}\right)\gamma'\right]\bar{i} + \left[v + \left(k_\beta v - \frac{1}{B}\frac{\partial w}{\partial \beta}\right)\gamma'\right]\bar{j} + \left[w + (\gamma' - \gamma)\right]\bar{k}$$

Note now that if the Kirchoff hypothese were all in force, the $\gamma' \doteq \gamma$ and the above expression would be the one more commonly encountered, namely;

Consider now the expression for γ'. Now the contraction of the normal is due to the variation of the normal strain where from elasticity considerations;

$$e_{\gamma\gamma} = \frac{\partial w}{\partial \gamma}$$

Thus the expression for γ' may be written in integral form as

$$\gamma' = \gamma + \int_0^{\gamma} e_{\gamma\gamma} d\gamma$$

Now without proof, note that $e_{\gamma\gamma}$ in the unloaded state must be zero. Furthermore, the strains $e_{\alpha\alpha}$, $e_{\alpha\beta}$, $e_{\beta\beta}$ were expression which in each of their terms contained either a displacement term or its derivative. Hence it would be reasonable to expect that $e_{\gamma\gamma}$ would be an expression such that each of its terms contained either a displacement term or its derivative. Hence, upon integration with respect to γ this situation would still be true. This can be most easily seen if $e_{\gamma\gamma}$ is written in power series as

$$e_{\gamma\gamma} = f_0(u,v,w) + f_1(u,v,w)\gamma + \cdots + f_n(u,v,w)\gamma^n + \cdots$$

Returning now to the expression for $\bar{\pi}_{p'p}$, note that linearization in the displacement functions leads following resulting expression for the displacement vector;

$$\bar{\pi}_{p'p} = \left[u + \left(k_\alpha u - \frac{1}{A}\frac{\partial w}{\partial \alpha}\right)\gamma\right]\bar{i} + \left[v + \left(k_\beta v - \frac{1}{B}\frac{\partial w}{\partial \beta}\right)\gamma\right]\bar{j} + \left[w + \int_0^{\gamma} e_{\gamma\gamma} d\gamma\right]\bar{k}$$

Letting $u(\gamma)$; $v(\gamma)$; $w(\gamma)$ represent the displacement components of a point away from the middle surface, the final result becomes;

$$u(\gamma) = u + \left(k_\alpha u - \frac{1}{A}\frac{\partial w}{\partial \alpha}\right)\gamma \ ; \ v(\gamma) = v + \left(k_\beta v - \frac{1}{B}\frac{\partial w}{\partial \beta}\right)\gamma \ ; \ w(\gamma) = w + \int_0^{\gamma} e_{\gamma\gamma} d\gamma$$

5. 2. Commentary

The Kirchoff hypotheses allow a solution for the displacement variation through the shell thickness. Ultimately, this displacement variation will allow an expression for the strain variation and hence a relation between the parameters of the deformed middle surface and the stress resultants. However, the Kirchoff hypotheses place some restrictions on shell analysis. Summarized, these restrictions are as follows:

a) Inability to properly account for the sheer stresses $\sigma_{\alpha\gamma}$ and $\sigma_{\beta\gamma}$. The implication immediately follows that the stress resultants $T_{\alpha\gamma}$ and $T_{\beta\gamma}$ cannot be solved explicity

b) Shell analysis cannot account for large normal stresses, $\sigma_{\gamma\gamma}$

c) Introduction of a basic error of magnitude ($k\delta$) which cannot be improved.

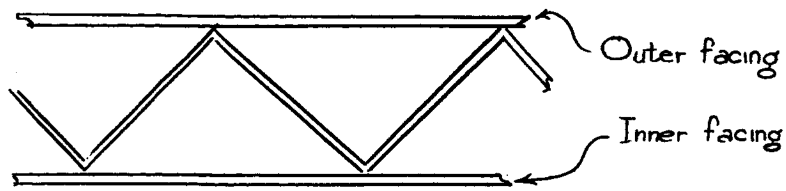
The failure to correctly account for the transverse sheer effects is a direct consequence of assuming non-warpage of the middle surface normal. In the report by Hildebrand, Reissner, Thomas[1], it is shown that by assuming $e_{\alpha\gamma}$ and $e_{\beta\gamma}$ and $e_{\gamma\gamma}$ identically zero, and further assuming that the displacements are given as;

$$u(\gamma) = u + \gamma u'$$
$$v(\gamma) = v + \gamma v'$$

the expressions for u' and v' result the same as was demonstrated in the present analysis. One difference between NACA TN 1833 and the present analysis should be noted. Whereas the reports assumption starts out with essentially a linearly truncated power series in γ for the tangential displacements, the present work shows that a linear variation in the displacements is in fact an exact expression once the non-deformability of the normal is assumed. Whether one formulation is more precise than the other is questionable. In fact it may be that the non-deformability of the normal is in fact a linear approximation to the truth of the matter.

There are two important consequences in neglecting the effect of transverse shear, one analytical and the other practical. From the analytical viewpoint, the neglect of the effects of the transverse shear stress will not enable the equations to be solved using this quantity as a variable. Hence as in the case of plates, the stress resultants, $T_{\alpha\gamma}$ and $T_{\beta\gamma}$ will be eliminated from the resulting system of differential equations. Furthermore, the inability to express the transverse sheer stress will also complicate the boundary conditions in that at a free edge it will be shown that there are four independent stress resultants rather than the five (i.e., $T_{\alpha\alpha}$, $T_{\alpha\gamma}$, $T_{\alpha\beta}$, $M_{\alpha\alpha}$, $M_{\alpha\gamma}$)

The practical implications of the neglect of transverse sheer are perhaps more important than the analytical ones. Thus shells with large surface loads or rapidly varying surface loads cannot be accurately analyzed using equations based on the Kirchoff hypotheses. But more importantly, a class of shells not treated in course but often found in shell applications, become subject to large analytical errors. This class of shells are termed structurally anisotropic or reinforced shells. As an example, consider a shell made up of two thin facing materials and having a corrugated inner construction, as shown on the next page.



In this instance, the shear effects on the inner construction especially in regard to buckling may be the single most important factor in predicting its failure. Yet the Kirchoff assumptions disregard these stresses and hence for this type of a shell, a great deal of modification in the present analysis must be made.

One important and last comment on the shear stresses. Through the effects of these stresses are discarded in the analysis, this is not to say that they will be assumed to be zero. In this respect, thin shell analysis is inconsistent but no more so than ordinary strength of materials in dealing with simple beam bending theory. Perhaps it would be best to say that straight lines normal to shell middle surface remaining undeformed truly implies that the effects of the transverse shear stresses is small because of the small magnitude of these stresses. Thus once the shell equations are solved and the stresses calculated, the corresponding transverse shear stresse may be calculated and in all events, it will be found that its variation over the cross section will be parabolic.

The question of error always arises whenever an approximation is made and hence in the case of shell analysis when the Kirchoff approximations are utilized. A great deal of research has gone into this question and the result is that the error is of magnitude ($k\delta$) where δ is the shell thickness and k the largest curvature of the shell and the quantity ($k\delta$) is composed to unity. Fortunately this error falls within the scope of definition of a thin shell. If it is remembered that a thin shell is one in which the quantity ($k\delta$) may be neglected in comparison with unity and a 5% error in analysis is tolerated, then the Kirchoff hypotheses yield an error of about 5% in ordinary shell analysis.

4. 3. Strain Variation WIithin the Shell.

When a surface was deformed, the parameters of length deformation were the quantities as strain and given as;

$$e_{\alpha\alpha} = \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta} v + k_\alpha w$$

$$e_{\beta\beta} = \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha} u + k_\beta w$$

$$e_{\alpha\beta} = \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right)$$

Ultimately, the problem is one of expressing the stress resultants in terms of either the displacements or the deformation functions. Hence, what is first needed is an expression for the above strains for any point within the interior of the shell. If such expressions can be found, then with the aid of Hooke's Law, the stress variation may be found and hence the stress resultant expressions integrated.

Now the above strain expressions are good for any surface. Since a shell of finite thickness is made up of an infinite number of surfaces, to find the strain variation it is only necessary to choose some reference surface for which A, B, k_α, k_β and u, v, and w are defined and then find the variation of these quantities from surface to surface. Let the middle surface be the reference surface. Then;

$$A(r) = A(1+k_\alpha r) \;;\; B(r) = B(1+k_\beta r);\; k_\alpha(r) = \frac{k_\alpha}{(1+k_\alpha r)};\; k_\beta(r) = \frac{k_\beta}{(1+k_\beta r)}$$

$$u(r) = u + \left(k_\alpha u - \frac{1}{A}\frac{\partial w}{\partial \alpha}\right) r$$

$$v(r) = v + \left(k_\beta v - \frac{1}{B}\frac{\partial w}{\partial \beta}\right) r$$

$$w(r) = w + \int_0^r e_{rr}\, dr$$

Consider now substituting into the strain expressions. For convenience, each of the terms will be separately handled;

1. Normal Strain $e_{\alpha\alpha}$;

Substituting;

$$e_{\alpha\alpha}(r) = \frac{1}{A(1+k_\alpha r)}\frac{\partial}{\partial \alpha}\left[u + \left(k_\alpha u - \frac{1}{A}\frac{\partial w}{\partial \alpha}\right) r\right] + \frac{1}{AB(1+k_\alpha r)(1+k_\beta r)}\left[v + \left(k_\beta v - \frac{1}{B}\frac{\partial w}{\partial \beta}\right) r\right]\frac{\partial}{\partial \beta}\left[A(1+k_\alpha r)\right] + \frac{k_\alpha}{(1+k_\alpha r)}\left[w + \int_0^r e_{rr}\, dr\right]$$

Factoring out $\frac{1}{(1+k_\alpha r)}$ and rearranging;

$$e_{\alpha\alpha}(r) = \frac{1}{(1+k_\alpha r)}\left\{\frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{A}\frac{\partial}{\partial \alpha}(k_\alpha u) r - \frac{1}{A}\frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial w}{\partial \alpha}\right) r + \frac{v}{AB(1+k_\beta r)}\frac{\partial}{\partial \beta}\left[A(1+k_\alpha r)\right] + \frac{k_\beta v r}{AB(1+k_\beta r)}\frac{\partial}{\partial \beta}\left[A(1+k_\alpha r)\right] - \frac{1}{AB(1+k_\beta r)}\left(\frac{1}{B}\frac{\partial w}{\partial \beta}\right) r \frac{\partial}{\partial \beta}\left[A(1+k_\alpha r)\right] + k_\alpha w + k_\alpha \int_0^r e_{rr}\, dr\right.$$

Now;

$$\frac{\partial}{\partial \beta}\left[A(1+k_\alpha r)\right] = \frac{\partial A}{\partial \beta} + \frac{\partial}{\partial \beta}(A k_\alpha) r$$

But by the condition of Codazzi, the above may be written as;

$$\frac{\partial}{\partial \beta}\left[A(1+k_\alpha r)\right] = \frac{\partial A}{\partial \beta} + k_\beta \frac{\partial A}{\partial \beta} r = (1+k_\beta r)\frac{\partial A}{\partial \beta}$$

Substituting and grouping terms;

$$e_{\alpha\alpha}(\gamma)=\frac{1}{(1+k_\alpha\gamma)}\left\{\left[\frac{1}{A}\frac{\partial u}{\partial\alpha}+\frac{1}{AB}\frac{\partial A}{\partial\beta}v+k_\alpha w\right]+\left[\frac{1}{A}\frac{\partial}{\partial\alpha}(k_\alpha u)+\frac{k_\beta}{AB}\frac{\partial A}{\partial\beta}v\right.\right.$$
$$\left.\left.-\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}\right)-\frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta}\right]\gamma+k_\alpha\int_0^\gamma e_{\gamma\gamma}d\gamma\right\}$$

The first bracketed term is nothing more than the middle surface strain while the second term is the curvature change of the surface in the α coordinate direction. Hence;

$$e_{\alpha\alpha}(\gamma)=\frac{1}{(1+k_\alpha\gamma)}\left(e_{\alpha\alpha}+K_\alpha\gamma\right)+\frac{k_\alpha}{(1+k_\alpha\gamma)}\int_0^\gamma e_{\gamma\gamma}d\gamma$$

2. Shear Strain

The shear strain for the middle surface has been given as;

$$e_{\alpha\beta}=\frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{u}{A}\right)+\frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{v}{B}\right)$$

An alternate form of expression can be derived upon expression as;

$$e_{\alpha\beta}=\frac{1}{B}\frac{\partial u}{\partial\beta}-\frac{1}{AB}\frac{\partial A}{\partial\beta}u+\frac{1}{A}\frac{\partial v}{\partial\alpha}-\frac{1}{AB}\frac{\partial B}{\partial\alpha}v$$

The latter form of the shear strain expression will be more convenient to use in deriving the expression for the shear strain variation. Note further the symmetry that exists in the expression. For convenience sake, let

$$G_1=\frac{1}{B}\frac{\partial u}{\partial\beta}-\frac{1}{AB}\frac{\partial A}{\partial\beta}u\ ;\quad G_2=\frac{1}{A}\frac{\partial v}{\partial\alpha}-\frac{1}{AB}\frac{\partial B}{\partial\alpha}v$$

Hence $e_{\alpha\beta}=G_1+G_2$ and for points away from the middle surface;

$$e_{\alpha\beta}(\gamma)=G_1(\gamma)+G_2(\gamma)$$

Because of the symmetry in G_1 and G_2, only one of these functions need be evaluated, say G_1. Thus;

$$G_1(\gamma)=\frac{1}{B(1+k_\beta\gamma)}\frac{\partial}{\partial\beta}\left[u+\left(k_\alpha u-\frac{1}{A}\frac{\partial w}{\partial\alpha}\right)\gamma\right]-\frac{1}{AB(1+k_\alpha\gamma)(1+k_\beta\gamma)}\left[u+\left(k_\alpha u-\frac{1}{A}\frac{\partial w}{\partial\alpha}\right)\gamma\right]\times$$
$$\times\frac{\partial}{\partial\beta}\left[A(1+k_\alpha\gamma)\right]$$

Expanding

$$G_1(\gamma) = \frac{1}{B(1+k_\beta\gamma)}\frac{\partial u}{\partial\beta} + \frac{1}{B(1+k_\beta\gamma)}\frac{\partial}{\partial\beta}\left(k_\alpha u - \frac{1}{A}\frac{\partial w}{\partial\alpha}\right)\gamma - \frac{u}{AB(1+k_\alpha\gamma)(1+k_\beta\gamma)}\frac{\partial}{\partial\beta}\left[A(1+k_\alpha\gamma)\right]$$

$$- \frac{\left(k_\alpha u - \frac{1}{A}\,\partial w/\partial\alpha\right)\gamma}{AB(1+k_\alpha\gamma)(1+k_\beta\gamma)}\frac{\partial}{\partial\beta}\left[A(1+k_\alpha\gamma)\right]$$

But as previously pointed out

$$\frac{\partial}{\partial\beta}\left[A(1+k_\alpha\gamma)\right] = (1+k_\beta\gamma)\frac{\partial A}{\partial\beta}$$

Combining terms, the resultant expression for $G_1(\gamma)$ becomes;

$$G_1(\gamma) = \frac{1}{(1+k_\beta\gamma)}\left[\frac{1}{B}\frac{\partial u}{\partial\beta}\right] + \frac{1}{(1+k_\alpha\gamma)}\left[-\frac{1}{AB}\frac{\partial A}{\partial\beta}u\right]$$

$$+ \frac{1}{(1+k_\beta\gamma)}\left[\frac{1}{A^2B}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} - \frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\partial\beta} + \frac{1}{B}\frac{\partial}{\partial\beta}(k_\alpha u)\right]\gamma$$

$$+ \frac{1}{(1+k_\alpha\gamma)}\left[\frac{1}{A^2B}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} - \frac{1}{AB}\frac{\partial A}{\partial\beta}k_\alpha u\right]\gamma$$

Forming the resultant expression for $e_{\alpha\beta}(\gamma)$;

$$e_{\alpha\beta}(\gamma) = \frac{1}{(1+k_\alpha\gamma)}\left[\frac{1}{A}\frac{\partial v}{\partial\alpha} - \frac{1}{AB}\frac{\partial A}{\partial\beta}u\right] + \frac{1}{(1+k_\beta\gamma)}\left[\frac{1}{B}\frac{\partial u}{\partial\beta} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}v\right]$$

$$+ \frac{1}{(1+k_\alpha\gamma)}\left[\frac{1}{A^2B}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} - \frac{1}{AB}\frac{\partial A}{\partial\beta}k_\alpha u + \frac{1}{AB^2}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta} - \frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\partial\beta} + \frac{1}{A}\frac{\partial}{\partial\alpha}(k_\beta v)\right]\gamma$$

$$+ \frac{1}{(1+k_\beta\gamma)}\left[\frac{1}{AB^2}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}k_\beta v + \frac{1}{A^2B}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} - \frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\partial\beta} + \frac{1}{B}\frac{\partial}{\partial\beta}(k_\alpha u)\right]\gamma$$

Consider putting the expression into a form involving only the middle surface deformations. Toward this end, consider defining four functions, ω_1, ω_2, γ_1, γ_2, as follows;

$$\omega_1 = \frac{1}{A}\frac{\partial v}{\partial\alpha} - \frac{1}{AB}\frac{\partial A}{\partial\beta}u \; ; \quad \omega_2 = \frac{1}{B}\frac{\partial u}{\partial\beta} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}v$$

$$\gamma_1 = \frac{1}{A^2B}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} + \frac{1}{AB^2}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta} - \frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\partial\beta} - \frac{1}{AB}\frac{\partial A}{\partial\beta}k_\alpha u + \frac{1}{A}\frac{\partial}{\partial\alpha}(k_\beta v)$$

$$\gamma_2 = \frac{1}{AB^2}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta} + \frac{1}{A^2B}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} - \frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\partial\beta} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}k_\beta v + \frac{1}{B}\frac{\partial}{\partial\beta}(k_\alpha u)$$

And thus the shear strain expression may be written as;

$$e_{\alpha\beta}(\gamma) = \frac{1}{(1+k_\alpha \gamma)}(\omega_1 + \Upsilon_1 \gamma) + \frac{1}{(1+k_\beta \gamma)}(\omega_2 + \Upsilon_2 \gamma)$$

Now note the following;

$$e_{\alpha\beta} = \omega_1 + \omega_2$$

Consider further the expressions for Υ_1 and Υ_2. Now note that the terms involving w are the same in Υ_1 as well as Υ_2 and further that these same terms are identical with those occuring in the expression for the twist Υ of the middle surface. The difference between Υ_1 and Υ_2 and Υ lies in the expressions involving the tangential displacements. Consider now dealing with the tangential displacement terms only in the following expression

$$\Upsilon_1 + k_\alpha \omega_2 = -\frac{k_\alpha}{AB}\frac{\partial A}{\partial \beta}u + \frac{1}{A}\frac{\partial}{\partial \alpha}(k_\beta v) + \frac{k_\alpha}{B}\frac{\partial u}{\partial \beta} - \frac{k_\alpha}{AB}\frac{\partial B}{\partial \alpha}v$$

Rewriting;

$$\Upsilon_1 + k_\alpha \omega_2 = -\frac{k_\alpha A}{BA^2}\frac{\partial A}{\partial \beta}u + \frac{k_\alpha A}{BA}\frac{\partial u}{\partial \beta} + \frac{1}{A}\frac{\partial}{\partial \alpha}\left(B k_\beta \frac{v}{B}\right) - \frac{k_\alpha}{AB}\frac{\partial B}{\partial \alpha}v$$

Or using the condition of Codazzi on the last term, and expanding the third;

$$\Upsilon_1 + k_\alpha \omega_2 = k_\alpha \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) + k_\beta \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right) + \frac{1}{AB}\frac{\partial}{\partial \alpha}(B k_\beta)v - \frac{1}{AB}\frac{\partial}{\partial \alpha}(B k_\beta)v$$

and hence;

$$\Upsilon_1 + k_\alpha \omega_2 = k_\alpha \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) + k_\beta \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right)$$

Thus the tangential displacement terms are the same as those occuring in the expression for the twist Υ. The same can be shown for the tangential displacement terms in the combination of $\Upsilon_2 + k_\beta \omega_1$. Hence;

$$\Upsilon = \Upsilon_1 + k_\alpha \omega_2 = \Upsilon_2 + k_\beta \omega_1$$

Consider now taking a common denominator of the expression for the shear strain.

$$e_{\alpha\beta}(\gamma) = \frac{1}{(1+k_\alpha\gamma)(1+k_\beta\gamma)}\left[(1+k_\beta\gamma)(\omega_1+\Upsilon_1\gamma) + (1+k_\alpha\gamma)(\omega_2+\Upsilon_2\gamma)\right]$$

Expanding;

$$e_{\alpha\beta}(\gamma) = \frac{1}{(1+k_\alpha\gamma)(1+k_\beta\gamma)}\left\{(\omega_1+\omega_2)+\left[(\Upsilon_1+k_\alpha\omega_2)+(\Upsilon_2+k_\beta\omega_1)\right]\gamma+(k_\beta\Upsilon_1+k_\alpha\Upsilon_2)\gamma^2\right\}$$

Substituting and noting that

$$k_\beta\Upsilon_1+k_\alpha\Upsilon_2 = k_\beta\Upsilon - k_\alpha k_\beta\omega_2 + k_\alpha\Upsilon - k_\alpha k_\beta\omega_1$$

or

$$k_\beta\Upsilon_1+k_\alpha\Upsilon_2 = (k_\alpha+k_\beta)\Upsilon - k_\alpha k_\beta e_{\alpha\beta}$$

the result becomes

$$e_{\alpha\beta}(\gamma) = \frac{1}{(1+k_\alpha\gamma)(1+k_\beta\gamma)}\left\{e_{\alpha\beta}+2\Upsilon\gamma+\left[(k_\alpha+k_\beta)\Upsilon-k_\alpha k_\beta e_{\alpha\beta}\right]\gamma^2\right\}$$

Or in altermate form;

$$e_{\alpha\beta}(\gamma) = \frac{1}{(1+k_\alpha\gamma)(1+k_\beta\gamma)}\left\{e_{\alpha\beta}(1-k_\alpha k_\beta\gamma^2)+2\Upsilon\left[1+\left(\frac{k_\alpha+k_\beta}{2}\right)\gamma\right]\gamma\right\}$$

Summary of Results

The strain variation through the shell has been found to be as follows;

$$\begin{cases} e_{\alpha\alpha}(\gamma) = \dfrac{1}{(1+k_\alpha\gamma)}(e_{\alpha\alpha}+K_\alpha\gamma) + \dfrac{k_\alpha}{(1+k_\alpha\gamma)}\displaystyle\int_0^\gamma e_{\gamma\gamma}\,d\gamma \\ e_{\beta\beta}(\gamma) = \dfrac{1}{(1+k_\beta\gamma)}(e_{\beta\beta}+K_\beta\gamma) + \dfrac{k_\beta}{(1+k_\beta\gamma)}\displaystyle\int_0^\gamma e_{\gamma\gamma}\,d\gamma \\ e_{\alpha\beta}(\gamma) = \dfrac{1}{(1+k_\alpha\gamma)(1+k_\beta\gamma)}\left\{e_{\alpha\beta}(1-k_\alpha k_\beta\gamma^2)+2\Upsilon\left[1+\left(\dfrac{k_\alpha+k_\beta}{2}\right)\gamma\right]\gamma\right\} \end{cases}$$

4. 4. Stress Variation

The stress strain relation at a point for an isotropic body is given as;

$$e_{\alpha\alpha} = \frac{\sigma_{\alpha\alpha}}{E} - \frac{\nu}{E}(\sigma_{\beta\beta} + \sigma_{\gamma\gamma})$$
$$e_{\beta\beta} = \frac{\sigma_{\beta\beta}}{E} - \frac{\nu}{E}(\sigma_{\alpha\alpha} + \sigma_{\gamma\gamma})$$
$$e_{\gamma\gamma} = \frac{\sigma_{\gamma\gamma}}{E} - \frac{\nu}{E}(\sigma_{\alpha\alpha} + \sigma_{\beta\beta})$$
$$e_{\alpha\beta} = \frac{2(1+\nu)}{E}\sigma_{\alpha\beta}$$

Since the Kirchoff hypothesis assumes a state of plane stress, then $\sigma_{\gamma\gamma} \approx 0$ and the above expressions may be considerably simplified. Thus solving for the stresses as functions of the strains, the result becomes;

$$\sigma_{\alpha\alpha} = \frac{E}{(1-\nu^2)}(e_{\alpha\alpha} + \nu e_{\beta\beta})$$
$$\sigma_{\beta\beta} = \frac{E}{(1-\nu^2)}(e_{\beta\beta} + \nu e_{\alpha\alpha})$$
$$\sigma_{\alpha\beta} = \frac{E}{2(1+\nu)} e_{\alpha\beta}$$
$$\sigma_{\gamma\gamma} = -\frac{\nu}{E}(\sigma_{\alpha\alpha} + \sigma_{\beta\beta})$$

Now the strain expressions, $e_{\alpha\alpha}$, $e_{\beta\beta}$ and $e_{\alpha\beta}$, are known at any point within the shell. Hence by substituting the strain expressions into the above stress-strain relation, the stress at any point within the shell may be obtained. Consider first finding the stress $\sigma_{\alpha\alpha}(\gamma)$.

Substituting the strain expressions;

$$\sigma_{\alpha\alpha}(\gamma) = \frac{E}{(1-\nu^2)}\left\{\left[\frac{e_{\alpha\alpha}}{(1+k_\alpha\gamma)} + \frac{\nu e_{\beta\beta}}{(1+k_\beta\gamma)}\right] + \left[\frac{K_\alpha}{(1+k_\alpha\gamma)} + \frac{\nu K_\beta}{(1+k_\beta\gamma)}\right]\gamma\right.$$
$$\left. + \left[\frac{k_\alpha}{(1+k_\alpha\gamma)} + \frac{\nu k_\beta}{(1+k_\beta\gamma)}\right]\int_0^\gamma e_{\gamma\gamma}\,d\gamma\right\}$$

The stress $\sigma_{\beta\beta}(\gamma)$ may be found by an interchange of subscripts with

with the previous expression. The shear strain $\sigma_{\alpha\beta}(\Upsilon)$ is given as;

$$\sigma_{\alpha\beta}(\Upsilon) = \frac{E}{2(1+\nu)}\left\{\frac{e_{\alpha\beta}(1-k_\alpha k_\beta \Upsilon^2)}{(1+k_\alpha\Upsilon)(1+k_\beta\Upsilon)} + \frac{2\Upsilon}{(1+k_\alpha\Upsilon)(1+k_\beta\Upsilon)}\left[1+\frac{(k_\alpha+k_\beta)\Upsilon}{2}\right]\Upsilon\right\}$$

The strain $e_{\Upsilon\Upsilon}(\Upsilon)$ may also be directly found by a substitution of the expressions for $e_{\alpha\alpha}(\Upsilon)$ and $e_{\beta\beta}(\Upsilon)$. Because of the complexity of development, the quantity will be evaluated separtely in the suceeding section.

<u>Summarizing;</u>

$$\left[\begin{aligned}
\sigma_{\alpha\alpha}(\Upsilon) &= \frac{E}{(1-\nu^2)}\left\{\left[\frac{e_{\alpha\alpha}}{(1+k_\alpha\Upsilon)} + \frac{\nu e_{\beta\beta}}{(1+k_\beta\Upsilon)}\right] + \left[\frac{K_\alpha}{(1+k_\alpha\Upsilon)} + \frac{\nu K_\beta}{(1+k_\beta\Upsilon)}\right]\Upsilon\right. \\
&\qquad \left. + \left[\frac{k_\alpha}{(1+k_\alpha\Upsilon)} + \frac{\nu k_\beta}{(1+k_\beta\Upsilon)}\right]\int_0^\Upsilon e_{\Upsilon\Upsilon}\,d\Upsilon\right\} \\
\sigma_{\beta\beta}(\Upsilon) &= \frac{E}{(1-\nu^2)}\left\{\left[\frac{e_{\beta\beta}}{(1+k_\beta\Upsilon)} + \frac{\nu e_{\alpha\alpha}}{(1+k_\alpha\Upsilon)}\right] + \left[\frac{K_\beta}{(1+k_\beta\Upsilon)} + \frac{\nu K_\alpha}{(1+k_\alpha\Upsilon)}\right]\Upsilon\right. \\
&\qquad \left. + \left[\frac{k_\beta}{(1+k_\beta\Upsilon)} + \frac{\nu k_\alpha}{(1+k_\alpha\Upsilon)}\right]\int_0^\Upsilon e_{\Upsilon\Upsilon}\,d\Upsilon\right\} \\
\sigma_{\alpha\beta}(\Upsilon) &= \frac{E}{2(1+\nu)}\left\{\left[\frac{(1-k_\alpha k_\beta\Upsilon^2)e_{\alpha\beta}}{(1+k_\alpha\Upsilon)(1+k_\beta\Upsilon)}\right] + \frac{2\Upsilon}{(1+k_\alpha\Upsilon)(1+k_\beta\Upsilon)}\left[1+\frac{(k_\alpha+k_\beta)\Upsilon}{2}\right]\Upsilon\right\}
\end{aligned}\right.$$

5. 5. Evaluation of $e_{rr}(r)$.

Now the expression for $e_{rr}(r)$ has been given as;

$$e_{rr}(r) = -\frac{\nu}{E}\left(\sigma_{\alpha\alpha}(r) + \sigma_{\beta\beta}(r)\right)$$

Substituting for $\sigma_{\alpha\alpha}(r)$ and $\sigma_{\beta\beta}(r)$ and combining terms;

$$e_{rr}(r) = \frac{-\nu}{(1-\nu^2)}\left\{\left[\frac{(1+\nu)e_{\alpha\alpha}}{(1+k_\alpha r)} + \frac{(1+\nu)e_{\beta\beta}}{(1+k_\beta r)}\right] + \left[\frac{(1+\nu)}{(1+k_\alpha r)}K_\alpha + \frac{(1+\nu)}{(1+k_\beta r)}K_\beta\right] r \right.$$

$$\left. + \left[\frac{(1+\nu)k_\alpha}{(1+k_\alpha r)} + \frac{(1+\nu)k_\beta}{(1+k_\beta r)}\right]\int_0^r e_{rr}\, dr\right\}$$

Simplifying and rearranging;

$$e_{rr}(r) + \frac{\nu}{(1-\nu)}\left[\frac{k_\alpha}{(1+k_\alpha r)} + \frac{k_\beta}{(1+k_\beta r)}\right]\int_0^r e_{rr}\, dr = \frac{-\nu}{(1-\nu)}\left[\frac{e_{\alpha\alpha}}{(1+k_\alpha r)} + \frac{e_{\beta\beta}}{(1+k_\beta r)}\right]$$

$$-\frac{\nu}{(1-\nu)}\left[\frac{K_\alpha}{(1+k_\alpha r)} + \frac{K_\beta}{(1+k_\beta r)}\right] r$$

Now the resulting equation is an integral equation of the Valterra type. However, the equation can be transformed to a linear differential equation in the following manner. From the definition of e_{rr};

$$e_{rr} = \frac{\partial w(r)}{\partial r}$$

and hence;

$$\int_0^r \frac{\partial w}{\partial r}\, dr = w(r) - w$$

where w represents the displacement of the middle surface. Hence the corresponding differential equation becomes;

$$\frac{dw(r)}{dr} + \frac{\nu}{(1-\nu)}\left[\frac{k_\alpha}{(1+k_\alpha r)} + \frac{k_\beta}{(1+k_\beta r)}\right] w = \frac{\nu}{(1-\nu)}\left[\frac{k_\alpha}{(1+k_\alpha r)} + \frac{k_\beta}{(1+k_\beta r)}\right] w$$

$$-\frac{\nu}{(1-\nu)}\left[\frac{e_{\alpha\alpha}}{(1+k_\alpha r)} + \frac{e_{\beta\beta}}{(1+k_\beta r)}\right] - \frac{\nu}{(1-\nu)}\left[\frac{K_\alpha}{(1+k_\alpha r)} + \frac{K_\beta}{(1+k_\beta r)}\right] r$$

Thus the resulting equation is a first order linear non-homogeneous differential equation. The general solution for this type of equation is known. (see Agnew, "Differential Equations", pp. 36). Thus defining the quantities p (γ) and q (γ) as follows;

$$p(\gamma) = \frac{\nu}{(1-\nu)}\left[\frac{k_\alpha}{(1+k_\alpha\gamma)} + \frac{k_\beta}{(1+k_\beta\gamma)}\right]$$

$$q(\gamma) = \frac{\nu}{(1-\nu)}\left[\frac{k_\alpha}{(1+k_\alpha\gamma)} + \frac{k_\beta}{(1+k_\beta\gamma)}\right]w - \frac{\nu}{(1-\nu)}\left[\frac{e_{\alpha\alpha}}{(1+k_\alpha\gamma)} + \frac{e_{\beta\beta}}{(1+k_\beta\gamma)}\right]$$
$$-\frac{\nu}{(1-\nu)}\left[\frac{K_\alpha}{(1+k_\alpha\gamma)} + \frac{K_\beta}{(1+k_\beta\gamma)}\right]\gamma$$

the differential equation becomes;

$$\frac{dw(\gamma)}{d\gamma} + p(\gamma)\,w(\gamma) = q(\gamma)$$

the resulting solution is given as;

$$w(\gamma) = C_0\, e^{-\int_0^\gamma p(t)dt} + e^{-\int_0^\gamma p(t)dt} + \int_0^\gamma e^{\int_0^t p(s)ds}\, q(t)\,dt$$

where C_0 is the constant of integration. Consider evaluating this constant. When $\gamma = 0$, w (γ) = w. Thus; substituting, the result is

$$C_0 = w$$

Consider then evaluating the solution. To ease the complexity due to algebra, the exponentials will be first be evaluated. Consider then the integral

$$\int_0^\gamma p(t)dt = \frac{\nu}{(1-\nu)}\int_0^\gamma \left[\frac{k_\alpha}{(1+k_\alpha t)} + \frac{k_\beta}{(1+k_\beta t)}\right]dt$$

Integrating;

$$\int_0^\gamma p(t)dt = \frac{\nu}{(1-\nu)}\ln\left[(1+k_\alpha\gamma)(1+k_\beta\gamma)\right]$$

Consider defining the mean curvature H and the Gaussian curvature K as the following;

$$H = \frac{k_\alpha + k_\beta}{2} \quad ; \quad K = k_\alpha k_\beta$$

Then the above integral may be written as;

$$\int_0^\gamma p(t)dt = \frac{\nu}{(1-\nu)} \ln\left[1 + 2H\gamma + K\gamma^2\right]$$

or more conveniently;

$$\int_0^\gamma p(t)dt = \ln\left[1 + 2H\gamma + K\gamma^2\right]^{\frac{\nu}{(1-\nu)}}$$

Now the above integral, with plus or minus signs, enters into all the terms with the exponentials. However, by a basic identity

$$e^{\ln x} = x$$

and hence;

$$e^{\int_0^\gamma p(t)dt} = (1 + 2H\gamma + K\gamma^2)^{\frac{\nu}{(1-\nu)}}$$

$$e^{-\int_0^\gamma p(t)dt} = (1 + 2H\gamma + K\gamma^2)^{-\frac{\nu}{(1-\nu)}}$$

Ultimately, what is desired is a polynomial expression in γ. Hence the above expressions will be expanded by the binomial theorem. Now it will be abritrarily stated that the resultant expression is to be truncated at powers of γ over three.

Consider first expanding the negative exponent expression;

$$\left[1 + (2H\gamma + K\gamma^2)\right]^{-\frac{\nu}{(1-\nu)}} = 1 - \frac{\nu}{(1-\nu)}(2H\gamma + K\gamma^2) + \left[\frac{-\nu}{(1-\nu)}\right]\left[\frac{-1}{(1-\nu)}\right]\frac{(2H\gamma + K\gamma^2)^2}{2}$$

$$+ \left[-\frac{\nu}{(1-\nu)}\right]\left[-\frac{1}{(1-\nu)}\right]\left[-\frac{(2-\nu)}{(1-\nu)}\right]\frac{(2H\gamma + K\gamma^2)^3}{6}$$

Expanding and truncating;

$$e^{-\int_0^{\gamma} p(t)dt} = 1 + \frac{\nu}{(1-\nu)}\left[-2H\right]\gamma + \frac{\nu}{(1-\nu)}\left[-K + \frac{2}{(1-\nu)}H^2\right]\gamma^2 + \frac{\nu}{(1-\nu)}\left[\frac{2\nu}{(1-\nu)}HK - \right.$$

$$\left. - \frac{4}{3}\frac{(2-\nu)}{(1-\nu)^2}H^3\right]\gamma^3$$

Expanding now the positive exponential;

$$\left[1 + (2H\gamma + K\gamma^2)\right]^{\nu/(1-\nu)} = 1 + \frac{\nu}{(1-\nu)}(2H\gamma + K\gamma^2) + \left[\frac{\nu}{(1-\nu)}\right]\left[-\frac{(1-2\nu)}{(1-\nu)}\right]\frac{(2H\gamma+K\gamma^2)^2}{2} +$$

$$+ \left[\frac{\nu}{(1-2\nu)}\right]\left[-\frac{(1-2\nu)}{(1-\nu)}\right]\left[-\frac{(2-3\nu)}{(1-\nu)}\right]\frac{(2H\gamma+K\gamma^2)^3}{6}$$

Expanding and truncating;

$$e^{\int_0^{\gamma} p(t)dt} = 1 + \frac{\nu}{(1-\nu)}\left[2H\right]\gamma + \frac{\nu}{(1-\nu)}\left[K - \frac{2(1-2\nu)}{(1-\nu)}H^2\right]\gamma^2 + \frac{\nu}{(1-\nu)}\left[-2\frac{(1-2\nu)}{(1-\nu)}HK + \right.$$

$$\left. + \frac{4(1-2\nu)(2-3\nu)}{3(1-\nu)^2}H^3\right]\gamma^3$$

Consider now the function q (t). This quantity enters into the integrand of the expression for w (γ). Hence upon integration, it will be raised one power in γ. As a consequence, it will only be necessary to maintain quadratic terms in its polynomial exression.

Now;

$$\frac{1}{(1+k_\alpha\gamma)} = 1 - k_\alpha\gamma + k_\alpha^2\gamma^2; \quad \frac{1}{(1+k_\beta\gamma)} = 1 - k_\beta\gamma + k_\beta^2\gamma^2$$

Combining terms

$$q(\gamma) = \frac{\nu}{(1-\nu)}\left[2H\omega - (e_{\alpha\alpha} + e_{\beta\beta})\right] + \frac{\nu}{(1-\nu)}\left[-4H^2\omega + 2K\omega + (k_\alpha e_{\alpha\alpha} + k_\beta e_{\beta\beta}) - (K_\alpha + K_\beta)\right]\gamma$$

$$+ \frac{\nu}{(1-\nu)}\left[8H^3\omega - 6KH\omega - (k_\alpha^2 e_{\alpha\alpha} + k_\beta^2 e_{\beta\beta}) + (k_\alpha K_\alpha + k_\beta K_\beta)\right]\gamma^2$$

Forming now the product and combining terms;

$$e^{\int_0^{\gamma} p(t)dt} q(\gamma) = \frac{\nu}{(1-\nu)}\left\{2H\omega - (e_{\alpha\alpha} + e_{\beta\beta})\right\} + \frac{\nu}{(1-\nu)}\left\{2\left[K - 2\frac{(1-2\nu)}{(1-\nu)}H^2\right]\omega + \right.$$

$$+ \left[k_\alpha - \frac{2\nu}{(1-\nu)} H\right] e_{\alpha\alpha} + \left[k_\beta - \frac{2\nu}{(1-\nu)} H\right] e_{\beta\beta} - (K_\alpha + K_\beta)\Big\} \gamma +$$

$$+ \frac{\nu}{(1-\nu)} \Big\{ \left[\frac{4(1-2\nu)^2(2+\nu)}{(1-\nu)^2} H^3 - \frac{6(1-2\nu)}{(1-\nu)} K H \right] \omega + \left[\frac{2\nu(1-2\nu)}{(1-\nu)^2} H^2 - \frac{(1-2\nu)}{(1-\nu)} k_\alpha^2 \right] e_{\alpha\alpha}$$

$$+ \left[\frac{2\nu(1-2\nu)}{(1-\nu)^2} H^2 - \frac{(1-2\nu)}{(1-\nu)} k_\beta^2 \right] e_{\beta\beta} + \left[k_\alpha - \frac{2\nu}{(1-\nu)} H\right] K_\alpha + \left[k_\beta - \frac{2\nu}{(1-\nu)} H\right] K_\beta \Big\} \gamma^2$$

Integrating;

$$\int_0^\gamma e^{\int_0^t p(s)ds} q(t)dt = \frac{\nu}{(1-\nu)} \Big\{ 2H\omega - (e_{\alpha\alpha} + e_{\beta\beta}) \Big\} \gamma + \frac{\nu}{(1-\nu)} \Big\{ \left[K - \frac{2(1-2\nu)}{(1-\nu)} H^2 \right] \omega$$

$$+ \left[\frac{k_\alpha}{2} - \frac{\nu}{(1-\nu)} H \right] e_{\alpha\alpha} + \left[\frac{k_\beta}{2} - \frac{\nu}{(1-\nu)} H \right] e_{\beta\beta} - \frac{(K_\alpha + K_\beta)}{2} \Big\} \gamma^2 + \frac{\nu}{(1-\nu)} \Big\{ \Big[\frac{4(1-2\nu)^2(2+\nu)}{3(1-\nu)^2} H^3 +$$

$$+ \frac{2(1-2\nu)}{(1-\nu)} K H \Big] \omega + \left[\frac{2\nu(1-2\nu)}{3(1-\nu)^2} H^2 - \frac{(1-2\nu)}{3(1-\nu)} k_\alpha^2 \right] e_{\alpha\alpha} + \left[\frac{2\nu(1-2\nu)}{3(1-\nu)^2} H^2 - \frac{(1-2\nu)}{3(1-\nu)} k_\beta^2 \right] e_{\beta\beta}$$

$$+ \left[\frac{k_\alpha}{3} - \frac{2\nu}{3(1-\nu)} H \right] K_\alpha + \left[\frac{k_\beta}{3} - \frac{2\nu}{3(1-\nu)} H \right] K_\beta \Big\} \gamma^3$$

Consider now forming the product;

$$e^{-\int_0^\gamma p(t)dt} \int_0^\gamma e^{\int_0^t p(s)ds} q(t)dt = \frac{\nu}{(1-\nu)} \Big\{ 2H\omega - (e_{\alpha\alpha} + e_{\beta\beta}) \Big\} \gamma + \frac{\nu}{(1-\nu)} \Big\{ \left[K - \frac{2H^2}{(1-\nu)} \right] \omega +$$

$$+ \left[\frac{k_\alpha}{2} + \frac{\nu}{(1-\nu)} H \right] e_{\alpha\alpha} + \left[\frac{k_\beta}{2} + \frac{\nu}{(1-\nu)} H \right] e_{\beta\beta} - \frac{(K_\alpha + K_\beta)}{2} \Big\} \gamma^2 + \Big\{ \frac{8\nu}{(1-\nu)} K H \omega +$$

$$+\left[\frac{8\nu}{(1-\nu)}+\frac{4(1-2\nu)^2(2+\nu)}{3(1-\nu)^2}\right]H^3\omega+\left[\frac{2\nu(1-2\nu)}{3(1-\nu)^2}H^2-\frac{(1-2\nu)}{3(1-\nu)}k_\alpha^2-\frac{\nu}{(1-\nu)}k_\alpha H+\right.$$

$$\left.+\frac{\nu}{(1-\nu)}K\right]e_{\alpha\alpha}+\left[\frac{2\nu(1-2\nu)}{3(1-\nu)^2}H^2-\frac{(1-2\nu)}{3(1-\nu)}k_\beta^2-\frac{\nu}{(1-\nu)}k_\beta H+\frac{\nu}{(1-\nu)}K\right]e_{\beta\beta}$$

$$\left.+\left[\frac{k_\alpha}{3}+\frac{\nu}{3(1-\nu)}H\right]\mathcal{K}_\alpha+\left[\frac{k_\beta}{3}+\frac{\nu}{3(1-\nu)}H\right]\mathcal{K}_\beta\right\}\gamma^3$$

The resultant expression for w (γ) may thus be formed. Combining terms;

$$\omega(\gamma)=\omega+\frac{\nu}{(1-\nu)}\left\{-(e_{\alpha\alpha}+e_{\beta\beta})\right\}\gamma+\frac{\nu}{(1-\nu)}\left\{\left[k_\alpha+\frac{2\nu}{(1-\nu)}H\right]e_{\alpha\alpha}\right.$$

$$\left.+\left[k_\beta+\frac{2\nu}{(1-\nu)}H\right]e_{\beta\beta}-(\mathcal{K}_\alpha+\mathcal{K}_\beta)\right\}\frac{\gamma^2}{2}+\frac{\nu}{(1-\nu)}\left\{\frac{30\nu}{(1-\nu)}HK\right.$$

$$+\left[\frac{24\nu}{(1-\nu)}+\frac{4(1-2\nu)^2(2+\nu)}{(1-\nu)^2}-\frac{4(2-\nu)}{(1-\nu)^2}\right]H^3\omega+\left[\frac{2\nu(1-2\nu)}{(1-\nu)^2}H^2-\frac{(1-2\nu)}{(1-\nu)}k_\alpha^2-\right.$$

$$\left.-\frac{3\nu}{(1-\nu)}k_\alpha H+\frac{3\nu}{(1-\nu)}K\right]e_{\alpha\alpha}+\left[\frac{2\nu(1-2\nu)}{(1-\nu)^2}H^2-\frac{(1-2\nu)}{(1-\nu)}k_\beta^2-\frac{3\nu}{(1-\nu)}k_\beta H+\frac{3\nu}{(1-\nu)}K\right]e_{\beta\beta}$$

$$\left.+\left[k_\alpha+\frac{\nu}{(1-\nu)}H\right]\mathcal{K}_\alpha+\left[k_\beta+\frac{\nu}{(1-\nu)}H\right]\mathcal{K}_\beta\right\}\frac{\gamma^3}{3}$$

The above may also be written in a somewhat more simplified form since the Poisson coefficients may be simplified. Thus;

$$\omega(\gamma)=\omega+\frac{\nu}{(1-\nu)}\left\{-(e_{\alpha\alpha}+e_{\beta\beta})\right\}\gamma+\frac{\nu}{(1-\nu)}\left\{\left[k_\alpha+\frac{2\nu}{(1-\nu)}H\right]e_{\alpha\alpha}+\right.$$

$$
\begin{aligned}
&+\left[k_\beta+\frac{2\nu}{(1-\nu)}H\right]e_{\beta\beta}-(\mathcal{K}_\alpha+\mathcal{K}_\beta)\Big\}\frac{\gamma^2}{2}+\frac{\nu}{(1-\nu)}\Big\{\frac{30\nu}{(1-\nu)}HK+\\
&+\frac{4\nu}{(1-\nu)}\left[6-\frac{(4-\nu)(2+\nu)}{(1-\nu)}\right]H^3\omega+\Big[\frac{2\nu(1-2\nu)}{(1-\nu)^2}H^2-\frac{(1-2\nu)}{(1-\nu)}k_\alpha^2-\\
&-\frac{3\nu}{(1-\nu)}k_\alpha H+\frac{3\nu}{(1-\nu)}K\Big]e_{\alpha\alpha}+\Big[\frac{2\nu(1-2\nu)}{(1-\nu)^2}H^2-\frac{(1-2\nu)}{(1-\nu)}k_\beta^2-\frac{3\nu}{(1-\nu)}k_\beta H+\\
&+\frac{3\nu}{(1-\nu)}K\Big]e_{\beta\beta}+\left[k_\alpha+\frac{\nu}{(1-\nu)}H\right]\mathcal{K}_\alpha+\left[k_\beta+\frac{\nu}{(1-\nu)}H\right]\mathcal{K}_\beta\Big\}\frac{\gamma^3}{3}
\end{aligned}
$$

The corresponding strain, $e_{\gamma\gamma}(\gamma)$ is then given as;

$$
\begin{aligned}
e_{\gamma\gamma}(\gamma)=\;&\frac{\nu}{(1-\nu)}\Big\{-(e_{\alpha\alpha}+e_{\beta\beta})\Big\}+\frac{\nu}{(1-\nu)}\Big\{\left[k_\alpha+\frac{2\nu H}{(1-\nu)}\right]e_{\alpha\alpha}+\left[k_\beta+\frac{2\nu H}{(1-\nu)}\right]e_{\beta\beta}-\\
&-(\mathcal{K}_\alpha+\mathcal{K}_\beta)\Big\}\gamma+\frac{\nu}{(1-\nu)}\Big\{\frac{30\nu}{(1-\nu)}HK+\frac{4\nu}{(1-\nu)}\left[6-\frac{(4-\nu)(2+\nu)}{(1-\nu)}\right]H^3\omega\\
&+\Big[\frac{2\nu(1-2\nu)}{(1-\nu)^2}H^2-\frac{(1-2\nu)}{(1-\nu)}k_\alpha^2-\frac{3\nu}{(1-\nu)}k_\alpha H+\frac{3\nu}{(1-\nu)}K\Big]e_{\alpha\alpha}+\Big[\frac{2\nu(1-2\nu)}{(1-\nu)^2}H^2-\\
&-\frac{(1-2\nu)}{(1-\nu)}k_\beta^2-\frac{3\nu}{(1-\nu)}k_\beta H+\frac{3\nu}{(1-\nu)}K\Big]e_{\beta\beta}+\left[k_\alpha+\frac{\nu}{(1-\nu)}H\right]\mathcal{K}_\alpha\\
&+\left[k_\beta+\frac{\nu}{(1-\nu)}H\right]\mathcal{K}_\beta\Big\}\gamma^2
\end{aligned}
$$

Comments:

Note that at $\gamma = 0$, the strain $e_{\gamma\gamma}$ is the value that would have been predicted at the middle surface from elasticity considerations, namely

$$e_{\gamma\gamma} = \frac{-\nu}{(1-\nu)}(e_{\alpha\alpha}+e_{\beta\beta})$$

Note also that the expression for w (γ) contains five basic quantities, namely, w, $e_{\alpha\alpha}$, $e_{\beta\beta}$, $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$ and these quantities all appear within the first three terms of the series (i.e., γ^2 coefficients). Inclusion of higher powers of γ merely repeat these terms, but their coefficients become involved with the curvature parameters, i.e., k_α, k_β, H or K. Thus the series for w(γ)

at least converges in the sense that higher powers of γ will contain smaller and smaller increments of the five basic quantities previously mentioned.

5. 6. Stress Resultant Calculation.

The stress resultants have been found to be;

$$T_{\alpha\alpha} = \int_{-\delta/2}^{\delta/2} \sigma_{\alpha\alpha}(1+k_\beta \gamma)d\gamma \qquad T_{\alpha\beta} = \int_{-\delta/2}^{\delta/2} \sigma_{\alpha\beta}(1+k_\beta \gamma)d\gamma$$

$$T_{\beta\beta} = \int_{-\delta/2}^{\delta/2} \sigma_{\beta\beta}(1+k_\alpha \gamma)d\gamma \qquad T_{\beta\alpha} = \int_{-\delta/2}^{\delta/2} \sigma_{\alpha\beta}(1+k_\alpha \gamma)d\gamma$$

$$M_{\alpha\alpha} = \int_{-\delta/2}^{\delta/2} \sigma_{\alpha\alpha}\gamma(1+k_\beta \gamma)d\gamma \qquad M_{\alpha\beta} = \int_{-\delta/2}^{\delta/2} \sigma_{\alpha\beta}\gamma(1+k_\beta \gamma)d\gamma$$

$$M_{\beta\beta} = \int_{-\delta/2}^{\delta/2} \sigma_{\beta\beta}\gamma(1+k_\alpha \gamma)d\gamma \qquad M_{\beta\alpha} = \int_{-\delta/2}^{\delta/2} \sigma_{\beta\alpha}\gamma(1+k_\alpha \gamma)d\gamma$$

The resulting expressions for the stresses may now be substituted into the above equations. Though the integrations will not be particularly difficult, as might be expected, they will be more complex than integrating a polynomial. It is at this point that the error analysis will be introduced into the resulting equations and in a sense, this section marks the beginning of the technical theory of shells. However even at this junction, there is no common agreement as to which simplification to make. In particular, there are three in common usage and defined as;

i) Love's first approximation theory

ii) Modified Love's first approximation theory

iii) Love's second approximation theory.

Only the first approximation will be used in the following development;

1. Love's First Approximation Theory

Love's first approximation theory is the classical one in that almost all of the literature cited adheres to its approximations. Thus it is found in Love's "Mathematical Theory of Elasticity", Timoshenko's, "Plates and Shells", and ultimately in Novozhilov's, "Theory of Thin Shells". The first approximation theory requires that all terms of order of magnitude ($k\delta$) in comparison with unity be neglected.

A consequence of the above approximation is that the expressions for stress and strain may be simplified since the denominators, (1 + $k_\alpha \gamma$) and (1 + $k_\beta \gamma$) may be taken as unity. Thus the stress expressions become;

$$\sigma_{\alpha\alpha}(\gamma) = \frac{E}{(1-\nu^2)} \left\{ (e_{\alpha\alpha} + \nu e_{\beta\beta}) + (K_\alpha + \nu K_\beta)\gamma + (k_\alpha + \nu k_\beta) \int_0^\gamma e_{\gamma\gamma}\, d\gamma \right\}$$

$$\sigma_{\beta\beta}(\gamma) = \frac{E}{(1-\nu^2)} \left\{ (e_{\beta\beta} + \nu e_{\alpha\alpha}) + (K_\beta + \nu K_\alpha)\gamma + (k_\beta + \nu k_\alpha) \int_0^\gamma e_{\gamma\gamma}\, d\gamma \right\}$$

$$\sigma_{\alpha\beta}(\gamma) = \frac{E}{2(1+\nu)} \left\{ e_{\alpha\beta} + 2\Upsilon\gamma \right\}$$

Neglecting terms of $k\delta$ in comparison to unity, the integral of $e_{\gamma\gamma}(\gamma)$ becomes.

$$\int_0^\gamma e_{\gamma\gamma}\, d\gamma = -\frac{\nu}{(1-\nu)}(e_{\alpha\alpha} + e_{\beta\beta})\gamma - \frac{\nu}{(1-\nu)}(K_\alpha + K_\beta)\frac{\gamma^2}{2}$$

Comparing the terms resulting from the integral with the remaining terms in the stress expressions, it must be concluded that the contributions of the integral is of order ($k\delta$) in comparison with unity and hence must be neglected in the first approximation theory. Thus the resultant expressions for the stresses become.

$$\sigma_{\alpha\alpha}(r) = \frac{E}{(1-\nu^2)}\left\{(e_{\alpha\alpha} + \nu e_{\beta\beta}) + (K_\alpha + \nu K_\beta)r\right\}$$

$$\sigma_{\beta\beta}(r) = \frac{E}{(1-\nu^2)}\left\{(e_{\beta\beta} + \nu e_{\alpha\alpha}) + (K_\beta + \nu K_\alpha)r\right\}$$

$$\sigma_{\alpha\beta}(r) = \frac{E}{2(1+\nu)}\left\{e_{\alpha\beta} + 2\tau r\right\}$$

The corresponding strain expressions become;

$$e_{\alpha\alpha}(r) = e_{\alpha\alpha} + K_\alpha r$$

$$e_{\beta\beta}(r) = e_{\beta\beta} + K_\beta r$$

$$e_{\alpha\beta}(r) = e_{\alpha\beta} + 2\tau r$$

$$e_{rr}(r) = -\frac{\nu}{(1-\nu)}(e_{\alpha\alpha} + e_{\beta\beta}) - \frac{\nu}{(1-\nu)}(K_\alpha + K_\beta)r$$

Note the simplicity of the above expressions and their relationship to the form that would have been derived for a flat plate. The form of the equations for the shell and the plate are identical. Thus Love's first approximation describes the conditions that exist in a shell that is so thin in comparison with its curvature that the curvature effects do not influence the stress strain relations. This is exactly the same phenomenon that is experienced in very thin curved beams where the classical beam bending formula is still retained.

One further simplification may be made and that is in regard to the curvature change and twist experiences. Now these quantities have been given as

$$K_\alpha = -\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta} + \frac{1}{A}\frac{\partial}{\partial\alpha}(k_\alpha u) + \frac{k_\alpha}{AB}\frac{\partial A}{\partial\beta}v$$

$$K_\beta = -\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha} + \frac{1}{B}\frac{\partial}{\partial\beta}(k_\beta v) + \frac{k_\beta}{AB}\frac{\partial B}{\partial\alpha}u$$

$$\gamma = -\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\partial\beta} + \frac{1}{A^2B}\frac{\partial B}{\partial\beta}\frac{\partial w}{\partial\alpha} + \frac{1}{AB^2}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta} + k_\alpha\frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{u}{A}\right) + k_\beta\frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{v}{B}\right)$$

Whereas the strain expressions have been given as;

$$e_{\alpha\alpha} = \frac{1}{A}\frac{\partial u}{\partial\alpha} + \frac{1}{AB}\frac{\partial A}{\partial\beta}v + k_\alpha w$$

$$e_{\beta\beta} = \frac{1}{B}\frac{\partial v}{\partial\beta} + \frac{1}{AB}\frac{\partial B}{\partial\alpha}u + k_\beta w$$

$$e_{\alpha\beta} = \frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{u}{A}\right) + \frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{v}{B}\right)$$

Comparing the tangential displacement terms in the curvature change and torsion expressions to those in the strain expressions, it is seen that they are of the same order of magnitude. Since the curvature change expressions in stress and strain equations are multiplied by γ, it must be concluded that the curvature change and torsion expressions may be simplified by the exclusion of the tangential displacement terms. Thus;

$$K_\alpha = -\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta}$$

$$K_\beta = -\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha}$$

$$\gamma = -\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\partial\beta} + \frac{1}{A^2B}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} + \frac{1}{AB^2}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta}$$

Consider now the evaluation of the stress resultants. Again, these expressions may be simplified in that the terms $(1+k_\alpha\gamma)$ and $(1+k_\beta\gamma)$ occurring in the integrals may be given the value of unity. Substituting and integrating, the results become;

$$
\begin{cases}
T_{\alpha\alpha} = \dfrac{E\delta}{(1-\nu^2)}\left(e_{\alpha\alpha} + \nu e_{\beta\beta}\right) & T_{\alpha\beta} = \dfrac{E\delta}{2(1+\nu)}\, e_{\alpha\beta} \\
T_{\beta\beta} = \dfrac{E\delta}{(1-\nu^2)}\left(e_{\beta\beta} + \nu e_{\alpha\alpha}\right) & T_{\beta\alpha} = \dfrac{E\delta}{2(1+\nu)}\, e_{\alpha\beta} \\
M_{\alpha\alpha} = \dfrac{E\delta^3}{12(1-\nu^2)}\left(K_\alpha + \nu K_\beta\right) & M_{\alpha\beta} = \dfrac{E\delta^3}{12(1+\nu)}\, \tau \\
M_{\beta\beta} = \dfrac{E\delta^3}{12(1-\nu^2)}\left(K_\beta + \nu K_\alpha\right) & M_{\beta\alpha} = \dfrac{E\delta^3}{12(1+\nu)}\, \tau
\end{cases}
$$

Comments:

Love's first approximation is the classical one as used in shell analysis. Reissner (NACA TN 1833) states that all the essential ingredients of shell theory are embodied in it. In his book, Novozhilov states that this approximation is due to Mushtari-Vlasov (See Novozhilov, pp. 85) and is to be used when the bending stresses are of the same order of magnitude as the membrane or in-plane stresses. However, whatever the resulting error, it will be the analysis on which all subsequent work will be based.

Certain interesting consequences of the first approximation result. Note in particular that the effect of the normal strain $e_{\gamma\gamma}$ vanishes. Hence if a first approximation derivation were to be initially stated, the contraction or expansion of the normal line segment could be neglected. In essence this is stating that a simultaneous plane stress and plane strain condition may be assumed without effecting the first approximation assumptions. However, this conclusion hold only for stress resultants. A further comment will be included for displacement formulations.

Another interesting observation is that within the scope of first approximation theory, $T_{\alpha\beta} = T_{\beta\alpha}$ and $M_{\alpha\beta} = M_{\beta\alpha}$. However, consider the last equilibrium equation, the one obtained by summing moments about the γ axis. This equation is given as;

$$T_{\alpha\beta} - T_{\beta\alpha} + k_\alpha M_{\alpha\beta} - k_\beta M_{\beta\alpha} = 0$$

Thus first approximation theory does not satisfy the above equation unless the principal curvatures are equal. Hence it must be concluded that the sixth equilibrium equation must be neglected in using first approximation theory.

The present development is based on a stress resultant formulation and for such a formulation, it has been found that the effect of the contraction or expansion of the normal may be neglected so far as first order theory is concerned. However consider the alternate formulation of the shell problem, namely in terms of middle surface displacements. Under such circumstances, inspection of the term w (γ) indicates that terms quadratic in γ must be maintained before the first approximation theory can be utilized to discard terms. In fact, for a first approximation theory, w (γ) must be taken as;

$$w(\gamma) = w - \frac{\nu}{(1-\nu)}(e_{\alpha\alpha} + e_{\beta\beta})\gamma - \frac{\nu}{(1-\nu)}(K_\alpha + K_\beta)\frac{\gamma^2}{2}$$

If w (γ) is chosen as just w, then the resulting displacement formulation will yield plane strain rather than plane stress solutions.

5. 7. Additional Comments on Curvature and Twist Simplifications

The quantities κ_α, κ_β and γ have been given as

$$\kappa_\alpha = -\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta} + \frac{1}{A}\frac{\partial}{\partial\alpha}(k_\alpha u) + \frac{k_\beta}{AB}\frac{\partial A}{\partial\beta}v$$

$$\kappa_\beta = -\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha} + \frac{1}{B}\frac{\partial}{\partial\beta}(k_\beta v) + \frac{k_\alpha}{AB}\frac{\partial B}{\partial\alpha}u$$

$$\gamma = -\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\partial\beta} + \frac{1}{A^2B}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} + \frac{1}{AB^2}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta} + k_\alpha\left(\frac{1}{B}\frac{\partial u}{\partial\beta} - \frac{1}{AB}\frac{\partial A}{\partial\beta}u\right)$$
$$+ k_\beta\left(\frac{1}{A}\frac{\partial v}{\partial\alpha} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}v\right)$$

In dealing with Love's first approximation, it had been pointed out that on a basis of a displacement comparison, the tangential displacement terms appearing in the above expressions might be omitted and the above equations could then be written in the simple form

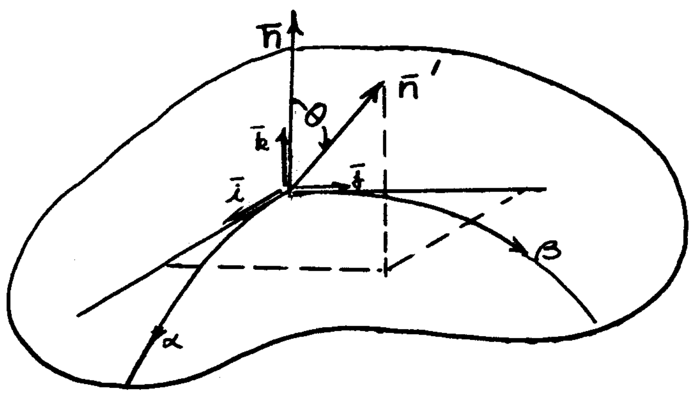
$$\kappa_\alpha = -\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta}$$

$$\kappa_\beta = -\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha}$$

$$\gamma = -\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\partial\beta} + \frac{1}{A^2B}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} + \frac{1}{AB^2}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta}$$

The above simplification will hold for those cases where the displacements are small and of the same order of magnitude as the strains. Thus the simplifications will certainly hold true for shallow shells and for those shells where the bending stresses will be the same order of magnitude as the in plane stresses. For many shells, however, the above approximation is too restrictive and it is the purpose of the present section to show that the curvature change and twist expressions should generally be used with the tangential displacement terms.

As a surface deforms, the deformed normal, $\bar{n}'$, will rotate relative to the undeformed normal, $\bar{n}$, so that if the two normal were placed at a common origin, say on the undeformed surface, the situation might appear as shown below.



Let the angle of rotation between the two normals be θ and assume that this angle is small. The expression for $\bar{n}'$ had been found in Chapter II and was given as;

$$\bar{n}' = \frac{1}{\sqrt{EG-F^2}}\left[\left(k_\alpha ABu - B\frac{\partial w}{\partial \alpha}\right)\bar{i} + \left(k_\beta ABv - A\frac{\partial w}{\partial \beta}\right)\bar{j} + AB(1+e_{\alpha\alpha}+e_{\beta\beta})\bar{k}\right]$$

The quantities E, F, and G were coefficients of the first quadratic form of the deformed middle surface and were found to be;

$$E = A^2(1+2e_{\alpha\alpha}) \;; \quad F = ABe_{\alpha\beta} \;; \quad G = B^2(1+2e_{\beta\beta})$$

Substituting and linearizing, the expression $\bar{n}'$ finally becomes;

$$\bar{n}' = \left(k_\alpha u - \frac{1}{A}\frac{\partial w}{\partial \alpha}\right)\bar{i} + \left(k_\beta v - \frac{1}{B}\frac{\partial w}{\partial \beta}\right)\bar{j} + \bar{k}$$

Now for sufficiently small values of the angle θ, this quantity may be treated as a vector quantity with components along the $\bar{i}$, $\bar{j}$ and $\bar{k}$ axis. Remembering that $\bar{n}$ and $\bar{n}'$ are both unit vectors, then;

$$\bar{\theta} = \bar{n} \times \bar{n}' = \bar{k} \times \left[\left(k_\alpha u - \frac{1}{A}\frac{\partial w}{\partial \alpha}\right)\bar{i} + \left(k_\beta v - \frac{1}{B}\frac{\partial w}{\partial \beta}\right)\bar{j} + \bar{k}\right]$$

Expanding;

$$\bar{\theta} = \left(\frac{1}{B}\frac{\partial w}{\partial \beta} - k_\beta v\right)\bar{i} + \left(k_\alpha u - \frac{1}{A}\frac{\partial w}{\partial \alpha}\right)\bar{j}$$

Thus the vector of rotation of the normal $\bar{n}'$ lies in the tangent plane to the undeformed surface and hence the plane of the vector $\bar{n}$ and $\bar{n}'$ are perpendicular to this plane. Note further that the $\bar{\theta}$ vector lies in the second quadrant so that the $\bar{i}$ component of this vector is in the negative $\bar{i}$ direction.

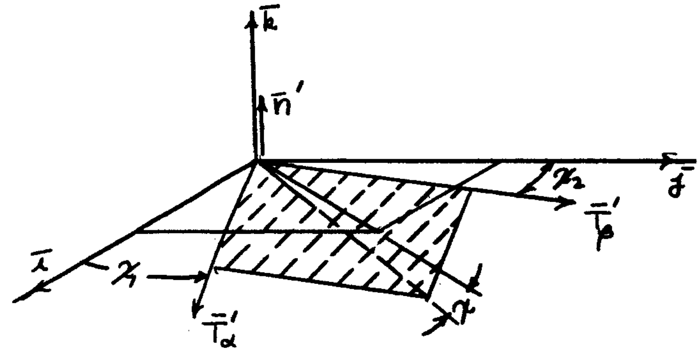
Define the scaler components of the rotation vector $\bar{\theta}$ as θ_α and θ_β, where;

$$\theta_\alpha = k_\beta v - \frac{1}{B}\frac{\partial w}{\partial \beta} \; ; \quad \theta_\beta = k_\alpha u - \frac{1}{A}\frac{\partial w}{\partial \alpha}$$

Hence;

$$\bar{\theta} = -\theta_\alpha \bar{i} + \theta_\beta \bar{j}$$

As a surface deforms and an element of that surface distorts and changes dimensions, the element also rotates about a normal to the surface. The figure below shows a rectangular element on an undeformed surface. The deformed element is superposed on the figure so that both elements lie on the tangent plane to the undeformed surface. The rotation of the element, designated as χ, will be measured by the relative angular displacements of the diagonals of the deformed and undeformed element.



Note that $\bar{T}_\alpha'$ and $\bar{T}_\beta'$ are the tangent vectors to the α and β curvalinear coordinate curves of the deformed surface and also that

$$2\chi = \chi_1 - \chi_2$$

Now as developed in Chapter 2;

$$\bar{T}_\alpha' = \bar{i} + \left(\frac{1}{A}\frac{\partial v}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}u\right)\bar{j} + \left(\frac{1}{A}\frac{\partial w}{\partial \alpha} - k_\alpha u\right)\bar{k}$$

$$\bar{T}_\beta' = \left(\frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}v\right)\bar{i} + \bar{j} + \left(\frac{1}{B}\frac{\partial w}{\partial \beta} - k_\beta v\right)\bar{k}$$

Hence;

$$\chi_1 = (\bar{i} \times \bar{T}_\alpha') \cdot \bar{k} = \frac{1}{A}\frac{\partial v}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}u$$

$$\chi_2 = (\bar{T}_\beta' \times \bar{j}) \cdot \bar{k} = \frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}v$$

Therefore the rotation of an element on the surface is given as;

$$2\chi = \frac{1}{A}\frac{\partial v}{\partial \alpha} + \frac{1}{B}\frac{\partial B}{\partial \alpha}v - \frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{1}{AB}\frac{\partial A}{\partial \beta}u$$

Now the expressions for K_α, K_β and γ may be written as;

$$K_\alpha = \frac{1}{A}\frac{\partial \theta_\beta}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}\theta_\alpha$$

$$\mathcal{K}_\beta = \frac{1}{B}\frac{\partial\theta_\alpha}{\partial\beta} + \frac{1}{AB}\frac{\partial B}{\partial\alpha}\theta_\beta$$

$$2\gamma = \frac{1}{A}\frac{\partial\theta_\alpha}{\partial\alpha} - \frac{1}{AB}\frac{\partial A}{\partial\beta}\theta_\beta + \frac{1}{B}\frac{\partial\theta_\beta}{\partial\beta} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}\theta_\alpha + (k_\beta - k_\alpha)\chi + He_{\alpha\beta}$$

where H is the mean curvature and $e_{\alpha\beta}$ the sheer stress. If it is now remembered that the curvature are change, $k_\alpha' - k_\alpha$, was given as;

$$k_\alpha' - k_\alpha = \mathcal{K}_\alpha - k_\alpha e_{\alpha\alpha}$$

then in terms of physical changes on the middle surface

$$k_\alpha' - k_\alpha = \frac{1}{A}\frac{\partial\theta_\beta}{\partial\alpha} + \frac{1}{AB}\frac{\partial A}{\partial\beta}\theta_\alpha - k_\alpha e_{\alpha\alpha}$$

$$k_\beta' - k_\beta = \frac{1}{B}\frac{\partial\theta_\alpha}{\partial\beta} + \frac{1}{AB}\frac{\partial B}{\partial\alpha}\theta_\beta - k_\beta e_{\beta\beta}$$

$$2\gamma = \frac{1}{A}\frac{\partial\theta_\alpha}{\partial\alpha} - \frac{1}{AB}\frac{\partial A}{\partial\beta}\theta_\beta + \frac{1}{B}\frac{\partial\theta_\beta}{\partial\beta} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}\theta_\alpha + (k_\beta - k_\alpha)\chi + He_{\alpha\beta}$$

Recalling now the simplifications that had been made in calculating curvature changes, it had been stated that the quantities $k_\alpha e_{\alpha\alpha}$, and $k_\beta e_{\beta\beta}$ could be neglected in comparison with $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$. Then attempting at consistency, the tangential displacement terms in $\mathcal{K}_\alpha$, $\mathcal{K}_\beta$ and had been discarded on the premise that they would be of the same order of magnitude as the tangential displacement terms occurring in the neglected strain terms.

Observing the above equations, it is noted that the tangential displacement terms occurring in $\mathcal{K}_\alpha$ and $k_\alpha e_{\alpha\alpha}$ measure two different physical conditions. In the first case, these terms enter into the rotation of the normal while in the latter they enter into the strain. If it is now assumed that although the strains are small, the rotations need not be providing they do not violate the bounds of linear theory, then the expressions K_α, K_β will measure the change in curvature and γ will measure the torsion, but these expressions must include the tangential displacement terms since these terms enter into the rotation expressions.

Summarizing, the expressions for K_α, K_β and γ, for reasonably large rotations of the middle surface, should be taken as

$$\begin{cases} K_\alpha = -\dfrac{1}{A}\dfrac{\partial}{\partial\alpha}\left(\dfrac{1}{A}\dfrac{\partial w}{\partial\alpha}\right) - \dfrac{1}{AB^2}\dfrac{\partial A}{\partial\beta}\dfrac{\partial w}{\partial\beta} + \dfrac{1}{A}\dfrac{\partial}{\partial\alpha}(k_\alpha u) + \dfrac{k_\beta}{AB}\dfrac{\partial A}{\partial\beta} v \\[2ex] K_\beta = -\dfrac{1}{B}\dfrac{\partial}{\partial\beta}\left(\dfrac{1}{B}\dfrac{\partial w}{\partial\beta}\right) - \dfrac{1}{A^2B}\dfrac{\partial B}{\partial\alpha}\dfrac{\partial w}{\partial\alpha} + \dfrac{1}{B}\dfrac{\partial}{\partial\beta}(k_\beta v) + \dfrac{k_\alpha}{AB}\dfrac{\partial B}{\partial\alpha} u \\[2ex] \gamma = -\dfrac{1}{AB}\dfrac{\partial^2 w}{\partial\alpha\partial\beta} + \dfrac{1}{A^2B}\dfrac{\partial A}{\partial\beta}\dfrac{\partial w}{\partial\alpha} + \dfrac{1}{AB^2}\dfrac{\partial B}{\partial\alpha}\dfrac{\partial w}{\partial\beta} + k_\alpha\left(\dfrac{1}{B}\dfrac{\partial u}{\partial\beta} - \dfrac{1}{AB}\dfrac{\partial A}{\partial\beta} u\right) \end{cases}$$

$$+ k_\beta\left(\frac{1}{A}\frac{\partial v}{\partial\alpha} - \frac{1}{AB}\frac{\partial B}{\partial\alpha} v\right)$$

CHAPTER VI

6. 1. Summary of Equations

Consider now a summary of the equilibrium equations, compatibility equations and stress resultant-deformation equations based on Love's first approximation.

Equilibrium Equations

$$
\left\{
\begin{aligned}
&\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BT_{\alpha\alpha})+\frac{\partial}{\partial\beta}(AT_{\beta\alpha})+\frac{\partial A}{\partial\beta}T_{\alpha\beta}-\frac{\partial B}{\partial\alpha}T_{\beta\beta}\right]+k_\alpha T_{\alpha\gamma}+q_\alpha=0\\
&\frac{1}{AB}\left[\frac{\partial}{\partial\beta}(AT_{\beta\beta})+\frac{\partial}{\partial\alpha}(BT_{\alpha\beta})+\frac{\partial B}{\partial\alpha}T_{\beta\alpha}-\frac{\partial A}{\partial\beta}T_{\alpha\alpha}\right]+k_\beta T_{\beta\gamma}+q_\beta=0\\
&\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BT_{\alpha\gamma})+\frac{\partial}{\partial\beta}(AT_{\beta\gamma})\right]-k_\alpha T_{\alpha\alpha}-k_\beta T_{\beta\beta}+q_\gamma=0\\
&\frac{1}{AB}\left[\frac{\partial}{\partial\beta}(AM_{\beta\beta})+\frac{\partial}{\partial\alpha}(BM_{\alpha\beta})-\frac{\partial A}{\partial\beta}M_{\alpha\alpha}+\frac{\partial B}{\partial\alpha}M_{\beta\alpha}\right]-T_{\beta\gamma}=0\\
&\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BM_{\alpha\alpha})+\frac{\partial}{\partial\beta}(AM_{\beta\alpha})-\frac{\partial B}{\partial\alpha}M_{\beta\beta}+\frac{\partial A}{\partial\beta}M_{\alpha\beta}\right]-T_{\alpha\gamma}=0\\
&T_{\alpha\beta}-T_{\beta\alpha}+k_\alpha M_{\alpha\beta}-k_\beta M_{\beta\alpha}=0
\end{aligned}
\right.
$$

Compatibility Equations

$$
\left\{
\begin{aligned}
&B\frac{\partial K_\beta}{\partial\alpha}+\frac{\partial B}{\partial\alpha}(K_\beta-K_\alpha)-A\frac{\partial\tau}{\partial\beta}-2\frac{\partial A}{\partial\beta}\tau+k_\beta\frac{\partial A}{\partial\beta}e_{\alpha\beta}+k_\alpha\left[A\frac{\partial e_{\alpha\beta}}{\partial\beta}+\frac{\partial A}{\partial\beta}e_{\alpha\beta}-B\frac{\partial e_{\beta\beta}}{\partial\alpha}-\frac{\partial B}{\partial\alpha}(e_{\beta\beta}-e_{\alpha\alpha})\right]=0\\
&A\frac{\partial K_\alpha}{\partial\beta}+\frac{\partial A}{\partial\beta}(K_\alpha-K_\beta)-B\frac{\partial\tau}{\partial\alpha}-2\frac{\partial B}{\partial\alpha}\tau+k_\alpha\frac{\partial B}{\partial\alpha}e_{\alpha\beta}+k_\beta\left[B\frac{\partial e_{\alpha\beta}}{\partial\alpha}+\frac{\partial B}{\partial\alpha}e_{\alpha\beta}-A\frac{\partial e_{\alpha\alpha}}{\partial\beta}-\frac{\partial A}{\partial\beta}(e_{\alpha\alpha}-e_{\beta\beta})\right]=0\\
&k_\beta K_\alpha+k_\alpha K_\beta+\frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\cdot\frac{1}{A}\left[B\frac{\partial e_{\beta\beta}}{\partial\alpha}+\frac{\partial B}{\partial\alpha}(e_{\beta\beta}-e_{\alpha\alpha})-\frac{A}{2}\frac{\partial e_{\alpha\beta}}{\partial\beta}-\frac{\partial A}{\partial\beta}e_{\alpha\beta}\right]+\frac{\partial}{\partial\beta}\cdot\frac{1}{B}\left[A\frac{\partial e_{\alpha\alpha}}{\partial\beta}+\right.\right.\\
&\left.\left.+\frac{\partial A}{\partial\beta}(e_{\alpha\alpha}-e_{\beta\beta})-\frac{B}{2}\frac{\partial e_{\alpha\beta}}{\partial\alpha}-\frac{\partial B}{\partial\alpha}e_{\alpha\beta}\right]\right\}=0
\end{aligned}
\right.
$$

Stress Resultant-Deformation Equations (Love's First Approximation)

$$
\left\{
\begin{array}{ll}
T_{\alpha\alpha} = \dfrac{E\delta}{(1-\nu^2)}\left(e_{\alpha\alpha} + \nu e_{\beta\beta}\right) & M_{\alpha\alpha} = \dfrac{E\delta^3}{12(1-\nu^2)}\left(\kappa_\alpha + \nu\kappa_\beta\right) \\[2ex]
T_{\alpha\beta} = \dfrac{E\delta}{2(1+\nu)}\, e_{\alpha\beta} & M_{\alpha\beta} = \dfrac{E\delta^3}{12(1+\nu)}\, \tau \\[2ex]
T_{\beta\beta} = \dfrac{E\delta}{(1-\nu^2)}\left(e_{\beta\beta} + \nu e_{\alpha\alpha}\right) & M_{\beta\beta} = \dfrac{E\delta^3}{12(1-\nu^2)}\left(\kappa_\beta + \nu\kappa_\alpha\right) \\[2ex]
T_{\beta\alpha} = \dfrac{E\delta}{2(1+\nu)}\, e_{\alpha\beta} & M_{\beta\alpha} = \dfrac{E\delta^3}{12(1+\nu)}\, \tau
\end{array}
\right.
$$

An alternate statement of the same equations wherin the deformations are expressed as functions of the stress resultants is given as follows.

$$
\left\{
\begin{array}{ll}
e_{\alpha\alpha} = \dfrac{1}{E\delta}\left(T_{\alpha\alpha} - \nu T_{\beta\beta}\right) & \kappa_\alpha = \dfrac{12}{E\delta^3}\left(M_{\alpha\alpha} - \nu M_{\beta\beta}\right) \\[2ex]
e_{\alpha\beta} = \dfrac{2(1+\nu)}{E\delta}\, T_{\alpha\beta} & \kappa_\beta = \dfrac{12}{E\delta^3}\left(M_{\beta\beta} - \nu M_{\alpha\alpha}\right) \\[2ex]
e_{\beta\beta} = \dfrac{1}{E\delta}\left(T_{\beta\beta} - \nu T_{\alpha\alpha}\right) & \tau = \dfrac{12(1+\nu)}{E\delta^3}\, M_{\alpha\beta}
\end{array}
\right.
$$

Strain Displacement Relations

$$
\left\{
\begin{array}{l}
e_{\alpha\alpha} = \dfrac{1}{A}\dfrac{\partial u}{\partial \alpha} + \dfrac{1}{AB}\dfrac{\partial A}{\partial \beta} v + k_\alpha w \,;\; e_{\beta\beta} = \dfrac{1}{B}\dfrac{\partial v}{\partial \beta} + \dfrac{1}{AB}\dfrac{\partial B}{\partial \alpha} u + k_\beta w \,;\; e_{\alpha\beta} = \dfrac{A}{B}\dfrac{\partial}{\partial \beta}\left(\dfrac{u}{A}\right) + \dfrac{B}{A}\dfrac{\partial}{\partial \alpha}\left(\dfrac{v}{B}\right) \\[3ex]
\kappa_\alpha = -\dfrac{1}{A}\dfrac{\partial}{\partial \alpha}\left(\dfrac{1}{A}\dfrac{\partial w}{\partial \alpha}\right) - \dfrac{1}{AB^2}\dfrac{\partial A}{\partial \beta}\dfrac{\partial w}{\partial \beta} + \dfrac{1}{A}\dfrac{\partial}{\partial \alpha}\left(k_\alpha u\right) + \dfrac{k_\beta}{AB}\dfrac{\partial A}{\partial \beta} v \\[2ex]
\kappa_\beta = -\dfrac{1}{B}\dfrac{\partial}{\partial \beta}\left(\dfrac{1}{B}\dfrac{\partial w}{\partial \beta}\right) - \dfrac{1}{A^2 B}\dfrac{\partial B}{\partial \alpha}\dfrac{\partial w}{\partial \alpha} + \dfrac{1}{B}\dfrac{\partial}{\partial \beta}\left(k_\beta v\right) + \dfrac{k_\alpha}{AB}\dfrac{\partial B}{\partial \alpha} u \\[2ex]
\tau = -\dfrac{1}{AB}\dfrac{\partial^2 w}{\partial \alpha \partial \beta} + \dfrac{1}{A^2 B}\dfrac{\partial A}{\partial \beta}\dfrac{\partial w}{\partial \alpha} + \dfrac{1}{AB^2}\dfrac{\partial B}{\partial \alpha}\dfrac{\partial w}{\partial \beta} + k_\alpha\left(\dfrac{1}{B}\dfrac{\partial u}{\partial \beta} - \dfrac{1}{AB}\dfrac{\partial A}{\partial \beta} u\right) + k_\beta\left(\dfrac{1}{A}\dfrac{\partial v}{\partial \alpha} - \dfrac{1}{AB}\dfrac{\partial B}{\partial \alpha} v\right)
\end{array}
\right.
$$

First and Second Quadratic Forms

$$\begin{cases} \mathrm{I} = A^2(1+2e_{\alpha\alpha})(d\alpha)^2 + 2ABe_{\alpha\beta}(d\alpha d\beta) + B^2(1+2e_{\beta\beta})(d\beta)^2 \\ \mathrm{II} = -A^2\left[k_\alpha(1+e_{\alpha\alpha})+\kappa_\alpha\right](d\alpha)^2 - 2AB\Upsilon(d\alpha d\beta) - B^2\left[k_\beta(1+e_{\beta\beta})+\kappa_\beta\right](d\beta)^2 \end{cases}$$

6. 2. Resultant Shell Equations Discussion

Two formulations of the shell equations are possible. One is a formulation whereby the resultant shell equations are exhibited in middle surface displacement form and the other whereby a middle surface deformation presentation is used. In the present work, the latter formulation will be utilized.

In attempting to use the middle surface deformation formulation of the shell equations, it would be wise to briefly review the shell problem. In dealing with the differential geometry of surfaces, it was pointed out that if the Gauss-Peterson-Codozzi conditions were satisfied and if the first and second quadratic forms of the surface were known, then a surface was uniquely specified up to its position in space. These arguements, when applied to a surface perturbed from some reference surface, led to the definitions of the three strains, $e_{\alpha\alpha}$, $e_{\alpha\beta}$, $e_{\beta\beta}$ and the curvature changes, κ_α, κ_β, Υ. Thus a total of six unknowns were necessary to define the perturbed surface. The Gauss-Peterson-Codozzi conditions yielded three relations involving the above six unknowns and hence the surface problem become one of third degree of indeterminacy.

A given shell was reduced to a surface problem by using stress resultants and these in turn were related to each other by means of the equilibrium equations. It had been hoped that the additional equations provided by the equilibrium conditions would ultimately yield the additional three equations relating the surface deformations so as to uniquely define the perturbed or deformed surface. However, the equilibrium equations were in stress resultant form whereas the compatibility equations and indeed the first and second quadratic forms of the surface were in deformation form. Hence additional equations relating the stress resultants and deformations had to be developed. It was at this point that the various assumptions were introduced thus transforming what up to then was a rigorous linear analysis to a technical analysis.

Two major limitations were imposed on the linear shell analysis in developing stress resultant deformation relations. The first limitation was the use of Kirchoff hypothesis which prescribed the displacement variation through the shell thickness and the second was the use of Love's first approximation theory which truncated resulting expressions. Thus an error was introduced into the shell analysis over that which ordinarily would be associated with linear analysis. The magnitude of the error was and is generally thought to be of order ($k\delta$).

Once the stress resultant deformation relations exist, then there exists a total of 16 unknowns, six deformations and ten stress resultants. However, there exists a total of seventeen equations, namely the three compatibility equations, the six equilibrium equations and the ten stress resultant deformation results. Thus, there exists one more equation than there are unknowns. However, inspection of the last of the equilibrium equations, the one found by taking moments about the $\bar{k}$ axis, is an algebraic relation relating the transverse shear stress resultants to the twisting moment stress resultants. This equation may then be thought of as an equation which is not linearly independent of the remaining sixteen equations in that it may be derived from the stress resultant deformation relations for the stress resultant involved in its form. Thus it may be concluded that there are only sixteen linearly independent equations and sixteen unknowns. Assuming suitable boundary conditions, the resulting system of equations is solvable.

The introduction of Love's first approximation theory has somewhat simplified the number of unknowns necessary to consider. Thus as has been found, $T_{\alpha\beta} = T_{\beta\alpha}$ and $M_{\alpha\beta} = M_{\beta\alpha}$. Hence, there are only a total of fourteen unknowns. The number of independent equations is now fourteen, the three compatibility equations, the five equilibrium equations (the sixth being discarded in that it's not linearly independent of the remaining equations), and the six stress resultant deformation relations. Note one thing, in dealing with the general problem it was mentioned that the sixth equilibrium equation could be discarded in that it could be derived from the stress resultant deformation equation relating transverse shear and twisting moment. In using

Love's first approximation theory, this equation must be discarded in that it cannot be satisfied.

The problem that is being now confronted is the manner in which the resultant shell equations are to be formulated. They may be stated in terms of strains and curvature changes or they may be formulated using stress resultants. From the mathematical view either formulation is acceptable. However, prevalence in literature dictates that the stress resultant formulation is the more desirable.

Whatever formulation is utilized, it is almost invariably true that the transverse shear stress resultants are eliminated from the system of equations. The reasons for this are two fold. One is that the use of the Kirchoff hypothesis in essence negates the existence of this stress resultants and secondly, stress resultants by themselves are not the end of shell analysis. Invariably, once the stress resultants have been calculated, either middle surface displacements or stresses are calculated from the stress resultant solution. Thus if the transverse sheer stress resultant were explicitly solved, then since there does not exist any stress resultant deformation relation for this variable, an auxiliary stress resultant system of equations would have to be solved in order to find the other stress resultants.

Since a stress resultant formulation of the resultant shell equations will be presented and since Love's first approximation theory will be utilized in expressing the relations between deformations and stress resultants, then a total of eight equations will be necessary. Now the equations of equilibrium provide five equations and thus it will be necessary to transform the three compatibility equations into stress resultant form.

6.3. Compatibility Equations in Terms of Stress Resultants.

In order to facilitate the transformation, each of the compatibility equations will be dealt with separately. Furthermore, if the first transformed compatibility equation is obtained, the second may be found by an interchange of subscripts. Thus, it is only necessary to transform the first and third of the compatibility equations.

1. First compatiblity equation.

This equation is given as

$$B\frac{\partial \kappa_\beta}{\partial \alpha}+\frac{\partial B}{\partial \alpha}(\kappa_\beta-\kappa_\alpha)-A\frac{\partial \gamma}{\partial \beta}-2\frac{\partial A}{\partial \beta}\gamma+k_\beta\frac{\partial A}{\partial \beta}e_{\alpha\beta}+k_\alpha\Big[A\frac{\partial e_{\alpha\beta}}{\partial \beta}+\frac{\partial A}{\partial \beta}e_{\alpha\beta}-$$

$$-B\frac{\partial e_{\beta\beta}}{\partial \alpha}-\frac{\partial B}{\partial \alpha}(e_{\beta\beta}-e_{\alpha\alpha})\Big]=0$$

Substituting the deformation-stress resultant expressions;

$$\frac{12}{E\delta^3}B\left(\frac{\partial M_{\beta\beta}}{\partial \alpha}-\nu\frac{\partial M_{\alpha\alpha}}{\partial \alpha}\right)+\frac{12(1+\nu)}{E\delta^3}\frac{\partial B}{\partial \alpha}(M_{\beta\beta}-M_{\alpha\alpha})-\frac{12(1+\nu)}{E\delta^3}A\frac{\partial M_{\alpha\beta}}{\partial \beta}$$

$$-\frac{24(1+\nu)}{E\delta^3}\frac{\partial A}{\partial \beta}M_{\alpha\beta}+\frac{2(1+\nu)}{E\delta}k_\beta\frac{\partial A}{\partial \beta}T_{\alpha\beta}+k_\alpha\Big[\frac{2(1+\nu)}{E\delta}A\frac{\partial T_{\alpha\beta}}{\partial \beta}+\frac{2(1+\nu)}{E\delta}\frac{\partial A}{\partial \beta}T_{\alpha\beta}-$$

$$-\frac{1}{E\delta}B\left(\frac{\partial T_{\beta\beta}}{\partial \alpha}-\nu\frac{\partial T_{\alpha\alpha}}{\partial \alpha}\right)-\frac{(1+\nu)}{E\delta}\frac{\partial B}{\partial \alpha}(T_{\beta\beta}-T_{\alpha\alpha})\Big]=0$$

Multiplying through by $E\delta^3/12$, the result becomes;

$$B\left(\frac{\partial M_{\beta\beta}}{\partial\alpha} - \nu\frac{\partial M_{\alpha\alpha}}{\partial\alpha}\right) + (1+\nu)\frac{\partial B}{\partial\alpha}(M_{\beta\beta}-M_{\alpha\alpha}) - (1+\nu)A\frac{\partial M_{\alpha\beta}}{\partial\beta} - 2(1+\nu)\frac{\partial A}{\partial\beta}M_{\alpha\beta}$$

$$+\frac{(1+\nu)}{6}\delta^2 k_\beta\frac{\partial A}{\partial\beta}T_{\alpha\beta} + \frac{\delta^2}{12}k_\alpha\left[2(1+\nu)A\frac{\partial T_{\alpha\beta}}{\partial\beta} + 2(1+\nu)\frac{\partial A}{\partial\beta}T_{\alpha\beta} - B\left(\frac{\partial T_{\beta\beta}}{\partial\alpha} - \nu\frac{\partial T_{\alpha\alpha}}{\partial\alpha}\right) -\right.$$

$$\left.-(1+\nu)\frac{\partial B}{\partial\alpha}(T_{\beta\beta}-T_{\alpha\alpha})\right] = 0$$

The above equation may be simplified but the basis of the simplifications lies not in the stress resultants but rather in the displacements. Note first that the force stress resultants are prefixed by a quantity $k\delta^2$ which indicates that so far as order of terms are concerned, the force stress resultants are at least of order $(k\delta)$ higher than the moment stress resultants. Note further that each force stress resultant has its corresponding image in the moment stress resultant. That is to say the structure of the first compatibility equations in looking at the stress resultant part is the same as that for the moment stress resultant.

If the above equation were expressed in terms of strains and curvature changes, note that the order of magnitude of all its terms would be the same. This is most readily seen by inspecting the original and given statement of the first compatibility equation. Now is using Love's first approximation, it has been shown that the curvature change and twist expressions could be simplified so as to contain only terms dependent on the rotations. Thus strain terms of the type $(k\ e)$ where k is the curvature and e is the strain could be discarded in comparison with the curvature change κ or the twist γ. Now each curvature change or twist term in the first compatibility equation has a direct counterpart in the strain terms and hence on the basis of consistency, these latter terms may be dropped. Thus the force stress resultants may be omitted and the first compatibility equation takes the form

$$B\frac{\partial M_{\beta\beta}}{\partial\alpha} - \nu B\frac{\partial M_{\alpha\alpha}}{\partial\alpha} + (1+\nu)\frac{\partial B}{\partial\alpha}M_{\beta\beta} - (1+\nu)\frac{\partial B}{\partial\alpha}M_{\alpha\alpha} - (1+\nu)A\frac{\partial M_{\alpha\beta}}{\partial\beta}$$
$$-2(1+\nu)\frac{\partial A}{\partial\beta}M_{\alpha\beta} = 0$$

Or rewriting;

$$\left(B\frac{\partial M_{\beta\beta}}{\partial\alpha} + \frac{\partial B}{\partial\alpha}M_{\beta\beta}\right) - \left(\nu B\frac{\partial M_{\alpha\alpha}}{\partial\alpha} + \nu\frac{\partial B}{\partial\alpha}M_{\alpha\alpha}\right) - \frac{\partial B}{\partial\alpha}\left(M_{\alpha\alpha} - \nu M_{\beta\beta}\right) -$$
$$-\left[(1+\nu)A\frac{\partial M_{\alpha\beta}}{\partial\beta} + (1+\nu)\frac{\partial A}{\partial\beta}M_{\alpha\beta}\right] - (1+\nu)\frac{\partial A}{\partial\beta}M_{\alpha\beta} = 0$$

Or more conveniently, in the final form

$$\frac{\partial}{\partial\alpha}(BM_{\beta\beta}) - \nu\frac{\partial}{\partial\alpha}(BM_{\alpha\alpha}) - \frac{\partial B}{\partial\alpha}(M_{\alpha\alpha} - \nu M_{\beta\beta}) - (1+\nu)\frac{\partial}{\partial\beta}(AM_{\alpha\beta}) - (1+\nu)\frac{\partial A}{\partial\beta}M_{\alpha\beta} = 0$$

By direct analogy, the second compatibility equation may be written.

2. Third compatibility equation.

The third compatibility equation is given as;

$$k_\beta\kappa_\alpha + k_\alpha\kappa_\beta + \frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\cdot\frac{1}{A}\left[B\frac{\partial e_{\beta\beta}}{\partial\alpha} + \frac{\partial B}{\partial\alpha}(e_{\beta\beta} - e_{\alpha\alpha}) - \frac{A}{2}\frac{\partial e_{\alpha\beta}}{\partial\beta} - \frac{\partial A}{\partial\beta}e_{\alpha\beta}\right]\right.$$
$$\left.+\frac{\partial}{\partial\beta}\cdot\frac{1}{B}\left[A\frac{\partial e_{\alpha\beta}}{\partial\beta} + \frac{\partial A}{\partial\beta}(e_{\alpha\alpha} - e_{\beta\beta}) - \frac{B}{2}\frac{\partial e_{\alpha\beta}}{\partial\alpha} - \frac{\partial B}{\partial\alpha}e_{\alpha\beta}\right]\right\} = 0$$

Note that in this case, simplifications of the type encountered in the first compatibility equation cannot be effected. The tangential displacement terms in the above equation occur in derivative form whereas the curvatures appear as algebraic structures. Thus an order of comparison argument cannot be made.

Substituting

$$\frac{12k_\beta}{E\delta^3}(M_{\alpha\alpha}-\nu M_{\beta\beta})+\frac{12k_\alpha}{E\delta^3}(M_{\beta\beta}-\nu M_{\alpha\alpha})+\frac{1}{ABE\delta}\frac{\partial}{\partial\alpha}\left\{\frac{1}{A}\left[B\frac{\partial}{\partial\alpha}(T_{\beta\beta}-\nu T_{\alpha\alpha})\right.\right.$$

$$\left.\left.+(1+\nu)\frac{\partial B}{\partial\alpha}(T_{\beta\beta}-T_{\alpha\alpha})-(1+\nu)\frac{\partial}{\partial\beta}(AT_{\alpha\beta})-(1+\nu)\frac{\partial A}{\partial\beta}T_{\alpha\beta}\right]\right\}+$$

$$+\frac{1}{ABE\delta}\frac{\partial}{\partial\beta}\left\{\frac{1}{B}\left[A\frac{\partial}{\partial\beta}(T_{\alpha\alpha}-\nu T_{\beta\beta})+(1+\nu)\frac{\partial A}{\partial\beta}(T_{\alpha\alpha}-T_{\beta\beta})-(1+\nu)\frac{\partial}{\partial\alpha}(BT_{\alpha\beta})-\right.\right.$$

$$\left.\left.-(1+\nu)\frac{\partial B}{\partial\alpha}T_{\alpha\beta}\right]\right\}=0$$

Multiplying through by $E\delta^3/12$ and recombining the terms;

$$k_\beta(M_{\alpha\alpha}-\nu M_{\beta\beta})+k_\alpha(M_{\beta\beta}-\nu M_{\alpha\alpha})+\frac{\delta^2}{12AB}\frac{\partial}{\partial\alpha}\left\{\frac{1}{A}\left[\frac{\partial}{\partial\alpha}(BT_{\beta\beta})-\nu\frac{\partial}{\partial\alpha}(BT_{\alpha\alpha})\right.\right.$$

$$\left.\left.-\frac{\partial B}{\partial\alpha}(T_{\alpha\alpha}-\nu T_{\beta\beta})-(1+\nu)\frac{\partial}{\partial\beta}(AT_{\alpha\beta})-(1+\nu)\frac{\partial A}{\partial\beta}T_{\alpha\beta}\right]\right\}+\frac{\delta^2}{12AB}\frac{\partial}{\partial\beta}\left\{\frac{1}{B}\left[\frac{\partial}{\partial\beta}(AT_{\alpha\alpha})-\right.\right.$$

$$\left.\left.-\nu\frac{\partial}{\partial\beta}(AT_{\beta\beta})-\frac{\partial A}{\partial\beta}(T_{\alpha\alpha}-\nu T_{\beta\beta})-(1+\nu)\frac{\partial}{\partial\beta}(AT_{\alpha\beta})-(1+\nu)\frac{\partial A}{\partial\beta}T_{\alpha\beta}\right]\right\}=0$$

Summary;

$$\left\{\begin{aligned}
&\frac{\partial}{\partial\alpha}(BM_{\beta\beta})-\nu\frac{\partial}{\partial\alpha}(BM_{\alpha\alpha})-\frac{\partial B}{\partial\alpha}(M_{\alpha\alpha}-\nu M_{\beta\beta})-(1+\nu)\frac{\partial}{\partial\alpha}(AM_{\alpha\beta})-(1+\nu)\frac{\partial A}{\partial\beta}M_{\alpha\beta}=0\\
&\frac{\partial}{\partial\beta}(AM_{\alpha\alpha})-\nu\frac{\partial}{\partial\beta}(AM_{\beta\beta})-\frac{\partial A}{\partial\beta}(M_{\beta\beta}-\nu M_{\alpha\alpha})-(1+\nu)\frac{\partial}{\partial\beta}(BM_{\alpha\beta})-(1+\nu)\frac{\partial B}{\partial\alpha}M_{\alpha\beta}=0\\
&k_\beta(M_{\alpha\alpha}-\nu M_{\beta\beta})+k_\alpha(M_{\beta\beta}-\nu M_{\alpha\alpha})+\frac{\delta^2}{12AB}\frac{\partial}{\partial\alpha}\left\{\frac{1}{A}\left[\frac{\partial}{\partial\alpha}(BT_{\beta\beta})-\nu\frac{\partial}{\partial\alpha}(BT_{\alpha\alpha})-\right.\right.\\
&\left.\left.-\frac{\partial B}{\partial\alpha}(T_{\alpha\alpha}-\nu T_{\beta\beta})-(1+\nu)\frac{\partial}{\partial\beta}(AT_{\alpha\beta})-(1+\nu)\frac{\partial A}{\partial\beta}T_{\alpha\beta}\right]\right\}+\frac{\delta^2}{12AB}\frac{\partial}{\partial\beta}\left\{\frac{1}{B}\left[\frac{\partial}{\partial\beta}(AT_{\alpha\alpha})-\right.\right.\\
&\left.\left.-\nu\frac{\partial}{\partial\beta}(AT_{\beta\beta})-\frac{\partial A}{\partial\beta}(T_{\beta\beta}-\nu T_{\alpha\alpha})-(1+\nu)\frac{\partial}{\partial\alpha}(BT_{\alpha\beta})-(1+\nu)\frac{\partial B}{\partial\alpha}T_{\alpha\beta}\right]\right\}=0
\end{aligned}\right.$$

6. 4. Resultant Differential Equations for A Shell.

Consider now summarizing the equations of compatibility and equilibrium.

(1) $$\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BT_{\alpha\alpha})+\frac{\partial}{\partial\beta}(AT_{\alpha\beta})+\frac{\partial A}{\partial\beta}T_{\alpha\beta}-\frac{\partial B}{\partial\alpha}T_{\beta\beta}\right]+k_\alpha T_{\alpha\gamma}+q_\alpha=0$$

(2) $$\frac{1}{AB}\left[\frac{\partial}{\partial\beta}(AT_{\beta\beta})+\frac{\partial}{\partial\alpha}(BT_{\alpha\beta})+\frac{\partial B}{\partial\alpha}T_{\alpha\beta}-\frac{\partial A}{\partial\beta}T_{\alpha\alpha}\right]+k_\beta T_{\beta\gamma}+q_\beta=0$$

(3) $$\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BT_{\alpha\gamma})+\frac{\partial}{\partial\beta}(AT_{\beta\gamma})\right]-k_\alpha T_{\alpha\alpha}-k_\beta T_{\beta\beta}+q_\gamma=0$$

(4) $$\frac{1}{AB}\left[\frac{\partial}{\partial\beta}(AM_{\beta\beta})+\frac{\partial}{\partial\alpha}(BM_{\alpha\beta})-\frac{\partial A}{\partial\beta}M_{\alpha\alpha}+\frac{\partial B}{\partial\alpha}M_{\alpha\beta}\right]-T_{\beta\gamma}=0$$

(5) $$\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BM_{\alpha\alpha})+\frac{\partial}{\partial\beta}(AM_{\alpha\beta})-\frac{\partial B}{\partial\alpha}M_{\beta\beta}+\frac{\partial A}{\partial\beta}M_{\alpha\beta}\right]-T_{\alpha\gamma}=0$$

(6) $$\frac{\partial}{\partial\alpha}(BM_{\beta\beta})-\nu\frac{\partial}{\partial\alpha}(BM_{\alpha\alpha})-(M_{\alpha\alpha}-\nu M_{\beta\beta})\frac{\partial B}{\partial\alpha}-(1+\nu)\frac{\partial}{\partial\beta}(AM_{\alpha\beta})-(1+\nu)M_{\alpha\beta}\frac{\partial A}{\partial\beta}=0$$

(7) $$\frac{\partial}{\partial\beta}(AM_{\alpha\alpha})-\nu\frac{\partial}{\partial\beta}(AM_{\beta\beta})-(M_{\beta\beta}-\nu M_{\alpha\alpha})\frac{\partial A}{\partial\beta}-(1+\nu)\frac{\partial}{\partial\alpha}(BM_{\alpha\beta})-(1+\nu)M_{\alpha\beta}\frac{\partial B}{\partial\alpha}=0$$

(8) $$(M_{\alpha\alpha}-\nu M_{\beta\beta})k_\beta+(M_{\beta\beta}-\nu M_{\alpha\alpha})k_\alpha+\frac{\delta^2}{12AB}\frac{\partial}{\partial\alpha}\left\{\frac{1}{A}\left[\frac{\partial}{\partial\alpha}(BT_{\beta\beta})-\nu\frac{\partial}{\partial\alpha}(BT_{\alpha\alpha})-\right.\right.$$

$$\left.\left.-\frac{\partial B}{\partial\alpha}(T_{\alpha\alpha}-\nu T_{\beta\beta})-(1+\nu)\frac{\partial}{\partial\beta}(AT_{\alpha\beta})-(1+\nu)\frac{\partial A}{\partial\beta}T_{\alpha\beta}\right]\right\}+\frac{\delta^2}{12AB}\frac{\partial}{\partial\beta}\left\{\frac{1}{B}\left[\frac{\partial}{\partial\beta}(AT_{\alpha\alpha})-\nu\frac{\partial}{\partial\beta}(AT_{\beta\beta})-\right.\right.$$

$$\left.\left.-\frac{\partial A}{\partial\beta}(T_{\beta\beta}-\nu T_{\alpha\alpha})-(1+\nu)\frac{\partial}{\partial\alpha}(BT_{\alpha\beta})-(1+\nu)\frac{\partial B}{\partial\alpha}T_{\alpha\beta}\right]\right\}=0$$

Note that there are as many unknowns as there are equations and hence a solution is possible. Now each of the stress resultants can be expressed in terms of middle surface displacements except for two, $T_{\alpha\gamma}$ and $T_{\beta\gamma}$. The scheme is now to eliminate these two stress resultants from the above corresponding system of equations.

Consider solving for $T_{\alpha\gamma}$ and $T_{\beta\gamma}$ from equations (4) and (5).

Thus;

$$T_{\alpha\gamma} = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BM_{\alpha\alpha}) + \frac{\partial}{\partial\beta}(AM_{\alpha\beta}) - \frac{\partial B}{\partial\alpha}M_{\beta\beta} + \frac{\partial A}{\partial\beta}M_{\alpha\beta}\right]$$

$$T_{\beta\gamma} = \frac{1}{AB}\left[\frac{\partial}{\partial\beta}(AM_{\beta\beta}) + \frac{\partial}{\partial\alpha}(BM_{\alpha\beta}) - \frac{\partial A}{\partial\beta}M_{\alpha\alpha} + \frac{\partial B}{\partial\alpha}M_{\alpha\beta}\right]$$

Further, consider eliminating $M_{\alpha\beta}$ from the above two equations. Since the above two equations are symmetric in the α and β subscripts and the terms associated with them, it is only necessary to deal with one equation, say the expression for $T_{\alpha\gamma}$. Now for the elimination of $M_{\alpha\beta}$ in the expression for $T_{\alpha\gamma}$ equation (6) will be utilized. Multiplying $T_{\alpha\gamma}$ by (1 + v) AB and making the substitution;

$$(1+\nu)ABT_{\alpha\gamma} = (1+\nu)\left[\frac{\partial}{\partial\alpha}(BM_{\alpha\alpha}) - \frac{\partial B}{\partial\alpha}M_{\beta\beta}\right] + \frac{\partial}{\partial\alpha}(BM_{\beta\beta}) - \nu\frac{\partial}{\partial\alpha}(BM_{\alpha\alpha}) -$$

$$- (M_{\alpha\alpha} - \nu M_{\beta\beta})\frac{\partial B}{\partial\alpha}$$

Expanding

$$(1+\nu)ABT_{\alpha\gamma} = B\frac{\partial M_{\alpha\alpha}}{\partial\alpha} + \frac{\partial B}{\partial\alpha}M_{\alpha\alpha} - \frac{\partial B}{\partial\alpha}M_{\beta\beta} + \nu B\frac{\partial M_{\alpha\alpha}}{\partial\alpha} + \nu\frac{\partial B}{\partial\alpha}M_{\alpha\alpha}$$

$$-\nu\frac{\partial B}{\partial\alpha}M_{\beta\beta} + B\frac{\partial M_{\beta\beta}}{\partial\alpha} + \frac{\partial B}{\partial\alpha}M_{\beta\beta} - \nu B\frac{\partial M_{\alpha\alpha}}{\partial\alpha} - \nu\frac{\partial B}{\partial\alpha}M_{\alpha\alpha}$$

$$-\frac{\partial B}{\partial\alpha}M_{\alpha\alpha} + \nu\frac{\partial B}{\partial\alpha}M_{\beta\beta}$$

Combining terms;

$$(1+\nu)ABT_{\alpha\gamma} = B\frac{\partial}{\partial\alpha}(M_{\alpha\alpha} + M_{\beta\beta})$$

or;

$$T_{\alpha\gamma} = \frac{1}{A(1+\nu)}\frac{\partial}{\partial\alpha}(M_{\alpha\alpha} + M_{\beta\beta})$$

and hence by analogy;

$$T_{\beta\gamma} = \frac{1}{B(1+\nu)}\frac{\partial}{\partial\beta}(M_{\alpha\alpha} + M_{\beta\beta})$$

Note that the above results correlate with those found in simple beam bending theory, namely that the transverse sheer force is directly dependent on the moment.

Consider now substituting the above relations into the remaining equations.

$$(1)\quad \frac{(1-\nu^2)}{AB}\left[\frac{\partial}{\partial\alpha}(BT_{\alpha\alpha})+\frac{\partial}{\partial\beta}(AT_{\alpha\beta})+\frac{\partial A}{\partial\beta}T_{\alpha\beta}-\frac{\partial B}{\partial\alpha}T_{\beta\beta}\right]+\frac{(1-\nu)k_\alpha}{A}\frac{\partial}{\partial\alpha}(M_{\alpha\alpha}+M_{\beta\beta})+(1-\nu^2)q_\alpha=0$$

$$(2)\quad \frac{(1-\nu^2)}{AB}\left[\frac{\partial}{\partial\beta}(AT_{\beta\beta})+\frac{\partial}{\partial\alpha}(BT_{\alpha\beta})+\frac{\partial B}{\partial\alpha}T_{\alpha\beta}-\frac{\partial A}{\partial\beta}T_{\alpha\alpha}\right]+\frac{(1-\nu)k_\beta}{B}\frac{\partial}{\partial\beta}(M_{\beta\beta}+M_{\alpha\alpha})+(1-\nu^2)q_\beta=0$$

$$(3)\quad \frac{(1-\nu)}{AB}\left\{\frac{\partial}{\partial\alpha}\left[\frac{B}{A}\frac{\partial}{\partial\alpha}(M_{\alpha\alpha}+M_{\beta\beta})\right]+\frac{\partial}{\partial\beta}\left[\frac{A}{B}\frac{\partial}{\partial\beta}(M_{\alpha\alpha}+M_{\beta\beta})\right]-(1-\nu^2)k_\alpha T_{\alpha\alpha}-(1-\nu^2)k_\beta T_{\beta\beta}+\right.$$
$$\left.+(1-\nu^2)q_\gamma=0\right.$$

$$(4)\quad \frac{\partial}{\partial\alpha}(BM_{\beta\beta})-\nu\frac{\partial}{\partial\alpha}(BM_{\alpha\alpha})-(M_{\alpha\alpha}-\nu M_{\beta\beta})\frac{\partial B}{\partial\alpha}-(1+\nu)\frac{\partial}{\partial\beta}(AM_{\alpha\beta})-(1+\nu)M_{\alpha\beta}\frac{\partial A}{\partial\beta}=0$$

$$(5)\quad \frac{\partial}{\partial\beta}(AM_{\alpha\alpha})-\nu\frac{\partial}{\partial\beta}(AM_{\beta\beta})-(M_{\beta\beta}-\nu M_{\alpha\alpha})\frac{\partial A}{\partial\beta}-(1+\nu)\frac{\partial}{\partial\alpha}(BM_{\alpha\beta})-(1+\nu)M_{\alpha\beta}\frac{\partial B}{\partial\alpha}=0$$

$$(6)\quad k_\beta(M_{\alpha\alpha}-\nu M_{\beta\beta})+k_\alpha(M_{\beta\beta}-\nu M_{\alpha\alpha})+\frac{\delta^2}{12AB}\frac{\partial}{\partial\alpha}\left\{\frac{1}{A}\left[\frac{\partial}{\partial\alpha}(BT_{\beta\beta})-\nu\frac{\partial}{\partial\alpha}(BT_{\alpha\alpha})\right.\right.$$
$$\left.\left.-\frac{\partial B}{\partial\alpha}(T_{\alpha\alpha}-\nu T_{\beta\beta})-(1+\nu)\frac{\partial}{\partial\beta}(AT_{\alpha\beta})-(1+\nu)\frac{\partial A}{\partial\beta}T_{\alpha\beta}\right]\right\}+\frac{\delta^2}{12AB}\frac{\partial}{\partial\beta}\left\{\frac{1}{B}\left[\frac{\partial}{\partial\beta}(AT_{\alpha\alpha})-\right.\right.$$
$$\left.\left.-\nu\frac{\partial}{\partial\beta}(AT_{\beta\beta})-\frac{\partial A}{\partial\beta}(T_{\beta\beta}-\nu T_{\alpha\alpha})-(1+\nu)\frac{\partial}{\partial\alpha}(BT_{\alpha\beta})-(1+\nu)\frac{\partial B}{\partial\alpha}T_{\alpha\beta}\right]\right\}=0$$

Note now that there are six equations in six unknowns.

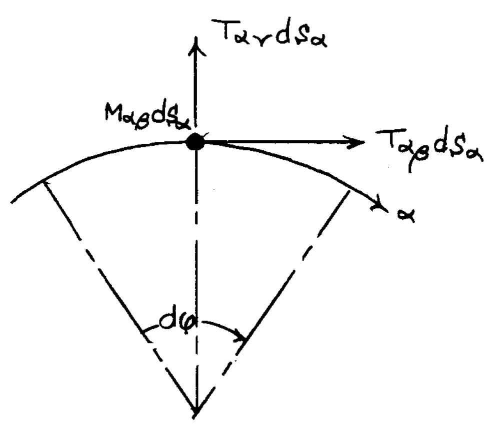
6. 5. Discussion of Boundary Conditions

What now has resulted is a set of six partial differential equations in terms of six unknowns, the quantities $T_{\alpha\alpha}$, $T_{\alpha\beta}$, $T_{\beta\beta}$,

$M_{\alpha\alpha}$, $M_{\alpha\beta}$, $M_{\beta\beta}$. Thus there are as many equations as there are unknowns and hence one requirement for a solution of the equation is satisfied. The other requirement is for suitable boundary conditions.

The use of the Kirchoff hypothesis in plate solutions had shown that in general, all boundary conditions on the displacements (or stress resultants) could not be satisfied and an equivalent transverse shear had to be developed. Since the Kirchoff hypotheses were also used in developing the present shell equations, a similar situation will also be found to be true.

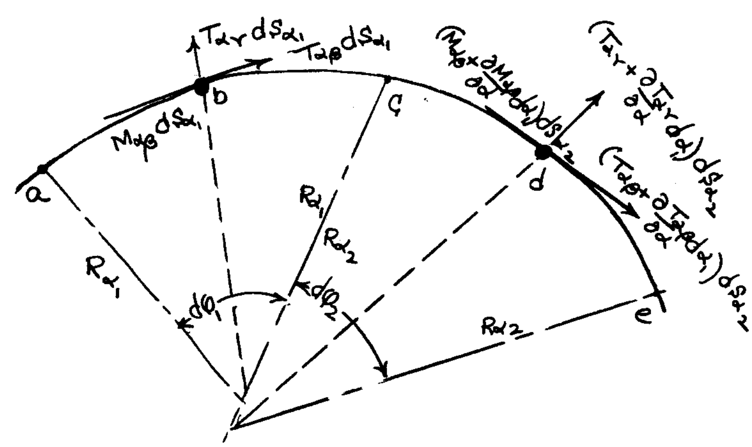
It is tacitly assumed that a free edge of a shell coincides with a principal curvalinear coordinate line, and further, that the free inner surface of the shell lies in a plane normal to the shell middle surface. Assume further that on this free edge, there exists a general stress resultant state. For arguments sake, let the edge of the shell coincide with some β = constant coordinate line. Then showing only the stress resultants $M_{\alpha\beta}$, $T_{\alpha\beta}$ and $T_{\alpha\gamma}$, the situation appears as shown.



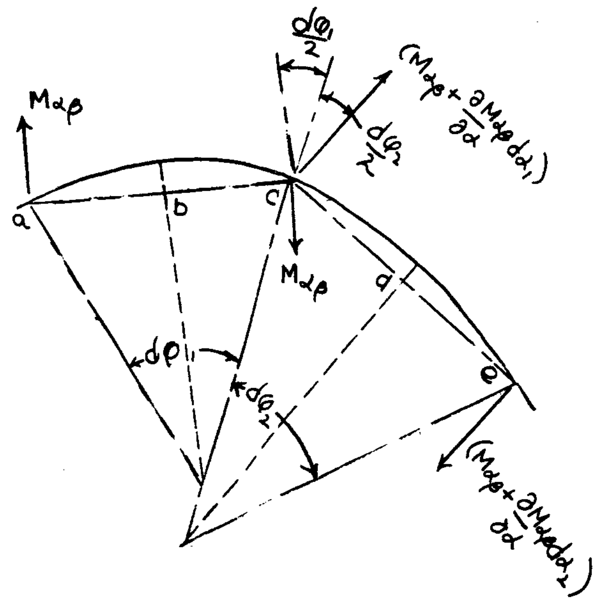
For a sufficiently small segment dS_α, the curvature of the principal coordinate line may be considered as constant. Now assume further that over this segment, the stress resultants, $T_{\alpha\gamma}$, $T_{\alpha\beta}$ and $M_{\alpha\beta}$ are constant. These corresponding forces and moments may be found by multiplying the stress resultants by the length increment dS_α and further, may be assumed to be acting at the center of the line segment as shown.

Consider now a sufficiently large segment of the α coordinate line and assume that it has been broken up into a series of segments each of which has a constant curvature but which may be different from the adjacent values. Furthermore, assume that for each segment there are the stress resultants and forces pictured in the above segment. Over each arc segment then, the stress resultants are assumed to be constant and of value equal to the stress resultant defined at the beginning of each arc segment. Thus in the sketch previously shown, the stress resultants indicated are the values found at the left end point of the arc.

Two adjacent arcs are pictured in the following sketch.



The end points for the first arc are a & c while its mid point is b. For the second arc, the end points are c & e while its mid point is d. Note that the curvatures of the two arc are different so that $k_{\alpha_2} = k_{\alpha_1} + \frac{\partial k_\alpha}{\partial \alpha} d\alpha_1$. Note further that the stress resultants for the second arc are defined at point c while for the first arc they are defined at point a. Consider now replacing the twisting moment, $M_{\alpha\beta}\, dS_{\alpha_1}$, by means of two forces equal in magnitude but opposite in direction. Let these forces be assumed to act perpendicular to the cords of each of the arcs and further let these forces be at the ends of the arcs. The situation is as shown. (The figure shows only the decomposition of $M_{\alpha\beta}\, dS_\alpha$)



At point c, consider the components of the resulting force along the chord and perpendicular to it. Now perpendicular to the chord, the resulting force component is;

$$\left(M_{\alpha\beta} + \frac{\partial M_{\alpha\beta}}{\partial \alpha} d\alpha_1\right) \cos\left(\frac{d\varphi_1 + d\varphi_2}{2}\right) - M_{\alpha\beta}$$

But for small angles, $\cos\left(\frac{d\varphi_1+d\varphi_2}{2}\right) \approx 1$, and hence

$$\frac{\partial M_{\alpha\beta}}{\partial \alpha} d\alpha_1$$

while parallel to the chord, the component is given as;

$$\left(M_{\alpha\beta} + \frac{\partial M_{\alpha\beta}}{\partial \alpha} d\alpha_1\right) \sin\left(\frac{d\varphi_1+d\varphi_2}{2}\right)$$

Again for small angles, $\sin\left(\frac{d\varphi_1+d\varphi_2}{2}\right) \approx \left(\frac{d\varphi_1+d\varphi_2}{2}\right)$. But for prinicpal curvalinear coordinates

$$d\varphi_1 = k_{\alpha_1} dS_{\alpha_1} \; ; \; d\varphi_2 = k_{\alpha_2} dS_{\alpha_2}$$

and further;

$$dS_{\alpha_1} = A\,d\alpha_1 \; ; \; dS_{\alpha_2} = \left(A + \frac{\partial A}{\partial \alpha} d\alpha_1\right) d\alpha_2$$

Substituting and simplifying;

$$d\varphi_1 = k_\alpha A\,d\alpha \; ; \; d\varphi_2 = k_\alpha A\,d\alpha$$

and hence;

$$\frac{d\varphi_1 + d\varphi_2}{2} = k_\alpha A\,d\alpha$$

Thus the horizontal component becomes;

$$\left(M_{\alpha\beta} + \frac{\partial M_{\alpha\beta}}{\partial \alpha} d\alpha_1\right) k_\alpha A\,d\alpha$$

Consider then the total resultant transverse and shear stress resultant forces. Letting these be designated as $T_{\alpha\gamma\,eff}$ and $T_{\alpha\beta\,eff}$, the result becomes;

$$T_{\alpha\gamma\,eff}\, dS_\alpha = T_{\alpha\gamma}\, dS_\alpha + \frac{1}{A}\frac{\partial M_{\alpha\beta}}{\partial \alpha} dS_\alpha$$

$$T_{\alpha\beta\,eff}\, dS_\alpha = T_{\alpha\beta}\, dS_\alpha + \left(M_{\alpha\beta} + \frac{\partial M_{\alpha\beta}}{\partial \alpha} d\alpha\right) k_\alpha dS_\alpha$$

Dividing through by dS_α and passing to the limit;

$$\left.\begin{aligned} T_{\alpha\gamma\,eff} &= T_{\alpha\gamma} + \frac{1}{A}\frac{\partial M_{\alpha\beta}}{\partial \alpha} \\ T_{\alpha\beta\,eff} &= T_{\alpha\beta} + k_\alpha M_{\alpha\beta} \end{aligned}\right] \qquad \beta = \text{constant coordinate line}$$

By direct analogy; ,

$$\left.\begin{aligned} T_{\beta\gamma\,eff} &= T_{\beta\gamma} + \frac{1}{B}\frac{\partial M_{\beta\alpha}}{\partial\alpha} \\ T_{\beta\alpha\,eff} &= T_{\beta\alpha} + k_{\beta} M_{\beta\alpha} \end{aligned}\right\} \qquad \alpha = \text{constant coordinat line}$$

6. 6. Commentary

Note that the derivation of the effective force stress resultants is a direct consequence of the Kirchoff hypothesis. However, it is not dependent on whether the first approximation, or its modification, or the second approximation is utilized. As a consequence, generality of results dictates that the distinction between $T_{\alpha\beta}$ and $T_{\beta\alpha}$ be maintained. The same is of course true for $M_{\alpha\beta}$ and $M_{\beta\alpha}$.

It should be specifically mentioned that the effective shears are used only for boundary conditions not for the interior of the shell. Note further that if the shell is a body of revolution and symmetrically loaded, then $M_{\alpha\beta} = M_{\beta\alpha} \equiv 0$ and hence the effective shears become the true values.

Thus on a given free edge of a shell, there are only four independent stress resultants to be evaluated rather then the five which would result if exactness of the stress resultants was postulated. The use of the effective sheers adds another approximation into shell theory, but this approximation yields errors of the same order of magnitude as the Kirchoff hypothesis.

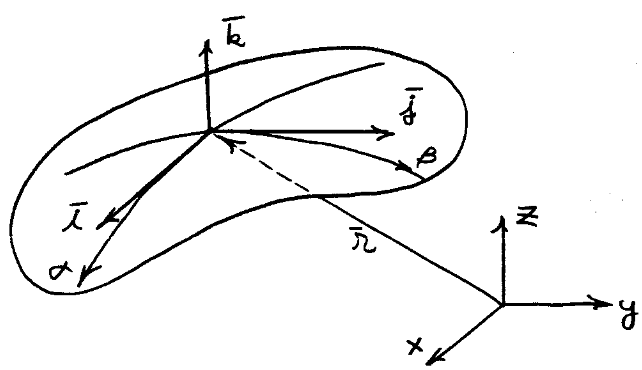
Addendum

In dealing with the material of Chapters 2 and 3, the derivatives of the unit vectors, $\bar{i}$, $\bar{j}$, and $\bar{k}$ are extensively used. However, nowhere in those chapters are these derivatives developed. To correct this oversight, the present addition is included.

Consider now a surface for which the α and β curves are the principal curvalinear coordinate curves. Let $\bar{i}$ and $\bar{j}$ be the tangents to the α and β curves and let $\bar{k}$ be given as

$$\bar{k} = \bar{i} \times \bar{j}$$

The situation is pictured below.



It is obvious that $\bar{k}$ is in the direction of the normal, $\bar{n}$, to the surface and further that;

$$d\bar{s}_\alpha = \frac{\partial \bar{r}}{\partial \alpha} d\alpha \; ; \quad d\bar{s}_\beta = \frac{\partial \bar{r}}{\partial \beta} d\beta$$

But in magnitude;

$$ds_\alpha = A\, d\alpha \; ; \; ds_\beta = B\, d\beta$$

and

$$d\bar{s}_\alpha = ds_\alpha \, \bar{i} \; ; \quad d\bar{s}_\beta = ds_\beta \, \bar{j}$$

Thus

$$\bar{i} = \frac{1}{A} \frac{\partial \bar{r}}{\partial \alpha} \quad ; \quad \bar{j} = \frac{1}{B} \frac{\partial \bar{r}}{\partial \beta}$$

A. 1. Differentiation of the $\bar{i}$ vector.

a. $\partial \bar{i} / \partial \alpha$

Since the differentiation of a vector will again yield a vector whose component can be resolved along the $(\bar{i}, \bar{j}, \bar{k})$ triad, then it will be convenient to find the components directly;

1. $\bar{i}$ component

The scalar component is given as

$$\left(\bar{i} \cdot \frac{\partial \bar{i}}{\partial \alpha}\right)$$

which may also be written as

$$\left(\bar{i} \cdot \frac{\partial \bar{i}}{\partial \alpha}\right) = \frac{\partial}{\partial \alpha}(\bar{i} \cdot \bar{i}) - \left(\frac{\partial \bar{i}}{\partial \alpha} \cdot \bar{i}\right)$$

Hence

$$\left(\bar{i} \cdot \frac{\partial \bar{i}}{\partial \alpha}\right) = 0$$

2. $\bar{j}$ component

The scalar component is given as

$$\left(\bar{j} \cdot \frac{\partial \bar{i}}{\partial \alpha}\right)$$

which may also be written as

$$\left(\bar{j} \cdot \frac{\partial \bar{i}}{\partial \alpha}\right) = \frac{\partial}{\partial \alpha}(\bar{i} \cdot \bar{j}) - \left(\bar{i} \cdot \frac{\partial \bar{j}}{\partial \alpha}\right)$$

and hence

$$\left(\bar{j} \cdot \frac{\partial \bar{i}}{\partial \alpha}\right) = -\left(\bar{i} \cdot \frac{\partial \bar{j}}{\partial \alpha}\right)$$

Consider now evaluating the derivative $(\partial \bar{j} / \partial \alpha)$. If $\bar{r}$ is a continuous position vector

$$\frac{\partial}{\partial \beta}\left(\frac{\partial \bar{r}}{\partial \alpha}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial \bar{r}}{\partial \beta}\right)$$

Substituting for $(\partial \bar{r} / \partial \beta)$ and $(\partial \bar{r} / \partial \alpha)$, then;

$$\frac{\partial}{\partial \beta}(A \bar{i}) = \frac{\partial}{\partial \alpha}(B \bar{j})$$

Expanding the right side of the above expression

$$\frac{\partial}{\partial \alpha}(A \bar{i}) = \bar{j} \frac{\partial B}{\partial \alpha} + B \frac{\partial \bar{j}}{\partial \alpha}$$

Hence solving for $\frac{\partial \bar{j}}{\partial \alpha}$;

$$\frac{\partial \bar{j}}{\partial \alpha} = \frac{1}{B}\frac{\partial}{\partial \beta}(A\bar{i}) - \frac{\bar{j}}{B}\frac{\partial B}{\partial \alpha}$$

Then returning to the original problem

$$\bar{j}\cdot\left(\frac{\partial \bar{i}}{\partial \alpha}\right) = -\frac{\bar{I}}{B}\cdot\frac{\partial}{\partial \beta}(A\bar{i}) + \frac{(\bar{I}\cdot\bar{j})}{B}\frac{\partial B}{\partial \alpha}$$

Or

$$\bar{j}\cdot\left(\frac{\partial \bar{I}}{\partial \alpha}\right) = -\frac{\bar{I}}{B}\cdot\frac{\partial}{\partial \beta}(A\bar{i})$$

Expanding the right side again

$$\bar{j}\cdot\left(\frac{\partial \bar{i}}{\partial \alpha}\right) = -\frac{1}{B}\frac{\partial A}{\partial \beta} - \frac{A}{B}\left(\bar{I}\cdot\frac{\partial \bar{I}}{\partial \beta}\right)$$

But

$$\bar{i}\cdot\frac{\partial \bar{I}}{\partial \beta} = \frac{\partial}{\partial \beta}(\bar{i}\cdot\bar{I}) - \left(\frac{\partial \bar{I}}{\partial \beta}\cdot\bar{I}\right)$$

and hence

$$\bar{i}\cdot\frac{\partial \bar{i}}{\partial \beta} = 0$$

Thus in final form

$$\left(\bar{j}\cdot\frac{\partial \bar{I}}{\partial \alpha}\right) = -\frac{1}{B}\frac{\partial A}{\partial \beta}$$

3. $\bar{k}$ component

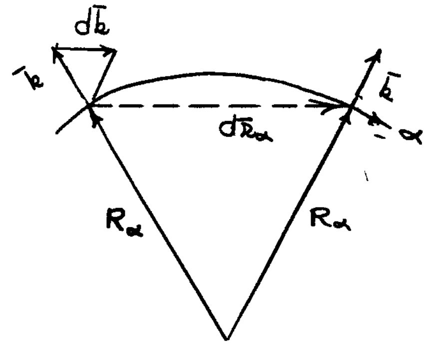
The $\bar{k}$ scalar component is given as

$$\bar{k}\cdot\left(\frac{\partial \bar{i}}{\partial \alpha}\right)$$

and which may be rewritten as

$$\bar{k}\cdot\left(\frac{\partial \bar{i}}{\partial \alpha}\right) = \frac{\partial}{\partial \alpha}(\bar{I}\cdot\bar{k}) - \bar{i}\cdot\left(\frac{\partial \bar{k}}{\partial \alpha}\right) = -\bar{I}\cdot\frac{\partial \bar{k}}{\partial \alpha}$$

Consider now evaluating the vector $\partial\bar{k}/\partial\alpha$. For this purpose consider a section of the α curvalinear curve and let the plane of the paper be normal to the surface. Since the α curve is a principal direction curve on the surface then there will be associated with the surface in that direction a quantity called the radius of curvature, R_α, which will be the reciprocal of the curvature of the surface, k_α. The portion of the curve is shown on the accompanying sketch.



Now by similar triangles

$$\frac{|d\bar{k}|}{|\bar{k}|} = \frac{A\,d\alpha}{R_\alpha}$$

Hence $\left|\frac{\partial \bar{k}}{\partial \alpha}\right| = \frac{A}{R_\alpha} = k_\alpha A$

Note that by the theorem of Rodrigues, $d\bar{k}$ must be in the same direction as $d\bar{r}_\alpha$ which in the limit is the $\bar{i}$ direction. Hence;

$$\frac{\partial \bar{k}}{\partial \alpha} = k_\alpha A\,\bar{i}$$

Returning then to the sought component

$$\bar{k}\cdot\left(\frac{\partial \bar{i}}{\partial \alpha}\right) = -\bar{i}\cdot\left(\frac{\partial \bar{k}}{\partial \alpha}\right) = -k_\alpha A$$

b. $\partial \bar{i}/\partial \beta$

1. $\bar{i}$ component

The scalar component is given as

$$\left(\bar{i}\cdot\frac{\partial \bar{i}}{\partial \beta}\right) = \frac{\partial}{\partial \beta}(\bar{i}\cdot\bar{i}) - \left(\frac{\partial \bar{i}}{\partial \beta}\cdot\bar{i}\right)$$

Hence

$$\left(\bar{i}\cdot\frac{\partial \bar{i}}{\partial \beta}\right) = 0$$

2. $\bar{j}$ component

This component is given as

$$\left(\bar{j} \cdot \frac{\partial \bar{i}}{\partial \beta}\right)$$

Now as has been shown

$$\frac{\partial}{\partial \alpha}(B\bar{j}) = \frac{\partial}{\partial \beta}(A\bar{i})$$

Expanding the right side of the above equation

$$\frac{\partial}{\partial \alpha}(B\bar{j}) = \bar{i}\frac{\partial A}{\partial \beta} + A\frac{\partial \bar{i}}{\partial \beta}$$

Solving for $\partial \bar{i}/\partial \beta$

$$\frac{\partial \bar{i}}{\partial \beta} = \frac{1}{A}\frac{\partial}{\partial \alpha}(B\bar{j}) - \frac{\bar{i}}{A}\frac{\partial A}{\partial \beta}$$

Hence;

$$\bar{j} \cdot \frac{\partial \bar{i}}{\partial \beta} = \frac{\bar{j}}{A} \cdot \frac{\partial}{\partial \alpha}(B\bar{j})$$

Expanding the right side

$$\left(\bar{j} \cdot \frac{\partial \bar{i}}{\partial \beta}\right) = \frac{(\bar{j}\cdot\bar{j})}{A}\frac{\partial B}{\partial \alpha} + \frac{B}{A}\left(\bar{j} \cdot \frac{\partial \bar{j}}{\partial \alpha}\right)$$

But by analogy with the $\bar{i}$ vector, $(\bar{j} \cdot \partial \bar{j}/\partial \alpha) = 0$ and thus

$$\left(\bar{j} \cdot \frac{\partial \bar{i}}{\partial \beta}\right) = \frac{1}{A}\frac{\partial B}{\partial \alpha}$$

3. $\bar{k}$ component

The $\bar{k}$ component is given as

$$\left(\bar{k} \cdot \frac{\partial \bar{i}}{\partial \beta}\right)$$

The above may be rewritten as

$$\left(\bar{k} \cdot \frac{\partial \bar{i}}{\partial \beta}\right) = \frac{\partial}{\partial \beta}(\bar{i}\cdot\bar{k}) - \left(\bar{i} \cdot \frac{\partial \bar{k}}{\partial \beta}\right) = -\bar{i} \cdot \frac{\partial \bar{k}}{\partial \beta}$$

However by analogy with the expression for ,

$$\frac{\partial \bar{k}}{\partial \beta} = k_\beta B\bar{j}$$

and thus

$$(\bar{k} \cdot \partial \bar{i}/\partial \beta) = 0$$

Now the remaining derivatives have either been calculated in the intermediate steps of the above development or may be found by an interchange of letters. The results are given in the following table.

	$\bar{i}$	$\bar{j}$	$\bar{k}$
$\partial\bar{i}/\partial\alpha$	–	$-\frac{1}{B}\,\partial A/\partial\beta$	$-k_\alpha A$
$\partial\bar{i}/\partial\beta$	–	$\frac{1}{A}\,\partial B/\partial\alpha$	–
$\partial\bar{j}/\partial\alpha$	$\frac{1}{B}\,\partial A/\partial\beta$	–	–
$\partial\bar{j}/\partial\beta$	$-\frac{1}{A}\,\partial B/\partial\alpha$	–	$-k_\beta B$
$\partial\bar{k}/\partial\alpha$	$k_\alpha A$	–	–
$\partial\bar{k}/\partial\beta$	–	$k_\beta B$	–