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A LATTICE ANALOGY FOR THE SOLUTION OF SOME NONLINEAR STRESS PROBLEMS

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Austin, Texas

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National Aeronautics and Space Administration

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Preface

This study involves the development of a method for solving stress problems in a continuum. A lattice representation of the continuum, more commonly called a lattice analogy, is used in the solution. The impetus for this study was the inability to solve rationally many stress problems occuring in soil and rock mechanics. Several classical problems in these two fields can be classified as two-dimensional stress problems, for example: long strip footing, retaining wall, culvert, and a well bore under fluid pressure with or without fluid leak off. These are only a few of the typical problems which can be considered to be two-dimensional. The method of solution developed herein is for two-dimensional problems. The extension of the lattice analogy technique to the general three-dimensional stress case is discussed and recommendations made for this extension.

Generally speaking, solutions to stress problems in the field of soil and rock mechanics should consider the nonlinear behavior of the continuum, especially when the solutions are concerned with both stress distributions and distortions at higher stress levels. The capability of considering nonlinearity, and also nonhomogeneity, of the continuum is incorporated in the method of solution. Infinitesimal strains are considered in the method of solution. For some problems in soils this consideration will make the method of solution presented herein inapplicable.

For the past several years soil-structure interaction problems have been studied at The University of Texas. These problems include laterally and axially loaded piles, flexible footings, and flexible retaining walls.

Methods for solving these problems have been developed whereby solutions may be obtained quite easily through the use of electronic digital computers. However, the accuracy of these solutions is dependent of the accuracy of the load-deformation characteristic used to define the soil. The term, load-deformation characteristic, means the resistant behavior offered by a soil as a structural element is deflected through it. Load-deformation characteristic is not synomymic with stress-strain characteristic. Much research is needed to develop procedures for predicting load-deformation characteristic of a soil from its stress-strain characteristic. Both experimental and analytical investigations will be involved. The possibility of using the lattice analogy technique as a tool for predicting the load-deformation characteristic of a soil from its stress-strain characteristic was primarily the reason for this study. Although load-deformation prediction was not addressed to directly in this study, this study will provide an analytical tool for future research activities in prediction of load-deformation characteristics.

ABSTRACT

The lattice analogy technique for solving plane stress problems presented by Hrennikoff in 1941 is extended to solve plane strain problems. Solutions to two-dimensional stress problems in linear homogeneous mediums are made feasible by the use of matrix algebra and an electronic digital computer.

Solutions of a plane stress problem and a plane strain problem are presented.

Extension of the method of solution to nonhomogeneous linear mediums for the two-dimensional stress case is made.

Extension of the method of solution to the general three-dimensional stress case is discussed. The required developments for this extension are enumerated and recommendations made.

A method for solving two-dimensional stress problems in nonlinear mediums is presented. The method uses the lattice analogy technique. Essentially, the procedure is to represent small blocks of the nonlinear medium by pseudo blocks of linear material. When subjected to the same state of strain, the two blocks will possess the same state of stress. The key to the solution is to find the particular set of pseudo linear blocks which will properly represent the behavior of the actual nonlinear medium under a given set of boundary conditions.

Solutions to two nonlinear stress problems are presented.

At the conclusion of this study, the method for solution of nonlinear stress problems is considered to be in an embryonic stage of development. Items for future study are enumerated and recommendations made.

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List of Symbols

σ_x, σ_y, σ_z	Normal stresses in the x, y, z directions, respectively.
τ_{xy}, τ_{xz}, τ_{yz}	Shearing stresses in the x-y, x-z, y-z planes, respectively.
σ_1, σ_2	Maximum and minimum principal stresses in the x-y plane.
θ_{1-x}	Angle from the major principal stress σ_1-direction to the x-direction.
ϵ_x, ϵ_y, ϵ_z	Normal strains in the x, y, z directions.
γ_{xy}, γ_{xz}, γ_{yz}	Shearing strains in the x-y, x-z, y-z planes, respectively.
ϵ_1, ϵ_2	Maximum and minimum principal strains in the x-y planes.
X, Y, Z	Body forces in the x, y, z directions, respectively.
c_{ij}	Elastic constants for general anisotropic linear medium.
E, ν	Elastic constants for an isotropic linear medium.
A, A_1, A_2	Structural elements, areas, of the lattice cells.
I	Structural elements, moment of inertia, of McCormick's lattice cell.
Δ	Translation of a lattice node or the extension of a finite block of material, used only during the development of the lattice cells.
P, S	Loads on lattice cell nodes, used only during development of lattice cells.
F	Bar forces in the various structural elements of a lattice cell, used only during development of the lattice cells.
SSW, SAFA etc.	Stiffness coefficients used in text and Fortran programs.
S, $\left[s_{ij} \right]$	Stiffness matrix
(x_i), (x)	Node movement vector.
(f_i), (f)	Node force vector.
w	Iteration factor (successive over-relaxation method) for solving linear simultaneous equations.

CHAPTER ONE

INTRODUCTION

Opening Remarks

This study involves the development of a method for solving stress problems in a continuum. A lattice representation of the continuum, more commonly called a lattice analogy, is used in the solution. The impetus for this study was the inability to solve rationally many stress problems occurring in soil and rock mechanics. Several classical problems in these two fields can be classified as two-dimensional stress problems, for example: long strip footing, retaining wall, culvert, and a well bore under fluid pressure with or without fluid leak off. These are only a few of the typical problems which can be considered to be two-dimensional. The method of solution developed herein is for two-dimensional problems. The extension of the lattice analogy technique to the general three-dimensional stress case is discussed and recommendations are made for this extension.

Generally speaking, solutions to stress problems in the field of soil and rock mechanics should consider the nonlinear behavior of the continuum, especially when the solutions are concerned with both stress distributions and distortions at higher stress levels. The capability of considering non-linearity, and also nonhomogeneity, of the continuum is incorporated in the method of solution.

Because mathematical difficulties make solutions of the differential equations which govern stress problems impossible in many cases, the engineer is impelled to seek a method of approach other than one of pure mathematical analysis. This was recognized by A. Hrennikoff in 1941.[7] Hrennikoff proposed the lattice analogy technique to solve stress problems in linear elastic mediums. The technique proved applicable, with some qualifications, to a variety of problems including two-dimensional stress, bending of plates, bending of cylindrical shells, and the general case of three-dimensional stress. However, as in other methods of solution based on approximating a continuous mathematical function by discrete lumped values, the solution of the resulting system of linear simultaneous equations by hand computation, namely relaxation schemes, presented a formidable task. Because of the inability to solve these large systems of equations, the application of the lattice analogy technique was impractical for many stress problems.

With the advent of the electronic digital computer, a considerably better means of computation became available. While this computational tool greatly increased the feasibility of numerical analysis procedures, there still exist many problems which completely tax the capability of computers. Improvements in methods for solving large systems of equations as well as improvements in computational hardware are currently being made. Since the ability to solve large systems of simultaneous equations will improve and since solutions of many problems are feasible using currently available means, the lattice analogy method of solution was deemed to be a practical approach to stress problems. The lattice analogy technique was broadened to consider the nonlinear behavior of a material. Several example problems are solved.

While the discussions and developments are in the context of soil and rock mechanics, the method of solution is equally applicable to problems in other fields.

As will be noted later in the development of the lattice analogy of a continuum, the structural configuration of an articulated framework used to represent a continuum is not arbitrary, although it is not unique. A wide range of articulated frameworks could be developed to approximate the deformability of a continuum. These frameworks would be composed of several simple structural elements; namely, beams, columns, beam-columns, plates and blocks. Lattice analogy is the descriptive term applied to any articulated framework consisting of beam, column, and beam-column structural elements. Finite plate analogy or block analogy are the descriptive terms for frameworks composed of plate or block elements.

Several analogies, both lattice and plate or block, have been proposed in the literature. The analogy proposed by Hrennikoff is the classic lattice analogy.[7] The analogy by Clough is an example of a finite plate analogy.[1] To date, no comprehensive evaluation of the relative merits of the several existing analogies has been reported. This is understandable since efficient means for computations have only recently become available. It is recommended that an evaluation be made in the near future.

A lattice analogy approach was used in this study. However, the method for solving nonlinear stress problems developed herein is not basically dependent on a particular analogy. If the evaluation study recommended above indicates another analogy has significant advantages over the particular analogy used herein, the employment of this analogy in the method of solution could be accomplished with little difficulty.

Objectives of this Study

The basic objective of this study is to develop a general method for solving two-dimensional stress problems in linear and nonlinear mediums. The lattice analogy technique is chosen as the basic analytical tool since random and nonhomogeneous boundary conditions and material descriptions can be handled. The steps involved in the development of this method of solution are listed below.

1. Development of the basic lattice analogy. This step involves developing the basic lattice configuration to be used and then deriving the system of equations which described the lattice representation of the medium. Developing the procedures for representing both linear and nonlinear mediums is required in this step.

2. Development of a means of solution based on this lattice analogy. This step basically involves writing a computer program which is as general as possible in application. The writing of this program involves (i) developing a general input capability in regard to boundary conditions, (ii) developing the means of generating the system of equations which describes the lattice representation of the problem, (iii) developing an efficient means for solving this system of equations, and (iv) developing the procedure for computing states of stress and strain throughout the medium based on the distortion of the loaded lattice.

3. Evaluation of the potential of the method of solution. This step involves solving a wide variety of stress problems to ascertain the general capability of the method of solution.

4. Recommendation for future work. This step involves making

concrete recommendations, based on the initial study, for improving the capability of the method of solution.

CHAPTER TWO

LATTICE ANALOGY OF A LINEAR MEDIUM

The solution of problems in elasticity generally involves the determination of the unique states of stress and of strain throughout a body which result from a particular set of boundary conditions. The familiar equations which govern the solutions are listed below for convenience. Identification of symbols are found in the list of symbols.

The three equations of static equilibrium are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0 \qquad \text{(1a)}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0 \qquad \text{(1b)}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0\,. \qquad \text{(1c)}$$

The six equations of compatibility are:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad \text{(2a)}$$

$$\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad \text{(2b)}$$

$$\frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \qquad \text{(2c)}$$

$$2\frac{\partial^2 \epsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left[-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right] \qquad \text{(2d)}$$

$$2\frac{\partial^2 \epsilon_x}{\partial z \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right] \tag{2e}$$

$$2\frac{\partial^2 \epsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left[\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right]. \tag{2f}$$

The six equations of Hooke's law in matrix form are:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \cdot \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \tag{3}$$

The equilibrium equations and compatibility equations are applicable to any linear or nonlinear continuous medium where only small strains are experienced. For an isotropic medium the twenty-one elastic constants in Hooke's law will reduce to only two independent constants, namely a modulus of elasticity E and Poisson's ratio ν. The medium is considered to be isotropic in the developments which follow. The resulting stress-strain relations will be as follows.

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_2 & 0 & 0 & 0 \\ c_2 & c_1 & c_2 & 0 & 0 & 0 \\ c_2 & c_2 & c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_3 \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \tag{4}$$

Where in terms of the elastic constants E and ν :

$$c_1 = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}$$

$$c_2 = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$c_3 = \frac{E}{2(1+\nu)} \quad .$$

For convenience and clarity in presenting the lattice analogy of a continuous medium, two-dimensional stress-strain problems will be considered in detail and example solutions will be presented. Two-dimensional stress-strain problems are classified as either plane stress or plane strain. Two lattice analogies for the plane stress problem will be referenced and illustrated in the next section. A lattice analogy for the plane strain problem will be developed following the plane stress analogy. An extension of the analogy to the general three-dimensional state of stress will be discussed following the two-dimensional treatment.

Lattice Analogy for Plane Stress Problems

The physical analogy between the behavior of a continuous medium and an articulated framework has been studied and reported by several researchers, dating from 1906. K. Wiehardt, 1906, and W. Reidel, 1927, studied the subject of plane stress analogy by framework methods in a somewhat restricted manner.[21, 15] A more comprehensive presentation of the lattice analogy for an isotropic linear medium was given by A. Hrennikoff, 1941.[7] During essentially the same period, D. McHenry reported a similar study to that of Hrennikoff.[12, 11]

Essentially, the lattice analogy method consists of representing a continuous body by a lattice consisting of simple structural elements which are grouped together to form individual lattice cells. The individual lattice cell is such that it possesses the same deformability characteristics under any type of uniform stress as that of a corresponding block of the continuous medium. When the dimensions of the lattice cell becomes infinitesimal, the lattice representation of the continuous body becomes rigorously equivalent to the continuous body. The lattice system is given the same external outline and the equivalent boundary conditions as that of the continuous body. Stress conditions on the body are represented by static equivalent loads at node points of the individual lattice cells. Distortion of the body is represented by movements of node points. The resulting frame analysis problem is solved by conventional stiffness methods.[(14)] The solution involves finding the resulting movement of each node point. The state of strain at the center of each lattice cell is based on various strain components which are expressed in finite difference form using adjacent node movements. The state of stress at the center of each lattice cell is then obtained from the known or assumed relationship between states of strain and states of stress.

In the developments to follow, the medium is assumed to be hyperelastic. That is, for a given state of strain there is only one unique state of stress.

The lattice analogy of a continuous medium will be illustrated by considering the conditions of plane stress. The condition of plane stress is used mainly for the sake of clarity; however, since this condition is often encountered in actual stress problems, the plane-stress analogy is a practical analytical tool. A problem will be solved at the end of this chapter to illustrate the plane stress analogy.

In the development of their lattice analogies, neither Hrennikoff nor McHenry stated fully the governing equations for plane stress. The conditions of plane stress are described by the condition $\sigma_z = \tau_{xz} = \tau_{yz} = 0$ and σ_x, σ_y, τ_{xy} are functions of x and y only. For a linear isotropic medium this stress condition implies that ϵ_x, ϵ_y, ϵ_z, γ_{xy} are independent of z and that γ_{xz}, γ_{yz} are zero. The governing equations, Eqs. 1, 2, and 4, will reduce to the following for plane stress.

The equations of static equilibrium for plane stress are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \tag{5a}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y = 0. \tag{5b}$$

The equations of compatibility for plane stress are:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{6a}$$

$$\frac{\partial^2 \epsilon_z}{\partial x^2} = 0 \tag{6b}$$

$$\frac{\partial^2 \epsilon_z}{\partial y^2} = 0 \tag{6c}$$

$$\frac{\partial^2 \epsilon_z}{\partial x \partial y} = 0. \tag{6d}$$

The equations for Hooke's law for plane stress are:

$$\sigma_x = \left(\frac{E}{1-\nu^2}\right)\left(\epsilon_x + \nu\epsilon_y\right) \tag{7a}$$

$$\sigma_y = \left(\frac{E}{1-\nu^2}\right)\left(\epsilon_y + \nu\epsilon_x\right) \tag{7b}$$

$$\epsilon_z = \left(-\frac{\nu}{E}\right)\left(\sigma_x + \sigma_y\right) \tag{7c}$$

$$\tau_{xy} = \left(\frac{E}{2(1+\nu)}\right)\gamma_{xy}. \tag{7d}$$

The equations involving the strain ϵ_z, Eqs. 6b, c, d, were not cited and were not used in the developments of the various lattice cells presented by Hrennikoff and McHenry. The apparent reason for this omission is that even though the solution of plane stress problems obtained by neglecting these additional compatibility equations are not exact, they are nevertheless very good approximations of plane stress problems.[18]

The structural configuration of a lattice cell used to represent a continuous block of material is not arbitrary, although it is not unique. There have been several lattice cells proposed in the literature, but the majority of these cells are restricted to one value of Poisson's ratio. The two lattice cells which are the most general in the application of problems dealing with plane stress in a linear isotropic material are those suggested by Hrennikoff and McCormick. Figure 1 presents the lattice cell suggested by Hrennikoff in 1941.[7] Figure 2 presents the lattice cell proposed by McCormick in 1963.[10] For Poisson's ratio equal to 1/3 these two lattice cells become identical in structural configuration.

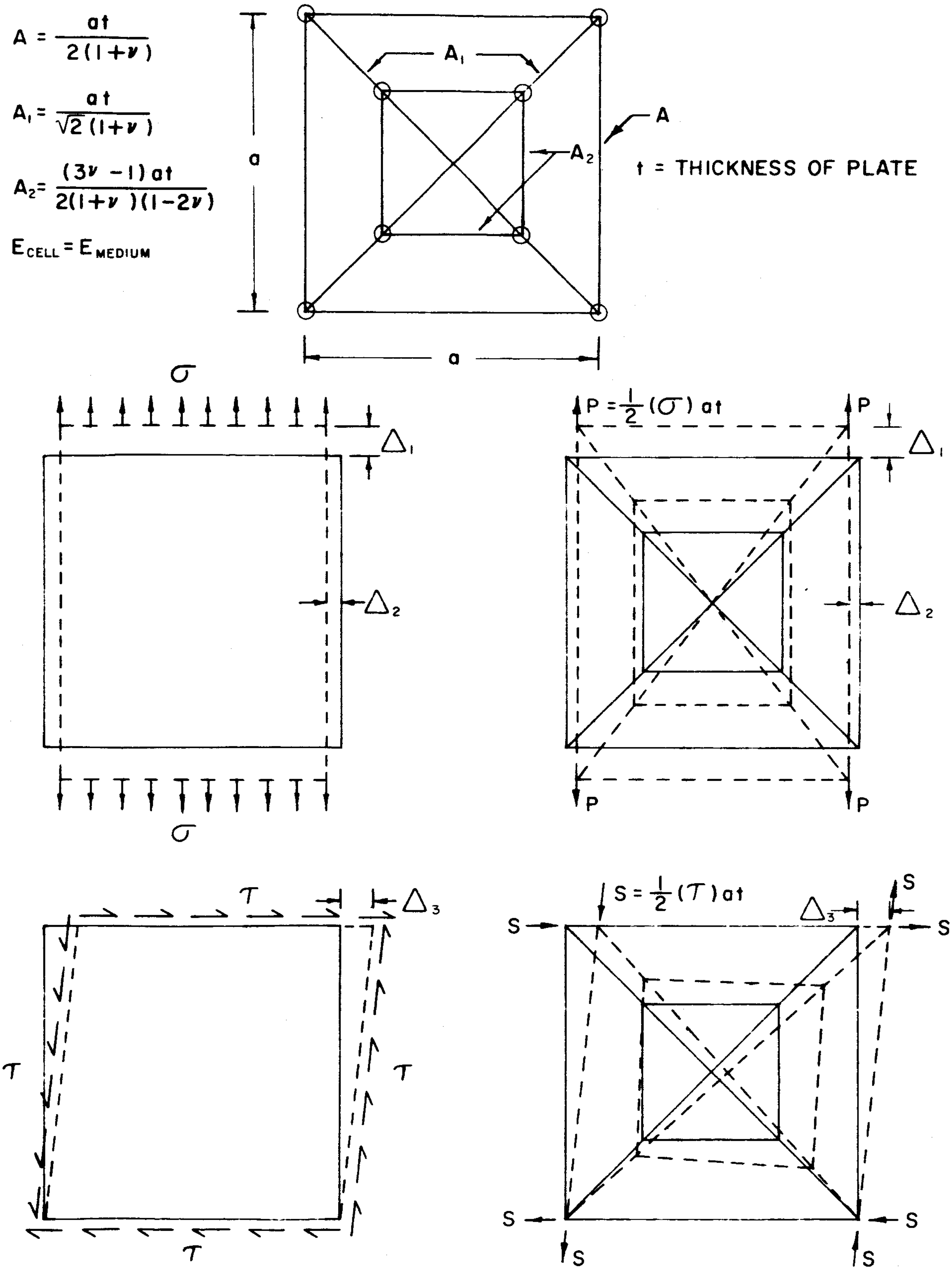


FIGURE 1 HRENNIKOFF PLANE STRESS LATTICE CELL

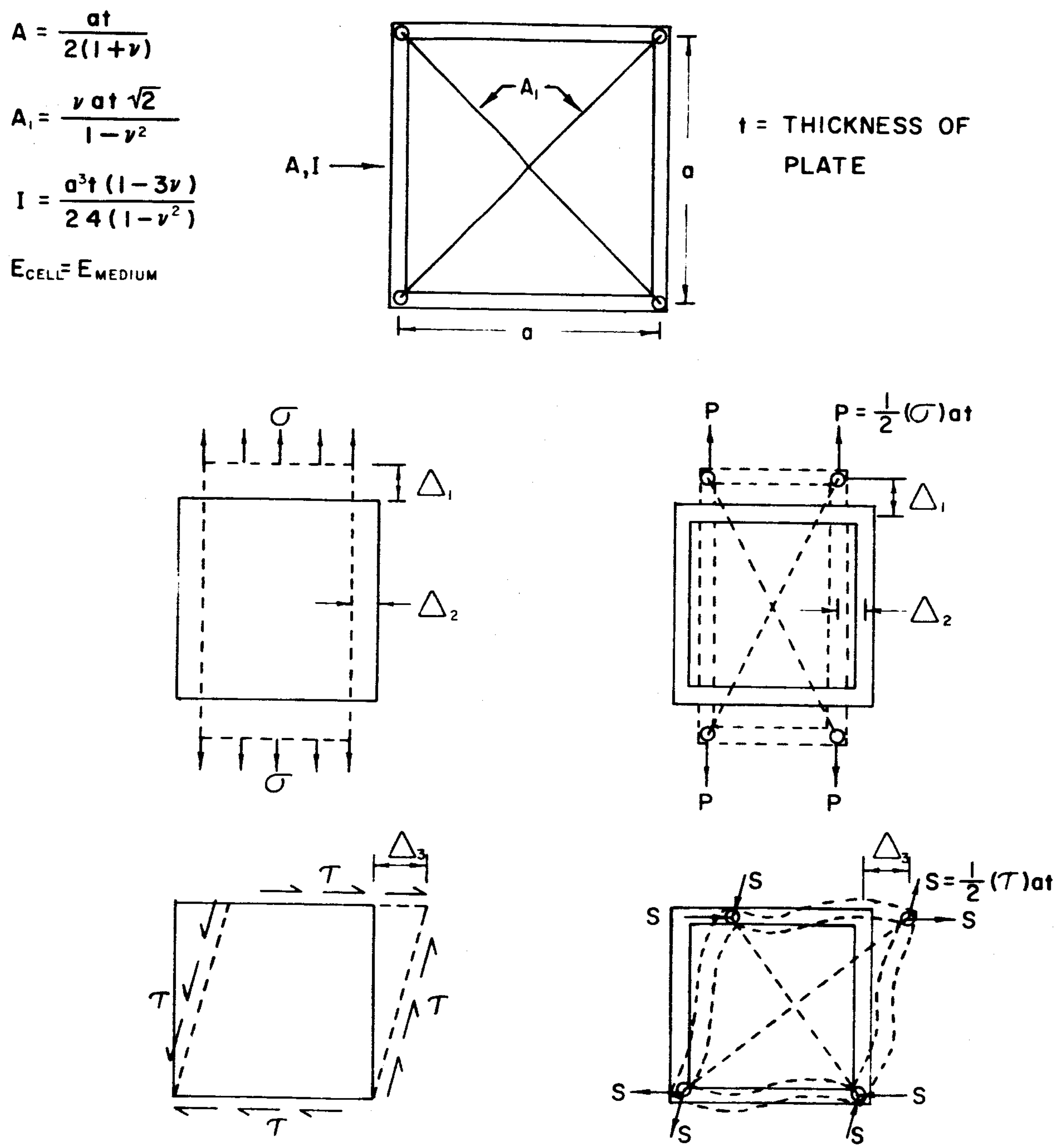


FIGURE 2 McCORMICK PLANE STRESS LATTICE CELL

Both Fig. 1 and Fig. 2 illustrate graphically the deformability equivalent between the lattice cells and the corresponding blocks of the continuous body for any arbitrary uniform stress condition. The lattice analogy of a continuous b ody employing either of these cells will be equivalent to the continuous body for any arbitrary uniform stress condition, regardless of the mesh size. If the stress condition is non-uniform in nature, the degree of approximation of the lattice analogy will be dependent upon the mesh size. The structural elements A, A_1, A_2 and I for both cells are expressed in terms of the Poisson's ratio ν, the side dimension of the cell a, and the thickness of the cell t. A, A_1, A_2 are cross-sectional areas of the structural members. I is the moment of inertia for the side members in the McCormick cell. The modulus of elasticity E is the same for both the continuous body and the lattice. The dimensions of the side elements A and I would be twice that given by the equations in Fig. 1 and Fig. 2.

In a discussion of McCormick's paper of 1963, Hrennikoff criticized the efficiency of the lattice cell proposed by McCormick.[6] Hrennikoff's criticism was that in McCormick's cell three components of movement of each node point are needed to describe the distortion of the lattice network, while in Hrennikoff's cell only two components of movement at each main node point are needed since the cell is simply connected. The degree of indeterminancy of a given lattice system based on Hrennikoff's cell would be 50 per cent less than the same lattice system based on McCormick's cell. In even a crude lattice system this difference is of computational importance.

In the initial stages of this study both McCormick's and Hrennikoff's cells were used. Solutions to several identical stress problems were made using both cells. The solutions agreed with each other. Since the purpose of this study was to develop a method for solving stress problems involving nonlinear mediums which automatically requires iterative procedures, the matter of computational efficiency was of primary importance. For this reason the Hrennikoff cell was used in developing a procedure for solving nonlinear stress problems.

The method for the solution of linear problems employing Hrennikoff's simply-connected model will be outlined in detail following the section which presents the simply-connected lattice analogy for plane strain problems.

Lattice Analogy for Plane Strain Problems

The literature review of this study revealed that the lattice analogy technique has not been developed for plane-strain problems. Since this type of strain condition is encountered quite often inclassical as well as practical stress problems, the extension of the lattice analogy to encompass plane-strain problems was deemed worthwhile. The development of a general lattice cell valid for any Poisson's ratio is presented here for the condition of plane strain.

The conditions of plane strain are such that the strain components $\epsilon_z = \gamma_{xz} = \gamma_{yz} = o$ and ϵ_x, ϵ_y, γ_{xy} are functions of x and y only. For a linear isotropic material this strain condition implies that σ_x, σ_y, τ_{xy}, σ_z are functions of x and y only, and that $\tau_{xz} = \tau_{yz} = o$ throughout the medium. The governing equations, Eqs. 1, 2, and 4, simplify as shown for plane-strain conditions.

The equations of static equilibrium for plane strain are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = o \tag{8a}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = o. \tag{8b}$$

The single equation of compatibility for plane strain is:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}. \tag{9}$$

The equations of Hooke's law written in the form of strain as a function of stress for plane strain are:

$$\epsilon_x = \left[\frac{1+\nu}{E}\right] \left[(1-\nu)\sigma_x - \nu\sigma_y\right] \tag{10a}$$

$$\epsilon_y = \left[\frac{1+\nu}{E}\right] \left[(1-\nu)\sigma_y - \nu\sigma_x\right] \tag{10b}$$

$$\gamma_{xy} = \left[\frac{2(1+\nu)}{E}\right] \tau_{xy}. \tag{10c}$$

The equivalent criteria for a lattice cell is that it possesses the same deformability under any uniform stress condition as that of a similar size block of continuous medium. Stated concisely the criterion of a lattice cell representation of a continuum is that it will deform identically with the continuous body under every possible uniform stress condition. This criterion for plane strain may be stated conveniently in terms of the following three conditions, although other equivalent formulations are possible. These three conditions closely parallel the conditions formulated by Hrennikoff in deriving his plane stress analogy. Figure 3 is used to illustrate these conditions.

1. If the lattice in Fig. 3a is loaded uniformly with normal loads P per node as shown in Fig. 3b in the x-direction and $\frac{\nu}{1-\nu}P$ in the y-direction, the resulting deformations of the lattice cell should be the same as that of the continuous body shown in Fig. 3b, that is $\Delta_1 = \Delta_3$. In other words the node deflections written in terms of the cell characteristics A, A_1, A_2 should be identical with that of the deformation of the continuous block. This criterion is stated below in equation form.

$$\Delta_1 = \epsilon_x a = \frac{\sigma_x (1+\nu)(1-2\nu) a}{(1-\nu)E} = \Delta_3 = \phi_1 (A, A_1, A_2, P). \tag{11a}$$

$$\epsilon_y a = o = \phi_2 (A, A_1, A_2, P). \tag{11b}$$

2. Reversing the loading condition above, two similar equations are produced.

$$\epsilon_x a = o = \phi_3 (A, A_1, A_2, P). \tag{11c}$$

$$\epsilon_y a = \frac{\sigma_y (1+\nu)(1-2\nu) a}{(1-\nu)E} = \phi_4 (A, A_1, A_2, P). \tag{11d}$$

3. If the lattice cell is loaded uniformly at node points with a shearing load S as defined in Fig. 3c, the resulting deformation of the lattice cell should be the same as that of the continuous body, that is $\Delta_2 = \Delta_4$. This criterion is shown below in equation form.

$$\Delta_2 = \gamma_{xy} a = \frac{2(1+\nu)}{E} \tau_{xy} a = \Delta_4 = \phi_5 (A, A_1, A_2, S). \tag{12}$$

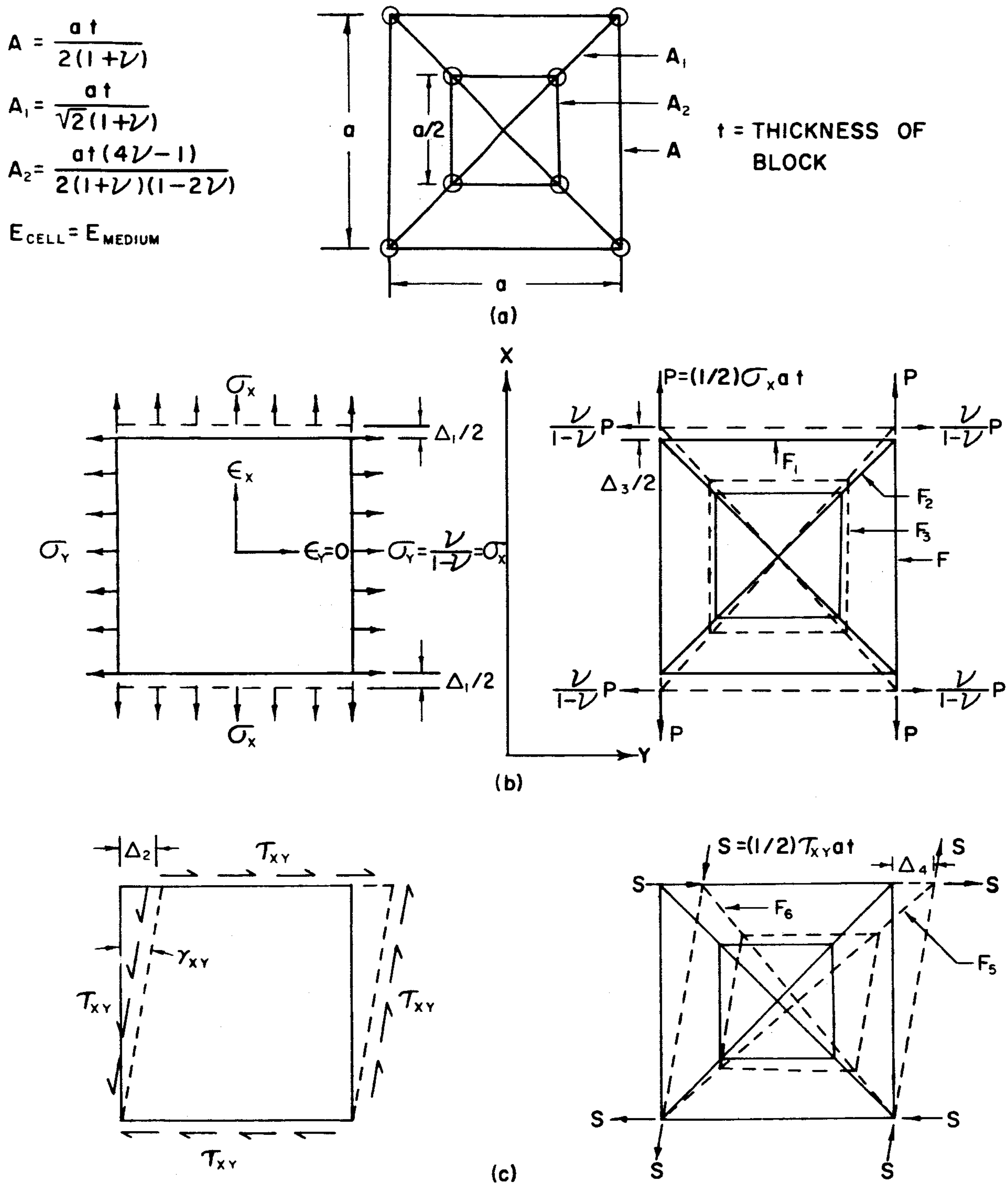


FIGURE 3 PLANE STRAIN LATTICE CELL

It should be noted that a proper combination of the three previous conditions will produce any conceivable state of uniform stress and consequently a lattice cell which obeys the previous conditions will be a valid analog of the continuum under any uniform stress conditions.

Equation 11c is not an independent equation, but follows from Eq. 11b by Betti's reciprocal theorem. Therefore, the number of independent equations involved in the three conditions is four, and it follows that a cell which possesses two axes of symmetry must in the general case possess four independent characteristics. Should the axes of symmetry be identical in the two directions, the number of necessary conditions reduces to three and Eq. 11d or Eq. 11a becomes superfluous. If the cell is deficient in characteristics by one, the condition of equivalent deformability is satisfied only for one particular value of Poisson's ratio, which would play the part of the missing characteristic. In order to incorporate both the plane stress and plane strain cells into one computer program, the geometrical configuration of the cell for plane strain was chosen to be identical to that of the Hrennikoff's plane stress cell. This cell has two axes of identical symmetry; therefore, only three characteristics A, A_1, A_2 are necessary for the cell to be completely general in regard to values of Poisson's ratio. The definitions of these characteristics are derived by writing expressions for the deformation of the lattice cell in terms of the cell characteristics and equating these expressions to those for the deformation of the continuous block as depicted by Eqs. 11 and 12. The solution of these expressions will yield the definition of the characteristics.

From Fig. 3b since there is no extension in the y-direction, the bar force F_1 is equal to 0 and for equilibrium in the y-direction at a corner node the force F_2 is equal to $\frac{\sqrt{2}\nu}{(1-\nu)}$ P or

$$F_2 = \left(\frac{\nu}{1-\nu}\right)\left(\frac{\sigma_x at}{\sqrt{2}}\right).$$

For equilibrium in the x-direction

$$F + \frac{F_2}{\sqrt{2}} = P$$

or

$$F = \left(\frac{\sigma_x at}{2}\right)\left(\frac{(1-2\nu)}{(1-\nu)}\right).$$

The elongation Δ_3 of the lattice cell in the x-direction is therefore

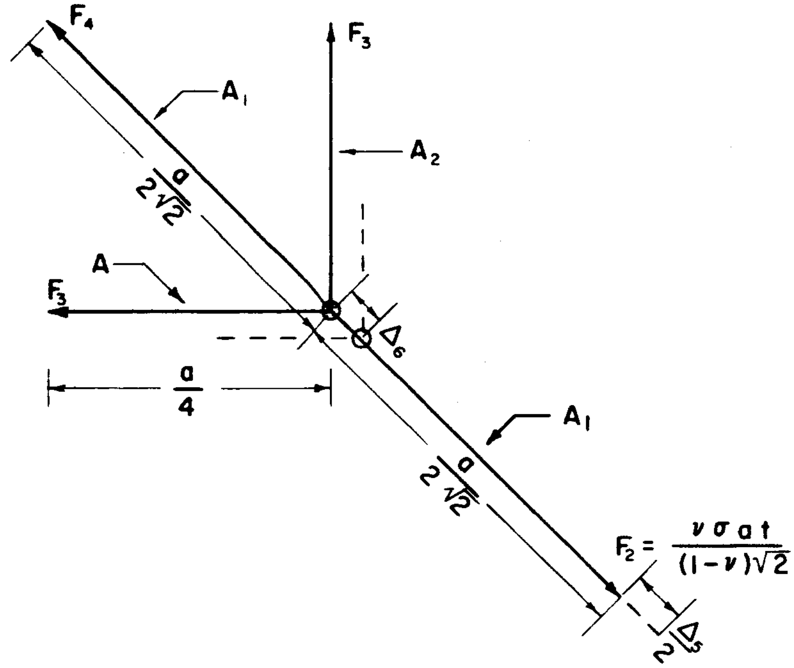
$$\Delta_3 = \frac{Fa}{AE} = \left(\frac{\sigma_x a^2 t}{2AE}\right)\left(\frac{(1-2\nu)}{(1-\nu)}\right)$$

Equating Δ_3 to Δ_1 yields the expression of the cell characteristic A.

$$\left(\frac{\sigma_x a^2 t}{2AE}\right)\left(\frac{(1-2\nu)}{(1-\nu)}\right) = \frac{\sigma_x (1+\nu)(1-2\nu)a}{(1-\nu)E} ,$$

$$A = \frac{at}{2(1+\nu)} . \tag{12}$$

The elongation of the diagonal members A_1 for small deformations with zero lateral contraction in the y-direction is $\Delta_5 = \frac{\Delta_3}{\sqrt{2}}$. Figure 4 is used to derive the expression of Δ_5 in terms of the characteristics A_1 and A_2 and the force F_2 . Equating this expression for Δ_5 to $\frac{\Delta_3}{\sqrt{2}}$ yields an equation for A_1 and A_2 .



$$F_4 = \frac{2\sqrt{2}\, A_1 E \Delta_6}{a}$$

$$F_3 = \frac{4 A_2 E \Delta_6}{a\sqrt{2}}$$

$$F_2 = F_4 + \sqrt{2} F_3 = \frac{(2\sqrt{2}\, A_1 + 4 A_2) E \Delta_6}{a}$$

$$\frac{\Delta_5}{2} = \Delta_6 + \frac{F_2\, a}{2\sqrt{2}\, A_1 E}$$

$$\frac{\Delta_5}{2} = \frac{F_2}{E}\left(\frac{a}{2\sqrt{2} A_1 + 4 A_2} + \frac{a}{2\sqrt{2} A_1 E}\right)$$

$$\Delta_5 = \frac{F_2\, a}{E}\left(\frac{1}{\sqrt{2} A_1} + \frac{1}{\sqrt{2} A_1 + 2 A_2}\right)$$

$$\Delta_5 = \frac{\nu \sigma a^2 t}{(1-\nu)\sqrt{2}\, E}\left(\frac{1}{\sqrt{2} A_1} + \frac{1}{\sqrt{2}\, A_1 + 2 A_2}\right)$$

FIGURE 4 DERIVATION OF PLANE STRAIN LATTICE CELL

$$\Delta_5 = \frac{\nu \sigma_x a^2 t}{(1-\nu)\sqrt{2}E}\left(\frac{1}{\sqrt{2}A_1} + \frac{1}{\sqrt{2A_1+2A_2}}\right) = \frac{\Delta_3}{\sqrt{2}} = \frac{\sigma_x a^2 t\,(1-2\nu)}{2\sqrt{2}\,AE(1-\nu)}.$$

The cell characteristic A is defined by Eq. 12. Substituting this expression for A into the above equation yields a relationship between A_1 and A_2.

$$\left(\frac{1}{\sqrt{2}A_1} + \frac{1}{\sqrt{2A_1+2A_2}}\right) = \frac{(1-2\nu)(1+\nu)}{\nu a t}. \tag{13}$$

In Fig. 3c the lattice cell is deformed by the shearing load S applied at node points. It should be noted that under the influence of the shearing distortions the elements A_2 are not strained and similarly the side elements A also are not strained. For static equilibrium at the node points $F_5 = F_6 = \sqrt{2}\,S$; F_5 being tension and F_6 being compression.

$$F_5 = F_6 = \frac{\tau_{xy} a t}{\sqrt{2}}.$$

An expression for Δ_4 is therefore

$$\Delta_4 = \left(\frac{F_5\sqrt{2}a}{A_1 E}\right)\sqrt{2} = \frac{2F_5 a}{A_1 E} = \frac{\sqrt{2}\tau_{xy} a^2 t}{A_1 E}.$$

Equating Δ_2 of the continuous medium to Δ_4 of the lattice cell yields the definition of A_1.

$$\Delta_2 = \gamma_{xy} a = \frac{2(1+\nu)\tau_{xy} a}{E} = \Delta_4 = \frac{\sqrt{2}\tau_{xy} a^2 t}{A_1 E}.$$

$$A_1 = \frac{at}{\sqrt{2}\,(1+\nu)}. \tag{14}$$

Substituting this expression for A_1 into Eq. 13 yields the definition of the characteristic A_2.

$$A_2 = \frac{at\,(4\nu-1)}{2(1+\nu)(1-3\nu)}\ . \tag{15}$$

The definition of the lattice cell developed here for plane strain is summarized in Fig. 3. Reiterating, the geometrical configuration of the above plane-strain lattice cell is identical with that of the plane-stress lattice cell presented by Hrennikoff, but the structural configuration is different. It should be noted here that only characteristic A_2 is different, that is, the center core of each lattice cell. Because of this geometrical similarity, the same computer program may be used to solve stress problems with the exception that in the generation of stiffness coefficients the proper definitions of the characteristic A_2 must be considered. In Appendix I the description of the computer program which employs both of these cells is presented. In the following section the method of solution will be outlined in detail, and the remarks will be equally applicable to either cell.

Method of Solution by the Lattice Analogy

The theory and procedure for solving stress problems in a linear, isotropic medium under the conditions of plane stress or plane strain by employing a lattice analogy were reviewed briefly in the preceeding sections. This section will detail the various steps involved in obtaining a solution by the lattice analogy technique. The method of solution outlined here will be concerned with the lattice cell shown in Figs. 1 and 3. When

Hrennikoff proposed the lattice cell in Fig. 1, there were no high speed computational facilities available to solve the resulting large system of simultaneous linear equations; therefore, to obtain solutions for example problems Hrennikoff solved the lattice systems by the method of successive joint displacements. This procedure resembled closely the method of moment distribution developed by Hardy Cross for frame solutions and was carried out by lengthy hand computations.

With the advent of the high-speed digital computer, a more efficient means of solving large systems of linear simultaneous equations became available. Improved methods for computer solutions are currently being developed and published in the literature. Hence, the use of the digital computer to solve the system of equations involved in the lattice analogy greatly increases the usefulness and the practicality of this type of stress analysis.

The various steps involved in the solution of a stress problem in a linear isotropic medium by the lattice analogy method are listed here in sequence. This outline of operation is in essence a general flow diagram of the computer program written for this method of solution.

1. The continuous body is represented by a lattice consisting of individual lattice cells and having the same geometrical boundaries as that of the body.

2. Boundary conditions in terms of deflections or loads at main node points are applied to the lattice to represent the boundary conditions of the continuous body.

3. The unique system of linear simultaneous equations resulting from the lattice representation and the applied boundary conditions is

generated. The resulting movements of the main node points are obtained from the solution of this system of equations.

4. From the obtained node movements, strain components are computed by finite difference techniques. These strain components are used to determine the states of strain at the center of each lattice cell.

5. From the states of strain, the states of stress are obtained at the center of each lattice cell by the known relationships between states of strain and states of stress.

Figure 5 presents in part the indexing used in representing a continuous body by a lattice analogy. For convenience in this presentation, the continuous body is considered to be rectangular in shape. With simple modifications to be discussed later, irregularly shaped bodies can be considered quite easily. As shown in Fig. 5, each cell is assigned two elastic constants E and ν . The structural elements of each cell A, A_1, A_2 are based solely on the side and thickness dimensions of the cell and these two elastic constants.

The movement of each main node is described by the two translations x_{2n-1}, x_{2n} and similarly the external loads (that is, loads simulating stress conditions on the boundary or body forces in the interior of the body) on the main node points are described by two forces f_{2n-1}, f_{2n} . This type of ordering shown in Fig. 5 will yield advantages in computational schemes. The chief advantage is that when the linear simultaneous equations are written in matrix form, the resulting "stiffness matrix" will be a multiple-diagonal-band matrix, which is an advantage from a computational standpoint. For reference, the main node points are ordered as indicated.

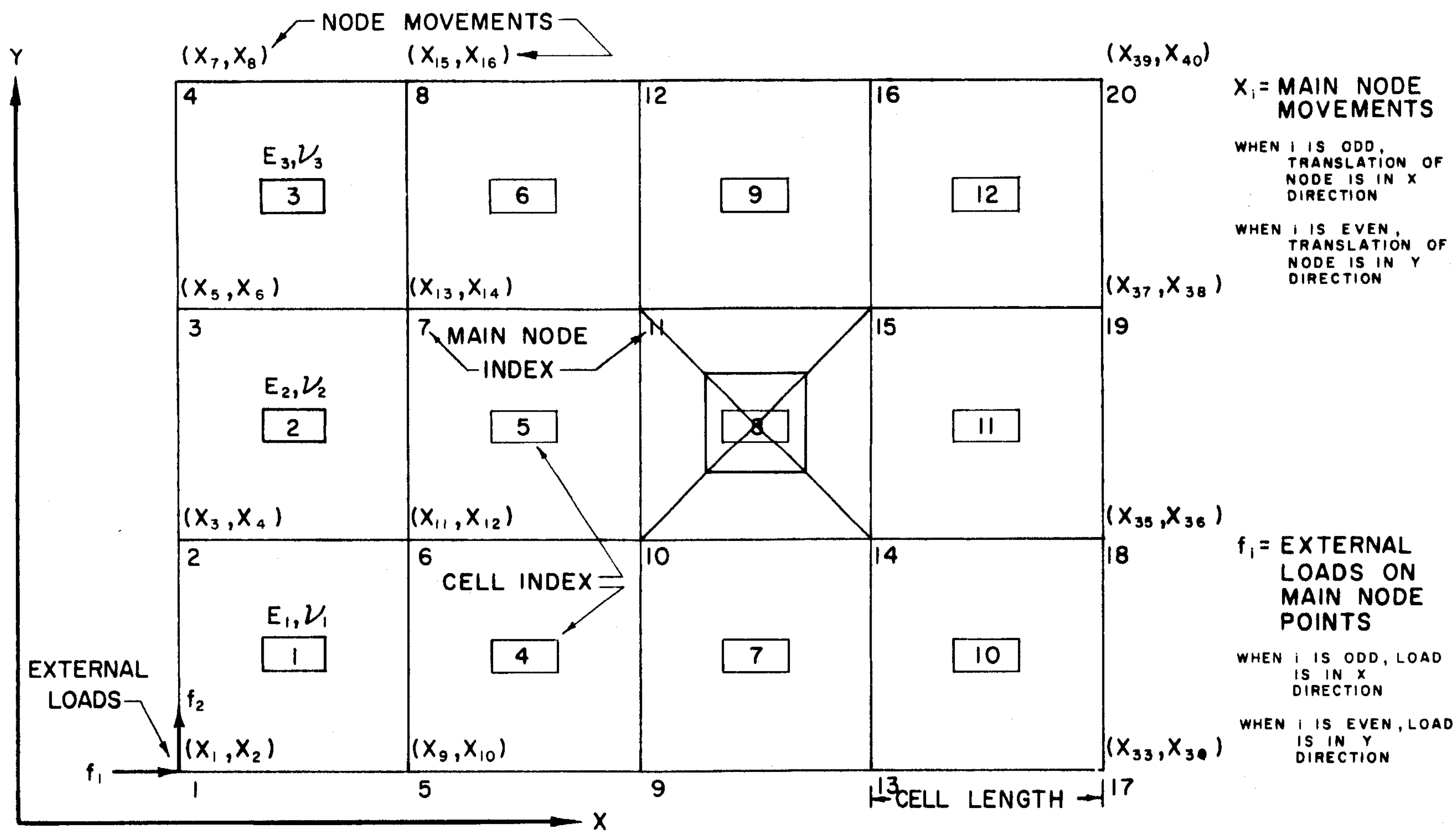


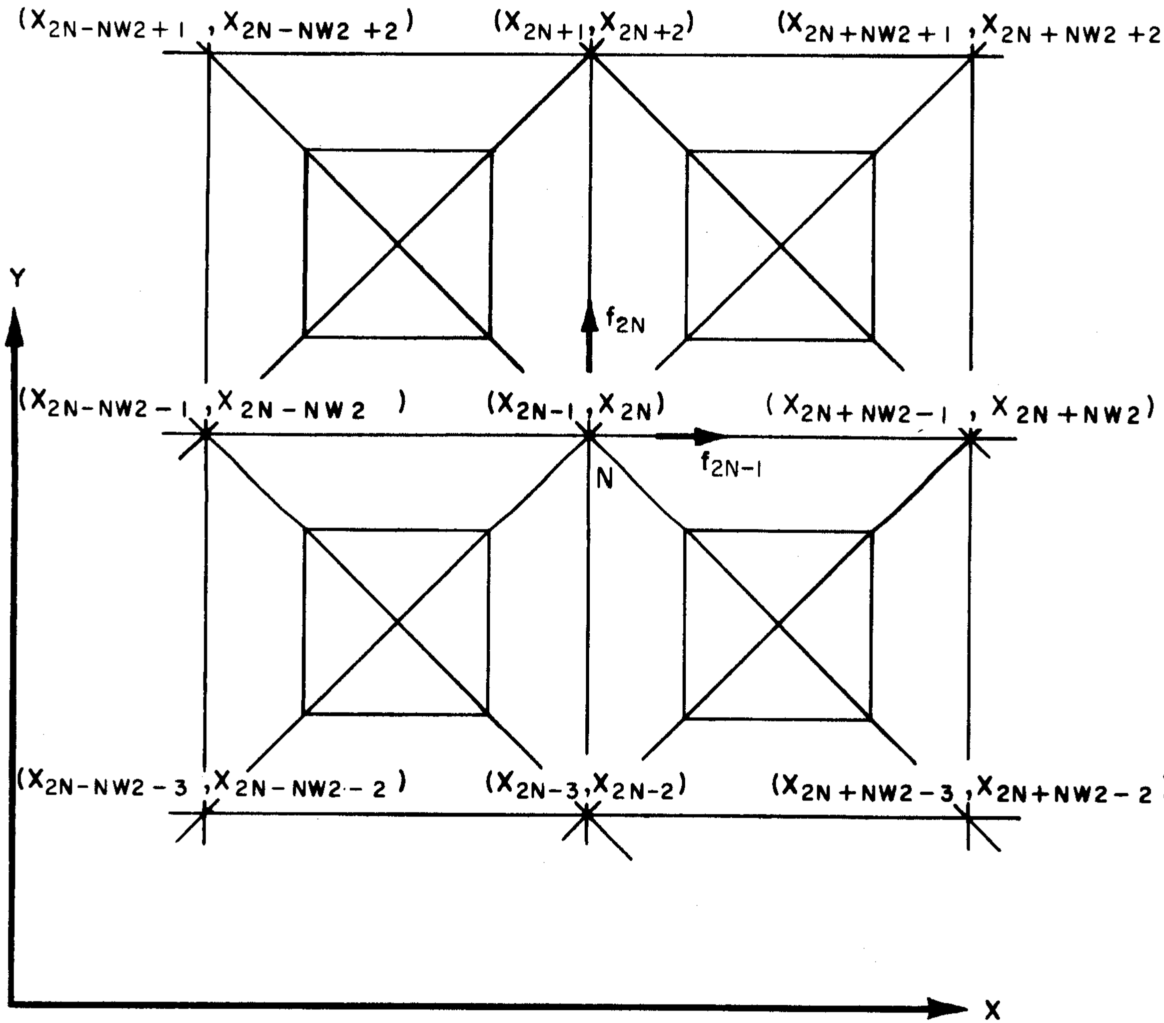
FIGURE 5 INDEXING EMPLOYED IN THE LATTICE ANALOGY OF A CONTINUOUS RECTANGULAR BODY FOR TWO-DIMENSIONAL STRESS PROBLEMS

As stated previously, conventional stiffness methods are used to solve for node movements resulting from the applied boundary conditions. Figure 6 is used to derive the two equilibrium equations, Eqs. 16a and 16b, which can be written at each main node point in terms of stiffness coefficients, node movements, and external loads. The indexing shown in Fig. 6 is consistent with that presented in Fig. 5. The various stiffness coefficients needed to write the two equilibrium equations are listed in Fig. 6. These stiffness coefficients are derived in Appendix I. By equating the sum of the individual forces resulting from the adjacent and central node movements and the external loads f_i to zero, the two equations of equilibrium are generated. The equation for forces in the x-direction is:

$$
\begin{aligned}
&(SSW)X_{2n-nw2-3} + (SSW)X_{2n-nw2-2} + (SW1)X_{2n-nw2-1} \\
&+ (-SW2)X_{2n-nw2} + (SNW)X_{2n-nw2+1} + (-SNW)X_{2n-nw2+2} \\
&+ (SS3)X_{2n-3} + (SS2)X_{2n-2} + (SCEN1)X_{2n-1} + (SCEN2)X_{2n} \\
&+ (SN3)X_{2n+1} + (-SN2)X_{2n+2} + (SSE)X_{2n+nw2-3} \\
&+ (-SSE)X_{2n+nw2-2} + (SE1)X_{2n+nw2-1} + (SE2)X_{2n+nw2} \\
&+ (SNE)X_{2n+nw2+1} + (SNE)X_{2n+nw2+2} + f_{2n-1} = 0. \qquad (16a)
\end{aligned}
$$

The equation for forces in the y-direction is:

$$
\begin{aligned}
&(SSW)X_{2n-nw2-3} + (SSW)X_{2n-nw2-2} + (SW2)X_{2n-nw2-1} \\
&+ (SW3)X_{2n-nw2} + (-SNW)X_{2n-nw2+1} + (SNW)X_{2n-nw2+2} \\
&+ (-SS2)X_{2n-3} + (SS1)X_{2n-2} + (SCEN2)X_{2n-1} + (SCEN1)X_{2n}
\end{aligned}
$$



STIFFNESS COEFFICIENTS

SSW, SNW, SNE, SSE, SS1, SW1,
SN1, SE1, SS2, SS2, SS2, SS2,
SS3, SS3, SS3, SS3, SCEN1, SCEN2

$$\sum \left[(s_{i,j})(x_i) \right] + \sum f_i = 0$$

FIGURE 6 SCHEME FOR GENERATING THE TWO EQUILIBRIUM EQUATIONS ABOUT AN INTERIOR NODE

$$+ (SN2)X_{2n+1} + (SN1)X_{2n+2} + (-SSE)X_{2n+nw2-3}$$

$$+ (SSE)X_{2n+nw2-2} + (-SE2)X_{2n+nw2-1} + (SE3)X_{2n+nw2}$$

$$+ (SNE)X_{2n+nw2+1} + (SNE)X_{2n+nw2+2} + f_{2n} = 0. \qquad (16b)$$

The above notation for stiffness coefficients, node movements, node loads, and the subscripting is the same as that used in the Fortran program, BODY 2, Appendix I. Where ever possible the same notation will be used in the text and in the Fortran programs. This is a departure from convention but it is considered to be an improvement.

For the purpose of later discussion, it is noted here that the equilibrium equations about an interior node point involves eighteen node movements.

The stiffness matrix equation, Eq. 17, is generated by writing these two equilibrium equations about each main node point:

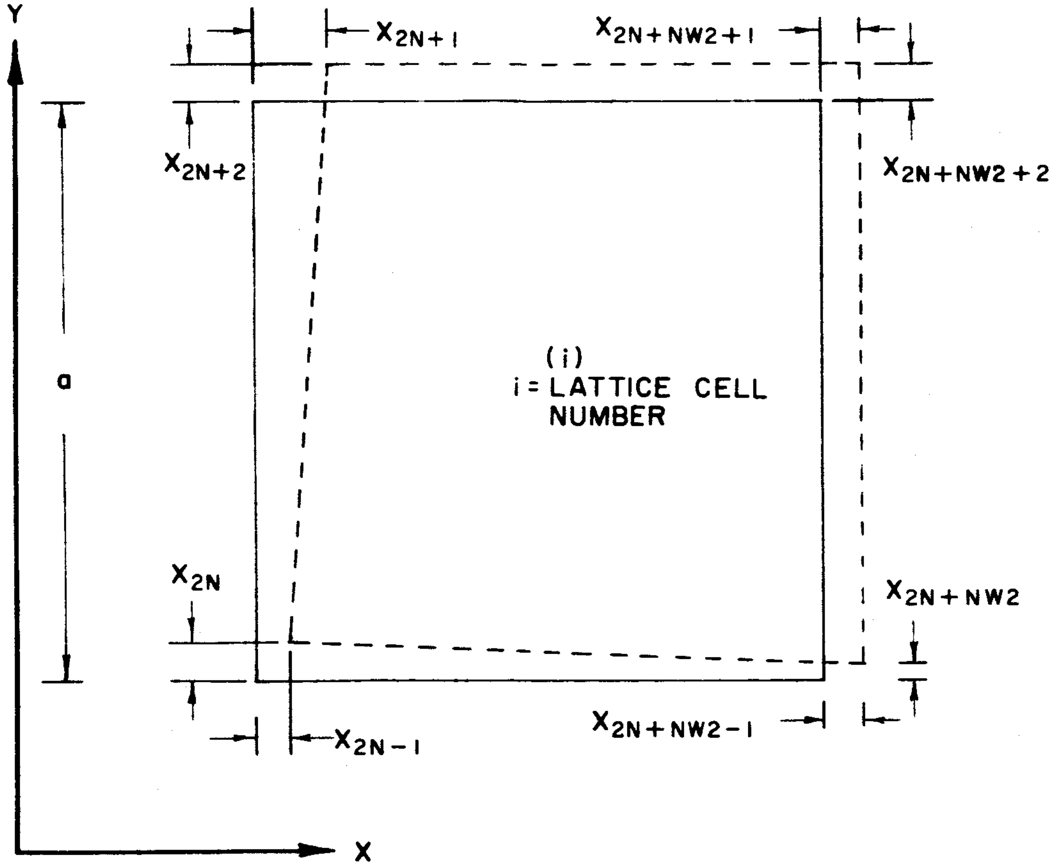
$$\left[S\right](x) = (f). \qquad (17)$$

The boundary conditions involving stresses or body forces are represented by the vector (f) . In order to specify boundary conditions involving distortions of the body, the corresponding node point movements are specified. To specify a node movement the equilibrium equations for forces in the direction of the particular node movement is deleted from the matrix equation. An equation is inserted that forces the movement of the node to be the desired value. The solution of this modified matrix equation yields the resulting node movements throughout the lattice. The method for solving this matrix equation is a direct elimination process which uses to full advantage the diagonally-banded characteristic of the stiffness matrix. Several possible means for solving diagonally-banded matrix equations will be discussed and reviewed in a later chapter for the purpose of making recommendations for extending the method of solution to more complex two-dimensional stress problems and to three-dimensional stress problems.

From the obtained node movements the state of strain at the center of each lattice cell is calculated by use of finite difference representations of various strain components. Figure 7 illustrates this scheme. From the strain components ϵ_{x_i}, ϵ_{y_i}, γ_{xy_i} principal strains ϵ_{1_i}, ϵ_{2_i} are computed along with an orientation angle θ_{1-x_i}, which is measured from direction of the maximum principal strain ϵ_1 to the x-direction. Counter-clockwise angles are positive. Sign conventions are given in Fig. 7.

The state of stress at each cell is computed from the known relation between strain and stress. Stresses that are obtained at the center of each cell are the principal stresses σ_{1_i} and σ_{2_i} and the normal stresses and shearing stresses in the xy-direction σ_{x_i}, σ_{y_i}, τ_{xy_i}.

In the computer program written for this study, non-rectangular-shaped bodies may be handled by simply specifying that certain lattice cells which lie outside the boundary of the body possesses a zero stiffness (E = 0). If this scheme becomes too inefficient, modification of the logic involved in the generation of the stiffness matrix may be necessary in order to represent irregularly shaped bodies. Since each cell can be assigned individual elastic constants, the method of solution will handle stress problems in nonhomogeneous mediums by simply representing piecewise, to a scale of the cell dimension, the nonhomogeneity of the medium.



$$\epsilon_{X_i} = \frac{X_{2N+NW2-1} + X_{2N+NW2+1} - X_{2N-1} - X_{2N+1}}{2a}$$

$$\epsilon_{Y_i} = \frac{X_{2N+2} + X_{2N+NW2+2} - X_{2N} - X_{2N+NW2}}{2a}$$

$$\gamma_{XY_i} = \frac{-X_{2N-1} - X_{2N} + X_{2N+1} - X_{2N+2} - X_{2N+NW2-1} + X_{2N+NW2}}{2a} + \frac{X_{2N+NW2+1} + X_{2N+NW2+2}}{2a}$$

SIGN CONVENTIONS

TENSION (+)

COMPRESSION (−)

SHEAR (+)

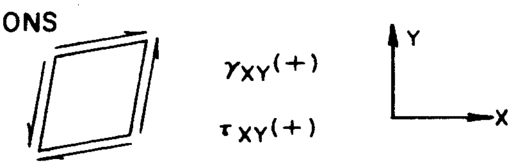


FIGURE 7 FINITE DIFFERENCE REPRESENTATIONS OF STRAIN COMPONENTS AT THE CENTER OF A LATTICE CELL

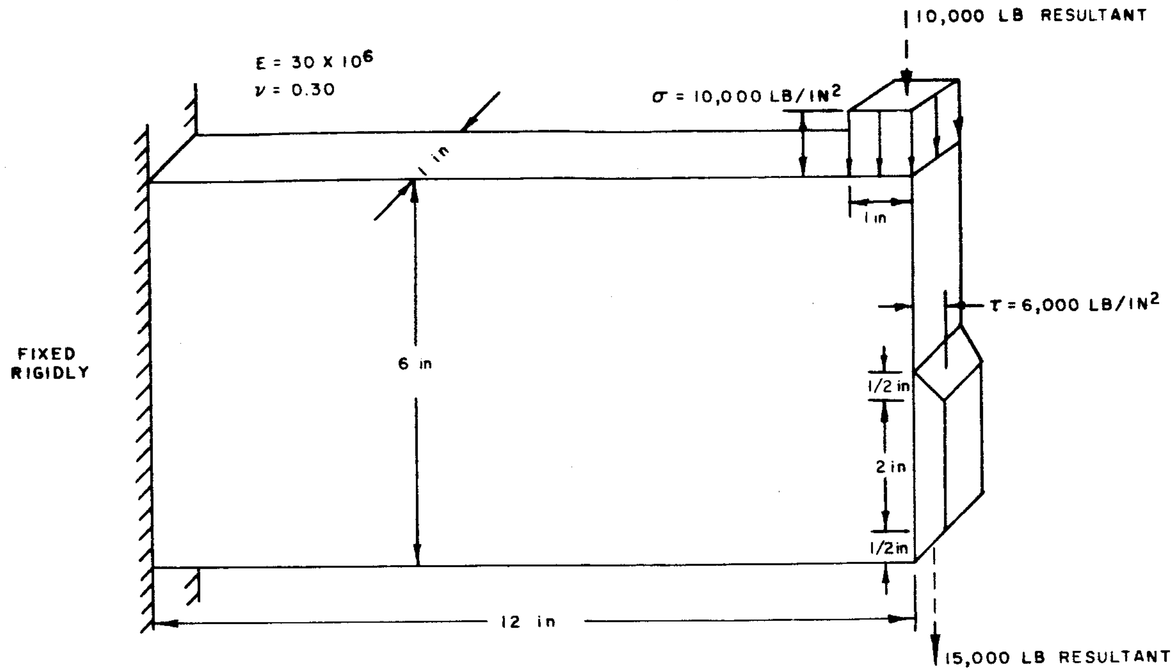
A listing of the computer program BODY 2 based on the lattice cells shown in Figs. 1 and 3 is presented in Appendix I. A program based on McCormick's lattice cell is presented in Appendix II. In the following section two example problems will be solved.

Illustrated Two-Dimensional Stress Problems of Plane Stress and Plane Strain

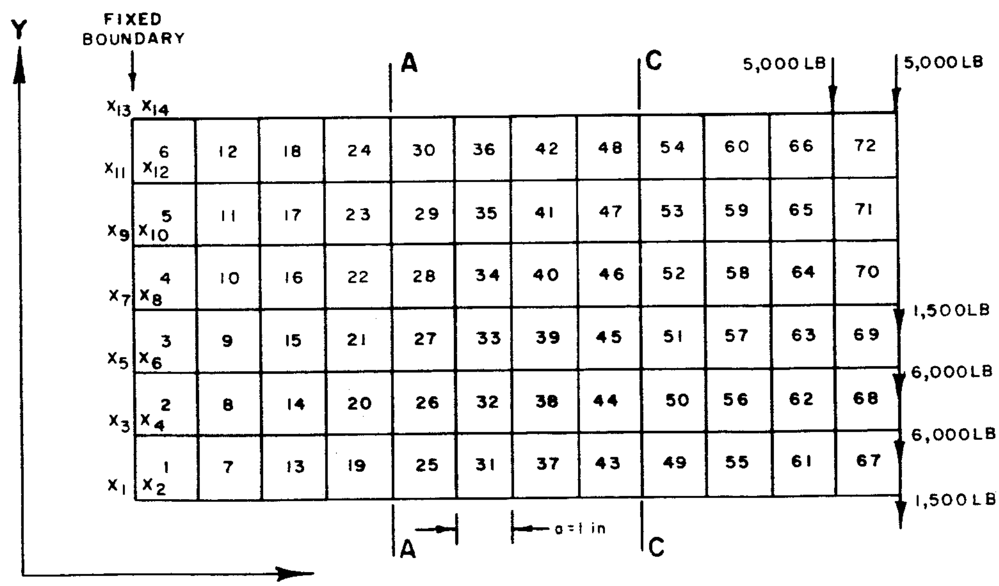
This section will illustrate the use of the lattice analogy by solving two problems, one of plane stress and one of plane strain, in linear isotropic mediums.

In addition to the problems illustrated here, several other problems were solved. A vertically loaded sheet-pile wall was analyzed. The sheet-pile problem is a soil-structure interaction problem. For this problem a lattice representing the soil medium was first generated. This lattice was modified by the addition of the sheet-pile wall as an additional structural member. This modified lattice was then solved. By similar means other two-dimensional soil-structure interaction problems could be analyzed. Several beams and short column problems were solved giving quite satisfactory results. Problems covering the three cases of uniform stress conditions have been solved and the solutions agree with the theoretical solutions, thus confirming the validity of the lattice cells and the computer program.

The first illustrative problem to be described is the cantilever beam shown in Fig. 8a. The left end of the beam is considered to be rigidly fixed and the right end is subjected to the loading shown. This is a problem of plane stress and the lattice proposed by Hrennikoff is applicable. The



BOUNDARY CONDITIONS AND OTHER DATA
(a)



BOUNDARY CONDITIONS $\begin{cases} \text{DISTORTIONS: } X(i) = 0 \quad \text{WHERE } i = 1,2,3,4,5,6,7,8,9,10,11,12,13,14 \\ \text{LOADS: } F_{168} = F_{182} = -5{,}000\text{ LB}\ ;\ F_{170} = F_{176} = -1{,}500\text{ LB}\ ;\ F_{172} = F_{174} = -6{,}000\text{ LB} \end{cases}$

LATTICE REPRESENTATION
(b)

FIGURE 8 CANTILEVER BEAM PROBLEM

ordering shown in Fig. 8b of lattice cells, node movements, and node loads is in accordance with the scheme presented in Fig. 5. The various boundary conditions which are imposed on the lattice to simulate the boundary conditions of the continuous body are listed in Fig. 8b. The fixed condition of the left end is represented by specifying all node movements along this edge of the lattice to be zero. The stress conditions shown in Fig. 8a is represented by static equivalent loadings shown in Fig. 8b.

The solution of the problem is illustrated in part by Fig. 9. The distorted lattice is shown. By magnifying the horizontal deflection scale the shear distortion of the vertical face A-A, which is not considered in conventional beam theory, is shown. Strains and stresses at the center of each lattice cell are computed from this distorted lattice by use of finite difference representations of strain components.

In order to demonstrate a method for studying stress concentrations, the lattice shown in Fig. 10a was employed. This lattice is the right portion of the cantilever beam shown in Fig. 8b. The size of the cells in Fig. 10b is one-half the size of the cells used in the initial analysis in Fig. 8. The reduction of the size of the lattice cell was made in order to reflect more properly the highly nonuniform stress conditions occurring in the right end of the cantilever beam. The node movements occurring along the vertical Section C-C obtained from the solution based on the lattice of Fig. 8 were used as boundary conditions for the problem in Fig. 10. The input and output data for this solution are given in Appendix III.

The resulting stress distributions obtained from this finer lattice are presented in part in Fig. 10. As can be seen, the stress distributions are considerably different than those which would have been calculated from

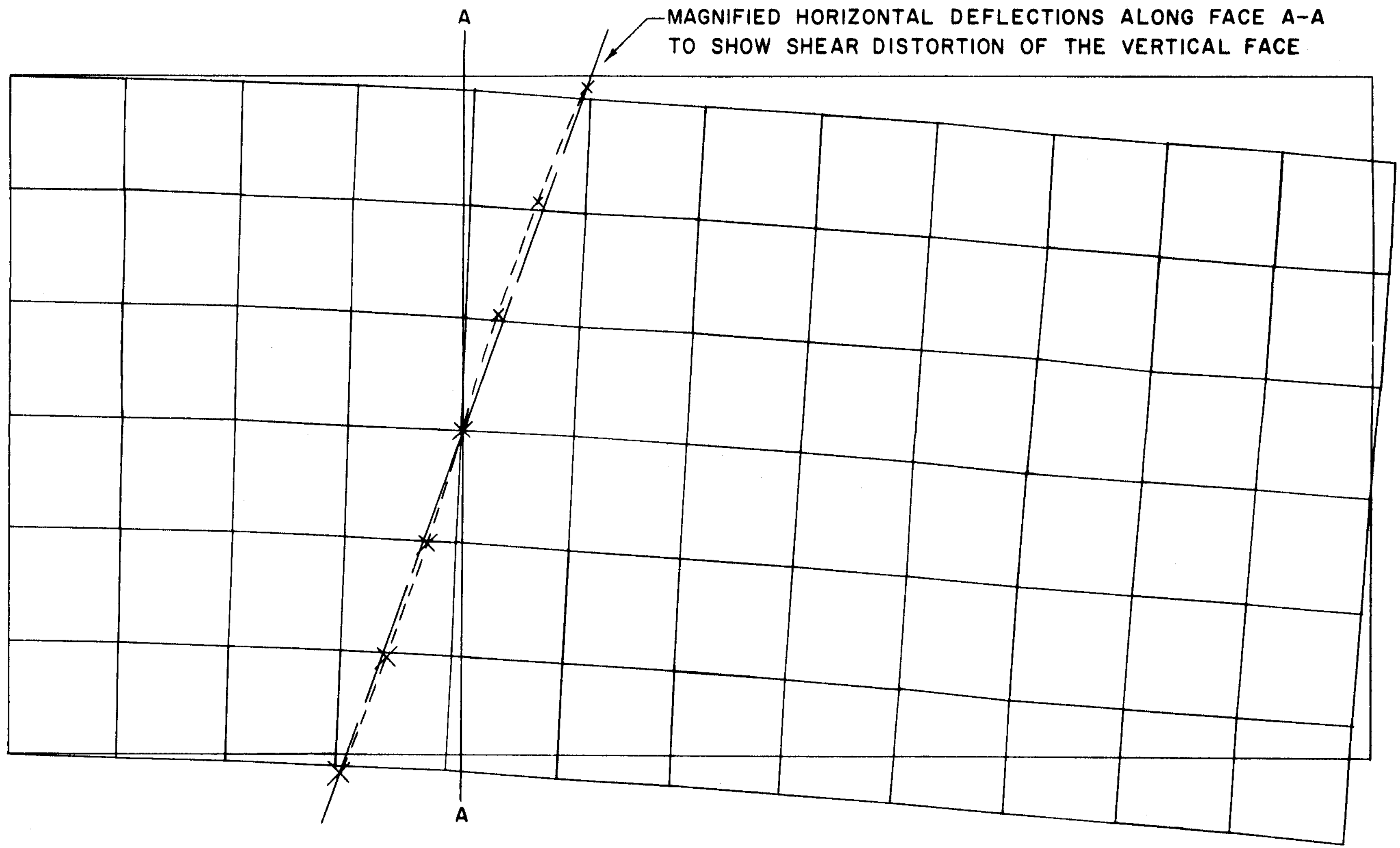


FIGURE 9 DISTORTED LATTICE FOR CANTILEVER BEAM PROBLEM

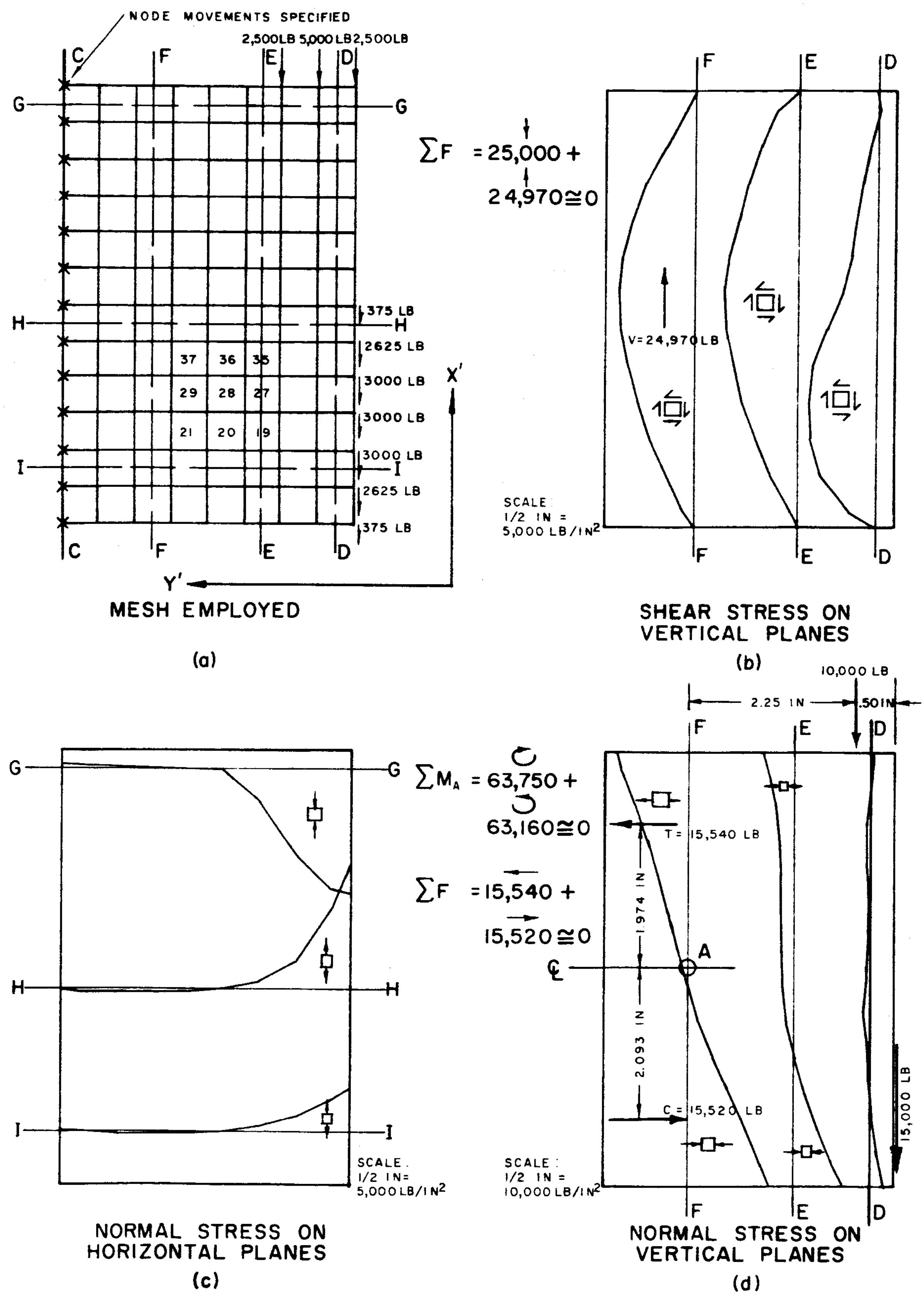


FIGURE 10 STRESS DISTRIBUTION AT POINTS OF LOADING

conventional beam theory. The stress distributions on the vertical face F-F were numerically integrated and the resulting forces are shown in Figs. 10b and 10d. Static equilibrium checks show that acceptable accuracy was obtained.

Compatability checks were made for the strain values at several locations in the right section of the beam and compatibility of strain values checked within 5 per cent. As an example, the compatibility check at cell No. 28 is shown below. Strain values at the adjacent cells are used to represent the compatibility equation, Eq. 6a, in finite difference form, and are taken from the output data of the problem in Appendix III. It should be noted that the x y axes are rotated 90 degrees counterclockwise to form the x' y' axes in the finer mesh solution shown in Fig. 10a. The compatibility equation is written with respect to this x' y' - reference system.

$$\frac{\partial^2 \epsilon_{x'}}{\partial y'^2} + \frac{\partial^2 \epsilon_{y'}}{\partial x'^2} = \frac{\partial^2 \gamma_{x'y'}}{\partial x' \partial y'}$$

$$\frac{\partial^2 \epsilon_{x'}}{\partial y'^2} + \frac{\partial^2 \epsilon_{y'}}{\partial x'^2} = \frac{\epsilon_{x'29} - 2\epsilon_{x'28} + \epsilon_{x'27}}{h^2} + \frac{\epsilon_{y'36} - 2\epsilon_{y'28} + \epsilon_{y'20}}{h^2}$$

$$= \left(\frac{1.95117250 - (2)(2.15110931) + 4.14514420}{(0.5)^2}\right) 10^{-5}$$

$$+ \left(\frac{-1.37979818 - (2)(-5.95871909) - 11.5039918}{(0.5)^2}\right) 10^{-5}$$

$$= +0.00003311 .$$

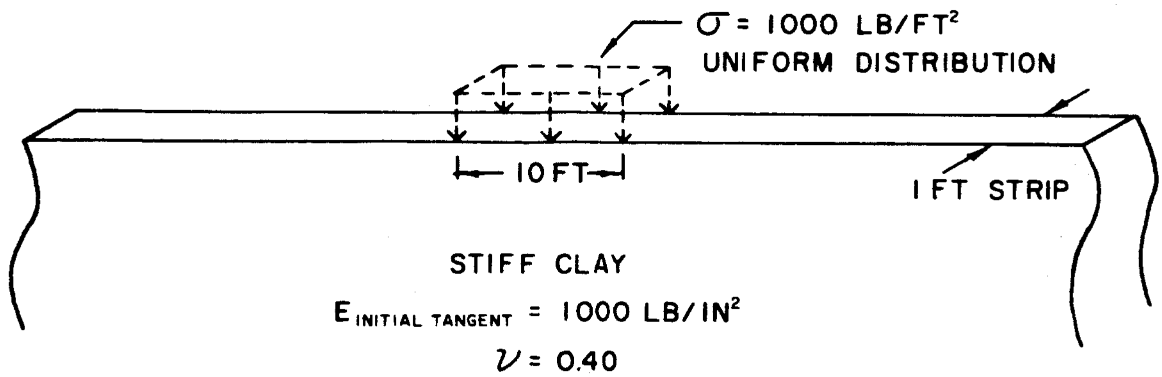
$$\frac{\partial^2 \gamma_{xy}}{\partial x' \partial y'} = \frac{\gamma_{x'y'19} + \gamma_{x'y'37} - \gamma_{x'y'21} - \gamma_{x'y'35}}{4h^2}$$

$$= \left(\frac{3.22355827 + 4.94522040 - 3.09887636 - 4.74811210}{(4)(0.5)^2}\right) 10^{-4}$$

$$= +0.00003218.$$

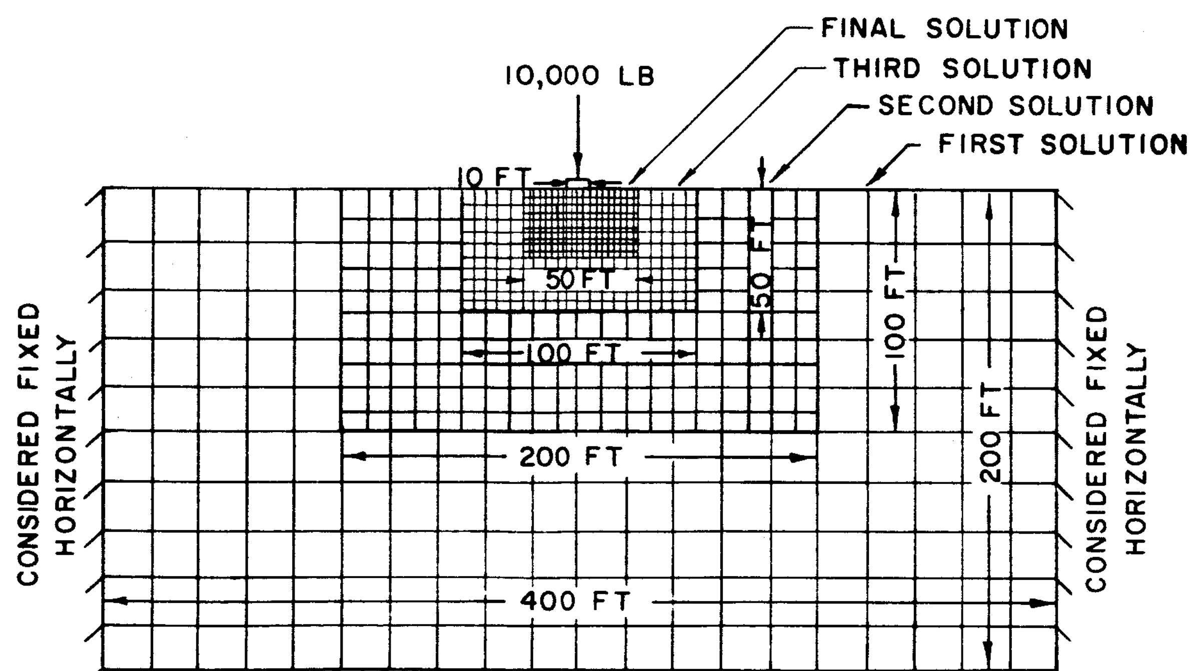
As can be seen, the error is small.

The second example of two-dimensional stress problems involving a linear isotropic medium is the long strip footing shown in Fig. 11a. Since the footing is relatively long compared to its width, the problem may be considered to be one of plane strain. It is desired to compute the initial pressure distribution beneath the uniformly distributed load. The technique of solution is shown in Fig. 11b. Four solutions were made to approximate the semi-infinite half space of the actual problem. In the crudest mesh, the fixed boundaries are 200 feet away from the loaded area or twenty times the width of the footing. Deflections of the cruder lattice were used as boundary conditions for the next finer lattice in identically the same manner as the cantilever beam problem. The results are illustrated in part by Fig. 12. The vertical pressure distribution obtained from the lattice analogy solution agrees within the accuracy of the pressures computed by conventional means, a solution of Boussinesq's equations in chart form for a linear isotropic medium.[18] The deflection of the ground surface and of the entire medium as well as the states of strain and stress throughout the medium are obtained from the lattice analogy solution. It should be readily appreciated that many other types of strip footing problems



BOUNDARY CONDITIONS AND OTHER DATA

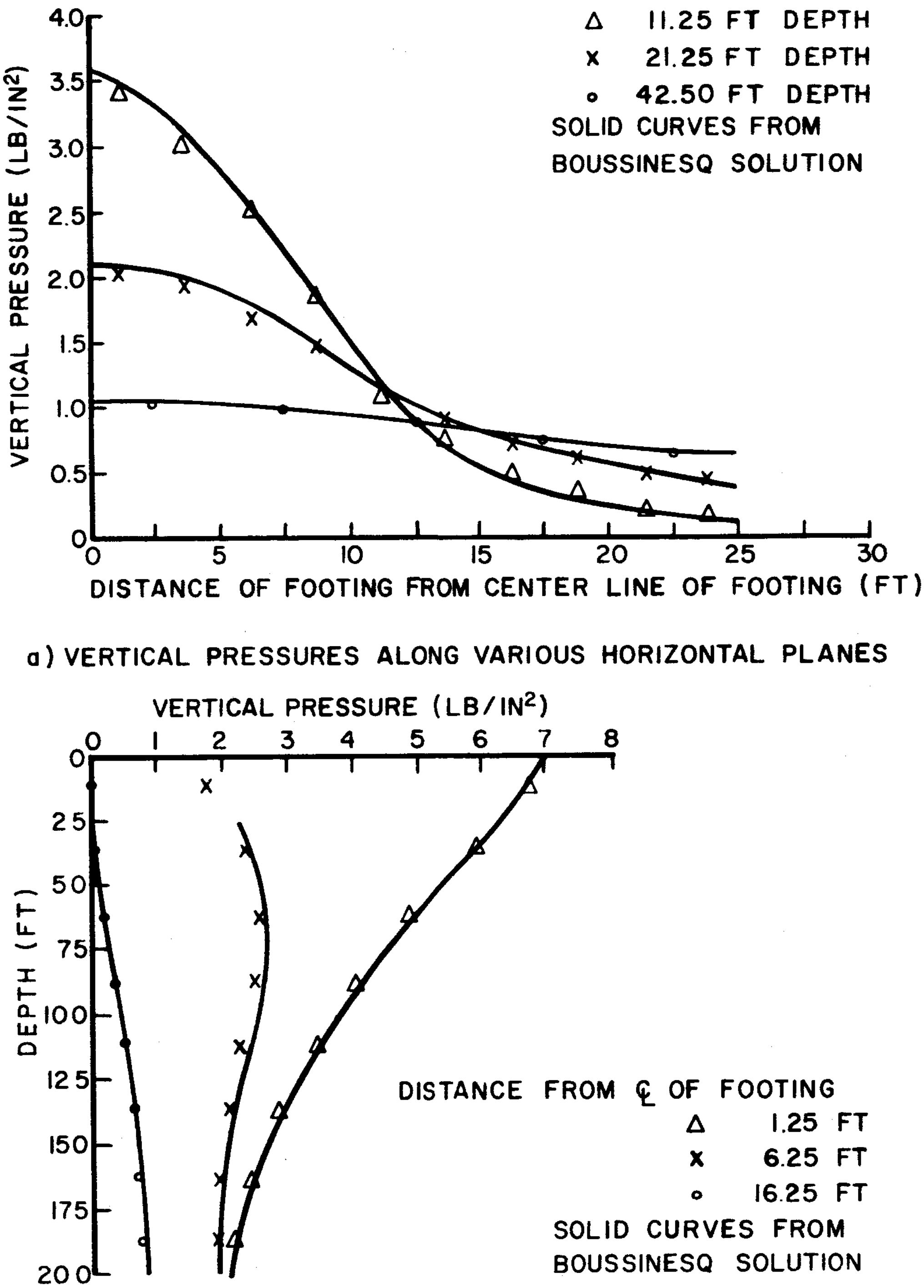
(a)



LATTICE REPRESENTATION

(b)

FIGURE 11 LONG STRIP FOOTING ON A STIFF CLAY



a) VERTICAL PRESSURES ALONG VARIOUS HORIZONTAL PLANES

b) VERTICAL PRESSURES ALONG VARIOUS VERTICAL PLANES

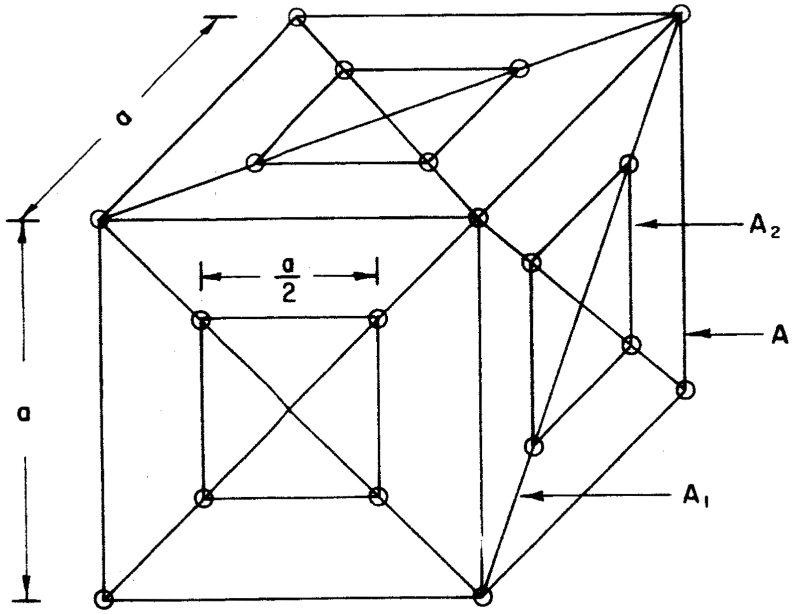
FIGURE 12 COMPARISON OF VERTICAL PRESSURES OBTAINED BY LATTICE SOLUTION AND BOUSSINESQ'S SOLUTION could be solved simply by altering the boundary conditions either in terms of deflections or pressures in the footing area.

Both theory and experience have shown that the distribution of vertical pressure is more or less independent of the physical properties of the medium. Conventional engineering practice is to assume the medium to be linear when calculating vertical pressures for consolidation predictions.[17] While vertical pressure distributions are not highly dependent on soil properties, the states of stress are dependent of soil properties and will vary appreciably from the linear case depending upon the nonlinearity of the soil. Since the consolidation of a soil is dependent on the states of stress and not just the vertical stress, the degree which the states of stress vary with nonlinearity needs to be studied. The method for solving two-dimensional stress problems in nonlinear mediums to be presented in the next chapter can be used in this regard. The above footing problem will be solved considering the soil medium as nonlinear.

Lattice Analogy for Three-Dimensional Stress Problems

In the preceding sections lattice analogy techniques were developed and used to solve stress problems for particular stress conditions (plane-stress and plane-strain) in linear isotropic mediums. This section will discuss briefly the extension of the analogy to the general three-dimensional stress problem.

An extension of the lattice analogy technique to the three-dimensional case was partially made when Hrennikoff proposed the three-dimensional lattice cell shown in Fig. 13 for a linear isotropic medium.[7] Since this lattice cell possesses three identical axes of symmetry, only three struc-



$$A = \frac{(1-3\nu)\,a^2}{4(1+\nu)(1-2\nu)} \; ; \; A_1 = \frac{a^2}{\sqrt{2}(1+\nu)} \; ; \; A_2 = \frac{(4\nu-1)\,a^2}{2(1+\nu)(1-3\nu)}$$

FIGURE 13 HRENNIKOFF THREE-DIMENSIONAL LATTICE CELL tural characteristics A, A_1, A_2 are necessary. In Hrennikoff's presentation only brief remarks were made in regard to the development of a complete method of solution based on this three-dimensional lattice cell. For a complete method of solution which is of practical use, the technique for generating the stiffness matrix equation and an efficient method for solving this matrix equation are required. While the generation of the stiffness matrix equation is primarily a straightforward extension of the two-dimensional cases cited above, it is nevertheless a step which needs some study in order to obtain a versatile and efficient method. The second step, the solution of the expanded matrix equation, is by far the most difficult aspect involved in this extension. Recommendations for solving the expanded matrix equation will be presented in the next chapter.

Since the lattice cell shown in Fig. 13 is simply connected, only three translational components of movement of each node point are required to describe the distortion of the lattice. In the two-dimensional stress case only two components were required. In writing the three general stiffness equations about an interior node, that is, the three equations of static equilibrium in terms of node movements and structural stiffnesses, the procedure used in Appendix I for the two-dimensional case is recommended. Again, only three stiffness coefficients SA, SAFA, SASP are needed to describe fully the deformability characteristics of an individual lattice cell. A two by two by two lattice representation of a cube is shown in Fig. 14. From this figure it is seen that a total of 45 node movements are involved in the three equilibrium equations about an interior node, while in the two-dimensional case only 18 node movements were involved.

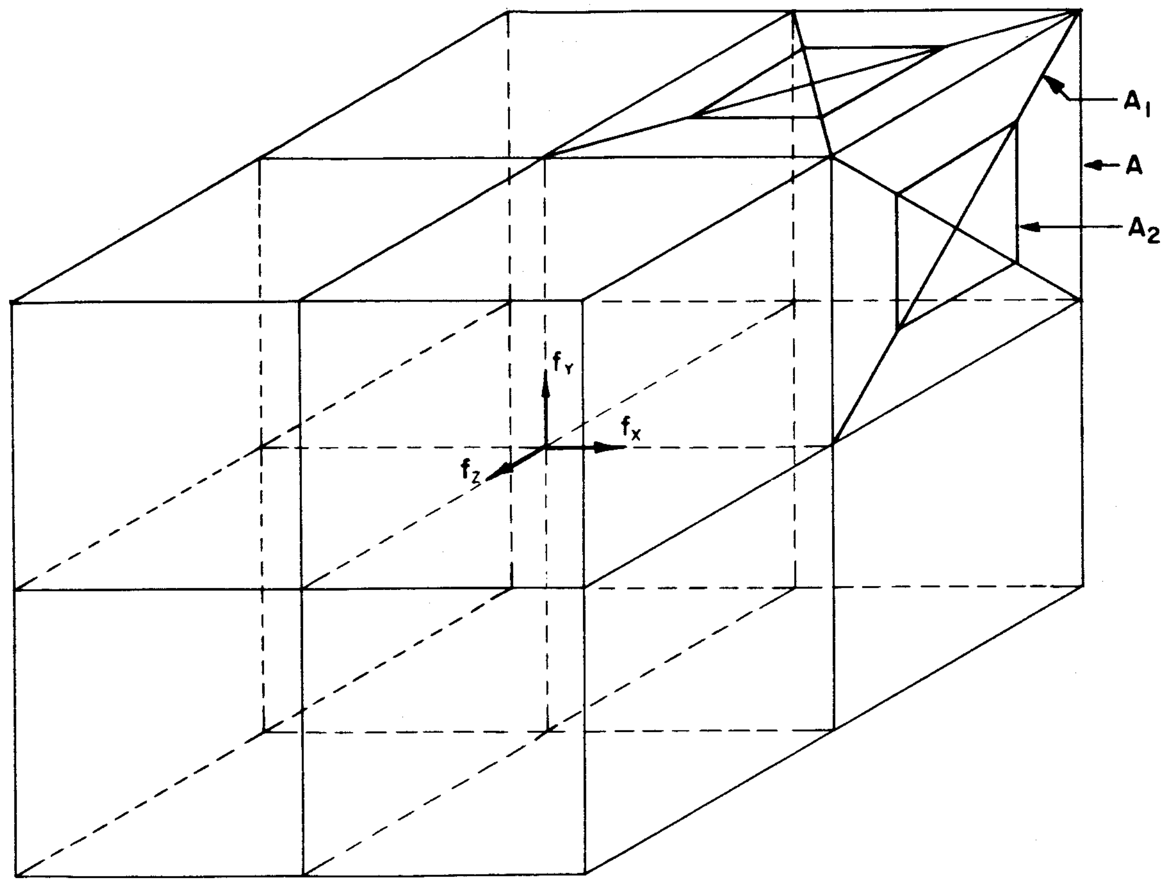


FIGURE 14 A TWO BY TWO BY TWO LATTICE REPRESENTATION OF A CUBE

The extension of the lattice analogy technique to the general three-dimensional stress case is deemed feasible and practical for some stress problems involving linear isotropic mediums. While this extension will not require additional theoretical developments of the analogy, it will require considerable formulation and programming and applications of improved methods for solving large systems of equations. Additional recommendations for the extension are presented in Chapter 4.

CHAPTER THREE

LATTICE ANALOGY OF TWO-DIMENSIONAL STRESS PROBLEMS IN NONLINEAR MEDIUMS

Many engineering materials have nonlinear stress-strain behavior. Solutions of stress problems by the theory of elasticity based on linearity assumption for such materials are approximations. The degree of approximation is dependent on the degree of nonlinearity of the material. In this section a method for solving two-dimensional stress problems, which will consider the nonlinearity of the medium, will be developed and two example problems will be solved. The method is based on the lattice analogy presented in the preceding chapter for stress problems in linear isotropic mediums. Essentially, the procedure consists of representing small sections of the nonlinear medium (sections being the size of the lattice cells) by pseudo-sections of linear isotropic material which are defined by two pseudo-elastic constants E and ν. When subjected to the same state of strain which exists in the nonlinear material, the pseudo-linear block will develop the same state of stress as that existing in the nonlinear material. In general this representation of a nonlinear material by a pseudo-linear material will be instantaneous in nature; that is, for each state of strain there exists a particular instantaneous pseudo-linear material representation. The key to the solution of a problem is to find the particular pseudo-linear material (E and ν for each lattice cell) which will properly represent the behavior of the actual nonlinear medium under a given set of boundary conditions.

For convenience in presenting the lattice analogy, the nonlinear medium is considered to be homogeneous and isotropic in its physical properties. The medium is also considered to be hyperelastic; that is, the state of stress is only a function of the state of strain and is not dependent upon the history of stress and strain. Stated more concisely, for every state of strain there exists a unique state of stress.

The above considerations are made only to facilitate the presentation of the basic theory and are not totally rigid limitations to the general lattice technique. Nonhomogeneity could be represented to a scale on the order of the lattice cell dimension. This would involve additional interpretation schemes in the method of solution (computer program). Nonhyperelasticity could be considered only if considerable logic and memory capabilities were included in the method of solution. Obviously, development of a capability to solve stress problems involving hyperelastic mediums has to precede consideration of the nonhyperelastic problem.

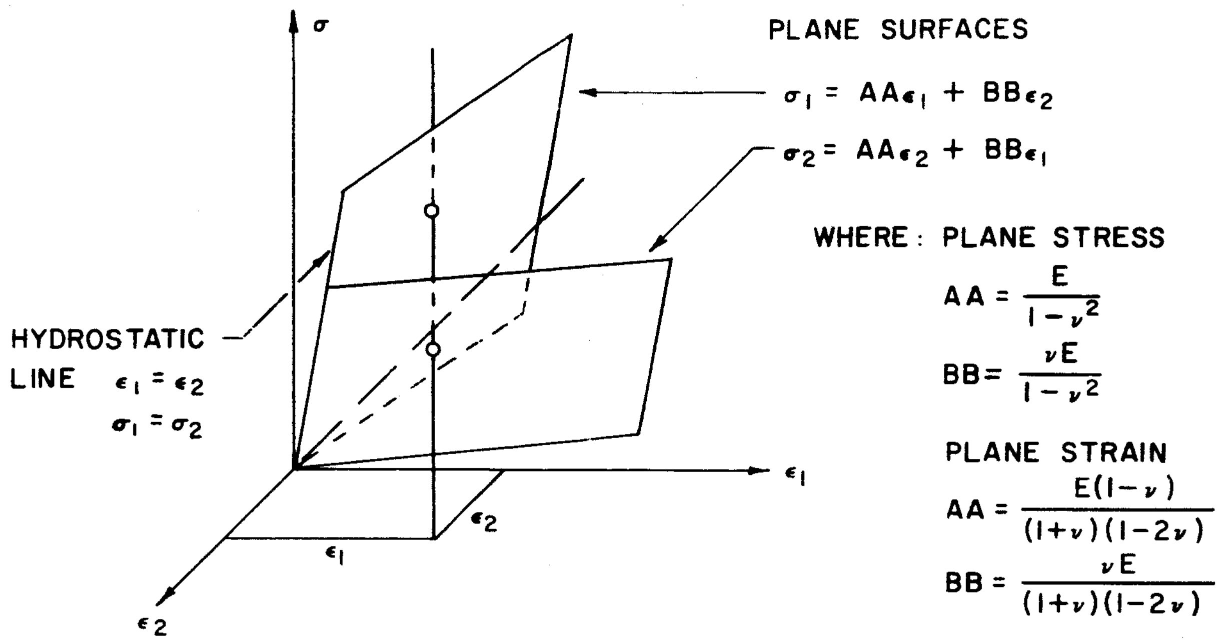
Representation of a Nonlinear Medium by a Pseudo-Linear Medium

As previously mentioned, the method for solving two-dimensional stress problems in nonlinear mediums by the lattice technique depends upon the ability to represent the behavior of the nonlinear medium under a given state of stress by a pseudo-linear isotropic material. In general this representation will be instantaneous in nature, but it will be sufficient in its approximation to describe the nonlinear medium over a small range of behavior. The range of approximation will be dependent upon the degree of nonlinearity.

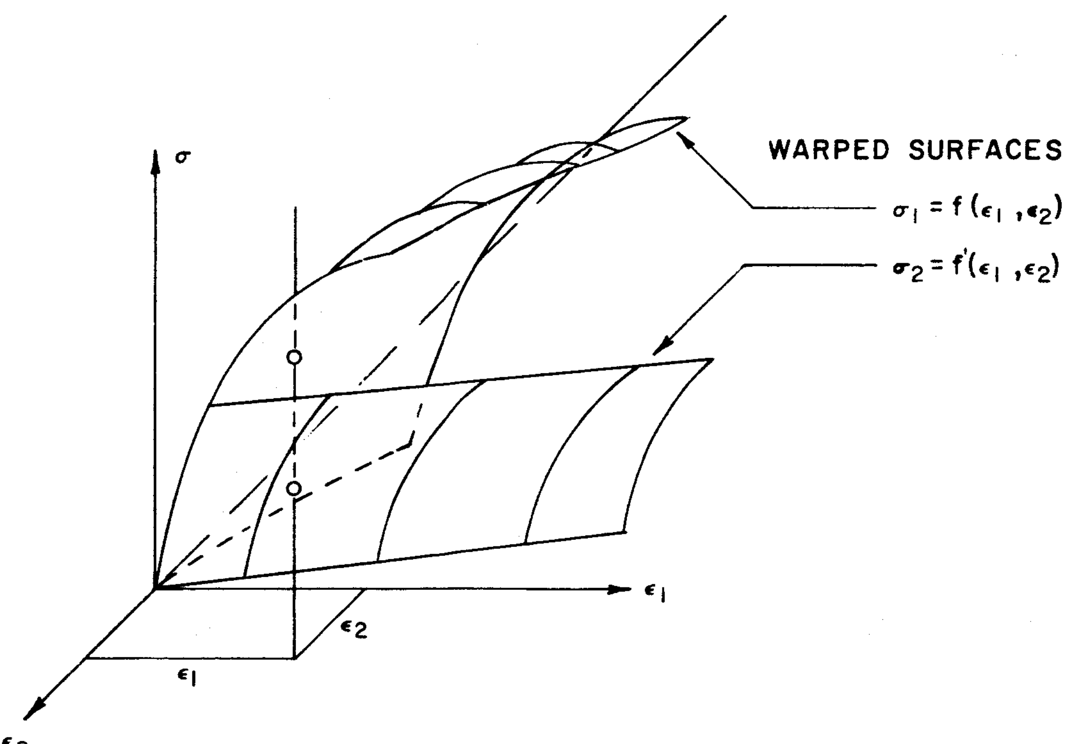
In order to illustrate better the method of solution to be presented in the next section, graphical representations of the stress-strain behavior of a linear and a nonlinear material under the conditions of plane-stress or of plane-strain are shown in Fig. 15. If the material is linear, the plane surfaces shown in Fig. 15a would describe the relationships between states of stress and states of strain. These two planes are simply a graphical representation of Hooke's Law as given by Eqs. 7a and 7b or Eqs. 10a and 10b. If the material is nonlinear, the two surfaces would not be plane, but rather would be warped as depicted in Fig. 15b.

Analytical expressions of the nonlinear functions $\sigma_1 = f(\epsilon_1, \epsilon_2)$ and $\sigma_2 = f'(\epsilon_2, \epsilon_1)$ in Fig. 15b for engineering materials have not been published to any extent in the literature. Quite obviously the prerequisite to an analytical solution of a nonlinear stress problem is a description of the stress-strain relationship. This stress-strain information would be either in the form of approximate analytical functions of the actual stress-strain relations, or in the majority of cases, in the form of numerical data from a series of experimental tests. A possible scheme for the latter type of information is illustrated in Fig. 16. The warped surface of the nonlinear material would be represented by discrete points of experimental data. This type of digitized information is readily adaptable as input information for a digital computer program.

The method of solution involves determining the unique pseudo-linear material which will possess the same state of stress as that of the nonlinear material for the same state of strain. The characteristics of the pseudo material can be considered as instantaneous and can change with each iteration. Since in an actual problem the states of stress and strain will vary

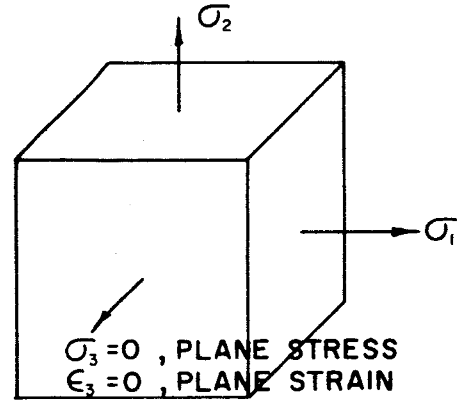


(a) STRESS - STRAIN RELATIONSHIPS FOR A LINEAR ISOTROPIC MATERIAL



(b) STRESS - STRAIN RELATIONSHIPS FOR A NONLINEAR ISOTROPIC MATERIAL

FIGURE 15 LINEAR AND NONLINEAR STRESS-STRAIN RELATIONSHIPS FOR TWO-DIMENSIONAL STRESS CONDITIONS

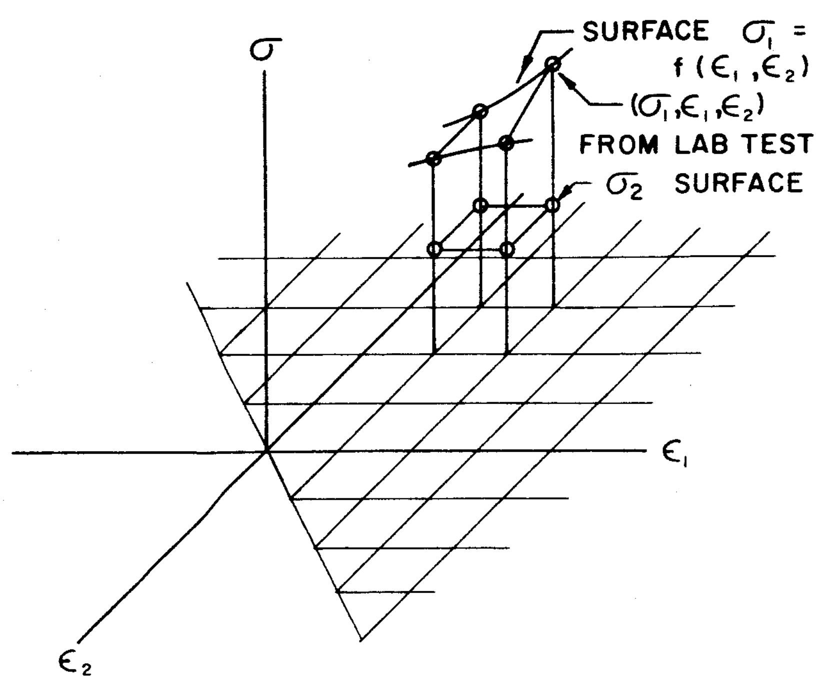


EXPERIMENTAL DATA

STATE OF STRESS	STATE OF STRAIN
$\sigma_1 , \sigma_2 \quad \sigma_1 > \sigma_2$	ϵ_1 , ϵ_2

LABORATORY TEST TO DEVELOP NUMERICAL STRESS – STRAIN DATA

(a)



NUMERICAL REPRESENTATION OF STRESS – STRAIN RELATIONSHIPS

(b)

FIGURE 16 NUMERICAL STRESS–STRAIN DATA NECESSARY FOR A SOLUTION OF A TWO–DIMENSIONAL STRESS PROBLEM IN A NONLINEAR MEDIUM throughout the medium, the nonlinear medium is represented piecewise by incremental sections (individual lattice cells) which are pseudo linear. The procedure for determining the pseudo-linear material, which will instantaneously possess the same deformability characteristics as that of the nonlinear material is as follows:

1. For a given state of strain in the nonlinear material ϵ_1 and ϵ_2 the resulting state of stress σ_1 and σ_2 is obtained from the stress-strain relationships such as shown in Fig. 15b. (The assumption is made that in the nonlinear isotropic material the principal directions of strain and stress are the same.)

2. The values of the elastic constants E and ν for the pseudo-linear materials are computed from the two equations of Hooke's Law, Eqs. 7a and 7b or Eqs. 10a and 10b.

3. With these elastic constants the structural elements of the lattice cell are computed from the equations which define the structural elements, Figs. 1 and 3.

In essence the elastic constants simply define the two secant planes which will intersect the warped surfaces of the nonlinear material at the instantaneous coordinates $(\sigma_1, \epsilon_1, \epsilon_2)$ and $(\sigma_2, \epsilon_1, \epsilon_2)$. By the above procedure the properties of each lattice cell in the entire lattice are obtained. Iterative procedures are required to find the unique lattice representation of the entire nonlinear medium for each set of boundary conditions.

Method for Solution of Two-Dimensional Stress Problems in Nonlinear Mediums

The theory and procedure for solving stress problems using a lattice analogy of a linear isotropic medium were presented in the preceding chapter. A technique of representing a nonlinear material by a pseudo-linear material was described in the immediate past section. By combining these procedures, a method for solving two-dimensional stress problems in nonlinear isotropic mediums will be outlined and illustrated below. Even though these procedures are essentially the same as those for a linear material, the steps to obtain a solution will be given here.

1. The continuous body is represented by a lattice consisting of individual lattice cells. The structural elements of these lattice cells are based on size and thickness dimensions of the cell a and t and the elastic constants E and ν for each cell. For the first iteration, elastic constants are assumed for each cell. In following iterations the elastic constants used are those obtained from the previous iteration.

2. Boundary conditions in terms of deflections or loads at main node points are applied to the lattice which was defined in Step 1.

3. The system of linear simultaneous equations $Sx = f$ resulting from the lattice representation and the applied boundary conditions is generated. The movements of the main node points are obtained from the solution of this system of equations.

4. From the obtained node movements, strain components are computed by finite difference methods. These strain components are

used to determine the states of strain at the center of each lattice cell.

5. The states of stress at the center of each lattice cell are determined by entering the stress-strain relationship, $\sigma_1 = f(\epsilon_1, \epsilon_2)$ and $\sigma_2 = f(\epsilon_1, \epsilon_2)$, Fig. 15b with the states of strain obtained in Step 4.

6. Tests for closure tolerance are made for main node point movements. If closure tolerances are not met, the procedure is to go to Step 7. If closure is satisfied, the procedure is to go to Step 8.

7. Revised pseudo-elastic constants E and ν are computed from Eqs. 7a and 7b or Eqs. 10a and 10b according to the stress condition by using the states of stress and strain obtained in Steps 4 and 5 above. After the computation of these revised elastic constants the procedure is to go to Step 1 for next iterative cycle.

8. Solution has converged to within the desired tolerance; therefore, the procedure is to tabulate results of last iteration for the final solution of the problem. In the program written for the above method of solution, the output information consists of the movement of each main node point x_1; the stress and strain components in the xy-directions σ_x, σ_y, τ_{xy}, ϵ_x, ϵ_y, γ_{xy}; and the principal stresses and strains σ_1, σ_2, ϵ_1, ϵ_2 and their orientation angle θ_{1-x}, measured from the 1-direction to the x-direction, counterclockwise being positive. Stress and strain components are given at each cell center.

The method for solution of two-dimensional stress problems involving nonlinear mediums detailed above will be illustrated in the next section by solving two example problems.

Illustrated Nonlinear Stress Problems

In the preceding sections a technique for using lattice analogy was developed for solving nonlinear stress problems. This section will illustrate the technique by presenting solutions to two problems.

Complete stress-strain relationships for nonlinear materials, that is, relationships which are valid over a wide range of stress states, are quite limited in the literature either in analytical or numerical form. Since this study is developmental in nature, a convenient analytical form, Eq. 18, is used to describe the nonlinear stress-strain behavior for the material in the illustrated problems.

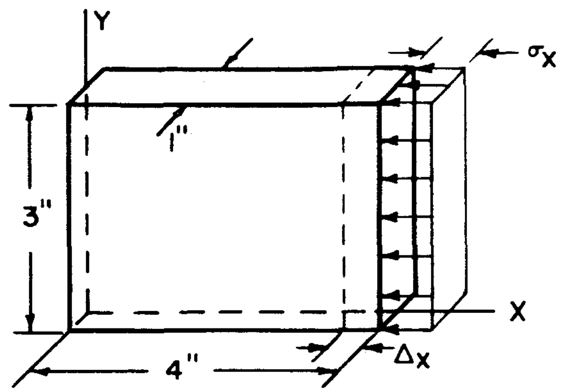
$$\sigma_1 = 25.0 \text{ Sin}(86.60\varepsilon_1 + 50.00\varepsilon_2).$$

$$\sigma_2 = 25.0 \text{ Sin}(86.60\varepsilon_2 + 50.00\varepsilon_1). \tag{18}$$

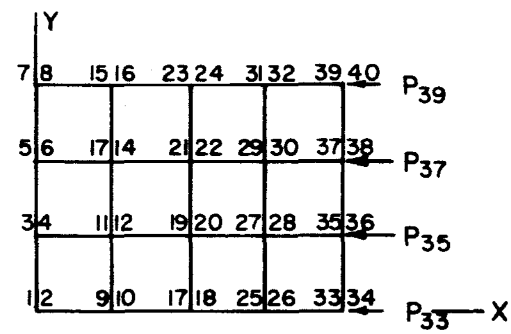
A graphical representation of Eq. 18 would be similar to the warped surfaces shown in Fig. 15b. The units of σ_1 and σ_2 are lb/in^2.

The first problem is one of several simple uniform stress problems which were solved in the process of developing and verifying the method of solution.

The problem is shown in Fig. 17. The rectangular block is loaded along its x-face either by a stress condition σ_x or a distortion Δ_x. The z-face is considered to remain plane and does not deflect in the z-direction. The y-face is a free surface. A summary of these boundary conditions is given in Fig. 17. The lattice solution and the exact solution for σ_x and Δ_x for eight different loading conditions are tabulated in



Z PROBLEM



LATTICE REPRESENTATION

BOUNDARY CONDITIONS AND DESCRIPTION

(i) PLANE STRAIN : $\epsilon_Z = \gamma_{XZ} = \gamma_{XY} = 0$

(ii) STRESS : $\sigma_Y = \tau_{XY} = 0$

(iii) DEFLECTION DATUM : $\Delta_X = 0$ AT $X=0$; $\Delta_Y = 0$ AT $X=0, Y=0$

(iv) SPECIFIED BOUNDARY CONDITIONS FOR INDIVIDUAL RUNS

σ_X AT X=4.0 FOR RUN NUMBERS 1,2,3,4,5,6

Δ_X AT X=4.0 FOR RUN NUMBERS 7,8

(v) MATERIAL DESCRIPTION : $\sigma_1 = 25.0 \sin(86.6\epsilon_1 + 50.0\epsilon_2)$

$\sigma_2 = 25.0 \sin(86.6\epsilon_2 + 50.0\epsilon_1)$

PARTIAL RESULTS OF LATTICE SOLUTION

RUN NO. SEE (v) ABOVE	EXACT SOLUTION		LATTICE SOLUTION			SPECIFIED CLOSURE TOLERANCE
	σ_X	Δ_X AT X=4.0	σ_X	Δ_X AT X=4.0	ITERATIONS	
1	-10.00	-.02851	-10.00	-.02851	5	10^{-6}
2	-20.00	-.06425	-20.00	-.06425	8	10^{-6}
3	-22.00	-.07454	-22.00	-.07454	17	10^{-8}
4	-24.00	-.08914	-24.00	-.08917	30	10^{-8}
5	-25.00	-.10883	-24.96	-.10489	20	* 10^{-6}
6	-25.00	-.10883	-24.99	-.10676	40	* 10^{-8}
7	-19.04	-.06000	-19.04	-.06000	9	10^{-8}
8	-24.80	-.10000	-24.80	-10000	20	10^{-8}

* CLOSURE TOLERANCE WAS NOT ACHIEVED

FIGURE 17 TYPICAL NONLINEAR MEDIUM TEST PROBLEM, PROBLEM STATEMENT AND PARTIAL RESULTS

Fig. 17. The exact solution for σ_x versus Δ_x for the above problem is in equation form.

$$\sigma_x = 25.0 \ \text{Sin} \left(\frac{25}{\sqrt{3}} \Delta_x \right)$$

Good agreement between the lattice and exact solutions is achieved for all loading conditions except for Runs 5 and 6 where σ_x = 25.0 lb/in.^{2}, a maximum. After 20 and 40 iterations the disagreements in the deflections Δ_x are approximately 4 and 2 per cent, respectively. The error in σ_x is smaller since at this stress level the magnitude of stress is essentially insensitive to strain. Figure 18 illustrates these points graphically.

The results tabulated in Fig. 17 are plotted in Fig. 18. The load deflection curve is seen to be nonlinear. The solutions for Runs 5 and 6 are seen to lie in the range where the load-deflection curve has approximately zero slope.

Iteration data for several runs are given in Fig. 19. The lattice solution for Run 6 converges relatively slowly compared to the other runs. Run 4, while having only a 4 per cent lower stress level, converges considerably faster than Run 6. This is due to the fact that the 4 per cent decrease in stress is sufficient to place the solution of Run 4 in a range where the slope of the load-deflection relationship is substantially larger than Run 6 as can be seen in Fig. 18. It is interesting to note that while the stress levels for Runs 6 and 8 are essentially equal, Run 6 required four times the number of iterations as Run 8 for the same accuracy in solutions. This is be-

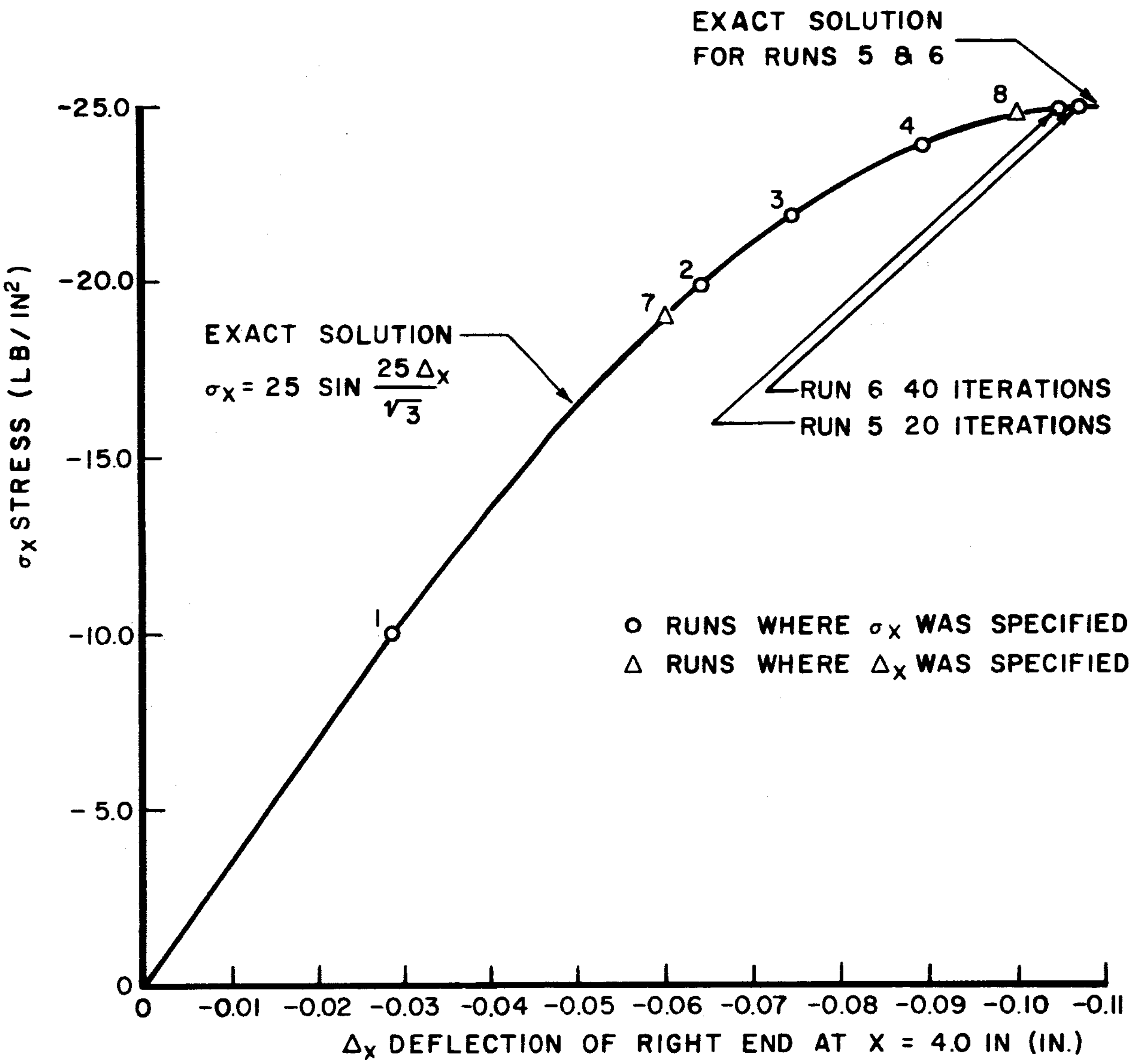


FIGURE 18 PARTIAL RESULTS OF THE NONLINEAR TEST PROBLEM SHOWN IN FIGURE 17 COMPARING EXACT AND LATTICE SOLUTIONS

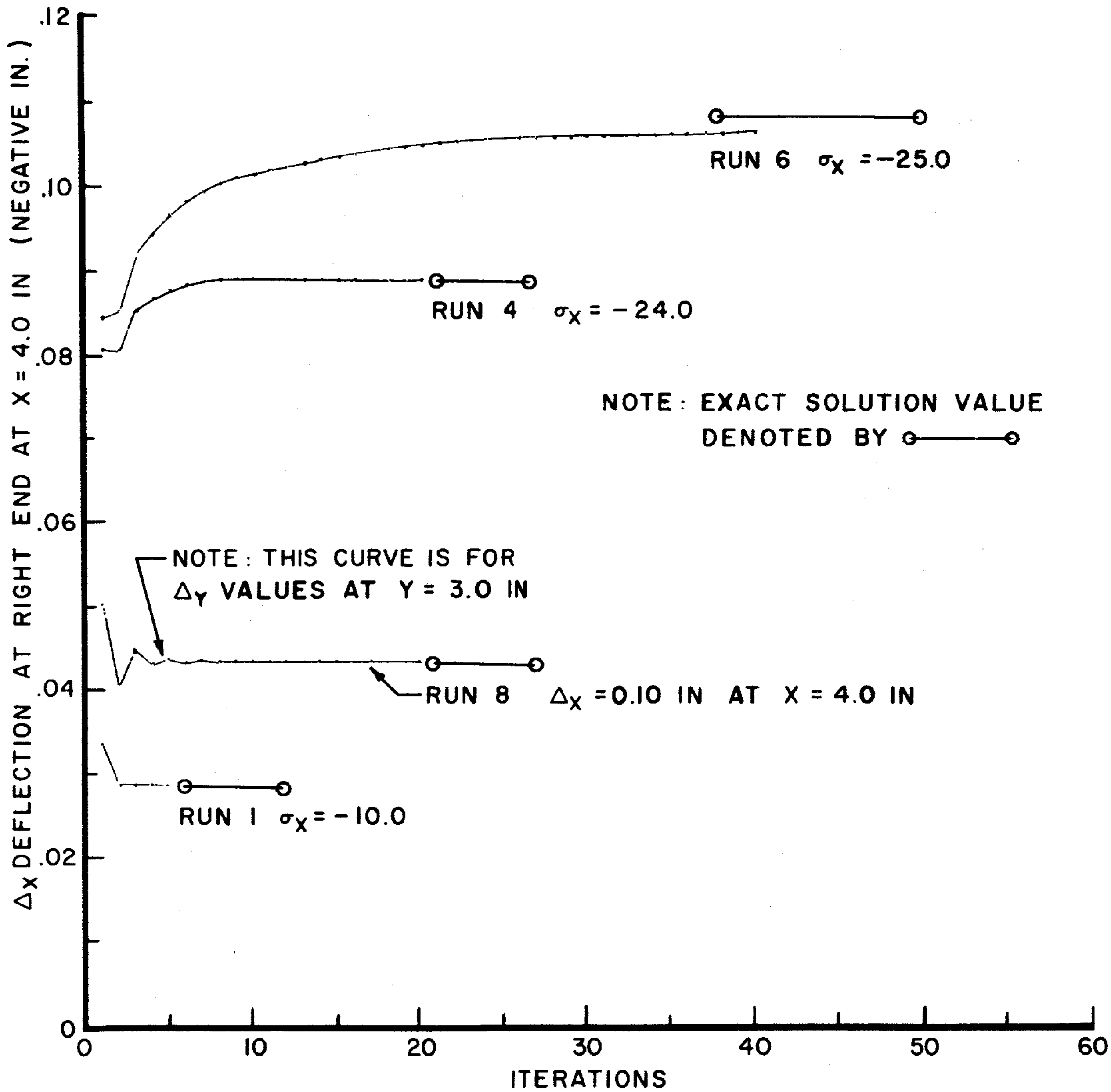


FIGURE 19 ITERATION DATA OF THE NONLINEAR TEST PROBLEM

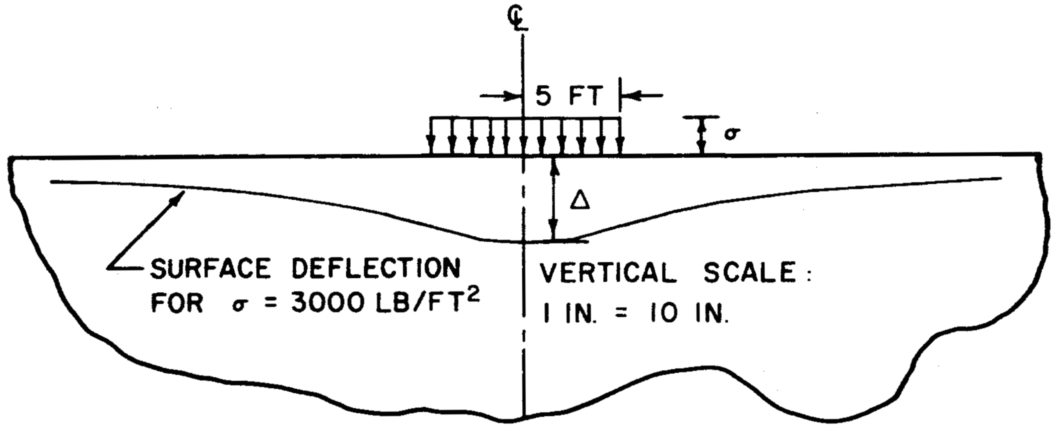
cause the deflection $\Delta_x = 0.10^{in}$ was specified as the loading condition for Run 8. Hence, closure tolerance for Δ_x deflections were satisfied on each iteration and only the Δ_y deflections needed to converge.

As previously mentioned, solutions of several uniform stress problems for nonlinear mediums were obtained by the lattice analogy technique. These problems were similar to the problem shown in Fig. 17. Boundary conditions for these problems involved both uniform distortions and stress conditions along the edges of the rectangular body. The solutions obtained were in agreement with known solutions. For these problems the level of stress was varied. In all cases when the state of stress entered the nonlinear region, the required number of iterations for solution increased considerably as in the above problem, Runs 5 and 6. While this convergent characteristic is readily appreciated and is common knowledge, it desires reiteration here.

The closure criterion used in this study was quite simple. The node deflections obtained for each iteration were compared with the deflections obtained from the immediately preceding iteration. If the differences were equal to or less than the specified closure tolerance, the solution was considered to be converged. This closure criterion could be misleading especially if the stress level is in the plastic range. For example, consider the iteration data for Run 6 as shown in Fig. 19. The solution is seen to be converging slowly to the exact solution. The difference between deflections for each iteration is seen to be quite small (Δ_x = 0.106707 in., 0.106758 in. for iteration 39 and 40, respectively) while the exact deflection value is 0.10883 in.

The second illustrative nonlinear problem is a long strip footing on a nonlinear foundation medium. The stress-strain properties of the foundation medium are defined by Eq. 18. Solutions were obtained for six different footing pressures by essentially the manner as for the linear footing problem in Chap. 2. The scheme of representing the semi-infinite half space by a crude lattice and then increasing the fineness of the lattice around the foooting area as shown in Fig. 11, Chap. 2, was used. In these solutions the material was assumed to be linearly homogenous for the first iteration; that is, the medium was described by single assumed values of E and ν. The same initial trial values of E and ν were assumed in the solution of each lattice spacing for all six footing pressures. Perhaps a more efficient procedure would have been to assume nonhomogeneous initial trial values of E and ν based on preceding solutions. This was not done in this problem because the time required for entering this data as input is lengthy. In future studies the merit of using initial trial values E and ν on values of preceding solutions should be studied.

Figures 20, 21, and 22 present, in part, the solution of the long strip footing on the nonlinear foundation medium described by Eq. 18. In the upper part of Fig. 20, the loading condition and the deflection of the foundation at the footing level are shown for the case where $\sigma = 3,000 \text{ lb/ft}^2 = 20.83 \text{ lb/in.}^2$. In the lower part of Fig. 20, the footing pressure versus centerline deflection curve is shown. The stress distributions for linear and nonlinear foundation media cases are compared in Fig. 21 and 22. These stress distributions are for



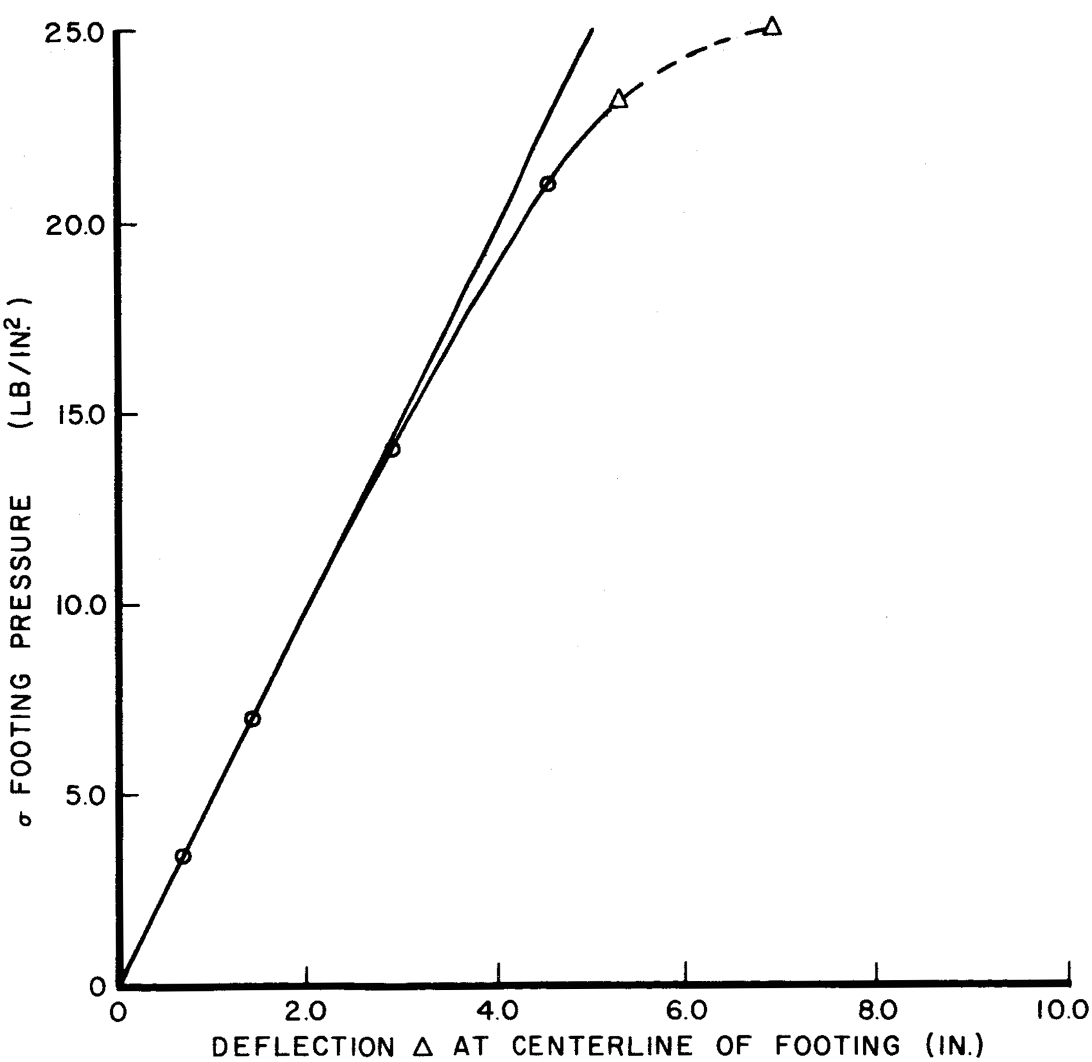


FIGURE 20 LONG STRIP FOOTING ON A NONLINEAR MEDIUM

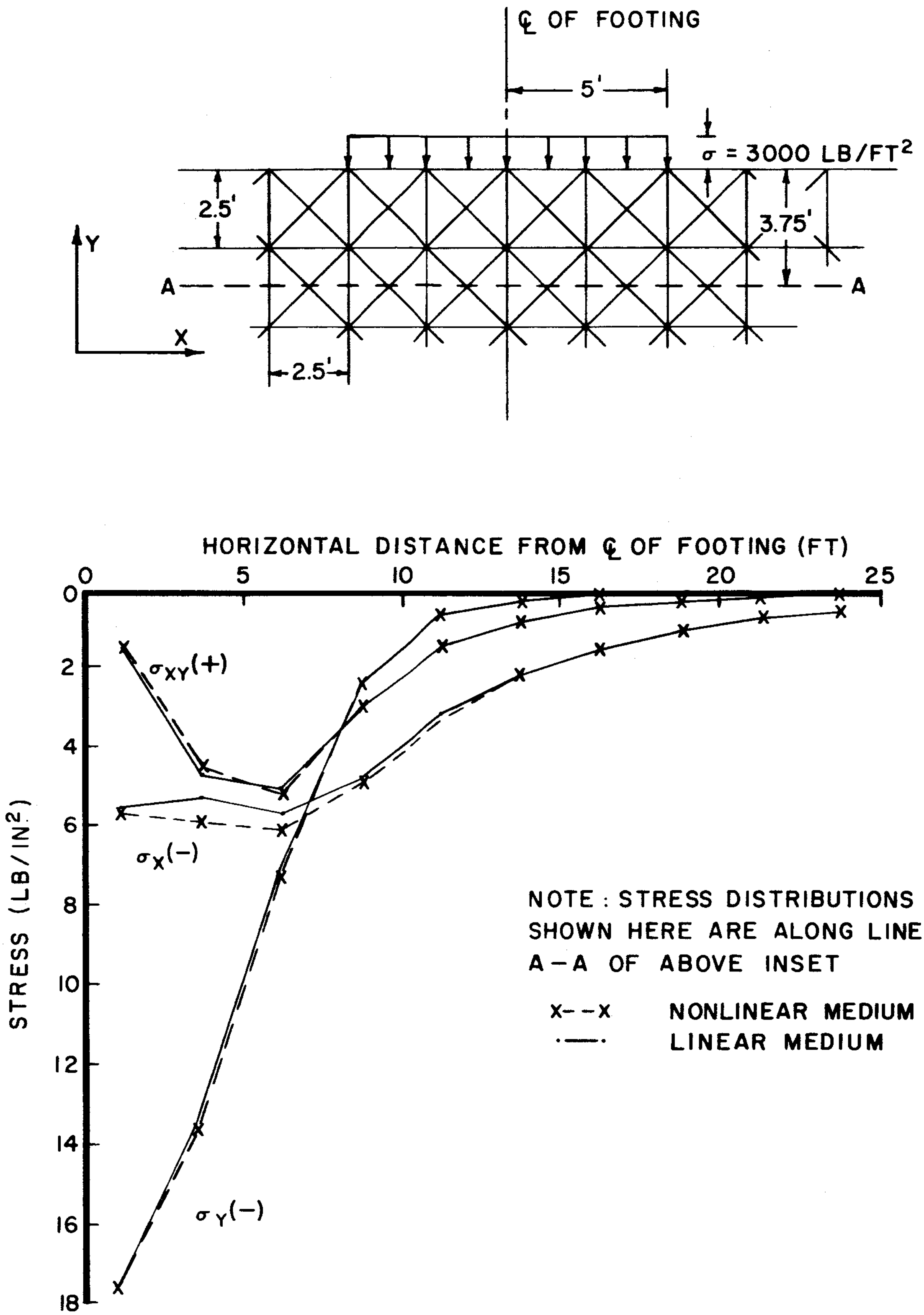


FIGURE 21 COMPARISON OF STRESS DISTRIBUTION FOR LINEAR AND NONLINEAR FOOTING PROBLEMS

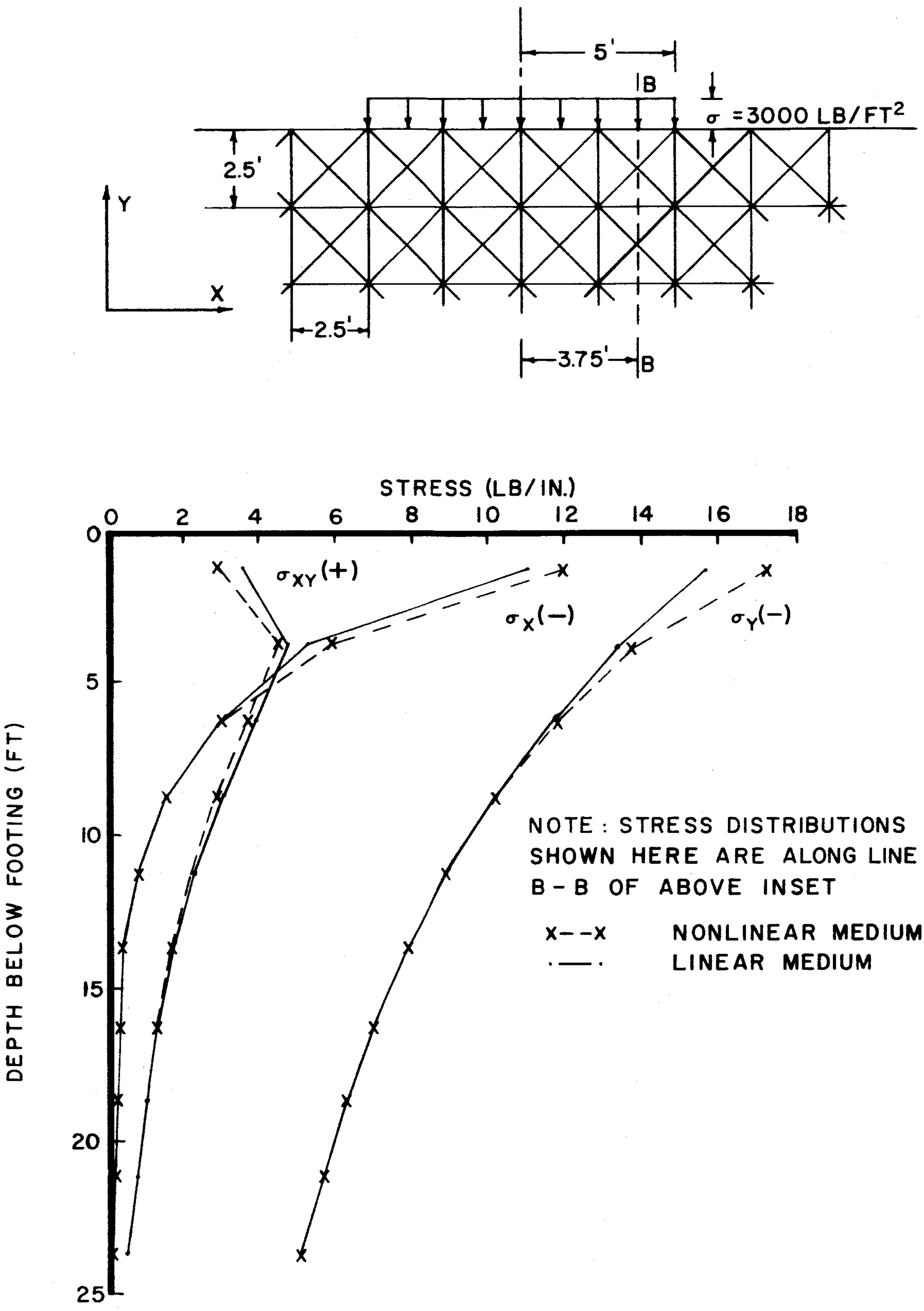


FIGURE 22 COMPARISON OF STRESS DISTRIBUTION FOR LINEAR AND NONLINEAR FOOTING PROBLEM

the 3,000 lb/ft^2 footing pressure and 2-1/2-ft lattice cell solutions. Variations between the stress distributions for the two cases are seen to occur only in the immediate area around the footing. If comparisons between stress distributions were made for higher footing pressures the variation would no doubt be greater. With the capacity of the computer program written for this initial study, completely satisfactory solutions for the higher footing pressures were not obtainable. Therefore, the stress distributions for the case of 3,000 lb/ft^2 footing pressure are given here.

For footing pressures of $\sigma = 3,000$ or 3,280 lb/ft^2 the vertical normal stress σ_y at the corner lattice cell (the lattice cell adjacent to the footing center-line and immediately below the surface) was larger than the applied footing pressure. This occurred in solutions based on 2.5-ft lattice cells. This size lattice cell is shown in Figs. 21 and 22. During the iterations involved in converging to a solution, the stress σ_y at the corner cell progressively grew with each iteration. When the nodal movements had converged to the specified closure tolerance, the stress σ_y was slightly larger than the applied footing pressure. For the case where the footing pressure was equal to 3,600 lb/ft^2, a converged solution was not obtainable for the closure tolerances specified. The reason for this failure can be explained by noting the stress-strain characteristics of the foundation medium, Eq. 18. The maximum possible normal stress the foundation medium could support is 25 lb/in.^{2} or 3,600 lb/ft^2. When the vertical stress σ_y passed this maximum point of the stress-strain surface, the corner lattice cell became progressively weaker and the solution therefore diverged. The deflection value shown in Fig. 20

for a footing pressure of 3,600 lb/ft^2 is the value given by the iteration cycle immediately preceding divergence of the solution.

Discussion of Errors Inherent in the Lattice Technique

In the preceding section the lattice analogy solution of the nonlinear footing problem yielded unrealistic vertical stresses σ_y in the immediate region beneath the loaded area. The reason for this unrealistic stress will be discussed here and recommendations made for further study.

To facilitate this discussion a brief review of the lattice technique for solving stress problems for continuous bodies will be given. The continuous body (referred to hereafter as body) is first represented by an articulated framework or lattice. The lattice possesses to an approximation the outline of the body. Statically equivalent loads are applied to external nodes to represent boundary conditions consisting of stresses. Statically equivalent loads are applied to the interior nodes to represent body forces. Node deflections are specified to represent boundary conditions consisting of distortions. Node deflections resulting from these loadings are obtained by stiffness methods. These node deflections are considered to represent at discrete points (node points) the distortion of the body. States of strain at discrete points in the body (centers of lattice cells) are obtained by using these node movements in finite difference expressions of strain components. States of stress at discrete points in the body (centers of lattice cells) are obtained by entering the stress-strains relationships for the body material with the states of strain.

From the above outline the accuracy of a lattice solution of a stress problem is seen to be basically dependent upon the ability of the lattice analogy to furnish, in sufficient detail and accuracy, the distortion of the stressed body.

As in other methods of solution of differential equations based on representing continuous functions by lumped parameters, it is not possible to formulate a general expression for the accuracy of a solution thus achieved for a given mesh size. The accuracy achieved by a given mesh size can be quite different depending upon the nature of the stress problem.

It is beneficial to list here the data which describe completely a stress problem: (1) the geometry of the body, (2) the material of the body, and (3) the boundary conditions on the body. All three types of data influence the requirement of mesh size. The usual procedure followed in establishing a feel for the accuracy of lumped parameter solutions is to study the cause and effect of each source of error separately. This will be the procedure here.

<u>Solution of the Stiffness Matrix Equation $\left[S\right](x) = (f)$:</u> The solution of the stiffness matrix equation yields the distortions at discrete points throughout the body. These distortions are used to evaluate states of strain and in turn states of stress throughout the body. Therefore, if the solution of the stiffness matrix equation is in error, the entire solution will be in error. The method for solving large systems of simultaneous equations will involve either direct elimination or some iterative technique. In either case the solution will be in error to some degree. In regard to the solution of the stiffness matrix equation, the accuracy should be such that the distortions can express in finite difference form the components of strain to acceptable accuracy. When node movements are substituted back into the

equations of equilibrium about the nodes, the equations should be satisfied at least to the same degree of accuracy as that desired for strains and stresses. This criterion can be considered to be a lower bound to the required accuracy of the solution of the stiffness matrix equation.

Geometry of the Body: The continuous body is represented by a lattice consisting of square lattice cells. Whenever the body is nonrectangular in shape or possesses irregularities in its boundary, the lattice representation can only possess to an approximation the same outline as that of the body. The degree of approximation decreases as the fineness of mesh size is increased. With regard to localized irregularities, there exists a saving factor for many cases in that the error due to the approximation of the irregularity is dominant only in a small region surrounding the irregularity. If detailed information is desired in the vicinity of the irregularity, solutions using progressively finer mesh sizes may be made in the region in question.

Material of the Body: In stress problems, stress-strain relationships for the material are required throughout the body. If these relationships are homogeneous throughout the body, there will be no source of error, in regard to material, created by the lattice representation. If the material is nonhomogeneous, approximations will exist in the lattice representation of the body and a source of error will be introduced. For stress problems involving nonlinear mediums, the pseudo-linear medium used to represent the nonlinear medium will be nonhomogeneous. Hence, a source of error will always exist in regard to material properties in nonlinear problems. Again, this error is minimized as the mesh size is made finer.

Boundary Conditions on the Body: The boundary conditions with respect to stresses or body forces are approximated in the lattice representation of the problem by statically equivalent loads at node points. The boundary conditions with respect to body distortions are represented in the lattice representation of the problem by node deflections. The source of errors from these procedures is readily apparent. Again, these errors are minimized as the mesh size is decreased. As in the error arising from the irregularities in the geometrical boundary of the body, the effects of these approximations of boundary conditions are in many cases dominant only in small regions surrounding the point of loading or distortion (Saint-Venant Principle). If detailed information is desired in the vicinity of the applied stresses or distorted boundaries, solutions using progressively finer mesh size may be made in the region in question.

Finite Difference Representation of Strains: The distribution of strains throughout the body are represented at discrete points by strains computed by finite difference methods using node movements. Even if the node movements are determined precisely by the lattice analogy, the strains components will still be approximations of the actual strains since the finite difference expressions are approximations of derivatives of distortions. Hence, the scheme for computing strains will always be a source of error. Again, as the mesh size decreases, the errors involved in the finite difference expressions are minimized.

The unrealistic vertical stress σ_y obtained in the nonlinear footing problem at the corner cell (corner cell being the cell adjacent to the footing centerline and immediately beneath the surface) was caused by the inability of the lattice to represent the highly variable conditions encountered in the immediate area of the footing. The nonhomogeneity of the pseudo-linear medium representing the nonlinear medium coupled with the highly nonuniform stress and strain distributions in the immediate area of the footing caused the highly variable conditions.

This conclusion is based on a brief analytical study. The study consisted of solving a similar footing problem to those above. The only difference being that this experimental problem consisted of a simple rectangular block with free edges and the medium was considered to be a nonhomogeneous linear medium instead of the homogeneous nonlinear medium described by Eq. 18. The width of the rectangular block was 2-1/2 times the length of the loaded area and its depth was 1-1/2 times the length of the loaded area. The nonhomogeneous linear medium was approximately the same as the pseudo-linear medium computed in the third iteration cycle of the solution for the 3,000 lb/ft^2 footing pressure case. For this case the modulus of elasticity E and the Poisson's ratio ν for the corner cell were 20 per cent less and 4 per cent larger, respectively, than for the adjacent cells. The magnitude of the vertical stress σ_y at the corner cell obtained in the solution of this experimental problem was approximately 8 per cent higher than the footing pressure. The solution of the similar footing problem on a linear homogeneous medium given in Chap. 2 yielded a σ_y stress for the corner cell which was slightly less than the applied footing pressure. This result indicates that the introduction of material nonhomogeneity was the cause for the unrealistic σ_y stress at the corner cell. The solution of the lattice was checked by checking the equilibrium about several nodes. The equilibrium checks showed that the nodes were in equilibrium with the external loads.

A computer program with a larger capacity than the one written for this initial study is felt necessary to investigate adequately the effect of nonhomogeneity. Means of increasing the program capacity are discussed in Chap. 4.

The stress-strain characteristics defined by Eq. 18 do not realistically represent an actual foundation medium. For example, the maximum normal stress that could exist in the medium is 25 $lb/in.^2$ regardless of the state of stress in the material. There is very little in the literature describing the actual shape of the stress-strain surfaces shown in Fig. 15b. While it is felt that Eq. 18 was adequate for this initial study, it is recommended that additional study be made to obtain more realistic stress-strain data.

CHAPTER FOUR

DISCUSSION OF COMPUTATIONAL METHODS

The theory and the techniques for solving stress problems in continuous bodies by representing these bodies by lattices of particular structural characteristics were presented in the preceding chapters. A lattice analogy of a continuous body results in a system of linear simultaneous equations represented by the stiffness matrix equation, Eq. 19.

$$\left[S\right](x) = (f) \tag{19}$$

The solution of this system of equations yields the movements of each lattice node point. As stated previously, the feasibility and usefulness of the lattice analogy technique depend greatly upon the development and usage of efficient and accurate methods for solving the above matrix equation. When considering the computational requirements involved in obtaining a solution of a stress problem in a nonlinear medium, the necessity of efficiency is readily appreciated since several or perhaps numerous iterative cycles are required in converging to the final solution. Each iteration in a nonlinear problem requires a solution of Eq. 19.

The solution of partial differential equations, especially the parabolic and elliptic types, by numerical procedures involves discrete approximations of these partial differential equations. Matrix equations in the form of Eq. 19 arise from these approximations. Numerous papers and texts have been written concerning the various methods of solution of these matrix equations. With the advent of the high speed electronic digital computers, new methods for solving large systems of linear simultaneous equations have been or are currently being developed. These methods are almost exclusively confined to cyclic iterative methods in contrast to non-cyclic iterative methods such as Southwell's relaxation method. It is the purpose of this chapter to enumerate and briefly review several methods of solution of Eq. 19 which are considered the most promising in the solution of the particular matrix equation derived from the lattice analogies and to list several computational schemes which will improve the efficiency of the programmed method of solution. The first section will briefly present the direct elimination procedure used to solve the matrix equation encountered in the development of the lattice analogy method. The second section will discuss and reference several possible methods of solution for the extension of the lattice analogy to three-dimensional problems or to finer lattice systems in two-dimensional stress problems.

It should be emphasized that solving Eq. 19 is by no means trivial. While there are several methods currently available for solving relatively large matrix equations, there exists in the literature little in the way of concrete recommendations about solving specific types and sizes of equation systems. Also, the more general recommendations are often contradictory from one study to the next.

For convenience in discussing the various methods of solution of Eq. 19, various properties of the coefficient matrix S are listed here.

1. The matrix S is a square multiple diagonally banded matrix. Some investigators classify the matrix S as a triple-diagonal-band matrix.

2. The matrix S is a real symmetric matrix. This is true even in nonlinear mediums since pseudo-linear mediums are used to represent the

nonlinear mediums at instantaneous states of stress and strain.

3. The matrix S is non-singular, that is the determinant of the matrix S is not equal to zero. Therefore, since the inverse of A exists there is a unique solution of the matrix Eq. 19 for each load vector f.

4. The elements $s_{i,i}$ along the main diagonal of the matrix S are non-zero and non-negative.

Direct Elimination Procedure to Solve Stiffness Matrix Equation

Since this study was primarily developmental in nature, a method for solving the matrix equation $Sx = f$ which would be easily programmed and would yield sufficiently accurate results for modest sized matrices was employed. In a study by White of several methods for solving linear simultaneous equations, conjugate gradients, Gauss-Sidell iterations, accelerated Gauss-Sidell iterations, and Gaussian elimination,, the technique of direct Gaussian elimination was found to be the most efficient and satisfactory method for solving diagonally banded matrices with sizes ranging up to 150 equations.[20] A direct elimination procedure was employed with satisfactory results by McCormick to solve the system of equations of his lattice analogy.[10] Based on these two papers, the direct elimination method was used in this study. Details of this method are presented below.

The stiffness matrix S is first triangularized by eliminating all nonzero elements below the main diagonal. The elimination proceeds column by column, using the main diagonal element in the i th row as the pivotal element. Only elements in the original band are considered in any column as it is known that only zero elements exist outside this band width. After triangularization, the matrix S is then diagonalized in a similar manner.

During both triangularization and diagonalization, operations performed on the stiffness elements are performed similarly on the load vector elements f_i. The unknown vector x is evaluated then by simple division. Because of the sparseness of non-zero elements even within the diagonal band, logic was included in the program to take full advantage of zero elements occurring in any row during the resolution process. Since this method of solution is programmed in FORTRAN language in BODY 2 Program, Appendix I, perhaps a more detailed illustration may be obtained by reviewing this section of the program. As an example of the speed of computation, consider the solution of the matrix equation resulting from the long strip footing problem. The stiffness matrix S involved 242 unknowns and a total band width of 51 elements. The time required to solve this matrix equation was 39 seconds.

McCormick's experience with a slightly modified direct elimination procedure to solve his stiffness matrix equation strongly indicated the suitability of this method of solution for two-dimensional stress problems. The time required for solutions were quite acceptable. McCormick records the computer time versus matrix size in a useful graph form. The direct elimination procedure was used to solve a 20 x 40 lattice system. For McCormick's lattice cell with $\nu \neq 1/3$, this fineness of lattice results in a stiffness matrix of 2, 583 unknowns and a total band width of 125 elements. The solution was programmed for the IBM 7090 and IBM 7094. It was, of course, necessary to use tapes in this program. The size of problems that can be handled by a given sized random access memory is limited by the total band width and not by the number of equations;

i.e., rows in the stiffness matrix. With a 32,000 word memory, the total band width is limited to approximately 320 elements. This means that the rectangular width of the lattice is limited to approximately 50 for McCormick's plane stress model. It is noteworthy to recall that the two-dimensional models which employ simply connected structural elements, Figs. 1 and 3, reduce the number of equations by 50% with respect to McCormick's model. Hence, even finer lattice systems could be solved in conjunction with this direct elimination method. While several computational aspects are briefly reviewed in McCormick's paper, the details of the method can be obtained from the references of Doolittle and Fox.[2, 4] From McCormick's reported experience, the direct elimination method of solution is judged quite satisfactory for solving the stiffness matrix equation which results from two-dimensional stress problems. Tezcan in a recent paper on the stiffness method for plane and space structures used a Gaussian elimination process in a manner similar to McCormick.[17] Tezcan gives a detailed outline of the computer program of the solution process and references a paper by Galletley.[5] Tezcan results parallel those of McCormick.

While the direct elimination procedures referenced above are considered to be satisfactory for two-dimensional stress problems, the use of this method in three-dimensional stress problems would not be feasible for any practical problem because of total band width requirements. This fact can be readily appreciated by noting Fig. 14. The most efficient type of ordering of node movements would lead to a band width on the order of 70 elements for a simple two by two by two lattice. Therefore, since storage and core requirements would quickly tax any currently available digital computer, other methods for solving matrix equations for three-dimensional problems are required.

Iterative Methods to Solve Stiffness Matrix Equation

In the above section the direct elimination scheme was shown to be a satisfactory means for solving the stiffness matrix equations resulting from two-dimensional stress problems. But the extension to three-dimensional stress problems was seen to tax storage facilities; hence, other methods of solution are required. The various iterative methods presented in the literature employed only the non-zero elements in the coefficient matrix S. If the procedure of "diagonal subscripting" the stiffness elements in an ordered stiffness matrix is used, the storage requirements will be considerably reduced.[20] Incorporating this scheme with an established iterative method, the solution of stiffness matrix equations describing three-dimensional lattice representation is considered possible.

An iterative method by Young, successive over-relaxation, is referenced as a possible means for solving Eq. 19.[22, 23] The successive over-relaxation method developed by Young is described by the iterative relationship in Eq. 20.

$$x^{(i+1)} = x^{(i)} + w\left[Lx^{(i+1)} + Ux^{(i)} + e - x^{(i)} \right] \tag{20}$$

In terms of the matrices of the stiffness matrix equation, Eq. 19, the various matrices in the above equations are defined as follows.

The vector $x^{(i+1)}$ is the deflection vector for the $(i+1)$ iteration.

The vector $x^{(i)}$ is the deflection vector for the (i) iteration.

The matrices L, U, e are most easily and clearly defined as follows. Let the stiffness matrix S be expressed as the sum of two matrices D and C.

$$S = D - C$$

where D is the diagonal matrix A_{ii} and C is a square matrix having zero elements on its main diagonal. The L and U matrices are given by the matrix equation

$$L + U = D^{-1}C$$

where L is a lower triangular matrix and U is a uppertriangular matrix. The vector e is defined by the matrix equation

$$e = D^{-1}f$$

The range of the iteration factor w is from 1 to 2. If w is equal to 1, the successive over-relaxation method of Eq. 20 will be equivalent to the Gauss-Sidell method.

Theoretical procedures are available for computing the optimum value of w for special types of matrices. Kahan states that the successive over-relaxation method can be employed with satisfactory results for matrices that are more general in nature[8]. Procedures for calculating w are reviewed in a text by Varga[19]. Since these procedures for calculating w involve considerable computation in themselves and apply to particular types of matrices, the procedure of using judgment and experience in selecting an over-relaxation factor is used in many cases.

If the node movements are consistently ordered and the technique of diagonal subscripting is utilized during the generation of the stiffness matrix S the successive over-relaxation method for solving the stiffness matrix Eq. 19 can be programmed with efficiency and should be relatively straight forward.

Another possible means of improving the capacity and efficiency of the method is to make full use of the symmetric property of the stiffness matrix S in the generation and solution of Eq. 19.

Another iterative method referenced as a possible method of solution of Eq. 19 is the Peaceman-Rachford iterative method, an alternating-direction implicit iterative method. [13, 3] This method of solution also involves obtaining iterative parameters. Methods for approximating these parameters are reviewed by Varga. [19]

CHAPTER FIVE

CONCLUSION

1. The lattice analogy technique is a powerful analytical tool for two-dimensional stress problems involving homogeneous linear mediums.

2. Computer times required for solutions of two-dimensional problems are quite acceptable for linear problems.

3. The preparation of input data for a lattice solution is a simple and straightforward task.

4. The ability of the lattice analogy to consider random boundary conditions and to approximate irregularly shaped bodies is a major advantage of this technique.

5. The procedure developed for solving two-dimensional stress problems involving nonhomogeneous linear mediums is useful and can be employed to obtain approximate solutions to some problems; however, additional study is needed on mesh-size requirements for nonhomogeneous materials.

6. The procedure for solving two-dimensional stress problems involving nonlinear mediums has been shown to converge to a correct solution where there is little variation in material properties. However, additional study of the effects of material nonhomogeneity and nonuniform stress distributions will be required before the method can become a useful tool for solving nonlinear stress problems. The method of solution is considered to be in an embryonic state of development at the present time.

7. The extension of the lattice techniques to the solution of three-dimensional stress problems is relatively straightforward.. The technique can be made into a useful tool when the ability is developed to solve large systems of linear simultaneous equations.

CHAPTER SIX

RECOMMENDATIONS

This chapter will enumerate the recommendations made in the preceding chapters. The recommendations are presented in the order in which they should be acted upon in the continuation of this study.

1. A comprehensive evaluation of the various lattice and plate analogies should be made. The most advantageous analogies should be used in future studies.

2. Improvements in the capacity of the computer program, BODY 2, should be made. This will mainly involve improving the ability to solve larger systems of simultaneous equations. Several detailed recommendations regarding these improvements are given in Chap. 4.

3. Studies should be made to understand the effects of material nonhomogeneity and nonuniform stress distribution. In particular, the effects of material inhomogeneity and stress nonuniformity on fineness of grid requirements should be studied. Chapter 3 discusses this problem.

4. A literature survey should be made to obtain stress-strain expressions or data as depicted in Fig. 15b. For experimental verification of the method of solution, experimental stress-strain data should be obtained for a convenient nonlinear material. Simple loading tests on bodies of this material could be made and the results compared with the analytical results obtained by lattice analogy solutions

of the same problems.

5. Studies should be made to understand better the convergent characteristics of the proposed method for solution of nonlinear stress problems. Convergent criteria should be studied. Computational procedures to aid in convergence should be developed.

6. An awareness should be maintained in regard to the ability to solve large systems of simultaneous equations. Whenever the ability is achieved, the lattice analogy technique should be extended to solve three-dimensional stress problems in linear mediums.

APPENDIX I

DETAILS OF A SIMPLY CONNECTED LATTICE ANALOGY FOR TWO-DIMENSIONAL STRESS PROBLEMS

APPENDIX I

As stated in the text, the technique for solving stress problems in a continuous body by a lattice analogy involves the generation and solution of a system of equations which describe the lattice representation of the continuous body. Appendix I will define the stiffness coefficients used in the stiffness matrix equation, Eq. 19. The simply connected lattice cells for plane stress and plane strain, Figs. 1 and 3 respectively, are geometrically identical, but the individual structural elements A, A_1, A_2 are defined differently. Hence, the stiffness coefficients depend on the particular stress condition. Equation 19 results from the writing of two equilibrium equations, Eq. 16, about each node point. These stiffness coefficients in Eq. 16 will be defined below. A computer program BODY 2 based on the lattice analogy will be described following the definition of the various stiffness coefficients.

Stiffness Coefficients

The stiffness coefficients listed in Fig. 6 and used in Eq. 16 will be defined in this section both for plane stress and plane strain lattice cells. A convenient way to develop the definitions of the various stiffness coefficients is shown in Fig. 23. The right top corner node of the lattice cell is deflected by the amount Δ such that there is the unit force F in the vertical member on the right side of the cell.

$$F = \frac{EA}{a} \Delta = 1$$

LATTICE CELL i

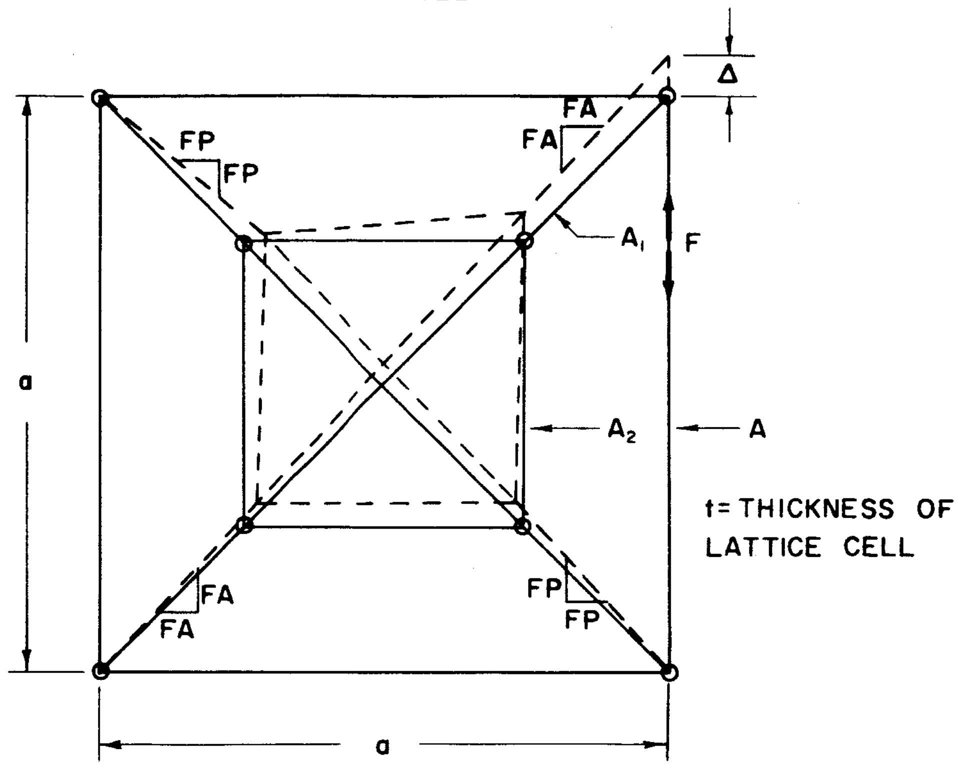


FIGURE 23 DERIVATION OF STIFFNESS COEFFICIENTS FOR PLANE STRESS OR PLANE STRAIN LATTICE CELLS, SIMPLY CONNECTED

By frame analysis, the resulting forces in the outer diagonal members due to the deflection Δ can be readily obtained. It is convenient to define these forces in terms of their horizontal and vertical components FP and FA. For the plane stress lattice cell, the resulting values of FP and FA due to the translation Δ are

$$FA = \frac{1}{4}\frac{(1+\nu)}{(1-\nu)} \frac{AE}{a} \Delta$$

$$FP = \frac{1}{4} \frac{(3\nu-1)}{(1-\nu)} \frac{AE}{a} \Delta.$$

For the plane strain lattice cell, the resulting values of FP and FA due to the translation Δ are

$$FA = \frac{1}{4(1-2\nu)} \frac{AE}{a} \Delta$$

$$FP = \frac{(4\nu-1)}{4(1-2\nu)} \frac{AE}{a} \Delta .$$

Three general stiffness coefficients for the lattice cell are defined using the above expression for horizontal and vertical forces resulting from a deflection of a node point.

For the plane stress lattice cell the three stiffness coefficients are

$$SA = \frac{AE}{a}$$

$$\text{SAFA} = \frac{(1+\nu)}{4(1-\nu)} \quad \frac{AE}{a}$$

$$\text{SAFP} = \frac{(3\nu-1)}{4(1-2\nu)} \quad \frac{AE}{a} \quad .$$

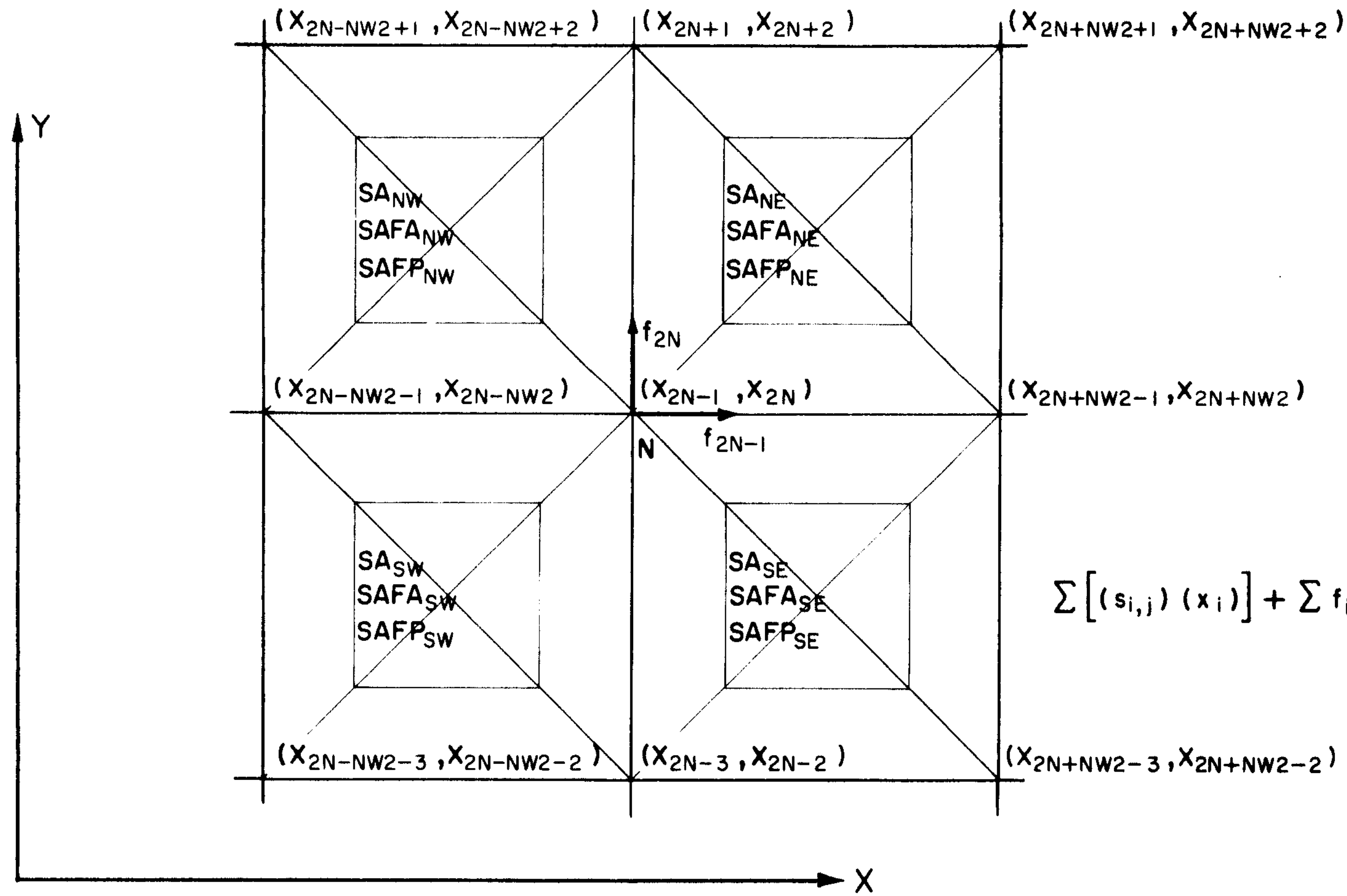
For the plane strain lattice cell the three stiffness coefficients are

$$\text{SA} = \frac{AE}{a}$$

$$\text{SAFA} = \frac{1}{4(1-2\nu)} \quad \frac{AE}{a}$$

$$\text{SAFP} = \frac{(4\nu-1)}{4(1-2\nu)} \quad \frac{AE}{a} \quad .$$

These three stiffness coefficients for each lattice cell are used to define the various stiffness coefficients listed in Fig. 6 and used in the two equilibrium equations, Eq. 16. For convenience Fig. 6 is redrawn here with slight modifications, Fig. 24. The three stiffness coefficients SA, SAFA, SAFP completely describe the stiffness of each lattice cell. In the development of the general stiffness matrix equation, Eq. 19, the two equilibrium equations about each interior node were first written in terms of these three stiffness coefficients. Because of the reoccurring combinations of these stiffness coefficients, it was advantageous to define an alternate set of stiffness coefficients, these being listed in Fig. 6 and used in Eqs. 16a and 16b. These various stiffness coefficients are defined below.



$$\sum\left[(s_{i,j})\,(x_i)\right] + \sum f_i = 0$$

FIGURE 24 DERIVATION OF THE TWO EQUILIBRIUM EQUATIONS ABOUT AN INTERIOR NODE IN TERMS OF NODE MOVEMENTS, NODE LOADS, AND STIFFNESSES

$$SSW = SAFA_{sw}$$

$$SNW = SAFA_{nw}$$

$$SNE = SAFA_{ne}$$

$$SSE = SAFA_{se}$$

$$SW1 = SAFP_{nw} + SAFP_{sw} + SA_{nw} + SA_{sw}$$

$$SW2 = -SAFP_{nw} + SAFP_{sw}$$

$$SW3 = -SAFP_{nw} - SAFP_{sw}$$

$$SN1 = SAFP_{nw} + SAFP_{ne} + SA_{nw} + SA_{ne}$$

$$SN2 = SAFP_{nw} - SAFP_{ne}$$

$$SN3 = -SAFP_{nw} - SAFP_{ne}$$

$$SE1 = SAFP_{ne} + SAFP_{se} + SA_{ne} + SA_{se}$$

$$SE2 = -SAFP_{ne} + SAFP_{se}$$

$$SE3 = -SAFP_{ne} - SAFP_{se}$$

$$SS1 = SAFP_{se} + SAFP_{sw} + SA_{se} + SA_{sw}$$

$$SS2 = SAFP_{sw} - SAFP_{se}$$

$$SS3 = -SAFP_{sw} - SAFP_{se}$$

$$SCEN1 = -(SAFA_{nw} + SAFA_{ne} + SAFA_{se} + SAFA_{sw} + SA_{nw} + SA_{ne} + SA_{se} + SA_{sw})$$

$$SCEN2 = SAFA_{nw} - SAFA_{ne} + SAFA_{se} - SAFA_{sw}$$

Computer Program

The computer program described here is based on the simply connected lattice cells shown in Figs. 1 and 3. The program BODY 2 will solve two-dimensional stress problems in linear or nonlinear mediums. Since the program is written in FORTRAN - 60 language, the comment cards in the program itself plus a general flow diagram are considered a sufficient description. The general flow diagram is shown in Fig. 25.

GENERAL FLOW DIAGRAM OF BODY 2 PROGRAM

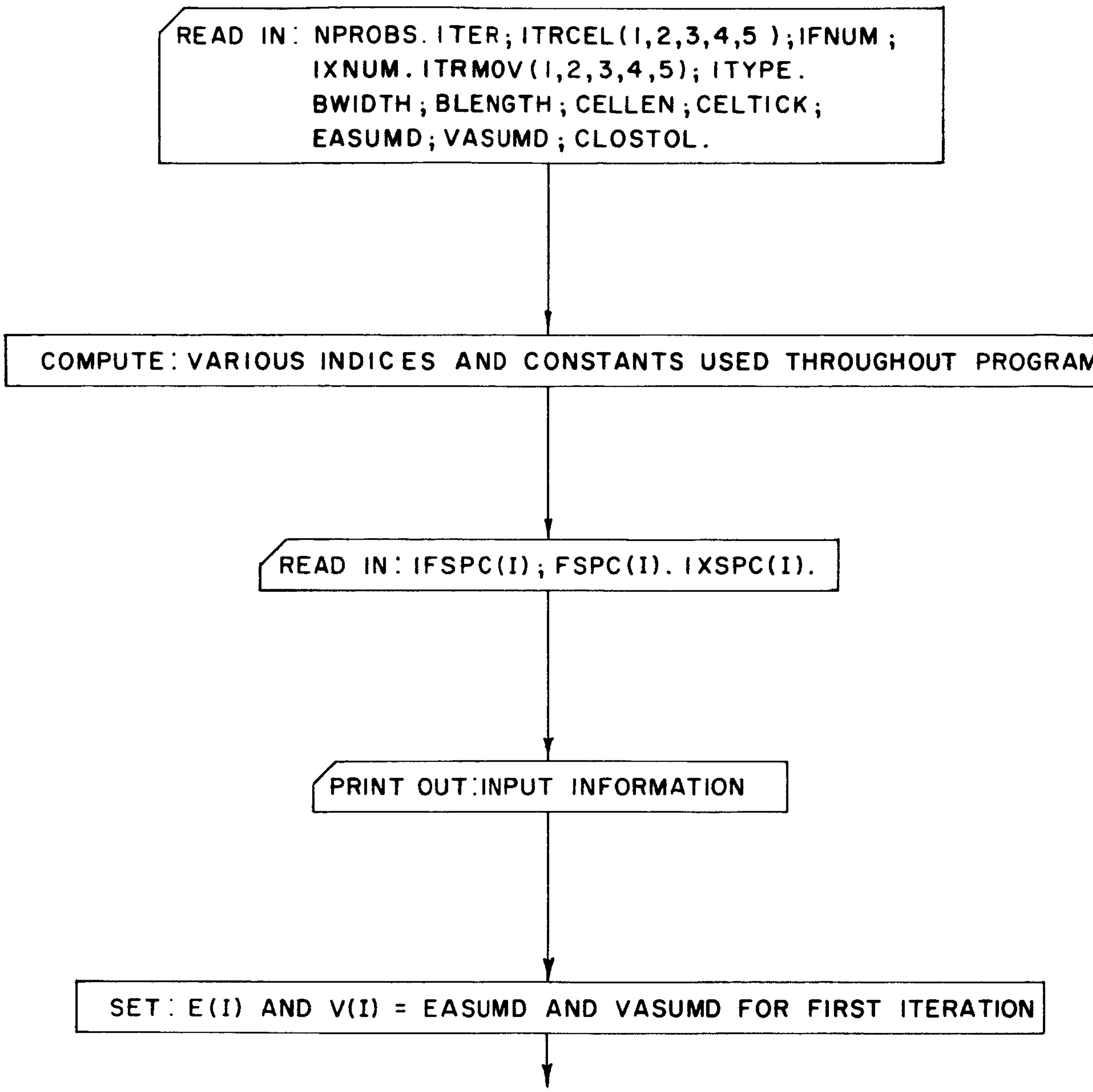
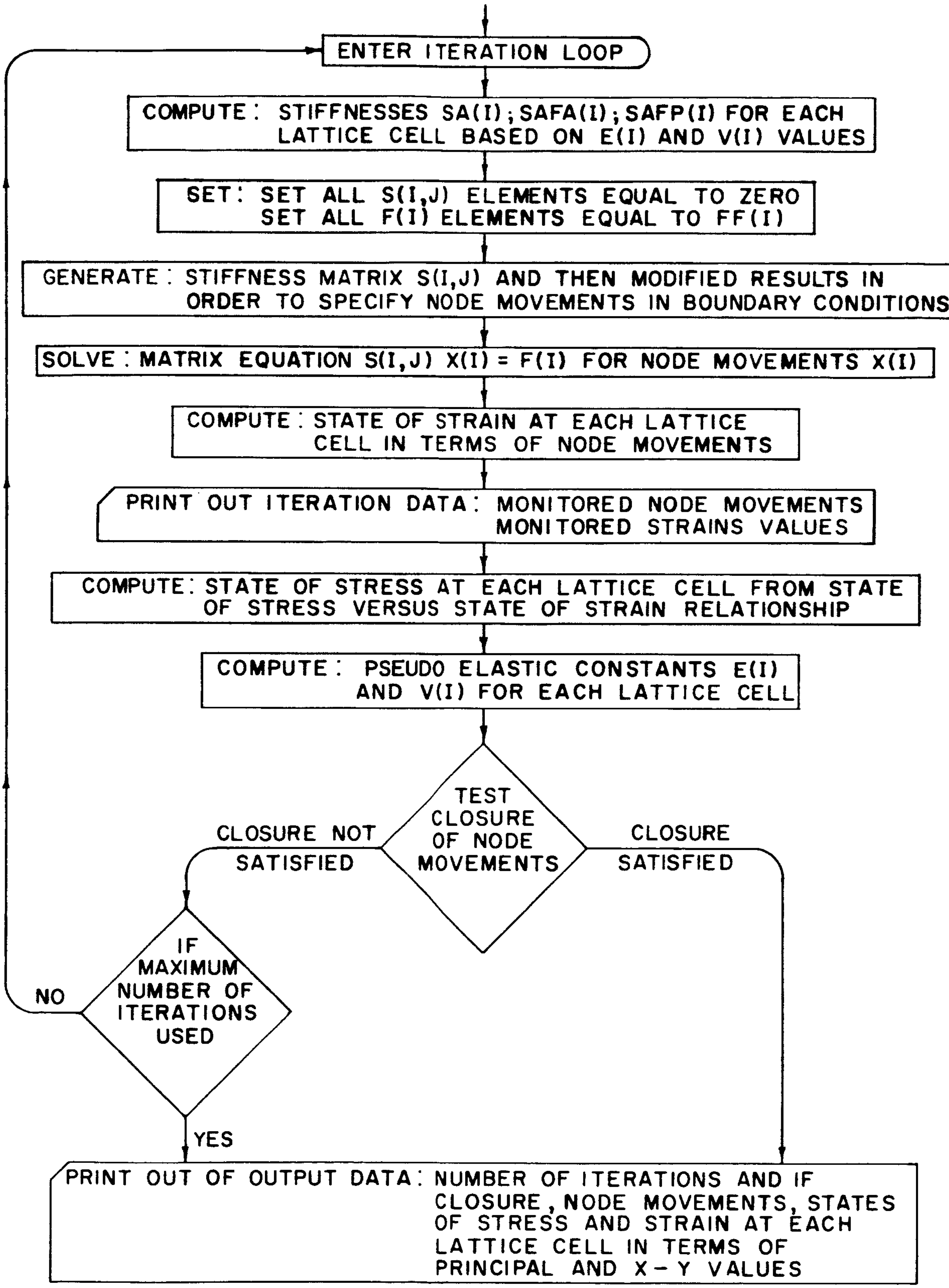


FIGURE 25 GENERAL FLOW DIAGRAM FOR BODY 2 PROGRAM



FLOW DIAGRAM FOR BODY 2 (CONT'D)

LISTING OF BODY 2 PROGRAM

```
..    SMITH R.E.  PROGRAM BODY 2  GENERAL SIMPLY CONNECTED LATTICE MODEL
      PROGRAM BODY2
      DIMENSION S(234,43),X(234),F(234),FF(234),FSPC(234),IFSPC(234),
     1           IXSPC(234),SA(96 ),SAFA(96 ),SAFP(96 ),E(96 ),V(96 ),
     2           STRESS1(96 ),STRESS2(96 ),STRESSX(96 ),STRESSY(96 ),
     3           STRESXY(96 ),STRAIN1(96 ),STRAIN2(96 ),STRAINX(96 ),
     4           STRAINY(96 ),STRANXY(96 ),THETA(96 ),XPREVUS(5),RUN(10)
      COMMON  S
C-----BODY 2 PROGRAM WILL SOLVE PROBLEMS OF PLANE STRESS OR PLANE STRAIN
C     USING THE SIMPLY CONNECTED LATTICE CELLS OF HRENNIKOFF AND SMITH.
C     THE MEDIUM MAY BE LINEAR OR NONLINEAR. MATERIAL PROPERTIES NECES-
C     SARY FOR A SOLUTION ARE  THE  RELATIONSHIP OF THE STATES OF STRESS
C     TO STATES OF STRAIN.
C--NOTE--IT IS USUALLY NECESSARY TO ALTER DIMENSION STATEMENT FOR EACH
C        PROBLEM
C
C                              NOTATIONS
C
C                           INPUT   DATA
C
C     NPROBS       = NUMBER OF STRESS PROBLEMS TO BE SOLVED.
C     ITER         = MAXIMUM NUMBER OF ITERATIONS ALLOWED FOR CLOSURE
C                    IN NONLINEAR STRESS PROBLEMS.
C     ITRCEL1--5   = MONITORED LATTICE CELLS DURING ITERATION. THE STATE
C                    OF STRAIN AT EACH MONITORED CELL IS PRINTED AFTER
C                    EACH ITERATION.
C     IFNUM        = TOTAL NUMBER OF SPECIFIED NODE MOVEMENTS OR NODE
C                    LOADS.
C     IXNUM        = TOTAL NUMBER OF SPECIFIED NODE MOVEMENTS.
C     ITRMOV1--5   = MONITORED NODE MOVEMENTS DURING ITERATION. ALSO
C                    THESE MOVEMENTS ARE USED IN CLOSURE TEST.
C     ITYPE        = SPECIFICATION OF STRESS CONDITION---FOR PLANE STRESS
C                    ITYPE = 0  FOR PLANE STRAIN ITYPE = NONZERO.
C     BWIDTH       = WIDTH OF RECTANGULAR BODY.
C     BLENGTH      = LENGTH OF RECTANGULAR BODY.
C     CELLEN       = SIDE DIMENSION OF THE LATTICE CELL.
C     CELTICK      = THICKNESS OF THE LATTICE CELL.
C     EASUMD       = THE ELASTIC CONSTANT E  ASSUMED FOR THE FIRST ITER-
C                    ATION.
C     VASUMD       = THE ELASTIC CONSTANT V  ASSUMED FOR THE FIRST ITER-
C                    ATION.
C     CLOSTOL      = CLOSURE TORERANCE FOR NODE MOVEMENTS ITRMOV1--5.
C     IFSPC        = INDEX OF SPECIFIED NODE MOVEMENTS OR LOADS. INDEXING
C                    SHOULD BE RIGHTLY ORDERED  THAT IS  2467 NOT 2746
C     FSPC(ISPC)   = VALUES OF SPECIFIED  NODE LOADS OR MOVEMEMTS.
C     IXSPC        = INDEX OF SPECIFIED NODE MOVEMENTS.
C                    NOTE---NUMBER OF IFSPC OR FSPC = IFNUM.
C                           NUMBER OF IXSPC = IXNUM.
C
C                    ADDITIONAL   NOTATIONS
C
C     NODEWID      = NUMBER OF NODE POINTS IN THE WIDTH OF THE BODY
C                    NODEWID = (BWIDTH/CELLEN) +1.
C     NODELEN      = NUMBER OF NODE POINTS IN THE LENGTH OF THE BODY
C                    NODELEN = (BLENGTH/CELLEN) +1.
```

```
C     NEQUAS        = NUMBER OF EQUATIONS IN THE MATRIX EQUATION SX=F.
C                     NEQUAS  = NODEWID*NODELEN*2
C     NTWID         = TOTAL WIDTH OF BAND OF NONZERO ELEMENTS IN THE
C                     STIFFNESS MATRIX S. NTWID= 4*NODEWID+ 7.
C     SA(I)           THE THREE MAIN STIFFNESS COEFFICIENTS FOR EACH
C     SAFA(I)       = CELL(I). USED TO DEFINE THE VARIOUS STIFFNESS
C     SAFP(I)         COEFFICIENTS IN THE STIFFNESS MATRIX S(I,J)
C     S(I,J)        = STIFFNESS ELEMENTS IN THE STIFFNESS MATRIX S.
C     X(I)          = ELEMENTS IN THE MOVEMENT VECTOR  X.
C     F(I)          = ELEMENTS IN THE LOAD VECTOR  F.
C     E(I)          = THE ELASTIC CONSTANT E ---ASSUMED FOR FIRST ITERATION
C                     ----PSEUDO VALUE FOR EACH CELL ON EACH ITERATION.
C     V(I)          = THE ELASTIC CONSTANT V--- ASSUMED FOR FIRST ITERATION
C                     ---- PSEUDO VALUE FOR EACH CELL ON EACH ITERATION.
C     STRESS1(I)    = NORMAL MAXIMUM PRINCIPAL STRESS FOR CELL(I).
C     STRESS2(I)    = NORMAL MINIMUM PRINCIPAL STRESS FOR CELL(I).
C     THETA(I)      = ANGLE FROM THE STRESS1 DIRECTION TO THE X-DIRECTION
C                     FOR CELL(I). CCW IS +
C     STRAIN1(I)    = MAXIMUM EXTENSION STRAIN FOR CELL(I)
C     STRAIN2(I)    = MINIMUM EXTENSION STRAIN FOR CELL(I)
C     STRESSX(I)
C     STRESSY(I)      STRESS AND STRAIN COMPONENTS IN THE X-Y DIRECTIONS
C     STRESXY(I)    = FOR CELL(I).
C     STRAINX(I)      STRESXY AND STRANXY ARE SHEAR COMPONENTS.
C     STRAINY(I)
C     STRANXY(I)
C
C                               OUTPUT INFORMATION
C
C     MONITORED NODE MOVEMENTS =X(ITRMOV1---5)
C     MONITORED STRAINS           =STRAIN1(ITRCEL1---5),STRAIN2(ITRCEL1--5)
C                                  THETA(ITRCEL1---5)
C     FINAL SOLUTION OF PROBLEM CONSISTING OF THE BELOW INFORMATION
C     NODE MOVEMENTS X(I).
C     STATE OF STRAIN AT EACH LATTICE CELL X-Y COMPONENTS AND PRINCIPAL
C     STRAINS.
C     STATE OF STRESS AT EACH LATTICE CELL X-Y COMPONENTS AND PRINCIPAL
C     STRESSES.
C
    1 FORMAT (1H1)
    2 FORMAT (10A8 )
    3 FORMAT (4(I10,1PE10.3))
    4 FORMAT (16I5)
    5 FORMAT (1P8E10.3)
    6 FORMAT (1X,10A8)
   20 FORMAT (// 20H          INPUT DATA//)
   21 FORMAT (   41H          MAXIMUM NUMBER OF ITERATIONS = I5 /)
   22 FORMAT (   31H          MONITORED CELLS ARE  5I5 /)
   23 FORMAT (   46H          NUMBER OF F(I) ELEMENTS SPECIFIED = I5/)
   24 FORMAT (   46H          NUMBER OF NODE MOVEMENTS SPECIFIED = I5/)
   25 FORMAT (   38H          WIDTH OF RECTANGULAR BODY = 1PE10.3/)
   26 FORMAT (   39H          LENGTH OF RECTANGULAR BODY = 1PE10.3/)
   27 FORMAT (   43H          SIDE DIMENSION OF LATTICE CELL = 1PE10.3/)
   28 FORMAT (   30H          THICKNESS OF BODY = 1PE10.3/ )
   29 FORMAT (   42H          MODULUS OF ELASTICITY ASSUMED = 1PE10.3/)
```

```
   30 FORMAT (   35H           POISSONS RATIO ASSUMED = 1PE10.3/ )
   31 FORMAT (// 60H           PRINT OUT OF SPECIFIED LOADS OR MOVEMENTS
     1AT NODES   )
   32 FORMAT (/  58H              LOAD                 MOVEMENT
     1 VALUE  //)
   33 FORMAT (11X,I5,34X,1PE10.3)
   34 FORMAT (33X,I5,12X,1PE10.3)
   35 FORMAT (   41H           MONITORED NODE MOVEMENTS ARE   5I5/)
   36 FORMAT (// 25H           ITERATION DATA   )
   37 FORMAT (// 26H           ITERATION NUMBER  I5 //)
   38 FORMAT (   39H           TABLE OF MONITORED MOVEMENTS  /)
   39 FORMAT (   36H            MOVEMENT             VALUE    )
   40 FORMAT (13X,I5,7X,1PE15.8)
   41 FORMAT (// 37H           TABLE OF MONITORED STRAINS   /)
   42 FORMAT (   76H           CELL NUMBER      MAX. STRAIN         MIN.
     1STRAIN            THETA  )
   43 FORMAT (13X,I5,7X,1PE15.8,5X,1PE15.8,5X,1PE15.8)
   47 FORMAT (   59H           CLOSURE TOLERANCE FOR MONITORED NODE MOVEM
     1ENTS = 1PE10.3 /)
  500 FORMAT  (/ 21H           OUTPUT DATA //)
  501 FORMAT  (  63H           SOLUTION CLOSED WITHIN TOLERANCE AT ITERAT
     1ION NUMBER I5 /)
  502 FORMAT  (  66H           SOLUTION DID NOT CLOSE AFTER SPECIFIED ITE
     1RATION NUMBER I5 /)
  503 FORMAT  (/ 33H           TABLE OF NODE MOVEMENTS /)
  504 FORMAT  (//52H           TABLE FOR THE STATE OF STRAIN AT EACH CELL
     1 /)
  505 FORMAT  (//52H           TABLE FOR THE STATE OF STRESS AT EACH CELL
     1 /)
  506 FORMAT (   76H           CELL NUMBER      MAX. STRESS         MIN.
     1STRESS            THETA  )
  507 FORMAT (// 64H           TABLE FOR STRAINS IN THE X-Y DIRECTIONS FO
     1R EACH CELL     / )
  508 FORMAT (   77H           CELL NUMBER        STRAINX              STR
     1AINY            STRANXY     )
  509 FORMAT (// 65H           TABLE FOR STRESSES IN THE X-Y DIRECTIONS F
     1OR EACH CELL     / )
  510 FORMAT (   77H           CELL NUMBER        STRESSX              STR
     1ESSY            STRESXY     )
      CALL TIME ( 1HP )
      READ 4,NPROBS
          DO 9999 NPROB = 1,NPROBS
      READ 2,  (RUN(I), I = 1, 10)
      READ 4, ITER, ITRCEL1, ITRCEL2, ITRCEL3, ITRCEL4, ITRCEL5, IFNUM,
     1        IXNUM
      READ 4, ITRMOV1,ITRMOV2, ITRMOV3, ITRMOV4, ITRMOV5, ITYPE
      READ 5, BWIDTH, BLENGTH, CELLEN, CELTICK, EASUMD, VASUMD, CLOSTOL
C-----COMPUTATION OF VARIOUS INDEXS TO BE USED THROUGHOUT PROGRAM.
      BWID          = (BWIDTH/CELLEN) + 1.0
      BLEN          = (BLENGTH/CELLEN) + 1.0
      NODEWID       = BWID
      NODELEN       = BLEN
      NEQUAS        = NODEWID*NODELEN*2
      NTWID         = NODEWID*4 + 7
      NWIDTH        = (NTWID-1)/2
```

```
      MD             = NWIDTH + 1
      NQMWID         = NEQUAS - NWIDTH
      NCELLS         = (NODEWID - 1)*(NODELEN - 1)
      NLENM1         = (NODELEN - 1)
      NWIDM1         = (NODEWID - 1)
      DENOM          = CELLEN*2.0
      NWIDP1         = NODEWID + 1
      NW2            = 2*NODEWID
      READ 3,  (IFSPC(I), FSPC(I), I = 1,IFNUM)
      READ 4,  (IXSPC(I), I = 1,IXNUM)
C-----PRINT OUT OF INPUT INFORMATION.
      PRINT 1
      PRINT 6, (RUN(I),I=1,10)
      PRINT 20
      PRINT 21, ITER
      PRINT 22, ITRCEL1,ITRCEL2,ITRCEL3,ITRCEL4,ITRCEL5
      PRINT 35, ITRMOV1,ITRMOV2,ITRMOV3,ITRMOV4,ITRMOV5
      PRINT 23, IFNUM
      PRINT 24, IXNUM
      PRINT 25, BWIDTH
      PRINT 26, BLENGTH
      PRINT 27, CELLEN
      PRINT 28, CELTICK
      PRINT 29, EASUMD
      PRINT 30, VASUMD
      PRINT 47, CLOSTOL
          DO 556 I=1,NEQUAS
      FF(I)          =  0.0
  556 CONTINUE
       J = 1
C-----SETTING LOAD VECTOR F(I) TO SPECIFIED CONDITIONS AND PRINTING OUT
C     NODE MOVEMENTS AND LOADS WHICH WERE SPECIFIED.
          DO 550  I = 1,IFNUM
      IF(IXSPC(J) - IFSPC(I))  554,555,554
  554 IFF            = IFSPC(I)
      FF(IFF)        = -FSPC(I)
          GO TO 550
  555 IXX            = IXSPC(J)
      FF(IXX)        = FSPC(I)
      J = J+1
  550 CONTINUE
      PRINT 31
      PRINT 32
      J = 1
          DO 551   I = 1,IFNUM
      IF (IXSPC(J) - IFSPC(I))  552,553,552
  552 IFF            = IFSPC(I)
      FFF            = -FF(IFF)
      PRINT 33, IFF,FFF
          GO TO 551
  553 IXX            = IXSPC(J)
      PRINT 34, IXX, FF(IXX)
      J = J + 1
  551 CONTINUE
      CALL TIME ( 1HP )
```

```
C-----SETTING THE ELASTIC CONSTANTS E AND V TO THE ASSUMED VALUES
C     EASUMD AND VASUMD FOR FIRST ITERATION
          DO 201 I = 1,NCELLS
      E(I)         = EASUMD
      V(I)         = VASUMD
  201 CONTINUE
      KOUNTER = 0
C-----BEGINNING OF ITERATION CYCLE
          DO  999  IT = 1,ITER
      ITERNUM      = IT
C-----COMPUTION OF THE THREE MAIN STIFFNESS COEFFICIENTS FOR EACH
C     LATTICE CELL
      CLT          = CELLEN*CELTICK
C-----LOGIC TO SPECIFY PLANE STRESS OR PLANE STRAIN PROBLEM   ITYPE = 0
C       FOR PLANE STRESS     ITYPE  = ANY INTERGER  FOR PLANE STRAIN
      IF  (ITYPE)  196,198,196
C-----DO LOOP 197 FOR PLANE STRAIN PROBLEMS
  196     DO 197  I = 1,NCELLS
      A            =  CLT/((1.0+V(I))*2.0)
      FA           = ((1.0)/(1.0-2.0*V(I)))*0.25
      FP           = ((4.0*V(I)-1.0)/(1.0-2.0*V(I)))*0.25
      SA(I)        = (A*E(I))/CELLEN
      SAFA(I)      =  FA*SA(I)
      SAFP(I)      =  FP*SA(I)
  197 CONTINUE
      GO TO 199
C-----DO LOOP 200 FOR PLANE STRESS PROBLEM
  198     DO  200   I = 1,NCELLS
      A            =  CLT/((1.0 + V(I))*2.0)
      FA           = ((V(I)+1.0)/(1.0-V(I)))*0.25
      FP           = ((3.0*V(I)-1.0)/(1.0-V(I)))*0.25
      SA(I)        = (A*E(I))/CELLEN
      SAFA(I)      =  FA* SA(I)
      SAFP(I)      = FP * SA(I)
  200 CONTINUE
C-----GENERATION OF THE STIFFNESS MATRIX S AND THE LOAD VECTOR F
  199    DO 202  I = 1,NEQUAS
         DO 202  J = 1,NTWID
      S(I,J)       =  0.0
  202 CONTINUE
         DO 203  I = 1,NEQUAS
      F(I)         = FF(I)
  203 CONTINUE
C-----COMPUTING STIFFNESS ELEMENTS FOR FIRST NODE --THAT IS FIRST CORNER
C     NODE
      N            = 1
      I1           = 2*N - 1
      I2           = I1+1
      SSW          =  0.0
      SNW          =  0.0
      SSE          =  0.0
      SW1          =  0.0
      SW2          =  0.0
      SW3          =  0.0
      SS1          =  0.0
```

```
      SS2          =  0.0
      SS3          =  0.0
      SNE          =  SAFA(N)
      SN1          =  SAFP(N)+SA(N)
      SN2          = -SAFP(N)
      SN3          = -SAFP(N)
      SE1          =  SAFP(N)+SA(N)
      SE2          = -SAFP(N)
      SE3          = -SAFP(N)
      SCEN1        = -(SAFA(N)+SA(N))
      SCEN2        = -SAFA(N)
      CALL  SMATRIX (SSW,SNW,SNE,SSE,SW1,SW2,SW3,SN1,SN2,SN3,SE1,SE2,SE3
     1          ,SS1,SS2,SS3,SCEN1,SCEN2,I1,I2,MD,NW2)
C-----COMPUTING STIFFNESS ELEMENTS FOR NODES ON FIRST COLUMN EXCLUDING
C     THE CORNER NODES
         DO 204    N = 2, NWIDM1
      I1           = 2*N - 1
      I2           = I1+1
      SSE          =  SAFA(N-1)
      SNE          =  SAFA(N)
      SN1          =  SAFP(N)+SA(N)
      SN2          = -SAFP(N)
      SN3          = -SAFP(N)
      SE1          =  SAFP(N)+SAFP(N-1)+SA(N)+SA(N-1)
      SE2          = -SAFP(N)+SAFP(N-1)
      SE3          = -SAFP(N)-SAFP(N-1)
      SS1          =  SAFP(N-1)+SA(N-1)
      SS2          = -SAFP(N-1)
      SS3          = -SAFP(N-1)
      SCEN1        =-(SAFA(N)+SAFA(N-1)
     1               +  SA(N)+  SA(N-1))
      SCEN2        = -SAFA(N)+SAFA(N-1)
      CALL  SMATRIX (SSW,SNW,SNE,SSE,SW1,SW2,SW3,SN1,SN2,SN3,SE1,SE2,SE3
     1          ,SS1,SS2,SS3,SCEN1,SCEN2,I1,I2,MD,NW2)
  204 CONTINUE
C-----COMPUTING STIFFNESS ELEMENTS FOR TOP NODE OF FIRST COLUMN
      N            = NODEWID
      I1           = 2*N - 1
      I2           = I1+1
      SNE          =  0.0
      SN1          =  0.0
      SN2          =  0.0
      SN3          =  0.0
      SSE          =  SAFA(N-1)
      SE1          =  SAFP(N-1)+SA(N-1)
      SE2          =  SAFP(N-1)
      SE3          = -SAFP(N-1)
      SS1          =  SAFP(N-1)+SA(N-1)
      SS2          = -SAFP(N-1)
      SS3          = -SAFP(N-1)
      SCEN1        =-(SAFA(N-1)+SA(N-1))
      SCEN2        =  SAFA(N-1)
      CALL  SMATRIX (SSW,SNW,SNE,SSE,SW1,SW2,SW3,SN1,SN2,SN3,SE1,SE2,SE3
     1          ,SS1,SS2,SS3,SCEN1,SCEN2,I1,I2,MD,NW2)
C-----COMPUTING STIFFNESS ELEMENTS FOR NODE OF INTERIOR COLUMNS
```

```
      NBOT          = NWIDP1
      INW           = NODEWID
      ISW           = NODEWID + 1
      INE           = 1
      ISE           = 2
C-----BOTTOM NODE INTERIOR COLUMNS
          DO 205   NCOLUMN = 2,NLENM1
      N             = NBOT
      I1            = 2*N-1
      I2            = I1+1
      SSE           =  0.0
      SSW           =  0.0
      SS1           =  0.0
      SS2           =  0.0
      SS3           =  0.0
      SNW           =  SAFA(N-INW)
      SNE           =  SAFA(N-INE)
      SW1           =  SAFP(N-INW)+SA(N-INW)
      SW2           = -SAFP(N-INW)
      SW3           = -SAFP(N-INW)
      SN1           =  SAFP(N-INW)+SAFP(N-INE)+SA(N-INW)+SA(N-INE)
      SN2           =  SAFP(N-INW)-SAFP(N-INE)
      SN3           = -SAFP(N-INW)-SAFP(N-INE)
      SE1           =  SAFP(N-INE)+SA(N-INE)
      SE2           = -SAFP(N-INE)
      SE3           = -SAFP(N-INE)
      SCEN1         =-(SAFA(N-INW)+SAFA(N-INE)
     1                 + SA(N-INW)+  SA(N-INE))
      SCEN2         =  SAFA(N-INW)-SAFA(N-INE)
      CALL  SMATRIX (SSW,SNW,SNE,SSE,SW1,SW2,SW3,SN1,SN2,SN3,SE1,SE2,SE3
     1           ,SS1,SS2,SS3,SCEN1,SCEN2,I1,I2,MD,NW2)
C-----INTERIOR NODES INTERIOR COLUMNS
      NBOTP1        = NBOT + 1
      NTOPM1        = NBOT + NODEWID - 2
          DO 206    N = NBOTP1,NTOPM1
      I1            =  2*N-1
      I2            =  I1+1
      SSW           =  SAFA(N-ISW)
      SNW           =  SAFA(N-INW)
      SNE           =  SAFA(N-INE)
      SSE           =  SAFA(N-ISE)
      SW1           =  SAFP(N-INW)+SAFP(N-ISW)+SA(N-INW)+SA(N-ISW)
      SW2           = -SAFP(N-INW)+SAFP(N-ISW)
      SW3           = -SAFP(N-INW)-SAFP(N-ISW)
      SN1           =  SAFP(N-INW)+SAFP(N-INE)+SA(N-INW)+SA(N-INE)
      SN2           =  SAFP(N-INW)-SAFP(N-INE)
      SN3           = -SAFP(N-INW)-SAFP(N-INE)
      SE1           =  SAFP(N-INE)+SAFP(N-ISE)+SA(N-INE)+SA(N-ISE)
      SE2           = -SAFP(N-INE)+SAFP(N-ISE)
      SE3           = -SAFP(N-INE)-SAFP(N-ISE)
      SS1           =  SAFP(N-ISW)+SAFP(N-ISE)+SA(N-ISW)+SA(N-ISE)
      SS2           =  SAFP(N-ISW)-SAFP(N-ISE)
      SS3           = -SAFP(N-ISW)-SAFP(N-ISE)
      SCEN1         =-(SAFA(N-INW)+SAFA(N-INE)+SAFA(N-ISE)+SAFA(N-ISW)
     1               +  SA(N-INW)+  SA(N-INE)+  SA(N-ISE)+  SA(N-ISW))
```

```
      SCEN2        =  SAFA(N-INW)-SAFA(N-INE)+SAFA(N-ISE)-SAFA(N-ISW)
      CALL  SMATRIX (SSW,SNW,SNE,SSE,SW1,SW2,SW3,SN1,SN2,SN3,SE1,SE2,SE3
     1          ,SS1,SS2,SS3,SCEN1,SCEN2,I1,I2,MD,NW2)
  206 CONTINUE
C-----TOP NODE INTERIOR COLUMNS
      N            = NBOT + NWIDM1
      I1           =  2*N-1
      I2           =  I1+1
      SNW          =  0.0
      SNE          =  0.0
      SN1          =  0.0
      SN2          =  0.0
      SN3          =  0.0
      SSW          =  SAFA(N-ISW)
      SSE          =  SAFA(N-ISE)
      SW1          =  SAFP(N-ISW)+SA(N-ISW)
      SW2          =  SAFP(N-ISW)
      SW3          = -SAFP(N-ISW)
      SE1          =  SAFP(N-ISE)+SA(N-ISE)
      SE2          =  SAFP(N-ISE)
      SE3          = -SAFP(N-ISE)
      SS1          =  SAFP(N-ISW)+SAFP(N-ISE)+SA(N-ISW)+SA(N-ISE)
      SS2          =  SAFP(N-ISW)-SAFP(N-ISE)
      SS3          = -SAFP(N-ISW)-SAFP(N-ISE)
      SCEN1        =-(SAFA(N-ISW)+SAFA(N-ISE)
     1                + SA(N-ISW)+  SA(N-ISE))
      SCEN2        = -SAFA(N-ISW)+SAFA(N-ISE)
      CALL  SMATRIX (SSW,SNW,SNE,SSE,SW1,SW2,SW3,SN1,SN2,SN3,SE1,SE2,SE3
     1          ,SS1,SS2,SS3,SCEN1,SCEN2,I1,I2,MD,NW2)
      NBOT         = NBOT + NODEWID
      INW          = INW + 1
      INE          = INE + 1
      ISW          = ISW + 1
      ISE          = ISE + 1
  205 CONTINUE
C-----COMPUTING STIFFNESS ELEMENTS FOR BOTTOM CORNER NODE LAST COLUMN
      N            = NBOT
      I1           = 2*N - 1
      I2           = I1+1
      SSW          =  0.0
      SSE          =  0.0
      SNE          =  0.0
      SS1          =  0.0
      SS2          =  0.0
      SS3          =  0.0
      SE1          =  0.0
      SE2          =  0.0
      SE3          =  0.0
      SNW          =  SAFA(N-INW)
      SW1          =  SAFP(N-INW)+SA(N-INW)
      SW2          = -SAFP(N-INW)
      SW3          = -SAFP(N-INW)
      SN1          =  SAFP(N-INW)+SA(N-INW)
      SN2          =  SAFP(N-INW)
      SN3          = -SAFP(N-INW)
```

```
      SCEN1         =-(SAFA(N-INW)+SA(N-INW))
      SCEN2         =  SAFA(N-INW)
      CALL   SMATRIX (SSW,SNW,SNE,SSE,SW1,SW2,SW3,SN1,SN2,SN3,SE1,SE2,SE3
     1          ,SS1,SS2,SS3,SCEN1,SCEN2,I1,I2,MD,NW2)
      M             = N + 1
      NLASTM1       = NODEWID*NODELEN - 1
C-----COMPUTING STIFFNESS ELEMENTS FOR INTERIOR NODES LAST COLUMN
          DO 207 N = M,NLASTM1
      I1            = 2*N - 1
      I2            = I1+1
      SSW           =  SAFA(N-ISW)
      SNW           =  SAFA(N-INW)
      SW1           =  SAFP(N-INW)+SAFP(N-ISW)+SA(N-INW)+SA(N-ISW)
      SW2           = -SAFP(N-INW)+SAFP(N-ISW)
      SW3           = -SAFP(N-INW)-SAFP(N-ISW)
      SN1           =  SAFP(N-INW)+SA(N-INW)
      SN2           =  SAFP(N-INW)
      SN3           = -SAFP(N-INW)
      SS1           =  SAFP(N-ISW)+SA(N-ISW)
      SS2           =  SAFP(N-ISW)
      SS3           = -SAFP(N-ISW)
      SCEN1         =-(SAFA(N-INW)+SAFA(N-ISW)
     1                 + SA(N-INW)+  SA(N-ISW))
      SCEN2         =  SAFA(N-INW)-SAFA(N-ISW)
      CALL   SMATRIX (SSW,SNW,SNE,SSE,SW1,SW2,SW3,SN1,SN2,SN3,SE1,SE2,SE3
     1          ,SS1,SS2,SS3,SCEN1,SCEN2,I1,I2,MD,NW2)
  207 CONTINUE
C-----COMPUTING STIFFNESS ELEMENTS FOR LAST NODE POINT
      N             = NODEWID*NODELEN
      I1            =  2*N-1
      I2            =  I1+1
      SNW           =  0.0
      SN1           =  0.0
      SN2           =  0.0
      SN3           =  0.0
      SSW           =  SAFA(N-ISW)
      SW1           =  SAFP(N-ISW)+SA(N-ISW)
      SW2           =  SAFP(N-ISW)
      SW3           = -SAFP(N-ISW)
      SS1           =  SAFP(N-ISW)+SA(N-ISW)
      SS2           =  SAFP(N-ISW)
      SS3           = -SAFP(N-ISW)
      SCEN1         =-(SAFA(N-ISW)+SA(N-ISW))
      SCEN2         = -SAFA(N-ISW)
      CALL   SMATRIX (SSW,SNW,SNE,SSE,SW1,SW2,SW3,SN1,SN2,SN3,SE1,SE2,SE3
     1          ,SS1,SS2,SS3,SCEN1,SCEN2,I1,I2,MD,NW2)
C-----END OF THE GENERATION OF THE GENERAL STIFFNESS MATRIX S. BELOW
C     DO LOOP SETS THE MAIN DIAGONAL ELEMENT S(I,MD) = 1.0 AND ALL OTHER
C     ELEMENTS S(I,J)= 0.0 ON ROW I FOR EACH SPECIFIED NODE MOVEMENT
C     X(I).
         DO 208    I = 1,IXNUM
      IXX           = IXSPC(I)
         DO 209    J = 1,NTWID
      S(IXX,J)      = 0.0
  209 CONTINUE
```

```
      S(IXX,MD)    = 1.0
  208 CONTINUE
      CALL TIME ( 1HP )
C-----BELOW ROUTINE SOLVES THE STIFFNESS MATRIX EQUATION SX=F BY
C     AN ELIMINATION PROCESS WHICH DOES NOT COMPUTE OUTSIDE ORIGINAL
C     DIAGONAL BAND OF NON ZERO ELEMENTS.
      DO 103  I = 1, NQMWID
          IP1     = I + 1
          IPN     = I + NWIDTH
          N       = 0
      DO 103  L   = IP1, IPN
          N       = N + 1
          M       = MD - N
      IF ( S(L,M) ) 104, 103, 104
  104     XM      = -S(L,M)/S(I,MD)
          F(L)    = F(L) + XM * F(I)
          MN      = M + NWIDTH
          LL      = L - I
      DO 103   MM = M, MN
          S(L,MM) = S(L,MM) + XM * S(I,MM+LL)
  103 CONTINUE
          II      =  NEQUAS - NWIDTH + 1
          NEQ     =  NEQUAS - 1
          NN      =  0
      DO 105   I = II, NEQ
          IP1      = I + 1
          N        = 0
        NN = NN + 1
      DO 105     L = IP1, NEQUAS
          N        = N + 1
          M        = MD - N
      IF ( S(L,M) ) 106, 105, 106
  106     XM       = -S(L,M)/S(I,MD)
          F(L)     =  F(L) + XM * F(I)
          MN     = M + NWIDTH - NN
          LL     =  L - I
      DO 105   MM = M, MN
          S(L,MM) = S(L,MM) + XM * S(I,MM+LL)
  105 CONTINUE
          IA      = NEQUAS + MD
      DO 108   I = MD, NEQUAS
          IB      = IA - I
          IBM1    = IB - 1
          IBMN    = IB - NWIDTH
          MB      = -1
      DO 108   L = IBMN, IBM1
          MB       = MB + 1
          MBB      = NTWID - MB
          XM       =   - S(L,MBB)/S(IB,MD)
          F(L)      = F(L) + XM * F(IB)
  108 CONTINUE
          IA       = NWIDTH + 2
          N       = -1
      DO 109   I = 2, NWIDTH
          IB      = IA - I
```

```
          IBM1    = IB - 1
          N       = N + 1
      DO 109   L = 1, IBM1
          NN      = NTWID - N - L
          XM      = - S(L,NN)/S(IB,MD)
          S(L,NN) = S(L,NN) + XM * S(IB,MD)
          F(L)    = F(L) + XM * F(IB)
  109 CONTINUE
      DO 110   I = 1, NEQUAS
          X(I) = F(I)/S(I,MD)
  110 CONTINUE
      CALL TIME ( 1HP )
C-----COMPUTION OF THE STATE OF STRAIN AT EACH LATTICE CELL(I)
      ICLBOT      = 1
      ICLTOP      = NWIDM1
      JJ          = 0
          DO 300  I = 1,NLENM1
      J            = JJ - 1
          DO 299  N = ICLBOT, ICLTOP
      J            = J + 2
      STRAINX(N)   = (X(J+NW2+2)+ X(J+NW2)-X(J)-X(J+2))/DENOM
      STRAINY(N)   = (X(J+NW2+3)- X(J+NW2+1)-X(J+1)+X(J+3))/DENOM
      STRANXY(N)   = (X(J+NW2+2)+X(J+NW2+3)-X(J+NW2)+X(J+NW2+1)-X(J)
     1                 -X(J+1)+X(J+2)-X(J+3))/DENOM
      EC          = (STRAINX(N) + STRAINY(N))/2.0
      ER          = ((((STRAINX(N)-STRAINY(N))**2  )/4.0) +
     1              ((STRANXY(N)**2  )/4.0))**0.5
      STRAIN1(N) = EC + ER
      STRAIN2(N) = EC - ER
          IF (STRANXY(N)) 312,310,312
  310     IF(STRAINX(N)-STRAINY(N)) 311,320,311
  312     IF (STRAINX(N)-STRAINY(N)) 311,321,311
  321     IF (STRANXY(N)) 313,311,314
  313 THETA(N)    = -45.0
          GO TO 299
  314 THETA(N)    =  45.0
          GO TO 299
  311 THETA2      = (ATANF(STRANXY(N)/(STRAINX(N)-STRAINY(N))))
      THETA2      = THETA2*57.2957795
          IF (STRANXY(N))301,301,302
  301     IF (STRAINX(N)-STRAINY(N)) 303,299,302
  303 THETA2      = THETA2 - 180.0
          GO TO 305
  302     IF (STRAINX(N)-STRAINY(N)) 304,299,305
  304 THETA2      = THETA2 + 180.0
          GO TO 305
  320 THETA2      = 0.0
  305 THETA(N)     = THETA2/2.0
  299 CONTINUE
      JJ         = JJ + NW2
      ICLBOT      = ICLBOT + NWIDM1
      ICLTOP      = ICLTOP + NWIDM1
  300 CONTINUE
C-----PRINT OUT OF ITERATION DATA
      KOUNTER  = KOUNTER + 1
```

```
          IF (KOUNTER - 2)    350, 353, 353
  350     IF (IT - 1)         351,351,352
  351 PRINT 1
      PRINT 6, (RUN(I), I = 1,10)
      PRINT 36
          GO TO 354
  352 PRINT 1
          GO TO 354
  353 KOUNTER = 0
  354 PRINT 37, IT
      PRINT 38
      PRINT 39
      PRINT 40, ITRMOV1, X(ITRMOV1)
      PRINT 40, ITRMOV2, X(ITRMOV2)
      PRINT 40, ITRMOV3, X(ITRMOV3)
      PRINT 40, ITRMOV4, X(ITRMOV4)
      PRINT 40, ITRMOV5, X(ITRMOV5)
      PRINT 41
      PRINT 42
      PRINT 43, ITRCEL1, STRAIN1(ITRCEL1), STRAIN2(ITRCEL1), THETA(ITRCE
     1L1)
      PRINT 43, ITRCEL2, STRAIN1(ITRCEL2), STRAIN2(ITRCEL2), THETA(ITRCE
     1L2)
      PRINT 43, ITRCEL3, STRAIN1(ITRCEL3), STRAIN2(ITRCEL3), THETA(ITRCE
     1L3)
      PRINT 43, ITRCEL4, STRAIN1(ITRCEL4), STRAIN2(ITRCEL4), THETA(ITRCE
     1L4)
      PRINT 43, ITRCEL5, STRAIN1(ITRCEL5), STRAIN2(ITRCEL5), THETA(ITRCE
     1L5)
C-----BELOW ROUTINE CALCULATES THE STATE OF STRESS AT EACH LATTICE CELL
C     WHICH CORRESPONDS TO THE STATE OF STRAINS COMPUTED ABOVE.
          DO 450   I = 1,NCELLS
C-----ANALYTICAL EXPRESSION OF THE MECHANICAL PROPERTIES ARE TO BE
C     WRITTEN HERE FOR THE PARTICULAR MEDIUM UNDER ANALYSIS.
C     THAT IS STRESS1= FUNCTION(STRAIN1,STRAIN2)AND
C     STRESS2= FUNCTION(STRAIN1,STRAIN2).
      STRESS1(I) =   32967032.90*(STRAIN1(I) + 0.30*STRAIN2(I))
      STRESS2(I) =   32967032.90*(STRAIN2(I) + 0.30*STRAIN1(I))
C-----BELOW ROUTINE COMPUTE THE PSEUDO ELASTIC CONSTANTS E AND V FOR
C     EACH CELL
C     ADDITIONAL LOGIC NEEDED FOR STRAIN1=STRAIN2  AS FOLLOWS  IF(ABSF
C     (STRAIN1(I)-STRAIN2(I)))447,448,447
  447 IF (STRAIN1(I)) 445,446,445
  446 BB             =  STRESS1(I)/STRAIN2(I)
      AA             =  STRESS2(I)/STRAIN2(I)
      GO TO 444
  448 GO TO 444
  445 BB             = (STRESS2(I)-STRAIN2(I)*STRESS1(I)/STRAIN1(I))/
     1                 (STRAIN1(I)-(STRAIN2(I)**2)/STRAIN1(I))
      AA             = (STRESS1(I)-STRAIN2(I)*BB)/STRAIN1(I)
C-----LOGIC TO DETERMINE IF PLANE STRESS OR PLANE STRAIN FOR COMPUTING
C     VALUES OF  E  AND  V
  444 IF(ITYPE)  442,443,442
C-----E AND V BELOW ARE FOR PLANE STRAIN PROBLEMS
  442 V(I)           = (BB)/(AA + BB)
```

```
      E(I)          = (AA*(1.0+V(I))*(1.0-2.0*V(I)))/(1.0-V(I))
      GO TO 450
C-----E AND V BELOW ARE FOR PLANE STRESS PROBLEMS
  443 V(I)          =  BB/AA
      E(I)          = AA*(1.0-V(I)**2)
  450 CONTINUE
C-----BELOW ROUTINE COMPUTES THE STRESS COMPONENTS IN THE X-Y DIRECTICNS
C     FOR EACH LATTICE CELL.
          DO 449   I = 1, NCELLS
      SCENTER       = (STRESS1(I)+STRESS2(I))/2.0
      SRADIUS       = (STRESS1(I)-STRESS2(I))/2.0
      THETA2        =  0.03490658504*THETA(I)
      STRESSX(I)    =  SCENTER+(SRADIUS)*(COSF(THETA2))
      STRESSY(I)    =  SCENTER-(SRADIUS)*(COSF(THETA2))
      STRESXY(I)    = (SRADIUS)*(SINF(THETA2))
  449 CONTINUE
      CALL TIME ( 1HP )
C-----CLOSURE TOLERANCE CHECK
      IF   (ABSF(X(ITRMOV1)-XPREVUS(    1    ))-CLOSTOL) 451,451,455
  451 IF   (ABSF(X(ITRMOV2)-XPREVUS(    2    ))-CLOSTOL) 452,452,455
  452 IF   (ABSF(X(ITRMOV3)-XPREVUS(    3    ))-CLOSTOL) 453,453,455
  453 IF   (ABSF(X(ITRMOV4)-XPREVUS(    4    ))-CLOSTOL) 454,454,455
  454 IF   (ABSF(X(ITRMOV5)-XPREVUS(    5    ))-CLOSTOL) 456,456,455
  455 XPREVUS(    1    ) = X(ITRMOV1)
      XPREVUS(    2    ) = X(ITRMOV2)
      XPREVUS(    3    ) = X(ITRMOV3)
      XPREVUS(    4    ) = X(ITRMOV4)
      XPREVUS(    5    ) = X(ITRMOV5)
  999 CONTINUE
C-----PRINT OUT OF FINAL SOLUTION
      PRINT 1
      PRINT 6, (RUN(I),I=1,10)
      PRINT 500
      PRINT 502, ITERNUM
      GO TO 459
  456 PRINT 1
      PRINT 6, (RUN(I),I=1,10)
      PRINT 500
      PRINT 501, ITERNUM
  459 PRINT 503
      PRINT 39
         DO 460 I=1,NEQUAS
      PRINT 40, I, X(I)
  460 CONTINUE
      PRINT 504
      PRINT 42
         DO 461 I=1,NCELLS
      PRINT 43, I, STRAIN1(I), STRAIN2(I), THETA(I)
  461 CONTINUE
      PRINT 505
      PRINT 506
         DO 462 I=1,NCELLS
      PRINT 43, I, STRESS1(I), STRESS2(I), THETA(I)
  462 CONTINUE
      PRINT 507
```

```
      PRINT 508
          DO 463  I = 1, NCELLS
      PRINT 43, I, STRAINX(I), STRAINY(I), STRANXY(I)
  463 CONTINUE
      PRINT 509
      PRINT 510
          DO 464 I = 1, NCELLS
      PRINT 43, I, STRESSX(I), STRESSY(I), STRESXY(I)
  464 CONTINUE
      CALL TIME ( 1HP )
 9999 CONTINUE
      END
C-----SUBROUTINE TO COMPUTE THE VARIOUS STIFFNESS COEFFICIENTS IN THE
C     STIFFNESS MATRIX S(I,J)
      SUBROUTINE  SMATRIX (SSW,SNW,SNE,SSE,SW1,SW2,SW3,SN1,SN2,SN3,SE1,S
     1                   E2,SE3,SS1,SS2,SS3,SCEN1,SCEN2,I1,I2,MD,NW2)
      DIMENSION S(234,43)
      COMMON     S
C-----EQUATION FOR X-DIRECTION
      S(I1,MD-NW2-2)     = SSW
      S(I1,MD-NW2-1)     = SSW
      S(I1,MD-NW2)       = SW1
      S(I1,MD-NW2+1)     =-SW2
      S(I1,MD-NW2+2)     = SNW
      S(I1,MD-NW2+3)     =-SNW
      S(I1,MD-2)         = SS3
      S(I1,MD-1)         = SS2
      S(I1,MD)           = SCEN1
      S(I1,MD+1)         = SCEN2
      S(I1,MD+2)         = SN3
      S(I1,MD+3)         =-SN2
      S(I1,MD+NW2-2)     = SSE
      S(I1,MD+NW2-1)     =-SSE
      S(I1,MD+NW2)       = SE1
      S(I1,MD+NW2+1)     = SE2
      S(I1,MD+NW2+2)     = SNE
      S(I1,MD+NW2+3)     = SNE
C-----EQUATION FOR Y-DIRECTION
      S(I2,MD-NW2-3)     = SSW
      S(I2,MD-NW2-2)     = SSW
      S(I2,MD-NW2-1)     = SW2
      S(I2,MD-NW2)       = SW3
      S(I2,MD-NW2+1)     =-SNW
      S(I2,MD-NW2+2)     = SNW
      S(I2,MD-3)         =-SS2
      S(I2,MD-2)         = SS1
      S(I2,MD-1)         = SCEN2
      S(I2,MD)           = SCEN1
      S(I2,MD+1)         = SN2
      S(I2,MD+2)         = SN1
      S(I2,MD+NW2-3)     =-SSE
      S(I2,MD+NW2-2)     = SSE
      S(I2,MD+NW2-1)     =-SE2
      S(I2,MD+NW2)       = SE3
      S(I2,MD+NW2+1)     = SNE
```

```
S(I2,MD+NW2+2)     = SNE
END
END
```

APPENDIX II

DETAILS OF McCORMICK'S LATTICE ANALOGY FOR PLANE STRESS PROBLEMS

APPENDIX II

As stated in the text, the technique for solving stress problems in a continuous body by a lattice analogy involves the generation and solution of the continuous body. Appendix II will present a computer program written for a lattice cell proposed by McCormick. This lattice cell, Fig. 2, possesses both flexural and axial structural elements and hence three components of movements (two translations and a rotation) of each node are needed to describe fully the distortion of a lattice. It follows that all three static equilibrium equations in a plane are needed to solve for these node movements. Since this characteristic of the McCormick lattice cell was found inefficiency relative to the simply connected models of Figs. 1 and 3, the McCormick model was used only in the preliminary stages of this study.

A computer program BODY 1 based on McCormick's lattice cell was written in FORTRAN 60 language for the CDC 1604. The program is essentially the same as the program written for the simply connected lattice cells. The only difference is in the generation of the stiffness matrix equation. The stiffness matrix equation for the McCormick's lattice possesses equilibrium equations for moments about each node where in the simply connected lattice only forces existed. Since this is the essential difference in the two programs and the programs contain descriptive comments cards, it is considered sufficient to present only the listing of the BODY 1 program.

LISTING OF BODY 1 PROGRAM

```
••    SMITH,R.E. PROGRAM BODY 1 CE010740 MCCORMICKS LATTICE CELL
      PROGRAM BODY1
      DIMENSION  S(240, 57), X(240), F(240), FF(240), FSPC( 50),
     1           IFSPC( 50), IXSPC( 50), A(100), B(100), D(100),
     2           STRESS1(100),STRESS2(100), THETA(100), STRAIN1(100),
     3           STRAIN2(100), STRAINX(100), STRAINY(100), STRANXY(100),
     4           STRESSX(100), STRESSY(100), STRESXY(100), RUN(10),
     5           XPREVUS(5), E(100), V(100)
      COMMON  S
C-----BODY 1 PROGRAM WILL SOLVE PROBLEMS OF PLANE STRESS USING
C     MCCORMICKS LATTICE CELL. THE MEDIUM MAY BE LINEAR OR NON-LINEAR.
C     MATERIAL PROPERTIES NECESSARY FOR A SOLUTION ARE THE RELATIONSHIP
C     OF STATES OF STRESS TO STATES OF STRAIN.
C-NOTE-IT IS USUALLY NECESSARY TO ALTER DIMENSION STATEMENT FOR
C     EACH PROBLEM.
C                                   NOTATIONS
C
C     NEQUAS       = NUMBER OF MOVEMENTS--THAT IS NUMBER OF EQUATION IN
C                    MATRIX EQUATION   SX=F   NEQUAS = NODEWID*NODELEN*3
C
C     NTWID        = TOTAL WIDTH OF BAND = 6*NODEWID + 9
C
C     NWIDTH       = BAND WIDTH OF MATRIX S(I,J) AND IS EQUAL TO THE NUM-
C                    BER OF ELEMENTS FROM THE MAIN DIAGONAL TO THE
C                    EXTREME NONZERO ELEMENT.  NOTE--- TOTAL WIDTH OF
C                    BAND WOULD BE = 2(NWIDTH) + 1 = NTWID
C
C     S(I,J)       = ELEMENTS IN THE STIFFNESS MATRIX S
C
C     X(I)         = ELEMENTS IN THE MOVEMENT MATRIX (VECTOR) X
C
C     FF(I)        = ELEMENTS IN THE CONSTANT VECTOR FF WHICH IS USED TO
C                    EQUATE THE VECTOR F TO AFTER EVERY ITERATION
C
C     F(I)         = ELEMENTS IN THE LOAD MATRIX (VECTOR) F
C                    WILL CHANGE IN VALUE DURING COMPUTATION--AFTER EACH
C                    ITERATION WILL RESET TO FF(I)
C
C     E(I)         = MODULUS OF ELASTICITY---A VALUE FOR EACH LATTICE
C                    CELL---ASSUMED VALUE FOR FIRST TRIAL---INTERPOLATED
C                    FROM CURVES ON ITERATION TRIALS--EASUMD  ASSUMED
C                    VALUE FOR FIRST TRIAL
C
C     V(I)         = POISSONS RATIO---A VALUE FOR EACH LATTICE CELL ---
C                    ASSUMED VALUE FOR FIRST TRIAL---INTERPOLATED FROM
C                    CURVES ON ITERATION TRIALS-VALUMD IS ASSUMED
C                    VALUE FOR FIRST TRIAL
C
C     STRESS1(I)   = NORMAL STRESS MAX. PRINCIPAL FOR LATTICE CELL  I
C     STRESS2(I)   = NORMAL STRESS MIN. PRINCIPAL FOR LATTICE CELL  I
C     THETA (I)    = ANGLE FROM 1 DIRECTION TO X DIRECTION FOR LATTICE
C                    CELL I IN DEGREES --- CCW IS +
C
C     STRAIN1(I)   = EXTENSION STRAIN MAX. PRINCIPAL FOR LATTICE CELL I
C     STRAIN2(I)   = EXTENSION STRAIN MIN. PRINCIPAL FOR LATTICE CELL I
```

```
C      STRAINX(I)   = EXTENSION STRAIN IN X-DIRECTION FOR LATTICE CELL I
C      STRAINY(I)   = EXTENSION STRAIN IN Y-DIRECTION FOR LATTICE CELL I
C      STRANXY(I)   = SHEAR STRAIN FOR X-Y ELEMENT FOR LATTICE CELL    I
C
C      NCELLS       = NUMBER OF LATTICE CELLS = (NODELEN-1)(NODELEN-1)
C
C      NODEWID      = NUMBER OF NODES IN THE WIDTH OF THE BODY
C
C      NODELEN      = NUMBER OF NODES IN THE LENGTH OF THE BODY
C
C      A(I)         = AREA OF SIDE ELEMENT OF LATTICE CELL I
C
C      B(I)         = MOMENT OF INERTIA OF SIDE ELEMENT OF LATTICE
C                     CELL I
C
C      D(I)         = AREA OF DIAGONAL ELEMENT OF LATTICE CELL I
C
C      ITER         = MAXIMUM NUMBER OF ITERATIONS TO BE ALLOWED IN
C                     OBTAINING A SOLUTION
C
C      RUN(I)       = IDENTIFICATION CARD
C      (INPUT)
C
C      BWIDTH       = WIDTH OF RECTANGULAR BODY
C      (INPUT)
C
C      BLENGTH      = LENGTH OF RECTANGULAR BODY
C      (INPUT)
C
C      CELLEN       = SIDE DIMENSION OF THE LATTICE CELL--UNIFORM THROUGH
C      (INPUT)        OUT BODY--(BWIDTH/CELLEN) AND (BLENGTH/CELLEN) = AN
C                     INTEGER
C
C      CELTICK      = THICKNESS OF BODY---UNIFORM THROUGHOUT BODY
C      (INPUT)
C
C      FSPC(I)      = ELEMENTS IN F(I) VECTOR WHICH ARE SPECIFIED EITHER
C      (INPUT)        FOR LOADS OR FOR MOVEMENT OF NODES.
C
C      IFSPC(I)     = THE INDEX I FOR FSPC(I)
C      (INPUT)
C
C      IFNUM        = NUMBER OF FSPC(I) ELEMENTS SPECIFIED
C      IFNUM        = NUMBER OF FSPC(I) SPECIFIED
C      (INPUT)
C
C      IXSPC(I)     = THE INDEX I FOR S(I,MD) WHERE MOVEMENTS OF NODES
C      (INPUT)        ARE SPECIFIED
C
C      IXNUM        = NUMBER OF SPECIFIED NODE MOVEMENTS
C      (INPUT)
C
C      ITRCEL1---5  = SPECIFICATION OF PARTICULAR CELLS TO BE MONITORED
C      (INPUT)        FOR STRESS AND STRAIN COMPONENTS DURING ITERATIONS
C
```

```
C      ITRMOV1---5 = SPECIFICATION OF PARTICULAR NODE MOVEMENTS TO BE
C      (INPUT)       MONITORED DURING ITERATION
C
C      XPREVUS(I)  = STORED NODE MOVEMENTS(SAME AS MONITORED MOVEMENTS)
C      (INPUT)       TO CHECK FOR CLOSURE.  XPREVUS IS X MOVEMENT FROM
C                    PREVIOUS ITERATION
C
C      CLOSTOL     = A VALUE OF TRANSLATION OF NODE POINT WHICH IS THE
C      (INPUT)       SPECIFIED CLOSURE TOLERANCE.
C
C-----NOTATIONS USED TO REPRESENT MEMBER STIFFNESSES
C
C      DSW         = DIAGONAL MEMBERS STIFFNESS.
C      DNW           EXAMPLE
C      DNE           DSW = (D(I)*E(I))/2.828*CELLEN    DSW(UNITS)=LB/IN
C      DSE           WHERE I IS NUMBER OF CELL WHICH CONTAINS DIA MEMBER
C
C      AS          = SIDE MEMBERS STIFFNESS. EXAMPLE
C      AW            AS   = (A(I)*E(I) + A(II)*E(II))/CELLEN
C      AW            WHERE  INDEX I AND II DENOTE CELL NUMBER ON EACH
C      AE            SIDE OF MEMBER AS                 AS(UNITS)=LB/IN
C
C      BS          = SIDE MEMBERS BENDING STIFFNESS.  EXAMPLE
C      BW            BS   = (B(I)*E(I) + B(II)*E(II))/CELLEN
C      BN            WHERE INDEX I AND II DENOTE CELL NUMBER ON EACH
C      BE            SIDE OF MEMBER BS                 BS(UNITS) =LB-IN/RAD
C
C      CELLEN2     = CELLEN SQUARED
C
C-----ALL FORMAT STATEMENTS ARE LISTED BELOW
C
    1 FORMAT (1H1)
    2 FORMAT (10A8 )
    3 FORMAT (4(I10,1PE10.3))
    4 FORMAT (10I5)
    5 FORMAT (1P8E10.3)
    6 FORMAT (1X,10A8)
   20 FORMAT (// 20H            INPUT DATA//)
   21 FORMAT (   41H            MAXIMUM NUMBER OF ITERATIONS = I5 /)
   22 FORMAT (   31H            MONITORED CELLS ARE  5I5 /)
   23 FORMAT (   46H            NUMBER OF F(I) ELEMENTS SPECIFIED = I5/)
   24 FORMAT (   46H            NUMBER OF NODE MOVEMENTS SPECIFIED = I5/)
   25 FORMAT (   38H            WIDTH OF RECTANGULAR BODY = 1PE10.3/)
   26 FORMAT (   39H            LENGTH OF RECTANGULAR BODY = 1PE10.3/)
   27 FORMAT (   43H            SIDE DIMENSION OF LATTICE CELL = 1PE10.3/)
   28 FORMAT (   30H            THICKNESS OF BODY = 1PE10.3/ )
   29 FORMAT (   42H            MODULUS OF ELASTICITY ASSUMED = 1PE10.3/)
   30 FORMAT (   35H            POISSONS RATIO ASSUMED = 1PE10.3/ )
   31 FORMAT (// 60H            PRINT OUT OF SPECIFIED LOADS OR MOVEMENTS
     1AT NODES  )
   32 FORMAT (/  58H               LOAD                  MOVEMENT
     1 VALUE  //)
   33 FORMAT (11X,I5,34X,1PE10.3)
   34 FORMAT (33X,I5,12X,1PE10.3)
   35 FORMAT (   41H            MONITORED NODE MOVEMENTS ARE   5I5/)
```

```
   36 FORMAT (// 25H          ITERATION DATA    )
   37 FORMAT (// 26H          ITERATION NUMBER  I5 //)
   38 FORMAT (   39H          TABLE OF MONITORED MOVEMENTS  /)
   39 FORMAT (   33H           MOVEMENT         VALUE   )
   40 FORMAT (13X,I5,7X,1PE10.3)
   41 FORMAT (// 37H          TABLE OF MONITORED STRAINS   /)
   42 FORMAT (   64H          CELL NUMBER    MAX. STRAIN    MIN. STRAIN
     1      THETA    )
   43 FORMAT (13X,I5,7X,1PE10.3,5X,1PE10.3,5X,1PE10.3)
   44 FORMAT (// 43H          TABLE OF MONITORED STRESS VALUES /)
   45 FORMAT (   51H          CELL NUMBER    MAX. STRESS    MIN. STRESS)
   46 FORMAT (13X,I5,7X,1PE10.3,5X,1PE10.3)
   47 FORMAT (   54H     CLOSURE TOLERANCE FOR MONITORED NODE MOVEMENTS
     1= 1PE10.3/)
  500 FORMAT  (/ 21H          OUTPUT DATA //)
  501 FORMAT  (  63H          SOLUTION CLOSED WITHIN TOLERANCE AT ITERAT
     1ION NUMBER I5 /)
  502 FORMAT  (  66H          SOLUTION DID NOT CLOSE AFTER SPECIFIED ITE
     1RATION NUMBER I5 /)
  503 FORMAT  (/ 33H          TABLE OF NODE MOVEMENTS /)
  504 FORMAT  (//52H          TABLE FOR THE STATE OF STRAIN AT EACH CELL
     1 /)
  505 FORMAT  (//52H          TABLE FOR THE STATE OF STRESS AT EACH CELL
     1 /)
  506 FORMAT  (  64H          CELL NUMBER    MAX.STRESS     MIN.STRESS
     1      THETA    )
  507 FORMAT (// 64H          TABLE FOR STRAINS IN THE X-Y DIRECTIONS FO
     1R EACH CELL     / )
  508 FORMAT (   65H          CELL NUMBER      STRAINX         STRAINY
     1     STRAINXY     )
  509 FORMAT (// 65H          TABLE FOR STRESSES IN THE X-Y DIRECTIONS F
     1OR EACH CELL     / )
  510 FORMAT (   65H          CELL NUMBER      STRESSX        STRESSY
     1     STRESXY     )
C
C
C-----READ IN OF INPUT DATA
C-----READ IN OF IDENTIFICATION CARD
      READ 2,  (RUN(I), I = 1, 10)
      READ 4, ITER, ITRCEL1, ITRCEL2, ITRCEL3, ITRCEL4, ITRCEL5, IFNUM,
     1         IXNUM
      READ 4, ITRMOV1,ITRMOV2, ITRMOV3, ITRMOV4, ITRMOV5
      READ 5, BWIDTH, BLENGTH, CELLEN, CELTICK, EASUMD, VASUMD, CLOSTOL
C-----COMPUTATION OF NODEWID, NODELEN,NEQUAS, NTWID, NWIDTH, NQMWID, NCELLS
C     NLENM1, NWIDM1, JP, DENOM, NWIDP1, CELLEN2, MD
C     TO BE USED THROUGHOUT PROGRAM
      BWID          = (BWIDTH/CELLEN) + 1.0
      BLEN          = (BLENGTH/CELLEN) + 1.0
      NODEWID       = BWID
      NODELEN       = BLEN
      NEQUAS        = NODEWID*NODELEN*3
      NTWID         = NODEWID*6 + 9
      NWIDTH        = (NTWID-1)/2
      MD            = NWIDTH + 1
      NQMWID        = NEQUAS - NWIDTH
```

```
      NCELLS         = (NODEWID - 1)*(NODELEN - 1)
      NLENM1        = (NODELEN - 1)
      NWIDM1        = (NODEWID - 1)
      JP            = NODEWID*3
      DENOM         = CELLEN*2.0
      NWIDP1         = NODEWID + 1
      CELLEN2        = CELLEN**2
      NW3            = 3*NODEWID
      READ 3,  (IFSPC(I), FSPC(I), I = 1,IFNUM)
      READ 4,  (IXSPC(I), I = 1,IXNUM)
C-----PRINT OUT OF INPUT DATA
      PRINT 1
      PRINT 6, (RUN(I),I=1,10)
      PRINT 20
      PRINT 21, ITER
      PRINT 22, ITRCEL1,ITRCEL2,ITRCEL3,ITRCEL4,ITRCEL5
      PRINT 35, ITRMOV1,ITRMOV2,ITRMOV3,ITRMOV4,ITRMOV5
      PRINT 23, IFNUM
      PRINT 24, IXNUM
      PRINT 25, BWIDTH
      PRINT 26, BLENGTH
      PRINT 27, CELLEN
      PRINT 28, CELTICK
      PRINT 29, EASUMD
      PRINT 30, VASUMD
      PRINT 47, CLOSTOL
C-----ROUTINE BELOW ESTABLISHES THE CONSTANT VECTOR FF(I)
         DO 550  I = 1,IFNUM
      IFF            = IFSPC(I)
      FF(IFF)        = FSPC(I)
  550 CONTINUE
C-----TO PRINT OUT LOAD AND MOVEMENT SPECIFICATION
      PRINT 31
      PRINT 32
      J = 1
          DO 551   I = 1,IFNUM
      IF (IXSPC(J) - IFSPC(I))  552,553,552
  552 IFF            = IFSPC(I)
      PRINT 33, IFF, FF(IFF)
          GO TO 551
  553 IXX            = IXSPC(J)
      PRINT 34, IXX, FF(IXX)
      J = J + 1
  551 CONTINUE
C-----SETTING E(I) AND V(I) FOR FIRST ITERATION
          DO 201 I = 1,NCELLS
      E(I)           = EASUMD
      V(I)           = VASUMD
  201 CONTINUE
C-----BELOW IS DO LOOP 999 INCLUDES ALL COMPUTATION INVOLVED IN THE
C     ITERATION PROCESS
          DO  999  IT = 1,ITER
      ITERNUM        = IT
C-----COMPUTATION OF THE STRUCTURAL ELEMENTS FOR EACH LATTICE CELL I
C     INFORMATION IN FROM INPUT AND OTHER SECTION OF PROGRAM
```

```
C     CELLEN , CELTICK, V(I), NODEWID, NODELEN
C-----INFORMATION OBTAINED FROM THIS SECTION
C     A(I), B(I), D(I)
C
C-----SINCE CELLEN * CELTICK IS USED IN ALL EQUATIONS IN DO LOOP BELOW,
C     WILL COMPUTE OUTSIDE LOOP--ALSO CELLEN**3.0*CELTICK/24.0
      CL3TD24        = (CELLEN**3.0)*CELTICK/24.0
      CLT            = (CELLEN * CELTICK)
          DO 200 I= 1,NCELLS
      A(I)           = CLT/((1.0 + V(I))*2.0)
      B(I)           = (CL3TD24)*(1.0-3.0*V(I))/(1.0-V(I)**2)
      D(I)           = (V(I)*CLT*1.414214)/(1.0-V(I)**2)
  200 CONTINUE
C-----GENERATION OF STIFFNESS MATRIX S(I,J) ELEMENTS AND THE VECTOR F(I)
C-----BELOW DO LOOP SETS ALL S(I,J) ELEMENTS TO ZERO BELOW EACH GENERATION
         DO 202  I = 1,NEQUAS
         DO 202  J = 1,NTWID
      S(I,J)         =  0.0
  202 CONTINUE
C-----BELOW DO LOOP SETS ALL F(I) ELEMENTS EQUAL TO CONSTANT VECTOR FF(I)
         DO 203  I = 1,NEQUAS
      F(I)           = FF(I)
  203 CONTINUE
C-----COMPUTING STIFFNESS ELEMENTS FOR FIRST CORNER NODE ---THAT IS
C     NODE NUMBER ONE
      N              = 1
      I1             = 3*N - 2
      I2             = I1 + 1
      I3             = I1 + 2
      DNW           =  0.0
      DSW            =  0.0
      DSE            =  0.0
      AS             =  0.0
      AW             =  0.0
      BS             =  0.0
      BW             =  0.0
      DNE            =  D(N)*E(N)/(2.828428*CELLEN)
      AE             = A(N)*E(N)/CELLEN
      AN             =  AE
      BE             =  B(N)*E(N)/CELLEN
      BN             =  BE
      CALL SMATRIX        (DSW,DSE,DNW,DNE,AS,AW,AE,AN,BS,BW,BN,BE,I1,I2,
     1                     I3, MD,NW3,CELLEN,CELLEN2)
C-----COMPUTING STIFFNESS ELEMENTS FOR NODES ON FIRST COLUMN EXCLUDING
C     THE CORNER NODES
          DO 204     N = 2, NWIDM1
      I1             = 3*N - 2
      I2             = I1 + 1
      I3             = I1 + 2
      DSE            =  D(N-1)*E(N-1)/(2.828428*CELLEN)
      DNE            =  D(N)*E(N)/(2.828428*CELLEN)
      AN             =  A(N)*E(N)/CELLEN
      AE             =  (A(N-1)*E(N-1)+A(N)*E(N))/CELLEN
      AS             =  A(N-1)*E(N-1)/CELLEN
      BN             =  B(N)*E(N)/CELLEN
```

```
      BS            = B(N-1)*E(N-1)/CELLEN
      BE            = (B(N)*E(N)+B(N-1)*E(N-1))/CELLEN
      CALL SMATRIX        (DSW,DSE,DNW,DNE,AS,AW,AE,AN,BS,BW,BN,BE,I1,I2,
     1                     I3, MD,NW3,CELLEN,CELLEN2)
  204 CONTINUE
C-----COMPUTING STIFFNESS ELEMENTS FOR TOP NODE ON FIRST COLUMN THAT IS
C     NODE NUMBER NODEWID
      N             = NODEWID
      I1            = 3*N - 2
      I2            = I1 + 1
      I3            = I1 + 2
      DNE           =  0.0
      AN            =  0.0
      BN            =  0.0
      DSE           =  D(N-1)*E(N-1)/(2.828428*CELLEN)
      AS            =  A(N-1)*E(N-1)/CELLEN
      AE            =  AS
      BS            =  B(N-1)*E(N-1)/CELLEN
      BE            =  BS
      CALL SMATRIX        (DSW,DSE,DNW,DNE,AS,AW,AE,AN,BS,BW,BN,BE,I1,I2,
     1                     I3, MD,NW3,CELLEN,CELLEN2)
C-----COMPUTING STIFFNESS ELEMENTS FOR INTERIOR COLUMNS
      NBOT          = NWIDP1
      INW           = NODEWID
      ISW           = NODEWID + 1
      INE           = 1
      ISE           = 2
          DO 205   NCOLUMN = 2,NLENM1
      N             = NBOT
C-----COMPUTING STIFFNESS ELEMENTS FOR BOTTOM NODES ON INTERIOR COLUMNS
      I1            = 3*N - 2
      I2            = I1 + 1
      I3            = I1 + 2
      DSW           =  0.0
      DSE           =  0.0
      AS            =  0.0
      BS            =  0.0
      DNE           =  D(N-INE)*E(N-INE)/(2.828428*CELLEN)
      DNW           =  D(N-INW)*E(N-INW)/(2.828428*CELLEN)
      AW            =  A(N-INW)*E(N-INW)/CELLEN
      AN            = (A(N-INW)*E(N-INW)+A(N-INE)*E(N-INE))/CELLEN
      AE            =  A(N-INE)*E(N-INE)/CELLEN
      BW            =  B(N-INW)*E(N-INW)/CELLEN
      BN            = (B(N-INW)*E(N-INW)+B(N-INE)*E(N-INE))/CELLEN
      BE            =  B(N-INE)*E(N-INE)/CELLEN
      CALL SMATRIX        (DSW,DSE,DNW,DNE,AS,AW,AE,AN,BS,BW,BN,BE,I1,I2,
     1                     I3, MD,NW3,CELLEN,CELLEN2)
C-----COMPUTING STIFFNESS ELEMENTS FOR INTERIOR NODES FOR INTERIOR COLUMNS
      NBOTP1        = NBOT + 1
      NTOPM1        = NBOT + NODEWID - 2
          DO 206   N = NBOTP1,NTOPM1
      I1            = 3*N - 2
      I2            = I1 + 1
      I3            = I1 + 2
      DSE           = D(N-ISE)*E(N-ISE)/(2.828428*CELLEN)
```

```
      DNE           = D(N-INE)*E(N-INE)/(2.828428*CELLEN)
      DNW           = D(N-INW)*E(N-INW)/(2.828428*CELLEN)
      DSW           = D(N-ISW)*E(N-ISW)/(2.828428*CELLEN)
      AS            = (A(N-ISW)*E(N-ISW)+A(N-ISE)*E(N-ISE))/CELLEN
      AW            = (A(N-ISW)*E(N-ISW)+A(N-INW)*E(N-INW))/CELLEN
      AN            = (A(N-INE)*E(N-INE)+A(N-INW)*E(N-INW))/CELLEN
      AE            = (A(N-INE)*E(N-INE)+A(N-ISE)*E(N-ISE))/CELLEN
      BS            = (B(N-ISW)*E(N-ISW)+B(N-ISE)*E(N-ISE))/CELLEN
      BW            = (B(N-ISW)*E(N-ISW)+B(N-INW)*E(N-INW))/CELLEN
      BN            = (B(N-INE)*E(N-INE)+B(N-INW)*E(N-INW))/CELLEN
      BE            = (B(N-INE)*E(N-INE)+B(N-ISE)*E(N-ISE))/CELLEN
      CALL SMATRIX        (DSW,DSE,DNW,DNE,AS,AW,AE,AN,BS,BW,BN,BE,I1,I2,
     1                     I3, MD,NW3,CELLEN,CELLEN2)
  206 CONTINUE
C-----COMPUTING STIFFNESS ELEMENTS FOR TOP NODE ON INTERIOR COLUMNS
      N             = NBOT + NWIDM1
      I1            = 3*N - 2
      I2            = I1 + 1
      I3            = I1 + 2
      DNW           =  0.0
      DNE           =  0.0
      AN            =  0.0
      BN            =  0.0
      DSW           =  D(N-ISW)*E(N-ISW)/(2.828428*CELLEN)
      DSE           =  D(N-ISE)*E(N-ISE)/(2.828428*CELLEN)
      AW            =  A(N-ISW)*E(N-ISW)/CELLEN
      AS            = (A(N-ISW)*E(N-ISW)+A(N-ISE)*E(N-ISE))/CELLEN
      AE            =  A(N-ISE)*E(N-ISE)/CELLEN
      BW            =  B(N-ISW)*E(N-ISW)/CELLEN
      BS            = (B(N-ISW)*E(N-ISW)+B(N-ISE)*E(N-ISE))/CELLEN
      BE            =  B(N-ISE)*E(N-ISE)/CELLEN
      CALL SMATRIX        (DSW,DSE,DNW,DNE,AS,AW,AE,AN,BS,BW,BN,BE,I1,I2,
     1                     I3, MD,NW3,CELLEN,CELLEN2)
      NBOT          = NBOT + NODEWID
      INW           = INW + 1
      INE           = INE + 1
      ISW           = ISW + 1
      ISE           = ISE + 1
  205 CONTINUE
C-----COMPUTING STIFFNESS ELEMENTS FOR LAST COLUMN
C-----COMPUTING STIFFNESS ELEMENTS FOR BOTTOM NODE OF LAST COLUMN
      N             = NBOT
      I1            = 3*N - 2
      I2            = I1 + 1
      I3            = I1 + 2
      DSW           =  0.0
      DNE           =  0.0
      DSE           =  0.0
      AS            =  0.0
      AE            =  0.0
      BS            =  0.0
      BE            =  0.0
      DNW           =  D(N-INW)*E(N-INW)/(2.828428*CELLEN)
      AW            =  A(N-INW)*E(N-INW)/CELLEN
      AN            =  AW
```

```
      BW             =  B(N-INW)*E(N-INW)/CELLEN
      BN             =  BW
      CALL SMATRIX         (DSW,DSE,DNW,DNE,AS,AW,AE,AN,BS,BW,BN,BE,I1,I2,
     1                      I3, MD,NW3,CELLEN,CELLEN2)
C-----COMPUTING STIFFNESS ELEMENTS FOR INTERIOR NODES ON LAST COLUMN
      M              = N + 1
      NLASTM1        = NODEWID*NODELEN - 1
          DO 207     N = M,NLASTM1
      I1             = 3*N - 2
      I2             = I1 + 1
      I3             = I1 + 2
      DNW            =  D(N-INW)*E(N-INW)/(2.828428*CELLEN)
      DSW            =  D(N-ISW)*E(N-ISW)/(2.828428*CELLEN)
      AS             =  A(N-ISW)*E(N-ISW)/CELLEN
      AN             =  A(N-INW)*E(N-INW)/CELLEN
      AW             =  AS + AN
      BN             =  B(N-INW)*E(N-INW)/CELLEN
      BS             =  B(N-ISW)*E(N-ISW)/CELLEN
      BW             =  BS + BN
      CALL SMATRIX         (DSW,DSE,DNW,DNE,AS,AW,AE,AN,BS,BW,BN,BE,I1,I2,
     1                      I3, MD,NW3,CELLEN,CELLEN2)
  207 CONTINUE
C-----COMPUTING STIFFNESS ELEMENTS FOR LAST NODE
      N              = NODEWID*NODELEN
      I1             = 3*N - 2
      I2             = I1 + 1
      I3             = I1 + 2
      DNW            =  0.0
      DSE            =  0.0
      DNE            =  0.0
      AN             =  0.0
      AE             =  0.0
      BN             =  0.0
      BE             =  0.0
      DSW            =  D(N-ISW)*E(N-ISW)/(2.828428*CELLEN)
      AW             =  A(N-ISW)*E(N-ISW)/CELLEN
      AS             =  AW
      BW             =  B(N-ISW)*E(N-ISW)/CELLEN
      BS             =  BW
      CALL SMATRIX         (DSW,DSE,DNW,DNE,AS,AW,AE,AN,BS,BW,BN,BE,I1,I2,
     1                      I3, MD,NW3,CELLEN,CELLEN2)
C-----END OF GENERATING GENERAL MATRIX  S(I,J)
C-----BELOW DO LOOP SETS THE DIAGONAL ELEMENT S(I,MD) = 1.0 AND ALL
C     OTHER ELEMENTS S(I,J) = 0.0 ON I ROW WHENEVER THE MOVEMENT X(I)
C     IS SPECIFIED
         DO 208      I = 1,IXNUM
      IXX            = IXSPC(I)
         DO 209      J = 1,NTWID
      S(IXX,J)       = 0.0
  209 CONTINUE
      S(IXX,MD)      = 1.0
  208 CONTINUE
C-----SOLUTION OF STIFFNESS MATRIX EQUATION FOR NODE MOVEMENTS X(I).
C     DIRECT ELIMINATION PROCESS WILL STOP IF A ZERO PIVOT IS ENCOUNTERED
C     INFORMATION USED FROM INPUT AND OTHER SECTION OF THIS PROGRAM
```

```
C      S(I,J),F(I),NEQUAS,NTWID,NTWIDTH,NQMWID
C-----INFORMATION OBTAINED FROM THIS SECTION
C      NODE MOVEMENTS VECTOR X(I)
       DO 103  I = 1, NQMWID
           IP1     = I + 1
           IPN     = I + NWIDTH
           N       = 0
       DO 103  L   = IP1, IPN
           N       = N + 1
           M       = MD - N
       IF ( S(L,M) ) 104, 103, 104
C NOTE---NEED TO ADD LOGIC STATEMENT FOR ZERO PIVOT
  104      XM      = -S(L,M)/S(I,MD)
           F(L)    = F(L) + XM * F(I)
           MN      = M + NWIDTH
           LL      = L - I
       DO 103   MM = M, MN
           S(L,MM) = S(L,MM) + XM * S(I,MM+LL)
  103  CONTINUE
C          END OF TRIANGULAR RESOLUTION OF ALL BUT LAST SECTION OF S(I,J)
           II      =  NEQUAS - NWIDTH + 1
           NEQ     =  NEQUAS - 1
           NN      =   0
       DO 105    I = II, NEQ
           IP1       = I + 1
           N         = 0
         NN = NN + 1
       DO 105     L = IP1, NEQUAS
           N         = N + 1
           M         = MD - N
       IF ( S(L,M) ) 106, 105, 106
C NOTE---NEED TO ADD LOGIC STATEMENT FOR ZERO PIVOT
  106      XM        = -S(L,M)/S(I,MD)
           F(L)      =  F(L) + XM * F(I)
           MN    = M + NWIDTH - NN
           LL    =  L - I
       DO 105   MM = M, MN
           S(L,MM) = S(L,MM) + XM * S(I,MM+LL)
  105  CONTINUE
       CALL TIME ( 1HP )
C          END OF TRIANGULAR RESOLUTION
C NOTE---WILL ELIMINATE COLUMN BY COLUMN IN THE TRIANGULAR MATRIX
C          S(I,J) IN ORDER TO OBTAIN ONLY A DIAGONAL MATRIX SAY
C          S(I,MD)  THEN WILL SOLVE FOR X(I).
           IA      = NEQUAS + MD
       DO 108    I = MD, NEQUAS
           IB      = IA - I
           IBM1    = IB - 1
           IBMN    = IB - NWIDTH
           MB      = -1
       DO 108    L = IBMN, IBM1
           MB       = MB + 1
           MBB      = NTWID - MB
C NOTE---NEED TO ADD LOGIC STATEMENT FOR ZERO PIVOT
           XM       =  - S(L,MBB)/S(IB,MD)
```

```
C NOTE---SINCE IT IS NOT NECESSARY TO GO THROUGH THE ELIMINATION OF COLUMN
C        ELEMENTS ABOVE PIVOT ELEMENTS EXCEPT TO STUDY COMPUTATIONAL
C        ERRORS, WILL NOT GO THROUGH THIS COMPUTATION
          F(L)      = F(L) + XM * F(IB)
  108 CONTINUE
C         END OF DIAGONALIZATION EXCEPT FOR LAST SECTION
          IA        = NWIDTH + 2
          N         = -1
      DO 109   I = 2, NWIDTH
          IB        = IA - I
          IBM1      = IB - 1
          N         = N + 1
      DO 109   L = 1, IBM1
          NN        = NTWID - N - L
          XM        = - S(L,NN)/S(IB,MD)
          S(L,NN) = S(L,NN) + XM * S(IB,MD)
          F(L)      = F(L) + XM * F(IB)
  109 CONTINUE
C NOTE---WILL NOW EVALUATE X(I) BY SIMPLE DIVISION
C         COMPUTING UNKNOWNS X(I)
      DO 110   I = 1, NEQUAS
          X(I) = F(I)/S(I,MD)
  110 CONTINUE
C-----TO COMPUTE THE STATE OF STRAIN AT THE CENTER OF EACH LATTICE CELL
C     FORM THE MOVEMENT OF THE NODE POINTS X(I)
      ICLBOT      = 1
      ICLTOP      = NWIDM1
      JJ          = 0
          DO 300  I = 1,NLENM1
      J           = JJ - 2
          DO 299  N = ICLBOT, ICLTOP
      J           = J + 3
C-----TO COMPUTE STRAINS WITH REFERENCE TO X-Y DIRECTIONS
      STRAINX(N) = (X(J+JP+3) + X(J+JP) - X(J) - X(J+3))/DENOM
      STRAINY(N) = (X(J+JP+4) - X(J+JP+1) - X(J+1) + X(J+4))/DENOM
      STRANXY(N) = (X(J+JP+3) + X(J+JP+4) - X(J+JP) + X(J+JP+1) - X(J)
     1              -X(J+1) + X(J+3) - X(J+4))/DENOM
C-----TO COMPUTE PRINCIPAL STRAINS
C     EC = STRAIN AT CENTER OF MOHRS CIRCLE
C     ER = RADIUS OF MOHRS CIRCLE
      EC          = (STRAINX(N) + STRAINY(N))/2.0
      ER          = ((((STRAINX(N)-STRAINY(N))**2  )/4.0) +
     1              ((STRANXY(N)**2  )/4.0))**0.5
      STRAIN1(N) = EC + ER
      STRAIN2(N) = EC - ER
C-----BELOW LOGIC IS TO TAKE CARE OF THE CASES OF  ATANF(Z) WHERE Z =
C     (0/0) OR (1/0)
          IF (STRANXY(N)) 312,310,312
  310     IF(STRAINX(N)-STRAINY(N)) 311,320,311
  312     IF (STRAINX(N)-STRAINY(N)) 311,321,311
  321     IF (STRANXY(N)) 313,311,314
  313 THETA(N)    = -45.0
          GO TO 299
  314 THETA(N)    =  45.0
          GO TO 299
```

```
  311 THETA2      = (ATANF(STRANXY(N)/(STRAINX(N)-STRAINY(N))))
C-----CHANGING THETA2 TO DEGREES
      THETA2      = THETA2*57.2957795
C-----BELOW LOGIC IS TO PLACE THETA2 IN CORRECTION QUADRANT SINCE
C     IN THE ATANF( ) ROUTINE ONLY ANGLES IN FIRST AND SECOND QUADRANT
C     ARE COMPUTED
          IF (STRANXY(N))301,301,302
  301     IF (STRAINX(N)-STRAINY(N)) 303,299,302
  303 THETA2      = THETA2 - 180.0
          GO TO 305
  302     IF (STRAINX(N)-STRAINY(N)) 304,299,305
  304 THETA2      = THETA2 + 180.0
          GO TO 305
  320 THETA2      = 0.0
  305 THETA(N)     = THETA2/2.0
  299 CONTINUE
      JJ          = JJ + JP
      ICLBOT      = ICLBOT + NWIDM1
      ICLTOP      = ICLTOP + NWIDM1
  300 CONTINUE
C-----PRINT OUT OF ITERATION DATA
C     PRINT OUT OF NODE MOVEMENT FOR EACH ITERATION---ONLY THOSE MOVE-
C     MENTS BEING MONITORED WILL BE PRINTED HERE
      PRINT 1
      PRINT 6, (RUN(I), I = 1,10)
      PRINT 36
      PRINT 37, IT
      PRINT 38
      PRINT 39
      PRINT 40, ITRMOV1, X(ITRMOV1)
      PRINT 40, ITRMOV2, X(ITRMOV2)
      PRINT 40, ITRMOV3, X(ITRMOV3)
      PRINT 40, ITRMOV4, X(ITRMOV4)
      PRINT 40, ITRMOV5, X(ITRMOV5)
C-----TO PRINT OUT MONITORED STRAINS VALUES
      PRINT 41
      PRINT 42
      PRINT 43, ITRCEL1, STRAIN1(ITRCEL1), STRAIN2(ITRCEL1), THETA(ITRCE
     1L1)
      PRINT 43, ITRCEL2, STRAIN1(ITRCEL2), STRAIN2(ITRCEL2), THETA(ITRCE
     1L2)
      PRINT 43, ITRCEL3, STRAIN1(ITRCEL3), STRAIN2(ITRCEL3), THETA(ITRCE
     1L3)
      PRINT 43, ITRCEL4, STRAIN1(ITRCEL4), STRAIN2(ITRCEL4), THETA(ITRCE
     1L4)
      PRINT 43, ITRCEL5, STRAIN1(ITRCEL5), STRAIN2(ITRCEL5), THETA(ITRCE
     1L5)
C-----COMPUTATION OF SECANT PLANE FOR EACH LATTICE CELL---E(I) V(I)
C     AND ALSO STATE OF STRESS
C     INFORMATION USED FROM INPUT AND OTHER SECTIONS OF PROGRAM
C     STATE OF STRESS VERSUS STATE OF STRAIN  THAT IS
C     STRAIN1(I) AND STRAIN2(I)
C-----INFORMATION OBTAINED FROM THIS SECTION--- E(I), V(I), STRESS1(I)
C     STRESS2(2)
C
```

```
C-----FOR PROGRAM BODY1 WILL USE HOOKES LAW FOR F  THAT IS WILL BE A
C      LINEAR MATERIAL---WILL SIMPLY WRITE VALUES OF E AND V IN PROGRAM
         DO 450   I = 1,NCELLS
C-----HOOKES LAW FOR SIGMA = F(E1,E2) BELOW  STEEL
      STRESS1(I)   = 1190.4762*(STRAIN1(I) + 0.4*STRAIN2(I))
      STRESS2(I)   = 1190.4762*(STRAIN2(I) + 0.4*STRAIN1(I))
C-----TO SOLVE FOR REVISED E(I) AND V(I)
      BB           = (STRESS2(I)-STRAIN2(I)*STRESS1(I)/STRAIN1(I))/
     1              (STRAIN1(I)-(STRAIN2(I)**2)/STRAIN1(I))
      AA           = (STRESS1(I)-STRAIN2(I)*BB)/STRAIN1(I)
      V(I)        = BB/AA
      E(I)        = AA*(1.0-V(I)**2)
  450 CONTINUE
C-----COMPUTATION OF STRESSES ON X-Y PLANES
C     INPUT DATA TO THIS SECTION ARE STRESS1(I), STRESS2(I), THETA(I)
C     OUTPUT OF THIS SECTION IS STRESSX(I), STRESSY(I), STRESXY(I)
         DO 449   I = 1, NCELLS
      SCENTER      = (STRESS1(I)+STRESS2(I))/2.0
      SRADIUS      = (STRESS1(I)-STRESS2(I))/2.0
      THETA2       =  0.03490658*THETA(I)
      STRESSX(I)   =  SCENTER+(SRADIUS)*(COSF(THETA2))
      STRESSY(I)   =  SCENTER-(SRADIUS)*(COSF(THETA2))
      STRESXY(I)   = (SRADIUS)*(SINF(THETA2))
  449 CONTINUE
C-----TEST ON CLOSURE BASED ON MOVEMENTS OF NODES
      IF  (ABSF(X(ITRMOV1)-XPREVUS(   1    ))-CLOSTOL) 451,451,455
  451 IF  (ABSF(X(ITRMOV2)-XPREVUS(   2    ))-CLOSTOL) 452,452,455
  452 IF  (ABSF(X(ITRMOV3)-XPREVUS(   3    ))-CLOSTOL) 453,453,455
  453 IF  (ABSF(X(ITRMOV4)-XPREVUS(   4    ))-CLOSTOL) 454,454,455
  454 IF  (ABSF(X(ITRMOV5)-XPREVUS(   5    ))-CLOSTOL) 456,456,455
C-----SETTING X( 1---5  ) = XPREVUS(ITRMOV--)
  455 XPREVUS(   1   ) = X(ITRMOV1)
      XPREVUS(   2   ) = X(ITRMOV2)
      XPREVUS(   3   ) = X(ITRMOV3)
      XPREVUS(   4   ) = X(ITRMOV4)
      XPREVUS(   5   ) = X(ITRMOV5)
C-----STATEMENT 999 IS CONTINUE STATEMENT OF ITERATION LOOP
C     STATEMENT 456 WILL BE START OF PRINT OUT OF OUTPUT
  999 CONTINUE
      PRINT 1
      PRINT 6, (RUN(I),I=1,10)
      PRINT 500
      PRINT 502, ITERNUM
      GO TO 459
  456 PRINT 1
      PRINT 6, (RUN(I),I=1,10)
      PRINT 500
      PRINT 501, ITERNUM
  459 PRINT 503
      PRINT 39
        DO 460 I=1,NEQUAS
      PRINT 40, I, X(I)
  460 CONTINUE
      PRINT 504
      PRINT 42
```

```
      DO 461 I=1,NCELLS
      PRINT 43, I, STRAIN1(I), STRAIN2(I), THETA(I)
  461 CONTINUE
      PRINT 505
      PRINT 506
      DO 462 I=1,NCELLS
      PRINT 43, I, STRESS1(I), STRESS2(I), THETA(I)
  462 CONTINUE
      PRINT 507
      PRINT 508
      DO 463  I = 1, NCELLS
      PRINT 43, I, STRAINX(I), STRAINY(I), STRANXY(I)
  463 CONTINUE
      PRINT 509
      PRINT 510
      DO 464 I = 1, NCELLS
      PRINT 43, I, STRESSX(I), STRESSY(I), STRESXY(I)
  464 CONTINUE
      END
      SUBROUTINE SMATRIX (DSW,DSE,DNW,DNE,AS,AW,AE,AN,BS,BW,BN,BE,I1,I2,
     1                    I3, MD,NW3,CELLEN,CELLEN2)
      DIMENSION  S(240,57)
      COMMON   S
C STIFFNESS ELEMENTS BELOW ARE FOR FORCES IN X-DIRECTION
      S(I1,MD-NW3-3)    = -DSW
      S(I1,MD-NW3-2)    = -DSW
      S(I1,MD-NW3)      = -AW
      S(I1,MD-NW3+3)    = -DNW
      S(I1,MD-NW3+4)    = DNW
      S(I1,MD-3)   = -12.0*BS/CELLEN2
      S(I1,MD-1)   = 6.0*BS/CELLEN
      S(I1,MD)     = (AE+AW+DNE+DNW+DSE+DSW+12.0*(BN+BS)/CELLEN2)
      S(I1,MD+1)   = (DNE+DSW-DSE-DNW)
      S(I1,MD+2)   = 6.0*(BS-BN)/CELLEN
      S(I1,MD+3)   = -12.0*BN/CELLEN2
      S(I1,MD+5)   = -6.0*BN/CELLEN
      S(I1,MD+NW3-3)    = -DSE
      S(I1,MD+NW3-2)     = DSE
      S(I1,MD+NW3)      = -AE
      S(I1,MD+NW3+3)    = -DNE
      S(I1,MD+NW3+4)    = -DNE
C STIFFNESS ELEMENTS BELOW ARE FOR FORCES IN Y-DIRECTION
      S(I2,MD-NW3-4)    = -DSW
      S(I2,MD-NW3-3)    = -DSW
      S(I2,MD-NW3)      = -12.0*BW/CELLEN2
      S(I2,MD-NW3+1)    = -6.0*BW/CELLEN
      S(I2,MD-NW3+2)    = DNW
      S(I2,MD-NW3+3)    = -DNW
      S(I2,MD-3)   = -AS
      S(I2,MD-1)   = (DSW-DNW+DNE-DSE)
      S(I2,MD)     = (DSW+DNW+DNE+DSE+AN+AS+12.0*(BE+BW)/CELLEN2)
      S(I2,MD+1)   = 6.0*(BE-BW)/CELLEN
      S(I2,MD+3)   = -AN
      S(I2,MD+NW3-4)    = DSE
      S(I2,MD+NW3-3)    = -DSE
```

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      S(I2,MD+NW3)       = -12.0*BE/CELLEN2
      S(I2,MD+NW3+1)     = 6.0*BE/CELLEN
      S(I2,MD+NW3+2)     = -DNE
      S(I2,MD+NW3+3)     = -DNE
C STIFFNESS ELEMENTS BELOW ARE FOR MOMENTS
      S(I3,MD-NW3-1)     = 6.0*BW/CELLEN
      S(I3,MD-NW3)       = 2.0*BW
      S(I3,MD-5)    = -6.0*BS/CELLEN
      S(I3,MD-3)    = 2.0*BS
      S(I3,MD-2)    = 6.0*(BS-BN)/CELLEN
      S(I3,MD-1)    = 6.0*(BE-BW)/CELLEN
      S(I3,MD)      = (BW+BN+BE+BS)*4.0
      S(I3,MD+1)    = 6.0*BN/CELLEN
      S(I3,MD+3)    = 2.0*BN
      S(I3,MD+NW3-1)     = -6.0*BE/CELLEN
      S(I3,MD+NW3)       = 2.0*BE
      END
      END
```

APPENDIX III

COMPUTER SOLUTION OF EXAMPLE PROBLEM

CANTILEVER BEAM

•• SMITH R.E. PROGRAM BODY 2 GENERAL SIMPLY CONNECTED LATTICE MODEL

CANTILEVER BEAM FINER MESH FOR STRESS DISTRIBUTION UNDER LOADS

INPUT DATA

MAXIMUM NUMBER OF ITERATIONS = 1

MONITORED CELLS ARE 1 2 3 4 5

MONITORED NODE MOVEMENTS ARE 1 2 3 4 5

NUMBER OF F(I) ELEMENTS SPECIFIED = 36

NUMBER OF NODE MOVEMENTS SPECIFIED = 26

WIDTH OF RECTANGULAR BODY = 4.000E+00

LENGTH OF RECTANGULAR BODY = 6.000E+00

SIDE DIMENSION OF LATTICE CELL = 5.000E-01

THICKNESS OF BODY = 1.000E+00

MODULUS OF ELASTICITY ASSUMED = 3.000E+07

POISSONS RATIO ASSUMED = 3.000E-01

CLOSURE TOLERANCE FOR MONITORED NODE MOVEMENTS = 1.000E+00

PRINT OUT OF SPECIFIED LOADS OR MOVEMENTS AT NODES

LOAD	MOVEMENT	VALUE
1		-3.750E+02
	17	-1.623E-02
	18	8.433E-03
19		-2.625E+03
	35	-1.617E-02
	36	6.940E-03
37		-3.000E+03
	53	-1.610E-02
	54	5.448E-03
55		-3.000E+03
	71	-1.607E-02
	72	4.056E-03
73		-3.000E+03
	89	-1.603E-02
	90	2.665E-03
91		-2.625E+03

	107	-1.602E-02
	108	1.329E-03
109		-3.750E+02
	125	-1.602E-02
	126	-7.474E-06
	143	-1.603E-02
	144	-1.340E-03
	161	-1.605E-02
	162	-2.672E-03
	179	-1.609E-02
	180	-4.058E-03
	197	-1.612E-02
	198	-5.444E-03
	215	-1.619E-02
	216	-6.936E-03
217		-2.500E+03
219		-5.000E+03
221		-2.500E+03
	233	-1.625E-02
	234	-8.428E-03

CANTILEVER BEAM FINER MESH FOR STRESS DISTRIBUTION UNDER LOADS

ITERATION DATA

ITERATION NUMBER 1

TABLE OF MONITORED MOVEMENTS

MOVEMENT	VALUE
1	-3.00269946E-02
2	9.69501243E-03
3	-2.81369407E-02
4	9.65786455E-03
5	-2.63032800E-02

TABLE OF MONITORED STRAINS

CELL NUMBER	MAX. STRAIN	MIN. STRAIN	THETA
1	8.03825628E-05	-1.21794494E-04	3.27890661E+01
2	6.61160891E-05	-1.62220941E-04	1.24394338E+01
3	6.84817200E-05	-2.11362318E-04	6.46671443E+00
4	7.88849298E-05	-2.57573902E-04	4.88383078E+00
5	9.19976205E-05	-3.04542868E-04	4.50047363E+00

CANTILEVER BEAM FINER MESH FOR STRESS DISTRIBUTION UNDER LOADS

OUTPUT DATA

SOLUTION CLOSED WITHIN TOLERANCE AT ITERATION NUMBER 1

TABLE OF NODE MOVEMENTS

MOVEMENT	VALUE
1	-3.00269946E-02
2	9.69501243E-03
3	-2.81369407E-02
4	9.65786455E-03
5	-2.63032800E-02
6	9.56239930E-03
7	-2.45278534E-02
8	9.43728342E-03
9	-2.27966100E-02
10	9.28991183E-03
11	-2.11019425E-02
12	9.12001506E-03
13	-1.94412616E-02
14	8.92452359E-03
15	-1.78164348E-02
16	8.69824659E-03
17	-1.62300000E-02
18	8.43300000E-03
19	-3.00316425E-02
20	7.87591240E-03
21	-2.81112035E-02
22	7.85055906E-03
23	-2.62734960E-02
24	7.79439829E-03
25	-2.44927054E-02
26	7.71170157E-03
27	-2.27553117E-02
28	7.60393794E-03
29	-2.10536846E-02
30	7.47173340E-03
31	-1.93857053E-02
32	7.31512206E-03
33	-1.77530956E-02
34	7.13553283E-03
35	-1.61700000E-02
36	6.94000000E-03
37	-2.99692888E-02
38	6.16011935E-03
39	-2.80753849E-02
40	6.13421061E-03
41	-2.62466714E-02
42	6.09977836E-03
43	-2.44670948E-02
44	6.04796864E-03

45	-2.27260843E-02
46	5.97540860E-03
47	-2.10185563E-02
48	5.88054324E-03
49	-1.93438639E-02
50	5.76281629E-03
51	-1.77057287E-02
52	5.62102602E-03
53	-1.61000000E-02
54	5.44800000E-03
55	-2.98697488E-02
56	4.52233977E-03
57	-2.80231520E-02
58	4.49570141E-03
59	-2.62182273E-02
60	4.47424714E-03
61	-2.44481139E-02
62	4.44616569E-03
63	-2.27077717E-02
64	4.40368582E-03
65	-2.09962989E-02
66	4.34315281E-03
67	-1.93156771E-02
68	4.26362564E-03
69	-1.76689648E-02
70	4.16588478E-03
71	-1.60700000E-02
72	4.05600000E-03
73	-2.97530660E-02
74	2.94325262E-03
75	-2.79595255E-02
76	2.91737884E-03
77	-2.61895474E-02
78	2.90796067E-03
79	-2.44353424E-02
80	2.89995111E-03
81	-2.26990322E-02
82	2.88284379E-03
83	-2.09855267E-02
84	2.85216527E-03
85	-1.93000754E-02
86	2.80656614E-03
87	-1.76485291E-02
88	2.74577321E-03
89	-1.60300000E-02
90	2.66500000E-03
91	-2.96147244E-02
92	1.40914987E-03
93	-2.78948627E-02
94	1.38614713E-03
95	-2.61665273E-02
96	1.39150405E-03
97	-2.44298100E-02
98	1.40029663E-03
99	-2.26986826E-02

100	1.40360597E-03
101	-2.09842828E-02
102	1.39801984E-03
103	-1.92953163E-02
104	1.38278185E-03
105	-1.76381773E-02
106	1.35842910E-03
107	-1.60200000E-02
108	1.32900000E-03
109	-2.94528674E-02
110	-8.33236413E-05
111	-2.78461009E-02
112	-9.70638765E-05
113	-2.61541327E-02
114	-8.37067224E-05
115	-2.44320797E-02
116	-6.47046586E-05
117	-2.27053046E-02
118	-4.67787847E-05
119	-2.09905903E-02
120	-3.21501591E-05
121	-1.92994182E-02
122	-2.10775896E-05
123	-1.76404382E-02
124	-1.30193909E-05
125	-1.60200000E-02
126	-7.47400000E-06
127	-2.93869444E-02
128	-1.55282941E-03
129	-2.78233336E-02
130	-1.55149392E-03
131	-2.61551167E-02
132	-1.53669676E-03
133	-2.44417489E-02
134	-1.51317543E-03
135	-2.27173098E-02
136	-1.48526705E-03
137	-2.10025219E-02
138	-1.45408995E-03
139	-1.93107435E-02
140	-1.41941623E-03
141	-1.76506344E-02
142	-1.38077940E-03
143	-1.60300000E-02
144	-1.34000000E-03
145	-2.94088773E-02
146	-3.03405973E-03
147	-2.78387904E-02
148	-3.01716715E-03
149	-2.61723496E-02
150	-2.99686650E-03
151	-2.44588505E-02
152	-2.96890444E-03
153	-2.27335683E-02
154	-2.93240938E-03

155	-2.10188374E-02
156	-2.88607890E-03
157	-1.93281566E-02
158	-2.82859776E-03
159	-1.76709138E-02
160	-2.75898213E-03
161	-1.60500000E-02
162	-2.67200000E-03
163	-2.94986343E-02
164	-4.55339400E-03
165	-2.78941636E-02
166	-4.52162521E-03
167	-2.62089551E-02
168	-4.49178967E-03
169	-2.44838065E-02
170	-4.45642145E-03
171	-2.27534907E-02
172	-4.41007606E-03
173	-2.10390254E-02
174	-4.34766344E-03
175	-1.93515174E-02
176	-4.26672825E-03
177	-1.76972399E-02
178	-4.16770747E-03
179	-1.60900000E-02
180	-4.05800000E-03
181	-2.96415801E-02
182	-6.13316709E-03
183	-2.79861899E-02
184	-6.08743507E-03
185	-2.62656130E-02
186	-6.04437784E-03
187	-2.45155833E-02
188	-5.99842895E-03
189	-2.27764550E-02
190	-5.93962308E-03
191	-2.10637858E-02
192	-5.85743565E-03
193	-1.93818843E-02
194	-5.74870902E-03
195	-1.77355533E-02
196	-5.61310369E-03
197	-1.61200000E-02
198	-5.44400000E-03
199	-2.98176886E-02
200	-7.79185515E-03
201	-2.81125696E-02
202	-7.73473899E-03
203	-2.63413248E-02
204	-7.67573779E-03
205	-2.45491454E-02
206	-7.61768561E-03
207	-2.28022082E-02
208	-7.54128593E-03
209	-2.10955064E-02

210	-7.43160843E-03
211	-1.94213920E-02
212	-7.29082720E-03
213	-1.77812496E-02
214	-7.12272607E-03
215	-1.61900000E-02
216	-6.93600000E-03
217	-2.99923834E-02
218	-9.51397656E-03
219	-2.82698176E-02
220	-9.47200468E-03
221	-2.64309993E-02
222	-9.42684137E-03
223	-2.45772340E-02
224	-9.34469526E-03
225	-2.28361812E-02
226	-9.22519558E-03
227	-2.11378505E-02
228	-9.07774740E-03
229	-1.94726388E-02
230	-8.90000615E-03
231	-1.78414508E-02
232	-8.68635497E-03
233	-1.62500000E-02
234	-8.42800000E-03

TABLE FOR THE STATE OF STRAIN AT EACH CELL

CELL NUMBER	MAX. STRAIN	MIN. STRAIN	THETA
1	8.03825628E-05	-1.21794494E-04	3.27890661E+01
2	6.61160891E-05	-1.62220941E-04	1.24394338E+01
3	6.84817200E-05	-2.11362318E-04	6.46671443E+00
4	7.88849298E-05	-2.57573902E-04	4.88383078E+00
5	9.19976205E-05	-3.04542868E-04	4.50047363E+00
6	1.06559925E-04	-3.54848682E-04	4.42437812E+00
7	1.22340925E-04	-4.09311705E-04	4.61746458E+00
8	1.28831939E-04	-4.66272130E-04	5.51301935E+00
9	2.28643137E-04	-1.81732899E-04	3.43226621E+01
10	1.34969184E-04	-1.62919032E-04	2.95210422E+01
11	9.65842493E-05	-1.78655538E-04	2.36095995E+01
12	8.72153185E-05	-2.12701022E-04	1.91816662E+01
13	9.24349701E-05	-2.55149157E-04	1.65125523E+01
14	1.03236747E-04	-3.00605261E-04	1.47756391E+01
15	1.13136369E-04	-3.45307488E-04	1.32064206E+01
16	1.33905212E-04	-3.85097195E-04	1.02829513E+01
17	3.03206685E-04	-2.03980882E-04	3.31217976E+01
18	2.09405460E-04	-1.84615011E-04	3.48605624E+01
19	1.57060525E-04	-1.89526677E-04	3.42240951E+01
20	1.32855450E-04	-2.10601846E-04	3.18353527E+01
21	1.25912018E-04	-2.40740346E-04	2.88456704E+01
22	1.28427421E-04	-2.75237370E-04	2.60740063E+01
23	1.37403776E-04	-3.11984297E-04	2.36738577E+01
24	1.41984555E-04	-3.58131486E-04	2.28190891E+01
25	3.31868601E-04	-2.04071437E-04	3.21258999E+01

26	2.54505819E-04	-1.93071791E-04	3.70126636E+01
27	2.12207341E-04	-2.06846905E-04	3.96682225E+01
28	1.89735497E-04	-2.27811595E-04	3.94002591E+01
29	1.78014281E-04	-2.49714078E-04	3.74986825E+01
30	1.72940078E-04	-2.71692444E-04	3.50392812E+01
31	1.70685393E-04	-2.93181680E-04	3.25999118E+01
32	1.73737519E-04	-3.03959761E-04	2.91445469E+01
33	3.34070094E-04	-1.79942262E-04	3.03287625E+01
34	2.71745632E-04	-1.88123966E-04	3.92461281E+01
35	2.52479038E-04	-2.23143532E-04	4.33264259E+01
36	2.40512687E-04	-2.48428804E-04	4.38466137E+01
37	2.30648910E-04	-2.65320124E-04	4.28111334E+01
38	2.23066492E-04	-2.77900638E-04	4.11663059E+01
39	2.18761149E-04	-2.88795948E-04	3.93037899E+01
40	2.19926221E-04	-3.09776739E-04	3.78657383E+01
41	3.01617536E-04	-1.27741652E-04	2.74109957E+01
42	2.71849169E-04	-1.91978681E-04	4.23749174E+01
43	2.78389341E-04	-2.40469758E-04	4.59757901E+01
44	2.77847989E-04	-2.65504346E-04	4.65892292E+01
45	2.72556181E-04	-2.76443043E-04	4.61468418E+01
46	2.65785071E-04	-2.80359808E-04	4.53275295E+01
47	2.59122401E-04	-2.81779763E-04	4.44739545E+01
48	2.52494362E-04	-2.78639005E-04	4.38334035E+01
49	1.71328343E-04	-9.50428013E-05	3.38478093E+01
50	2.51373718E-04	-2.01436164E-04	4.54030906E+01
51	2.84235765E-04	-2.52365620E-04	4.78436443E+01
52	2.96219568E-04	-2.72059735E-04	4.84112782E+01
53	2.97600377E-04	-2.75731472E-04	4.84935041E+01
54	2.94688374E-04	-2.72198981E-04	4.84957740E+01
55	2.91177788E-04	-2.66004224E-04	4.85162192E+01
56	2.85495596E-04	-2.59366978E-04	4.85062998E+01
57	8.78683779E-05	-1.07030038E-04	5.32904308E+01
58	2.08402410E-04	-2.05994298E-04	4.97074108E+01
59	2.67636829E-04	-2.50487898E-04	4.97669652E+01
60	2.93281749E-04	-2.62238370E-04	5.00680076E+01
61	3.03110792E-04	-2.58177291E-04	5.06551744E+01
62	3.07080051E-04	-2.48653895E-04	5.15460696E+01
63	3.10127928E-04	-2.39567961E-04	5.26983700E+01
64	3.22383210E-04	-2.34901048E-04	5.37749218E+01
65	7.45309663E-05	-1.70999762E-04	7.10589570E+01
66	1.69006002E-04	-2.10848492E-04	5.59852695E+01
67	2.37381379E-04	-2.35612637E-04	5.26551129E+01
68	2.72367189E-04	-2.34405227E-04	5.22986659E+01
69	2.90345306E-04	-2.21712682E-04	5.34498353E+01
70	3.03387151E-04	-2.08519634E-04	5.54109846E+01
71	3.16174061E-04	-1.97224576E-04	5.75832789E+01
72	3.18676022E-04	-1.88312520E-04	6.06252386E+01
73	8.76214862E-05	-2.45092838E-04	7.99557324E+01
74	1.43548420E-04	-2.19339832E-04	6.38161888E+01
75	2.04803130E-04	-2.11920724E-04	5.70191875E+01
76	2.39612206E-04	-1.89202068E-04	5.59463934E+01
77	2.64880252E-04	-1.68004810E-04	5.81888234E+01
78	2.92292928E-04	-1.57758319E-04	6.14752300E+01
79	3.23439409E-04	-1.57493580E-04	6.45495088E+01
80	3.66824413E-04	-1.56326654E-04	6.57846881E+01

81	1.04673090E-04	-3.04313101E-04	8.61698807E+01
82	1.35687168E-04	-2.35720316E-04	7.24880072E+01
83	1.77996032E-04	-1.83268917E-04	6.30911477E+01
84	2.02368955E-04	-1.26478704E-04	6.31325683E+01
85	2.41464959E-04	-1.07073881E-04	6.78372997E+01
86	2.89351976E-04	-1.11072514E-04	7.16124022E+01
87	3.36380519E-04	-1.17878154E-04	7.44431425E+01
88	3.77183372E-04	-1.37049961E-04	7.82420731E+01
89	9.96701665E-05	-3.32524875E-04	-8.78967385E+01
90	1.14410874E-04	-2.57168741E-04	8.04413195E+01
91	1.65093730E-04	-1.42658487E-04	7.34758505E+01
92	2.01532927E-04	-6.76951539E-05	8.16827367E+01
93	2.61289235E-04	-8.04806601E-05	8.36631425E+01
94	3.22760117E-04	-9.78284787E-05	8.42391331E+01
95	3.86624451E-04	-1.16319984E-04	8.43515960E+01
96	4.52134229E-04	-1.27254328E-04	8.36654513E+01

TABLE FOR THE STATE OF STRESS AT EACH CELL

CELL NUMBER	MAX. STRESS	MIN. STRESS	THETA
1	1.44541367E+03	-3.22021071E+03	3.27890661E+01
2	5.75268354E+02	-4.69404773E+03	1.24394338E+01
3	1.67242569E+02	-6.29069677E+03	6.46671443E+00
4	5.31678858E+01	-7.71126670E+03	4.88383078E+00
5	2.09261580E+01	-9.13000819E+03	4.50047363E+00
6	3.47210202E+00	-1.06444188E+04	4.42437812E+00
7	-1.49204317E+01	-1.22838273E+04	4.61746458E+00
8	-3.64275815E+02	-1.40974466E+04	5.51301935E+00
9	5.74032750E+03	-3.72988871E+03	3.43226621E+01
10	2.83824642E+03	-4.03609704E+03	2.95210422E+01
11	1.41717323E+03	-4.93451418E+03	2.36095995E+01
12	7.71593797E+02	-6.14955252E+03	1.91816662E+01
13	5.23853507E+02	-7.49731865E+03	1.65125523E+01
14	4.30390184E+02	-8.88904077E+03	1.47756391E+01
15	3.14641406E+02	-1.02648322E+04	1.32064206E+01
16	6.05803968E+02	-1.13711747E+04	1.02829513E+01
17	7.97843145E+03	-3.72589703E+03	3.31217976E+01
18	5.07761396E+03	-4.01516615E+03	3.48605624E+01
19	3.30337983E+03	-4.69478635E+03	3.42240951E+01
20	2.29697462E+03	-5.62896299E+03	3.18353527E+01
21	1.76999717E+03	-6.69121123E+03	2.88456704E+01
22	1.51174320E+03	-7.80359814E+03	2.60740063E+01
23	1.44423584E+03	-8.92625817E+03	2.36738577E+01
24	1.13884977E+03	-1.04022896E+04	2.28190891E+01
25	8.92243416E+03	-3.44541288E+03	3.21258999E+01
26	6.48080050E+03	-3.84791359E+03	3.70126636E+01
27	4.95010777E+03	-4.72037483E+03	3.96682225E+01
28	4.00193469E+03	-5.63376745E+03	3.94002591E+01
29	3.39890301E+03	-6.47175143E+03	3.74986825E+01
30	3.01425311E+03	-7.24649739E+03	3.50392812E+01
31	2.72739195E+03	-7.97723281E+03	3.25999118E+01
32	2.72141509E+03	-8.30236831E+03	2.91445469E+01
33	9.23365106E+03	-2.62817254E+03	3.03287625E+01
34	7.09808049E+03	-3.51429485E+03	3.92461281E+01

35	6.11657072E+03	-4.85933475E+03	4.33264259E+01
36	5.47200151E+03	-5.81126365E+03	4.38466137E+01
37	4.97976502E+03	-6.46567422E+03	4.28111334E+01
38	4.60537256E+03	-6.95540737E+03	4.11663059E+01
39	4.35568236E+03	-7.35717373E+03	3.93037899E+01
40	4.18658900E+03	-8.03732547E+03	3.78657383E+01
41	8.68005628E+03	-1.22823268E+03	2.74109957E+01
42	7.06337028E+03	-3.64034935E+03	4.23749174E+01
43	6.79939825E+03	-5.17427327E+03	4.59757901E+01
44	6.53395664E+03	-6.00494340E+03	4.65892292E+01
45	6.25131655E+03	-6.41789632E+03	4.61468418E+01
46	5.98935588E+03	-6.61398749E+03	4.53275295E+01
47	5.75566392E+03	-6.72669372E+03	4.44739545E+01
48	5.56821958E+03	-6.68870429E+03	4.38334035E+01
49	4.70820339E+03	-1.43882302E+03	3.38478093E+01
50	6.29481986E+03	-4.15463897E+03	4.54030906E+01
51	6.87448611E+03	-5.50862276E+03	4.78436443E+01
52	7.07477959E+03	-6.03935816E+03	4.84112782E+01
53	7.08398690E+03	-6.14674809E+03	4.84935041E+01
54	7.02292351E+03	-6.05909236E+03	4.84957740E+01
55	6.96845673E+03	-5.88958971E+03	4.85162192E+01
56	6.84677480E+03	-5.72697691E+03	4.85062998E+01
57	1.83822087E+03	-2.65943488E+03	5.32904308E+01
58	4.83310286E+03	-4.72989809E+03	4.97074108E+01
59	6.34583932E+03	-5.61088515E+03	4.97669652E+01
60	7.07506278E+03	-5.74463227E+03	5.00680076E+01
61	7.43926171E+03	-5.51354021E+03	5.06551744E+01
62	7.66430382E+03	-5.16032571E+03	5.15460696E+01
63	7.85464416E+03	-4.83064558E+03	5.26983700E+01
64	8.30482074E+03	-4.55558522E+03	5.37749218E+01
65	7.65858388E+02	-4.90023534E+03	7.10589570E+01
66	3.48631169E+03	-5.27956125E+03	5.59852695E+01
67	5.49552486E+03	-5.41972164E+03	5.26551129E+01
68	6.66084465E+03	-5.03390340E+03	5.22986659E+01
69	7.37906049E+03	-4.43766231E+03	5.34498353E+01
70	7.93949212E+03	-3.87374138E+03	5.54109846E+01
71	8.47274798E+03	-3.37491287E+03	5.75832789E+01
72	8.64337141E+03	-3.05636418E+03	6.06252386E+01
73	4.64625327E+02	-7.21339753E+03	7.99557324E+01
74	2.56307047E+03	-5.81127381E+03	6.38161888E+01
75	4.65583228E+03	-4.96087205E+03	5.70191875E+01
76	6.02807427E+03	-3.86763975E+03	5.59463934E+01
77	7.07072998E+03	-2.91892529E+03	5.81888234E+01
78	8.07578348E+03	-2.31001454E+03	6.14752300E+01
79	9.10520886E+03	-1.99324474E+03	6.45495088E+01
80	1.05470247E+04	-1.52569220E+03	6.57846881E+01
81	4.41071199E+02	-8.99707166E+03	8.61698807E+01
82	2.14190351E+03	-6.42903844E+03	7.24880072E+01
83	4.05545133E+03	-4.28143210E+03	6.30911477E+01
84	5.42061574E+03	-2.16817639E+03	6.31325683E+01
85	6.90141081E+03	-1.14179318E+03	6.78372997E+01
86	8.44055676E+03	-8.00008396E+02	7.16124022E+01
87	9.92363976E+03	-5.59252689E+02	7.44431425E+01
88	1.10791775E+04	-7.87745591E+02	7.82420731E+01
89	-2.87789285E+00	-9.97660963E+03	-8.78967385E+01

90	1.22835993E+03	-7.34655427E+03	8.04413195E+01
91	4.03174232E+03	-3.07023191E+03	7.34758505E+01
92	5.97443014E+03	-2.38525576E+02	8.16827367E+01
93	7.81796824E+03	-6.90293300E+01	8.36631425E+01
94	9.67290902E+03	-3.29816540E+01	8.42391331E+01
95	1.15954436E+04	-1.09664466E+01	8.43515960E+01
96	1.36469647E+04	2.76459572E+02	8.36654513E+01

TABLE FOR STRAINS IN THE X-Y DIRECTIONS FOR EACH CELL

CELL NUMBER	STRAINX	STRAINY	STRANXY
1	2.10892863E-05	-6.25012171E-05	1.84087452E-04
2	5.55211645E-05	-1.51626017E-04	9.60616715E-05
3	6.49320068E-05	-2.07812605E-04	6.26343558E-05
4	7.64462443E-05	-2.55135217E-04	5.70813497E-05
5	8.95560720E-05	-3.02101319E-04	6.20390745E-05
6	1.03814045E-04	-3.52102802E-04	7.09769729E-05
7	1.18895447E-04	-4.05866228E-04	8.53211798E-05
8	1.23339238E-04	-4.60779428E-04	1.13816680E-04
9	9.81723156E-05	-5.12620770E-05	3.82201325E-04
10	6.26431670E-05	-9.05930149E-05	2.55452661E-04
11	5.24351549E-05	-1.34506444E-04	2.02014305E-04
12	5.48379717E-05	-1.80323675E-04	1.86141911E-04
13	6.43557078E-05	-2.27069894E-04	1.89435592E-04
14	7.69697808E-05	-2.74338294E-04	1.99175811E-04
15	8.92083699E-05	-3.21379489E-04	2.03932293E-04
16	1.17366877E-04	-3.68558860E-04	1.82317515E-04
17	1.51772905E-04	-5.25471020E-05	4.64211766E-04
18	8.06769690E-05	-5.58865200E-05	3.69597816E-04
19	4.74250123E-05	-7.98911645E-05	3.22355827E-04
20	3.72935228E-05	-1.15039918E-04	3.07826960E-04
21	4.05700303E-05	-1.55398358E-04	3.09887636E-04
22	5.04441773E-05	-1.97254126E-04	3.18733140E-04
23	6.49506010E-05	-2.39531122E-04	3.30515539E-04
24	6.67638787E-05	-2.82910809E-04	3.57552328E-04
25	1.80309304E-04	-5.25121403E-05	4.82727563E-04
26	9.23064704E-05	-3.08724425E-05	4.30293706E-04
27	4.14514420E-05	-3.60910068E-05	4.11817472E-04
28	2.15110931E-05	-5.95871909E-05	4.09595707E-04
29	1.95117250E-05	-9.12115214E-05	4.13148777E-04
30	2.63739344E-05	-1.25126301E-04	4.18026025E-04
31	3.60375097E-05	-1.58533796E-04	4.21087483E-04
32	6.04357479E-05	-1.90657990E-04	4.06382365E-04
33	2.03004347E-04	-4.88765143E-05	4.48067778E-04
34	8.76829108E-05	-4.06124576E-06	4.50625185E-04
35	2.85524916E-05	7.83014457E-07	4.74811210E-04
36	5.88186504E-06	-1.37979818E-05	4.88545274E-04
37	1.59343199E-06	-3.62646466E-05	4.94522040E-04
38	6.00297335E-06	-6.08371190E-05	4.96488134E-04
39	1.51108825E-05	-8.51456811E-05	4.97556859E-04
40	2.03517920E-05	-1.10202310E-04	5.13362301E-04
41	2.10618857E-04	-3.67429729E-05	3.50943639E-04
42	6.11564096E-05	1.87140786E-05	4.61881937E-04
43	1.01249407E-05	2.77946422E-05	5.18558142E-04

44	-8.89156718E-06	2.12352094E-05	5.42516486E-04
45	-1.29293553E-05	9.04249401E-06	5.48559373E-04
46	-1.04093210E-05	-4.16541673E-06	5.46109186E-04
47	-6.36281493E-06	-1.62945471E-05	5.40810976E-04
48	-2.26093016E-06	-2.38837129E-05	5.30693046E-04
49	8.86902794E-05	-1.24047375E-05	2.46441442E-04
50	2.17832476E-05	2.81543066E-05	4.52765060E-04
51	-1.06532543E-05	4.25233992E-05	5.33960007E-04
52	-2.16744174E-05	4.58342508E-05	5.64255213E-04
53	-2.39368178E-05	4.58057233E-05	5.69074149E-04
54	-2.32568968E-05	4.57462904E-05	5.62672048E-04
55	-2.15214659E-05	4.66950298E-05	5.52990329E-04
56	-2.01961748E-05	4.63247922E-05	5.40786636E-04
57	-3.73897396E-05	1.82280794E-05	1.86794140E-04
58	-3.26896934E-05	3.50978046E-05	4.08814734E-04
59	-3.43344618E-05	5.14833925E-05	5.10968227E-04
60	-3.33600628E-05	6.44034413E-05	5.46849979E-04
61	-3.25740739E-05	7.75075754E-05	5.50387448E-04
62	-3.37287038E-05	9.21548597E-05	5.41288784E-04
63	-3.76924927E-05	1.08252460E-04	5.29967584E-04
64	-4.02793730E-05	1.27761535E-04	5.31345460E-04
65	-1.45130166E-04	4.86613710E-05	1.50765309E-04
66	-9.19786867E-05	5.01361968E-05	3.52268075E-04
67	-6.15615272E-05	6.33302691E-05	4.56207604E-04
68	-4.48784986E-05	8.28404611E-05	4.90414262E-04
69	-4.01104694E-05	1.08743093E-04	4.89944895E-04
70	-4.35488118E-05	1.38416329E-04	4.78473869E-04
71	-4.96869229E-05	1.68636409E-04	4.64664485E-04
72	-6.63261030E-05	1.96689605E-04	4.33428332E-04
73	-2.34972167E-04	7.75008153E-05	1.14277980E-04
74	-1.48684180E-04	7.28927686E-05	2.87387438E-04
75	-8.84346987E-05	8.13171042E-05	3.80582575E-04
76	-5.47411278E-05	1.05151267E-04	3.97889562E-04
77	-4.77245990E-05	1.44600042E-04	3.87815303E-04
78	-5.51272124E-05	1.89661821E-04	3.77656530E-04
79	-6.86802864E-05	2.34626116E-04	3.73231519E-04
80	-6.83134003E-05	2.78811160E-04	3.91396958E-04
81	-3.02488193E-04	1.02848182E-04	5.45172219E-05
82	-2.02091571E-04	1.02058422E-04	2.13157925E-04
83	-1.09273959E-04	1.04001074E-04	2.91592393E-04
84	-5.93152981E-05	1.35205550E-04	2.65146040E-04
85	-5.74738483E-05	1.91864926E-04	2.43535414E-04
86	-7.12284004E-05	2.49507862E-04	2.39724889E-04
87	-8.52041035E-05	3.03706468E-04	2.34732845E-04
88	-1.15696346E-04	3.55829757E-04	2.05180543E-04
89	-3.31942737E-04	9.90880285E-05	-3.17022581E-05
90	-2.46922381E-04	1.04164513E-04	1.21693889E-04
91	-1.17763048E-04	1.40198291E-04	1.67831386E-04
92	-6.20615892E-05	1.95899362E-04	7.70707928E-05
93	-7.63170992E-05	2.57125674E-04	7.49838518E-05
94	-9.35908402E-05	3.18522479E-04	8.40082110E-05
95	-1.11447854E-04	3.81752321E-04	9.85225469E-05
96	-1.20201135E-04	4.45081036E-04	1.27071508E-04

TABLE FOR STRESSES IN THE X-Y DIRECTIONS FOR EACH CELL

CELL NUMBER	STRESSX	STRESSY	STRESXY
1	7.71072913E+01	-1.85190433E+03	2.12408598E+03
2	3.30770096E+02	-4.44954947E+03	1.10840390E+03
3	8.53261091E+01	-6.20878031E+03	7.22704105E+02
4	-3.10947430E+00	-7.65498934E+03	6.58630958E+02
5	-3.54172698E+01	-9.07366476E+03	7.15835474E+02
6	-5.98943526E+01	-1.05810524E+04	8.18965071E+02
7	-9.44314539E+01	-1.22043163E+04	9.84475151E+02
8	-4.91030463E+02	-1.39706920E+04	1.31326939E+03
9	2.72946239E+03	-7.19023595E+02	4.41001529E+03
10	1.16918448E+03	-2.36703510E+03	2.94753071E+03
11	3.98347969E+02	-3.91568892E+03	2.33093429E+03
12	2.44242560E+01	-5.40238298E+03	2.14779128E+03
13	-1.24129469E+02	-6.84933567E+03	2.18579529E+03
14	-1.75770576E+02	-8.28288001E+03	2.29818243E+03
15	-2.37543189E+02	-9.71264762E+03	2.35306492E+03
16	2.24150088E+02	-1.09895208E+04	2.10366363E+03
17	4.48380576E+03	-2.31271335E+02	5.35628961E+03
18	2.10695647E+03	-1.04450866E+03	4.26459018E+03
19	7.73329549E+02	-2.16473607E+03	3.71949031E+03
20	9.16993659E+01	-3.42368774E+03	3.55184953E+03
21	-1.99433313E+02	-4.72178074E+03	3.57562657E+03
22	-2.87870124E+02	-6.00398481E+03	3.67769007E+03
23	-2.27760515E+02	-7.25426182E+03	3.81364083E+03
24	-5.97011999E+02	-8.66642788E+03	4.12560378E+03
25	5.42491192E+03	5.21093656E+01	5.56993342E+03
26	2.73773860E+03	-1.04851695E+02	4.96492737E+03
27	1.00958703E+03	-7.79854095E+02	4.75174006E+03
28	1.19833051E+02	-1.75166581E+03	4.72610431E+03
29	-2.58848288E+02	-2.81400013E+03	4.76710127E+03
30	-3.68042500E+02	-3.86420177E+03	4.82337722E+03
31	-3.79866894E+02	-4.86997396E+03	4.85870172E+03
32	1.06758824E+02	-5.68771204E+03	4.68902729E+03
33	6.20905689E+03	3.96421636E+02	5.17001283E+03
34	2.85047925E+03	7.33306399E+02	5.19952136E+03
35	9.49035032E+02	3.08200941E+02	5.47859089E+03
36	5.74440844E+01	-3.96706232E+02	5.63706085E+03
37	-3.06130613E+02	-1.17977858E+03	5.70602354E+03
38	-4.03785569E+02	-1.94624924E+03	5.72870924E+03
39	-3.43939180E+02	-2.65755219E+03	5.74104068E+03
40	-4.18974751E+02	-3.43176172E+03	5.92341117E+03
41	6.58008676E+03	8.71736839E+02	4.04934968E+03
42	2.20122967E+03	1.22179126E+03	5.32940697E+03
43	6.08681317E+02	1.01644366E+03	5.98336317E+03
44	-8.31100353E+01	6.12123272E+02	6.25980561E+03
45	-3.36811225E+02	1.70231455E+02	6.32953123E+03
46	-3.84360860E+02	-2.40270758E+02	6.30125984E+03
47	-3.70917988E+02	-6.00111812E+02	6.24012665E+03
48	-3.10748702E+02	-8.09736002E+02	6.12338131E+03
49	2.80117115E+03	4.68209217E+02	2.84355510E+03
50	9.96578226E+02	1.14360267E+03	5.22421223E+03
51	6.93549031E+01	1.29650845E+03	6.16107701E+03
52	-2.61235459E+02	1.29665689E+03	6.51063707E+03

53	-3.36102226E+02	1.27334103E+03	6.56624019E+03
54	-3.14275047E+02	1.27810620E+03	6.49236978E+03
55	-2.47679901E+02	1.32654693E+03	6.38065765E+03
56	-2.07650675E+02	1.32744856E+03	6.23984580E+03
57	-1.05235107E+03	2.31137060E+02	2.15531700E+03
58	-7.30561056E+02	8.33765824E+02	4.71709308E+03
59	-6.22728926E+02	1.35768310E+03	5.89578723E+03
60	-4.62825181E+02	1.79325569E+03	6.30980745E+03
61	-3.07312131E+02	2.23303362E+03	6.35062440E+03
62	-2.00513601E+02	2.70449171E+03	6.24563981E+03
63	-1.71980930E+02	3.19597951E+03	6.11501059E+03
64	-6.43157981E+01	3.81355132E+03	6.13090916E+03
65	-4.30324467E+03	1.68867727E+02	1.73959972E+03
66	-2.53641190E+03	7.43162335E+02	4.06463163E+03
67	-1.40315758E+03	1.47896080E+03	5.26393389E+03
68	-6.60209680E+02	2.28715093E+03	5.65862610E+03
69	-2.46842024E+02	3.18824020E+03	5.65321033E+03
70	-6.67224115E+01	4.13247315E+03	5.52085234E+03
71	2.98021864E+01	5.06803292E+03	5.36151329E+03
72	-2.41293018E+02	5.82830025E+03	5.00109614E+03
73	-6.97984359E+03	2.31071384E+02	1.31859208E+03
74	-4.18075876E+03	9.32555428E+02	3.31600890E+03
75	-2.11119453E+03	1.80615477E+03	4.39133740E+03
76	-7.64694983E+02	2.92512950E+03	4.59103341E+03
77	-1.43228126E+02	4.29503282E+03	4.47479196E+03
78	5.83956241E+01	5.70737332E+03	4.35757534E+03
79	5.62928022E+01	7.05567132E+03	4.30651752E+03
80	5.05382886E+02	8.51594966E+03	4.51611875E+03
81	-8.95495840E+03	3.98957938E+02	6.29044868E+02
82	-5.65299046E+03	1.36585554E+03	2.45951452E+03
83	-2.57385615E+03	2.34787538E+03	3.36452761E+03
84	-6.18251646E+02	3.87069099E+03	3.05937738E+03
85	2.82295263E+00	5.75679467E+03	2.81002401E+03
86	1.19471151E+02	7.52107721E+03	2.76605640E+03
87	1.94763858E+02	9.16962321E+03	2.70845591E+03
88	-2.94969859E+02	1.05864018E+04	2.36746780E+03
89	-9.96317568E+03	-1.63118478E+01	-3.65795286E+02
90	-7.11009980E+03	9.91905463E+02	1.40416025E+03
91	-2.49572179E+03	3.45723221E+03	1.93651599E+03
92	-1.08520235E+02	5.84442480E+03	8.89278378E+02
93	2.70528433E+01	7.72188607E+03	8.65198291E+02
94	6.48100057E+01	9.57511736E+03	9.69325512E+02
95	1.01467325E+02	1.14830098E+04	1.13679862E+03
96	4.39225574E+02	1.34841987E+04	1.46620971E+03

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