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A METHOD FOR DETERMINING AN OPTIMUM SHAPE OF A CLASS OF THIN SHELLS OF REVOLUTION

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
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DEPARTMENT OF THEORETICAL AND APPLIED MECHANICS
UNIVERSITY OF ILLINOIS
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Department of Theoretical and Applied Mechanics
University of Illinois
Urbana, Illinois

SUMMARY

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This third report under the current grant is concerned with a method for determining an optimum shape of a convex shell of revolution with respect to volume, weight and length.

The technique used depends on replacing the class of functions, over which the shape may range, by the parameters b/a, α and β in the equation

$$\left|\frac{x}{a}\right|^{\alpha} + \left|\frac{y}{b}\right|^{\beta} = 1$$

where a, b, α and β are positive constants not necessarily integers, with α and β equal to or greater than unity. The bodies of revolution are generated by revolving the line, described by the above equation, about the x-axis.

The procedure is illustrated for a thin shell which will fit within the space defined by a circular cylinder of radius b and length 2a. The shell is optimized, in terms of α and β, with respect to volume and weight. The FORTRAN program used to achieve these results is presented in Appendix B.

Author

INTRODUCTION

1. Statement of the Problem.

The previous reports under the current grant, [1,2] * stated a future objective of the project as being the optimum contour design of a class of shells. This third report is directed toward achieving that objective in terms of enclosed volume and shell weight for thin shells of revolution.

Optimization can be treated in several ways. A general formulation of the optimization of the design of thin shells of revolution might include the determination of the shell shape as well as the variation of the shell thickness along meridional lines. A less general approach involves assigning the shape and varying the shell thickness [3, 4]. The current report treats an alternate approach. Here a uniform thickness is maintained, but the meridional lines which define the geometry are permitted to vary in accordance with the relation

$$\left|\frac{x}{a}\right|^{\alpha} + \left|\frac{y}{b}\right|^{\beta} = 1 \qquad (3.1)^{**}$$

where a, b, α and β are positive constants, not necessarily integers.

The use of Eq. (3.1) permits an optimization of shape which is limited to the choice of the parameters α and β for a shell of length 2a and of radius b. The body of this report is limited to the variation of α and β for fixed length and fixed diameter, but Appendix A presents a mathematical formulation which permits the length to vary as well as α and β.

The achievement of the stated objective depends on a suitable failure criterion. One criterion could involve a complete stress analysis of the shell including varying thickness. Others could include thick walled shells or buckling. However, in illustrating the method, the shells have been restricted to thin, constant thickness walls with internal pressure loading. Further the failure is assumed to occur either on the central plane circle normal to the x-axis at x = 0 or along a meridian. Thus separate computer programs which involve the complete stress analysis of the shell have not been used.

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*Numbers in brackets refer to the References.

**The notation (3.1) is adopted to aid in cross-referencing equations from the first two reports under the grant [1, 2].

The techniques described can be applied in a manner which would permit the direct inclusion of one of the existing computer programs on the stress analysis of shells [5, 6, 7] . These auxilliary computer programs would provide the thickness requirement or the variation in thickness of the shell when incorporated into the proper location within the FORTRAN program presented in this report. In this way the optimized shell would be based on a more realistic failure criterion than is actually reported.

2. Symbols

a	half length of the shell, $[L]$*
b	radius of the shell in the equatorial plane, $[L]$
x	horizontal coordinate of the first quadrant of Eq. (3. 1), $[L]$
y	vertical coordinate of the first quadrant of Eq. (3. 1), $[L]$
g	acceleration due to gravity, $[LT^{-2}]$
V_{xa}	volume of the shell, $[L^3]$
W	weight of the shell, $[MLT^{-2}]$
A	surface area of the shell, $[L^2]$
A_a	area enclosed by first quadrant of Eq. (3. 1), $[L^2]$
L	arc length in the first quadrant of Eq. (3. 1), $[L]$
t	thickness of the shell, $[L]$
V_{min}	preassigned minimum allowable volume of the shell, $[L^3]$
W_{max}	preassigned maximum allowable weight of the shell, $[MLT^{-2}]$
a_{max}	preassigned maximum half length of the shell, $[L]$
V_{cyl}	volume of cylinder with radius b, length $2a$, $[L^3]$
W_{cyl}	weight of cylindrical shell with radius b, length $2a$, $[MLT^{-2}]$
v	ratio of V_{xa}/V_{min}, $[1]$
w	ratio of W/W_{max}, $[1]$
ℓ	ratio of a/a_{max}, $[1]$
h^2	$\left(\frac{b\alpha}{a\beta}\right)^2$, $[1]$

*The dimensional notation $[L]$ indicates a length while $[M]$ indicates mass, $[T]$ indicates time and $[1]$ indicates a dimensionless quantity.

p_o	uniform internal pressure on shells, $[ML^{-1}T^{-2}]$
k_o	preselected limiting value for the ratio $\Delta\alpha/F_\alpha$ or $\Delta\beta/F_\beta$ of iteration, $[1]$
α	exponent of the absolute value of x/a, $[1]$
β	exponent of the absolute value of y/b, $[1]$
α	(as a subscript) indicates partial differentiation with respect to α, $[1]$
β	(as a subscript) indicates partial differentiation with respect to β, $[1]$
ρ	mass density, $[ML^{-3}]$
λ	non-negative weighting function of v, $[1]$
μ	non-negative weighting function of w, $[1]$
ν	non-negative weighting function of ℓ, $[1]$
σ_o	yield stress of the shell material, $[ML^{-1}T^{-2}]$
η_o^2	preselected limiting value for the maximum change in $(\Delta v^2 + \Delta w^2)$ to be allowed in one iteration step, $[1]$
J_1 through J_7	integrals as defined in Eqs. (3.33)
$I(\epsilon)$, $K(\epsilon)$	improper integrals as defined in Eqs. (3.35) and (3.36)

3. Acknowledgment

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The investigation was part of the work of the Engineering Experiment Station of which Professor Ross J. Martin is Director and was conducted in the Department of Theoretical and Applied Mechanics of which Professor Thomas J. Dolan is Head, with Will J. Worley as Principal Investigator.

The authors wish to acknowledge the assistance of Charles Cecil Fretwell, formerly Instructor in Theoretical and Applied Mechanics, University of Illinois, in the early stages of the numerical programming and the assistance of undergraduate students: Messrs. Tom E. Breuer and Edward H. Stredde with various phases of the project.

The suggestion that the length variation be included as a parameter in the optimization procedure was made by Melvin G. Rosche, Space Vehicle Structures Program, NASA, Washington, D. C.

Both the ILLIAC II and the IBM 7094 computer facilities were used. The ILLIAC II was constructed in the Digital Computer Laboratory, now known as the Department of Computer Science, University of Illinois with support from the Atomic Energy Commission, grant USAEC AT(11-1)-415, and from the Office of Naval Research, grant NONOR-1832 (15). The IBM 7094 computer facility is partially supported by the National Science Foundation under grant NSF GP700.

DISCUSSION OF THE METHOD

Otpimization with respect to enclosed volume and shell weight, for a shell of revolution defined by the meridian curve Eq. (3.1) is achieved by considering the exponents α and β as parameters.

Then the volume and weight may be expressed as

$$V_{xa} = V_{xa}(\alpha, \beta) \tag{3.2}$$

$$W = \rho g A(\alpha, \beta) t(\alpha, \beta) \tag{3.3}$$

where ρ is the mass density of the material of the shell, g is the gravitational acceleration, while A represents the area of the middle surface of the shell and t is the thickness. The thickness is maintained constant over the entire shell and is small compared to the radius b and to the length a.

Both the volume V_{xa} and the surface area A depend only on the geometrical shape of the shell, which is controlled by the parameters α and β for the fixed cylindrical volume. The thickness t depends on the geometrical shape of the shell, on the load condition and on the failure criteria. Therefore the mode of the failure of the shell, under a specified load condition, must be defined for the evaluation of the thickness t, before optimization can be achieved.

Let the primary design requirements, to be fulfilled for the shell, be

$$V_{xa} \geq V_{min} \quad , \qquad W \leq W_{max}$$

where V_{min} and W_{max} are preassigned limits. It is further assumed that at least one set of values (α, β) will satisfy the primary requirements. Otherwise the material of the shell, the assumed mode of failure, the load conditions, or the dimensions a and b have to be modified in order to determine an optimum shape.

To facilitate the calculations, the ratios of the volume and weight are introduced as

$$v = \frac{V_{xa}}{V_{min}} \quad , \qquad w = \frac{W}{W_{max}} \tag{3.4}$$

The differential of a function is then defined as

$$dF = \lambda \, dv - \mu \, dw \tag{3.5}$$

where λ and μ are non-negative weighting functions of v and w, which define the relative importance of increasing in volume and of decreasing in weight. These weighting functions are defined in terms of current volume and weight. As long as it is possible to select dv and dw, consistent with the constraints of the problem, such that dF is positive, one has not achieved the optimum shape. Thus one seeks the values of α and β for which the differential dF is either zero or negative.

While superior shapes may exist, the above criteria will assure an optimum shape within the limitations of Eq. (3.1) and with the imposed constraints on volume and weight.

To determine the values of α and β for which F yields the extreme value, one may write Eq. (3.5) in the form

$$dF = F_\alpha \, d\alpha + F_\beta \, d\beta \tag{3.6}$$

with

$$F_\alpha = \lambda \, v_\alpha - \mu \, w_\alpha \tag{3.7}$$

$$F_\beta = \lambda \, v_\beta - \mu \, w_\beta \tag{3.8}$$

where the subscripts α and β indicate the partial differentiation with respect to α and β.

If Eq. (3.6) is an exact differential, then in principle one need only look among the solutions of $F_\alpha = F_\beta = 0$ for the optimum shape. Because of the complex nature of the equations for v_α, v_β, w_α and w_β, it is difficult to determine whether Eq. (3.6) is exact. Even if Eq. (3.6) were exact, the analytical solution of $F_\alpha = F_\beta = 0$ would be extremely difficult to obtain. The following iterative procedure is therefore used in the evaluation of $F_\alpha = F_\beta = 0$.

A shape defined by a set of α and β consistent with the primary requirements is selected first. This defines the shape of the shell middle surface. Therefore, the volume and the surface area of the shell can be calculated and the required thickness computed consistent with the assumed mode failure of the shell. Once the volume and weight are computed, values of λ and μ, which were defined by the design criterion, are established. Hence the values of F_α and F_β are determined by Eqs. (3.7) and (3.8). The shape is then modified by incrementing α and β in accordance with the path of the steepest ascent

$$d\alpha : d\beta = F_\alpha : F_\beta \tag{3.9}$$

The iterative procedure is repeated until F_α and F_β are both essentially zero.

To determine the incremental size $\Delta\alpha$ and $\Delta\beta$ for the steps in the iteration, let a constant k be defined from Eq. (3.9) as

$$\frac{\Delta\alpha}{F_\alpha} = \frac{\Delta\beta}{F_\beta} = k \tag{3.10}$$

Therefore

$$dv = v_\alpha\, d\alpha + v_\beta\, d\beta = k(v_\alpha F_\alpha + v_\beta F_\beta) \tag{3.11}$$

$$dw = w_\alpha\, d\alpha + w_\beta\, d\beta = k(w_\alpha F_\alpha + w_\beta F_\beta) \tag{3.12}$$

In order to limit the size of the increments of Δv and Δw, and of $\Delta\alpha$ and $\Delta\beta$, the constant k is selected in the following way

$$k = \begin{cases} k_1 \text{ if } k_1 < k_o \\ k_o \text{ if } k_1 > k_o \end{cases} \tag{3.13}$$

The constant k_1 is determined from Eqs. (3.11) and (3.12) consistent with the assigned increments of Δv and Δw, and is evaluated as follows

$$\eta_o^2 = \Delta v^2 + \Delta w^2 = k_1^2 \left[(v_\alpha F_\alpha + v_\beta F_\beta)^2 + (w_\alpha F_\alpha + w_\beta F_\beta)^2 \right]$$

from which

$$k_1 = \eta_o \Big/ \left[(v_\alpha F_\alpha + v_\beta F_\beta)^2 + (w_\alpha F_\alpha + w_\beta F_\beta)^2 \right]^{1/2} \tag{3.14}$$

where η_o^2 is a preselected limiting value for the maximum change of $(\Delta v^2 + \Delta w^2)$ to be allowed in one iteration step, while k_o is a preselected limiting value for the ratio $\Delta\alpha/F_\alpha$ or $\Delta\beta/F_\beta$ for each step of iteration. The process is then repeated with a new set of values of α and β formed by adding the increments $\Delta\alpha$ and $\Delta\beta$ to the previous values. The iteration process terminates when the value $(F_\alpha^2 + F_\beta^2)$ is less than a preassigned accuracy parameter.

The mathematical formulation of the more general problem which permits the length to vary as well as α and β is presented in Appendix A.

MATHEMATICAL FORMULATION

In the process of iteration, as described in the previous sections, the values of v, v_α, v_β, w, w_α and w_β for a given set of values of α and β must be calculated. From Eqs. (3.3) and (3.4), w_α and w_β may be written as

$$w_\alpha = \frac{\rho g}{W_{max}} (A \; t_\alpha + A_\alpha \; t) \tag{3.15}$$

$$w_\beta = \frac{\rho g}{W_{max}} (A \; t_\beta + A_\beta \; t) \tag{3.16}$$

The symbols used in the iteration procedure, described earlier in the report, are defined by the following integrals. The notation in these integrals is consistent with that used in the previous reports under the current research grant [1, 2] .

$$v = \frac{2 \pi b^2}{V_{min}} \int_0^a \left[1 - \left(\frac{x}{a}\right)^\alpha \right]^{2/\beta} dx$$

$$= \frac{2 \pi ab^2}{V_{min}} \int_0^1 (1 - X^\alpha)^{2/\beta} \, dX \tag{3.17}$$

$$v_\alpha = \frac{-2 \pi ab^2}{V_{min}} \left(\frac{2}{\beta}\right) \int_0^1 (1 - X^\alpha)^{(2-\beta)/\beta} X^\alpha \log X \, dX \tag{3.18}$$

$$v_\beta = \frac{-2 \pi ab^2}{V_{min}} \left(\frac{2}{\beta^2}\right) \int_0^1 (1 - X^\alpha)^{2/\beta} \log (1 - X^\alpha) \, dX \tag{3.19}$$

$$A = 4 \pi b \int_0^a \left[1 - \left(\frac{x}{a}\right)^\alpha\right]^{1/\beta} \left\{1 + \left(\frac{b\alpha}{a\beta}\right)^2 \left(\frac{x}{a}\right)^{2(\alpha-1)} \left[1 - \left(\frac{x}{a}\right)^\alpha\right]^{2(1-\beta)/\beta}\right\}^{1/2} dx$$

$$= 4 \pi ab \int_0^1 (1 - X^\alpha)^{1/\beta} \left[1 + \left(\frac{b\alpha}{a\beta}\right)^2 X^{2(\alpha - 1)} (1 - X^\alpha)^{2(1-\beta)/\beta}\right]^{1/2} dX \tag{3.20}$$

Let $$F(X, \alpha, \beta) = 1 + \left(\frac{b\alpha}{a\beta}\right)^2 X^{2(\alpha - 1)} (1 - X^{\alpha})^{2(1-\beta)/\beta}$$

and $$h^2 = \left(\frac{b\alpha}{a\beta}\right)^2$$

then $$A_{\alpha} = 4 \pi ab \left\{ \left(\frac{h^2}{\alpha}\right) \int_0^1 F(X, \alpha, \beta)^{-1/2} X^{2(\alpha-1)} (1 - X^{\alpha})^{(3-2\beta)/\beta} dX \right.$$

$$- \frac{1}{\beta} \int_0^1 F(X, \alpha, \beta)^{-1/2} X^{\alpha} (1 - X^{\alpha})^{(1-\beta)/\beta} \log X \, dX$$

$$+ h^2 \int_0^1 F(X, \alpha, \beta)^{-1/2} X^{2(\alpha-1)} (1 - X^{\alpha})^{(3-2\beta)/\beta} \log X \, dX$$

$$\left. - h^2 \left(\frac{2 - \beta}{\beta}\right) \int_0^1 F(X, \alpha, \beta)^{-1/2} X^{(3\alpha-2)} (1 - X^{\alpha})^{3(1-\beta)/\beta} \log X \, dX \right\} \quad (3.21)$$

and

$$A_{\beta} = -4 \pi ab \left\{ \left(\frac{h^2}{\beta}\right) \int_0^1 F(X, \alpha, \beta)^{-1/2} X^{2(\alpha-1)} (1 - X^{\alpha})^{(3-2\beta)/\beta} dX \right.$$

$$+ \frac{1}{\beta^2} \int_0^1 F(X, \alpha, \beta)^{-1/2} (1 - X^{\alpha})^{1/\beta} \log (1 - X^{\alpha}) \, dX$$

$$\left. + 2\left(\frac{h^2}{\beta^2}\right) \int_0^1 F(X, \alpha, \beta)^{-1/2} X^{2(\alpha-1)} (1 - X^{\alpha})^{(3-2\beta)/\beta} \log (1 - X^{\alpha}) \, dX \right\} \quad (3.22)$$

The next step consists of the determination of the thickness t and the values of t_{α} and t_{β}. These values should ideally be determined from a limit analysis, but since this would constitute a major undertaking in itself [6, 7] , the following simple failure criterion is adopted. It is assumed that under a uniform internal pressure, p_o, the shell will fail by general yielding either along a longitudinal plane or around the equatorial plane. If σ_o is the yield stress for the shell material, failure along a longitudinal plane requires a thickness given by

$$t_1 = \left(\frac{p_o}{\sigma_o}\right) \frac{A_a}{L} \tag{3.23}$$

while failure around the equatorial plane requires a thickness given by

$$t_2 = \frac{1}{2}\left(\frac{p_o}{\sigma_o}\right) b \tag{3.24}$$

where A_a is the area enclosed by the first quadrant of Eq. (3. 1) and L is the complete arc length in the first quadrant of Eq. (3. 1). The design thickness t should be either t_1 or t_2, whichever is larger. If $t_2 \geq t_1$ then t equals t_2, a constant; therefore $t_\alpha = t_\beta = 0$. For $t_2 < t_1$, then by Eq. (3.23).

$$t_\alpha = \frac{p_o}{\sigma_o}\left[\frac{(A_a)_\alpha}{L} - \frac{A_a L_\alpha}{L^2}\right] \tag{3.25}$$

$$t_\beta = \frac{p_o}{\sigma_o}\left[\frac{(A_a)_\beta}{L} - \frac{A_a L_\beta}{L^2}\right] \tag{3.26}$$

where

$$A_a = b\int_o^a \left[1 - \left(\frac{x}{a}\right)^\alpha\right]^{1/\beta} dx = ab\int_o^1 (1 - X^\alpha)^{1/\beta}\, d\, X \tag{3.27}$$

$$(A_a)_\alpha = \frac{\partial A_a}{\partial \alpha} = -\left(\frac{ab}{\beta}\right)\int_o^1 (1 - X^\alpha)^{(1-\beta)/\beta} X^\alpha \log X\, dX \tag{3.28}$$

$$(A_a)_\beta = \frac{\partial A_a}{\partial \beta} = -\left(\frac{ab}{\beta^2}\right)\int_o^1 (1 - X^\alpha)^{1/\beta} \log (1 - X^\alpha)\, dX \tag{3.29}$$

$$L = \int_0^a \left\{ 1 + \left(\frac{b\alpha}{a\beta}\right)^2 \left(\frac{x}{a}\right)^{2(\alpha-1)} \left[1 - \left(\frac{x}{a}\right)^{\alpha}\right]^{2(1-\beta)/\beta} \right\}^{1/2} dX$$

$$= a \int_0^1 F(X, \alpha, \beta)^{1/2} \, dX \tag{3.30}$$

$$L_\alpha = a \left\{ \frac{h^2}{\alpha} \int_0^1 F(X, \alpha, \beta)^{-1/2} X^{2(\alpha-1)} (1 - X^\alpha)^{2(1-\beta)/\beta} \, dX \right.$$

$$+ h^2 \int_0^1 F(X, \alpha, \beta)^{-1/2} X^{2(\alpha-1)} (1 - X^\alpha)^{(2-3\beta)/\beta} \log X \, dX$$

$$\left. - \frac{h^2}{\beta} \int_0^1 F(X, \alpha, \beta)^{-1/2} X^{3\alpha-2} (1 - X^\alpha)^{(2-3\beta)\beta} \log X \, dX \right\} \tag{3.31}$$

$$L_\beta = a \left\{ - \frac{h^2}{\beta} \int_0^1 F(X, \alpha, \beta)^{-1/2} X^{2(\alpha-1)} (1 - X^\alpha)^{2(1-\beta)/\beta} \, dX \right.$$

$$\left. - \frac{h^2}{\beta^2} \int_0^1 F(X, \alpha, \beta)^{-1/2} X^{2(\alpha-1)} (1 - X^\alpha)^{2(1-\beta)/\beta} \log (1 - X^\alpha) \, dX \right\} \tag{3.32}$$

All of the integrals which appear in the above equations may be collected into seven groups, by defining the following convergent but sometimes improper integrals in notations as

$$J_1 (p, q) = \int_0^1 (1 - u^p)^q \, du$$

$$J_2 (p, q) = \int_0^1 (1 - u^p)^q \log (1 - u^p) \, du$$

$$J_3(p, q) = \int_0^1 (1 - u^p)^{q-1} u^p \log u \, du$$

$$J_4(p, q, s) = \int_0^1 (1 - u^p)^s \left[1 + h^2 u^{2(p-1)} (1 - u^p)^{2(q-1)}\right]^{1/2} du$$

$$J_5(p, q, s) = \int_0^1 \left[1 + h^2 u^{2(p-1)} (1 - u^p)^{2(q-1)}\right]^{-1/2} u^{2(p-1)} (1 - u^p)^{s-2} du \tag{3.33}$$

$$J_6(p, q, r, s) = \int_0^1 \left[1 + h^2 u^{2(p-1)} (1-u^p)^{2(q-1)}\right]^{-1/2} u^{2(r-1)} (1 - u^p)^{s-2} \log(1-u^p) \, du$$

$$J_7(p, q, s, m) = \int_0^1 \left[1 + h^2 u^{2(p-1)} (1 - u^p)^{2(q-1)}\right]^{-1/2} u^m (1 - u^p)^{s-3} \log u \, du$$

with $h^2 = \left(\frac{b\alpha}{a\beta}\right)^2$ and $p, r \geq 1$, $q, s, m \geq 0$.

Then the equations used in the calculation of v, w and their derivatives can be expressed by Eqs. (3.33) as

$$v = \frac{2 \pi ab^2}{V_{min}} J_1(\alpha, \frac{2}{\beta})$$

$$v_\alpha = - \frac{2 \pi ab^2}{V_{min}} (\frac{2}{\beta}) J_3(\alpha, \frac{2}{\beta})$$

$$v_\beta = - \frac{2 \pi ab^2}{V_{min}} (\frac{2}{\beta^2}) J_2(\alpha, \frac{2}{\beta})$$

$$
\left.
\begin{aligned}
A &= 4\pi ab \; J_4(\alpha, \tfrac{1}{\beta}, \tfrac{1}{\beta}) \\
A_\alpha &= 4\pi ab \left[\frac{h^2}{\alpha} \; J_5(\alpha, \tfrac{1}{\beta}, \tfrac{3}{\beta}) - \frac{1}{\beta} \; J_7(\alpha, \tfrac{1}{\beta}, \tfrac{1}{\beta} + 2, \alpha) \right. \\
&\quad \left. + h^2 \; J_7(\alpha, \tfrac{1}{\beta}, \tfrac{3}{\beta} + 1, \; 2\alpha - 2) - h^2 \left(\tfrac{2}{\beta} - 1\right) \; J_7(\alpha, \tfrac{1}{\beta}, \tfrac{3}{\beta}, \; 3\alpha - 2) \right] \\
A_\beta &= -4\pi ab \left[\frac{h^2}{\beta} \; J_5(\alpha, \tfrac{1}{\beta}, \tfrac{3}{\beta}) + \frac{1}{\beta^2} \; J_6(\alpha, \tfrac{1}{\beta}, \; 1, \; \tfrac{1}{\beta} + 2) \right. \\
&\quad \left. + 2\left(\frac{h}{\beta}\right)^2 \; J_6(\alpha, \tfrac{1}{\beta}, \alpha, \tfrac{3}{\beta}) \right] \\
A_a &= ab \; J_1(\alpha, \tfrac{1}{\beta}) \\
(A_a)_\alpha &= ab(-\tfrac{1}{\beta}) \; J_3(\alpha, \tfrac{1}{\beta}) \\
(A_a)_\beta &= ab\left(-\frac{1}{\beta^2}\right) \; J_2(\alpha, \tfrac{1}{\beta}) \\
L &= a \; J_4(\alpha, \tfrac{1}{\beta}, \; 0) \\
L_\alpha &= a\left[\frac{h^2}{\alpha} \; J_5(\alpha, \tfrac{1}{\beta}, \tfrac{2}{\beta}) + h^2 \; J_7(\alpha, \tfrac{1}{\beta}, \tfrac{2}{\beta}, \; 2\alpha - 2) \right. \\
&\quad \left. - \frac{h^2}{\beta} \; J_7(\alpha, \tfrac{1}{\beta}, \tfrac{2}{\beta}, \; 3\alpha - 2) \right] \\
L_\beta &= -a\left[\frac{h^2}{\beta} \; J_5(\alpha, \tfrac{1}{\beta}, \tfrac{2}{\beta}) + \frac{h^2}{\beta^2} \; J_6(\alpha, \tfrac{1}{\beta}, \alpha, \tfrac{2}{\beta}) \right]
\end{aligned}
\right\} \qquad (3.34)
$$

NUMERICAL INTEGRATION

The analytical expressions for the integrals in Eqs.(3. 33) are not available except for J_1, J_2 and J_3 which may be expressed in terms of gamma functions and the derivatives of gamma functions, the psi functions. Since the above functions also involve series expansions or numerical integration , all of the integrals in Eqs.(3. 33) are evaluated numerically using Simpson's Rule. In this process, special consideration is given to improper integrals of the following two types.

$$I(\epsilon) = \int_0^{\epsilon} f(\xi)\, \xi^{\delta}\, d\xi \qquad \delta > -1 \qquad (3.35)$$

$$K(\epsilon) = \int_0^{\epsilon} f(\xi)\, \xi^{\delta} \log \xi \, d\xi \qquad \delta > -1 \qquad (3.36)$$

with $f(\xi)$ continuous in the interval $0 \leq \xi \leq \epsilon$. For ϵ small enough, replace $f(\xi)$ with a parabola

$$f(\xi) = b_o + b_1 \left(\frac{\xi}{\epsilon}\right) + b_2 \left(\frac{\xi}{\epsilon}\right)^2 \qquad (3.37)$$

where

$$b_o = f(0)$$
$$b_1 = 4\, f(\tfrac{\epsilon}{2}) - f(\epsilon) - 3\, f(0)$$
$$b_2 = 2\, f(\epsilon) - 4\, f(\tfrac{\epsilon}{2}) + 2\, f(0)$$

Then the improper integrals $I(\epsilon)$ and $K(\epsilon)$ can be approximated as

$$I(\epsilon) = \int_0^{\epsilon} \left[b_o + b_1 \left(\frac{\xi}{\epsilon}\right) + b_2 \left(\frac{\xi}{\epsilon}\right)^2 \right] \xi^{\delta}\, d\xi$$

$$= \frac{\epsilon^{\delta+1}}{\delta + 1} \left[b_o + \frac{\delta + 1}{\delta + 2}\, b_1 + \frac{\delta + 1}{\delta + 3}\, b_2 \right] \qquad (3.38)$$

$$K(\epsilon) = \int_0^{\epsilon} \left[b_o + b_1 \left(\frac{\xi}{\epsilon}\right) + b_2 \left(\frac{\xi}{\epsilon}\right)^2 \right] \xi^{\delta} \log \xi \, d\xi$$

$$= \frac{\epsilon^{\delta+1}}{\delta+1} \left\{ \log \epsilon \left[b_o + \frac{\delta+1}{\delta+2} b_1 + \frac{\delta+1}{\delta+3} b_2 \right] - \frac{1}{\delta+1} \left[b_o + \left(\frac{\delta+1}{\delta+2}\right)^2 b_1 + \left(\frac{\delta+1}{\delta+3}\right)^2 b_2 \right] \right\} \qquad (3.39)$$

Therefore the improper integrals J_2 to J_7 listed in Eqs.(3.33) can be expressed in terms of $I(\epsilon)$, $K(\epsilon)$ and a proper integral. They are derived as follows.

$$J_2(p, q) = \int_0^{1-\eta} (1 - u^p)^q \log (1 - u^p) \, du + \int_{1-\eta}^{1} (1 - u^p)^q \log (1 - u^p) \, du$$

Let $\xi = 1 - u^p$

then $du = -\frac{1}{p} (1 - \xi)^{(1-p)/p} \, d\xi$

and $J_2(p, q) = \int_0^{1-\eta} (1-u^p)^q \log (1 - u^p) \, du + \frac{1}{p} \int_0^{1-(1-\eta)^p} (1 - \xi)^{(1-p)/p} \xi^q \log \xi \, d\xi$

Using Eq. (3.39) along with the definitions of Eq. (3.37), J_2 is approximated as

$$J_2(p, q) = \int_0^{1-\eta} (1 - u^p)^q \log (1 - u^p) \, du + \frac{1}{p} K(\epsilon) \Bigg|^{*}_{\substack{\epsilon = 1-(1-\eta)^p \\ f(\xi) = (1-\xi)^{(1-p)/p} \\ \delta = q}} \qquad (3.40)$$

In a similar manner, by the approximations of $I(\epsilon)$ and $K(\epsilon)$, other integrals yield;

$$J_3(p, q) = \int_{\epsilon}^{1-\eta} (1 - u^p)^{q-1} u^p \log u \, du + \frac{1}{p^2} I(\epsilon) \Bigg|_{\substack{\epsilon = 1-(1-\eta)^p \\ f(\xi) = (1-\xi)^{1/p} \log (1-\xi) \\ \delta = q-1}} + K(\epsilon) \Bigg|_{\substack{\epsilon = \epsilon \\ f(\xi) = (1-\xi^p)^{q-1} \\ \delta = p}}$$

*This notation indicates that the integral is evaluated at the indicated values of ϵ, $f(\xi)$ and δ.

$$J_4(p, q, s) = \int_0^{1-\eta} (1 - u^p)^s \left[1 + h^2 u^{2(p-1)} (1 - u^p)^{2(q-1)}\right]^{1/2} du$$

$$+ \frac{h}{p} I(\epsilon) \Bigg| \begin{array}{l} \epsilon = 1-(1-\eta)^p \\ f(\xi) = \left[\frac{1}{h^2} (1 - \xi)^{2(1-p)/p} \xi^{2(1-q)} + 1\right]^{1/2} \\ \delta = q + s - 1 \end{array}$$

$$J_5(p, q, s) = \int_0^{1-\eta} \left[1 + h^2 u^{2(p-1)} (1 - u^p)^{2(q-1)}\right]^{-1/2} u^{2(p-1)} (1 - u^p)^{s-2} du$$

$$+ \frac{1}{p} I(\epsilon) \Bigg| \begin{array}{l} \epsilon = 1-(1-\eta)^p \\ f(\xi) = \left[\xi^{2(1-q)} + h^2 (1-\xi)^{2(p-1)/p}\right]^{-1/2} (1 - \xi)^{(p-1)/p} \\ \delta = s - q - 1 \end{array} \qquad (3.41)$$

$$J_6(p, q, r, s) = \int_0^{1-\eta} \left[1 + h^2 u^{2(p-1)} (1-u^p)^{2(q-1)}\right]^{-1/2} u^{2(r-1)} (1-u^p)^{s-2}$$

$$\log(1-u^p)\, du$$

$$+ \frac{1}{p} K(\epsilon) \Bigg| \begin{array}{l} \epsilon = 1-(1-\eta)^p \\ f(\xi) = \left[\xi^{2(1-q)} + h^2 (1-\xi)^{2(p-1)/p}\right]^{-1/2} (1-\xi)^{(2r-p-1)/p} \\ \delta = s - q - 1 \end{array}$$

$$J_7(p, q, s, m) = \int_{\epsilon}^{1-\eta} \left[1 + h^2 u^{2(p-1)} (1-u^p)^{2(q-1)}\right]^{-1/2} (1-u^p)^{s-3} u^m \log u \, du$$

$$+ K(\epsilon) \left| \begin{array}{l} \epsilon = \epsilon \\ f(\xi) = \left[(1-\xi^p)^{2(1-q)} + h^2 \xi^{2(p-1)}\right]^{-1/2} (1-\xi^p)^{s-q-2} \\ \delta = m \end{array} \right.$$

$$+ \frac{1}{p^2} I(\epsilon) \left| \begin{array}{l} \epsilon = 1 - (1-\eta)^p \\ f(\xi) = \left[\xi^{2(1-q)} + h^2 (1-\xi)^{2(p-1)/p}\right]^{-1/2} (1-\xi)^{(m+1-p)/p} \\ \qquad \frac{1}{\xi} \log(1-\xi) \\ \delta = s - q - 1 \end{array} \right.$$

The integrals J_1 to J_7 therefore involve only proper integrals; thus numerical integration by Simpson's rule can be applied. The FORTRAN programs for the evaluation of J_1 through J_7 by means of a digital computer are written in subfunction form as listed in Appendix B.

NUMERICAL COMPUTATIONS

Several characteristics of the design problem have to be defined before the numerical iterations can be performed. One is the magnitude of the required values for V_{min} and W_{max}, the others include the weighting functions λ and μ.

1. The Ranges of the Volumes and Weights of Shells

Among the shells of revolution which may be generated by revolving the meridian curve, Eq. (3.1), about the x-axis, the range of shapes of interest lie between the cylindrical shell for which the exponents α and β are both large, and the conical shell for which the exponents α and β both equal unity. The volume of the cylinder with radius b and length 2a is $V_{cyl} = 2\pi ab^2$, and the volume of the double cone, having apexes at -a and at +a, with corresponding base diameter, 2b, is $V_{cone} = \frac{2}{3}\pi ab^2$. If the required minimum volume V_{min} is written as $V_{min} = C_1(2\pi ab^2)$, then C_1 must lie between 1/3 and 1.

In order to determine the weight for the two limiting cases, the thickness variation must be considered as well as the surface area. The surface areas for the cylindrical and conical shells are

$$A_{cyl} = 4\pi ab\left[1 + \frac{1}{2}\left(\frac{b}{a}\right)\right]$$

$$A_{cone} = 4\pi ab\left[\frac{1}{2}\sqrt{1 + \left(\frac{b}{a}\right)^2}\right]$$

Using Eqs. (3.23) and (3.24), the thicknesses required for the shell of cylindrical type, based on two different failure criteria, are expressed as

$$t_1 = \frac{1}{1 + \left(\frac{b}{a}\right)}\left(\frac{p_o}{\sigma_o}\right)b$$

$$t_2 = \frac{1}{2}\left(\frac{p_o}{\sigma_o}\right)b$$

therefore

$$t = \begin{cases} \frac{1}{2}\left(\frac{p_o}{\sigma_o}\right)b & \text{for } \left(\frac{b}{a}\right) \geq 1 \\ \frac{1}{1 + \left(\frac{b}{a}\right)}\left(\frac{p_o}{\sigma_o}\right)b & \text{for } \left(\frac{b}{a}\right) < 1 \end{cases}$$

Similarly, the thicknesses required for the conical type shell are

$$t_1 = \frac{1}{2} \frac{1}{\sqrt{1+(\frac{b}{a})^2}} \left(\frac{p_o}{\sigma_o}\right) b$$

$$t_2 = \frac{1}{2} \left(\frac{p_o}{\sigma_o}\right) b$$

Since $\frac{1}{\sqrt{1+(\frac{b}{a})^2}} \leq 1$ for any value of $(\frac{b}{a})$, it follows that

$$t = \frac{1}{2} \left(\frac{p_o}{\sigma_o}\right) b \qquad \text{for any ratio } (\frac{b}{a}) .$$

Then the weights of the cylindrical and the conical shells become

$$W_{cyl} = \begin{cases} \frac{1}{2} \left[1 + \frac{1}{2}(\frac{b}{a})\right] \left[4\pi ab^2 \left(\frac{p_o}{\sigma_o}\right) \rho g\right] & \text{for } \frac{b}{a} \geq 1 \\ \frac{1}{1+(\frac{b}{a})} \left[1 + \frac{1}{2}(\frac{b}{a})\right] \left[4\pi ab^2 \left(\frac{p_o}{\sigma_o}\right) \rho g\right] & \text{for } \frac{b}{a} < 1 \end{cases} \qquad (3.42)$$

$$W_{cone} = \frac{1}{4} \sqrt{1+(\frac{b}{a})^2} \left[4\pi ab^2 \left(\frac{p_o}{\sigma_o}\right) \rho g\right]$$

By writing the primary required limiting weight as

$$W_{max} = C_2 \left[4\pi ab^2 \left(\frac{p_o}{\sigma_o}\right) \rho g\right]$$

then C_2 has to be in the range

$$\frac{1}{4} \sqrt{1+(\frac{b}{a})^2} \leq C_2 \leq \frac{1}{2} \left[1 + \frac{1}{2}(\frac{b}{a})\right] \quad \text{for } (\frac{b}{a}) \geq 1$$

$$\frac{1}{4} \sqrt{1+(\frac{b}{a})^2} \leq C_2 \leq \frac{1}{1+(\frac{b}{a})} \left[1 + \frac{1}{2}(\frac{b}{a})\right] \quad \text{for } (\frac{b}{a}) < 1$$

2. Weighting Functions λ and μ

The functions λ and μ which define the relative importance of the variation of the volume and the weight of the shells, are preassigned according to the design criterion. Any functions in terms of V and W can be assigned in the problem. One such set of functions is defined as

$$\lambda_1 = \frac{(V_{cyl} - V)^p}{(V - V_{min})^q}, \qquad \mu_1 = \frac{W^m}{(W_{max} - W)^n} \tag{3.43}$$

The shapes of the functions in Eq. (3.43) appear in Fig. 1, for $p = q = m = n = 1$. From the characteristics of the functions λ and μ, one can predict that when the volume is close to V_{min} or when the weight is close to W_{max}, a small increment of V or W will produce a large change of dF as defined in Eq. (3.6). If dF is considered as the slope of a surface F, then the surface has a positive slope along the edge where V is close to V_{min} and has a negative slope along the edge where W is close to W_{max}. Thus it follows that there must exist a maximum value of F, that is $dF = 0$, in the assigned range $V > V_{min}$ and $W < W_{max}$.

The functions λ and μ may also be defined as

$$\lambda_2 = \frac{\lambda_1}{\sqrt{\lambda_1^2 + \mu_1^2}} \qquad \mu_2 = \frac{\mu_1}{\sqrt{\lambda_1^2 + \mu_1^2}} \tag{3.44}$$

where λ_1 and μ_1 are obtained from Eq. (3.43). If one divides λ_1 and μ_1 by $\sqrt{\lambda_1^2 + \mu_1^2}$, the magnitude of λ_2 and μ_2 will be limited to the range of 0 to 1. The ratios λ_2/μ_2 and λ_1/μ_1, remain the same. The characteristics of the functions are as shown in Fig. 2; the values of λ_2 and μ_2 outside the range of V_{min} and W_{max} are arbitrary set equal to 0 and to 1 respectively.

Another form of λ and μ consists of straight lines, which define a linear variation of V and W as

$$\lambda_3 = \frac{V_{cyl} - V}{V_{cyl} - V_{min}} \qquad \mu_3 = \frac{W}{W_{max}} \tag{3.45}$$

The above three definitions of λ and μ are applied in the numerical examples of this report. Subroutine programs for the calculation of λ and μ are attached to the main iteration program as listed in Appendix B.

3. Numerical Examples

The functions of λ and μ for the first example are chosen as in Fig. 1, that is

$$\lambda = \frac{V_{cyl} - V}{V - V_{min}}, \qquad \mu = \frac{W}{W_{max} - W}$$

Let the required minimum volume of the shell be 0.6 of the volume of cylindrical shell, thus $V_{min} = 0.6\ (2\pi ab^2)$, and let the required maximum weight be 0.8 that of the cylindrical shell under the same load condition. Since the thickness required for the cylindrical shell is dependent on the ratio of b/a, the weights for the cylindrical shells with different ratios b/a are given by Eq. (3.42) as

$$W_{cyl} = \begin{Bmatrix} 0.75 \\ 1.00 \\ 0.90 \end{Bmatrix} (4\pi\rho g p_o a b^2/\sigma_o) \quad \begin{matrix} \text{for } b/a = 1.0 \\ \text{for } b/a = 2.0 \\ \text{for } b/a = 0.25 \end{matrix}$$

therefore the values of W_{max} are chosen as $0.6(4\pi\rho g p_o ab^2/\sigma_o)$, $0.8(4\pi\rho g p_o ab^2/\sigma_o)$ and $0.72\ (4\pi\rho g p_o ab^2/\sigma_o)$ for the ratios b/a = 1.0, 2.0, and 0.25 respectively.

With all the requirements set, the iterative calculations are performed with the aid of digital computers. The FORTRAN programs for the iteration procedure are listed in Appendix B.

Choosing the starting values $\alpha = 1.5$, $\beta = 1.5$ with the fineness ratio a/b = 1.0, and the limiting value $\eta_o = 0.1$, $k_o = 0.3$, results in the output listed in Table I. From steps 1 to 7 in the Table, the results listed are presented for each iteration; from step 8 on, the results are presented for every other iteration. From these results it is seen that the values of α and β increase rapidly in each of the first six iterations and then change slowly. The same pattern is apparent for the slopes F_α and F_β.

Table II shows the iteration results with the same parameters as in Table I, but with the starting condition $\alpha = 2.0$, $\beta = 3.0$. The results in steps 1 to 8 are listed for each iteration while after step 8, they are listed for every fourth iteration. The iterated values of α and β decrease rapidly in the first three steps and then change slowly.

The results in Tables I and II, indicate that the shape defined by $\alpha = 1.5$, $\beta = 1.5$ lies on one side of the ridge, Fig. 3, while the shape defined by $\alpha = 2.0$, $\beta = 3.0$ lies on the other side of the ridge. During the iteration process, the successively improving values of α and β climb to the ridge rapidly according to the path of steepest ascent, and then progress slowly along the ridge due to the small variation of slope along the ridge.

The phenomena observed in the above results may be verified or described more clearly by the exact integration of the function dF of Eq. (3.6) using the assigned functions λ and μ. Rewriting the weighting functions λ and μ in dimensionless terms v and w, one obtains

$$\lambda = \frac{c - v}{v - 1} \qquad\qquad \mu = \frac{w}{1 - w}$$

where $$c = \frac{V_{cyl}}{V_{min}}$$

Then $$dF = \frac{c - v}{v - 1}\, dv - \frac{w}{1 - w}\, dw$$

which, after integration, yields

$$F = (c - 1) \log (v - 1) + \log (1 - w) - (v + w)$$

The function F is in terms of v and w, which can be represented by the integrals with parameters α and β. The relative variation of F with respect to α and β is plotted as a three dimensional surface in Fig. 3. The surface has the shape of a mountain range with the projection of the ridge shown in the α - β plane in Fig. 3. The peak of the ridge is located near the point $\alpha = 2.65$, $\beta = 1.55$.

Changing the values of V_{min}, W_{max} and the reciprocal of fineness ratio, b/a, results in little change in the shape of the surface F, but does produce a slight shift in the location of the ridge. The projections of the ridges on the α - β plane, with different combinations of V_{min}, W_{max}, and b/a, are plotted in Fig. 4. The shift in the ridge is in the same sense as the change in V_{min} or in W_{max}.

The results in Tables III and IV show the iterative calculations for the case $V_{min} = 0.6\ (2 \pi ab^2)$, $W_{max} = 0.8\ (4 \pi \rho g p_o ab^2/\sigma_o)$ with the ratio b/a = 2.0 and $k_o = 0.3$. While the initial values for α and β are different in Tables III and IV, it is noted that they converge to the same values of α and β after successive iterations.

As a second example, λ_2 and μ_2 of Fig. 2 are chosen as the weighting functions. The results of each iterated calculation with three different starting values are listed in Table V. The preassigned values for computations are $V_{min} = 0.6\ (2 \pi ab^2)$, $W_{max} = 0.6\ (4 \pi \rho g p_o ab^2/\sigma_o)$ and b/a = 1.0. The values of α and β reach the ridge rapidly after several iterations regardless of the starting point.

Table VI gives the iterated results for the weighting functions λ_3 and μ_3, which vary linearly with V and W, as defined by Eq. (3.45). The other preassigned values for computations are the same as for Table V. The results indicate that both the slopes F_α and F_β are within the limit 0.005 after ten iterations for all of the different starting values.

Since the iteration procedure is controlled by the slope of the function F, the rate of convergence is mainly dependent on the weighting functions λ and μ. For the currently assigned functions, the results in the above tables indicate that the values α and β converge rapidly to the region where the ordered pair (α, β) lies near the projection of the ridge and then change slowly along the ridge. Due to the small variation of the slope along the ridge, any point located on the projection of the ridge on the α - β plane constitutes a good shape with respect to volume and weight.

As another example, the functions λ and μ may be considered as constants. In this case, the problem becomes one of determining the relative maximum of the function $F = v - w$. Since the variation of the thickness is very small due to the change of values α, β, a shape which is nearly optimum may be achieved by assigning a specific value of volume in determining the values of α, β for minimum shell surface or by assigning a specific value of surface area in determining the values of α, β for maximum volume.

The shapes to fulfill the above requirement can be determined with the aid of data from previous reports [1, 2]. The surface in Fig. 5 represents the volume variation with respect to α and β. The heavy curve on this surface represents the volumes of shells for which the surface area is equal to a preassigned value. From the projection of this curve on a vertical plane, the values of α and β for the maximum volume for the defined surface area can be established. In a similar manner, the surface in Fig. 6 represents the area variation with respect to α and β. The heavy curve on the surface represents the areas of shells for which the shell volume is equal to a preassigned value. The projection of this curve on a vertical plane indicates the area variation among shells having a constant volume.

APPENDIXES

A. Iteration Procedures with Varying Shell Length

Similar to the Eqs. (3.2) and (3.3), the volume and weight of the shells of revolution may be taken as the functions of three parameters α, β and a,

$$V_{xa} = V_{xa} (\alpha, \beta, a)$$

$$W = \rho g A (\alpha, \beta, a) \; t (\alpha, \beta, a)$$

Here the shape requirements to be fulfilled for the shell are

$$V_{xa} \geq V_{min} \qquad W \leq W_{max} \qquad a \leq a_{max}$$

In this case the dimensionless forms are defined as

$$v = \frac{V_{xa}}{V_{min}} \qquad w = \frac{W}{W_{max}} \qquad \ell = \frac{a}{a_{max}}$$

and the differential of a function is formed as

$$dF = \lambda \, d v - \mu \, d w - \nu \, d \ell$$

where λ, μ and ν are the functions defining the relative importance of increases in volume and decreases in weight and length. The differential dF also can be written as

$$dF = F_{\alpha} \, d\alpha + F_{\beta} \, d\beta + F_{\ell} \, d\ell$$

with

$$F_{\alpha} = \lambda \, v_{\alpha} - \mu \, w_{\alpha}$$

$$F_{\beta} = \lambda \, v_{\beta} - \mu \, w_{\beta}$$

$$F_{\ell} = \lambda \, v_{\ell} - \mu \, w_{\ell} - \nu$$

The iteration steps will then follow path of steepest ascent, as defined by

$$d\alpha : d\beta : d\ell = F_{\alpha} : F_{\beta} : F_{\ell}$$

The iterative procedure is repeated until F_{α}, F_{β} and F_{ℓ} are essentially zero.

B. FORTRAN Programs

1. Programs for FUNCTIONS FJ1 through FJ7

Since the integrals J_1 through J_7 listed in Eqs. (3.33) appear in the calculations of the main iteration program many times, they are computed in separate FUNCTIONS attached to the main program. Simpson's rule is used to evaluate the above integrals with the approximation techniques discussed in the section on Numerical Integration.

Among the input arguments for the FUNCTION programs, the values P, Q, R, S and T are the exponents in the integrals. They are dependent on the values of α and β. The quantities ETAI and EPS are two small numbers assigned in the calculation of the two improper integrals $I(\epsilon)$ and $K(\epsilon)$ of Eqs. (3.35) and (3.36). The value FK2 represents the term $\left(\frac{b\alpha}{a\beta}\right)^2$ and the value ACC is the accuracy required for the relative difference between two successive approximations in the Simpson's rule integration routine. In the previous numerical examples, the value assigned to ETAI and to EPS is 0.01 while the value assigned to ACC is 0.0001.

2. Program for SUBROUTINE FMULAM

The SUBROUTINE FMULAM is written to compute either the values λ_1, μ_1, as Eq. (3.43) or the values λ_2, μ_2 as Eq. (3.44), which is controlled by the number NC. The outputs defined by FLAM and FMU represent λ_1 and μ_1 for NC = 1 and λ_2 and μ_2 for NC = 2. The input arguments P, Q, FM and FN are the same as the exponents p, q, m and n of Eq. (3.43).

3. The Main Iteration Program

The main purpose of the program is to compute the increments of $\Delta\alpha$ and $\Delta\beta$ along the path of the steepest ascent from the current assigned values α and β. The computations are repeated for the new calculated α and β until they reach a point where the absolute values F_α and F_β, as in Eqs. (3.7) and (3.8), are less than a preassigned small number QEPS.

The input data of FP, FQ, FM, FN and NC listed on the first data card are supplied for the calculation of functions λ and μ. The constants EPS, ETAI and ACC on the second data card are the numbers assigned to the FUNCTIONS J_1 through J_7 in order to compute the integrals. The values PO and SIGO represent internal pressure p_o and the yield stress σ_o and are used to calculate the thickness t. Input data ETAO and FKV are assigned to limit the step size of α and

β in each iteration, and represent η_0 and k_0 in Eqs. (3.14) and (3.13). The two integers NRVWP and NRAB are the number of the sets of V_{min}, W_{max} and the number of sets of the ratio BOA (b/a) to be calculated in the program.

The input values VMIN and WMAX are two dimensionless numbers which represent the preassigned allowable minimum volume and maximum weight. The true value of the minimum volume is VMIN $\cdot$ $(2 \pi ab^2)$ and the true value of maximum weight is WMAX $\cdot$ $(4 \pi \rho g p_0 ab^2/\sigma_0)$.

For the output, the results of each iteration are printed using the symbols DA, DB, ATIL and BTIL to represent $\Delta\alpha$, $\Delta\beta$, F_α and F_β respectively.

PROGRAM FOR EXAMPLES

```
$      FORTRAN IBM
$      PUNCH OBJECT
$      GO
      READ INPUT TAPE 7, 1, FP, FQ, FM, FN, NC, EPS, ETAI, ACC, NRVWP     SHELL001
    1 FORMAT (4F15.5, I10 / 3E20.8 / I10)                                 SHELL002
      READ INPUT TAPE 7, 2, PO, SIGO, QEPS, ETAO, FKV                     SHELL003
    2 FORMAT (2E20.8/ 3F20.5)                                             SHELL004
    3 DO 69 INRVWP = 1, NRVWP                                             SHELL005
    4 READ INPUT TAPE 7, 5, VMIN, WMAX, NRAB                              SHELL006
    5 FORMAT ( 2E20.8 /I10)                                               SHELL007
    9 DO 50 INRAB = 1, NRAB                                               SHELL008
   10 READ INPUT TAPE 7, 11, ALPHA, BETA, BOA                             SHELL009
   11 FORMAT ( 3E20.8)                                                    SHELL010
      WRITE OUTPUT TAPE 6, 12, VMIN, WMAX, BOA                            SHELL011
   12 FORMAT (8H1VMIN = ,F7.3,2X,7HWMAX = ,F7.3,3X, 4HBOA=, F6.3, //)     SHELL012
       NCONT =0                                                           SHELL013
C                                                                         SHELL014
C    TO LIMIT ALPHA AND BETA BOTH LARGER THAN ONE                         SHELL015
C                                                                         SHELL016
   14 IF (ALPHA - 1.0) 15, 17, 17                                         SHELL017
   15 ALPHA = 1.0                                                         SHELL018
   17 IF (BETA - 1.0) 18, 19, 19                                          SHELL019
   18 BETA = 1.0                                                          SHELL020
   19 FK2 = (BOA*ALPHA/BETA)**2                                           SHELL021
       ONOB = 1.0/BETA                                                    SHELL022
       TWOB = 2.0/BETA                                                    SHELL023
       THOB = 3.0/BETA                                                    SHELL024
       NCONT = NCONT +1                                                   SHELL025
C                                                                         SHELL026
C    TO DETERMINE THE VALUE OF T= T1 OR T2                                SHELL027
C                                                                         SHELL028
      AR= FJ1(ALPHA,ONOB,ACC)                                             SHELL029
      SL = FJ4(ALPHA,ONOB,0.0,FK2,ETAI,ACC)                               SHELL030
      RR = AR/ SL                                                         SHELL031
      IF ( RR-0.5 ) 20, 20, 21                                            SHELL032
C                                                                         SHELL033
C    T2 LARGER THAN T1                                                    SHELL034
   20 T = 0.5 *PO/SIGO                                                    SHELL035
      TA = 0.                                                             SHELL036
      TB = 0.                                                             SHELL037
      GO TO 24                                                            SHELL038
C                                                                         SHELL039
C    T1 LARGER THAN T2                                                    SHELL040
   21 T = RR *PO /SIGO                                                    SHELL041
      ARA= -(ONOB)*FJ3(ALPHA,ONOB,EPS,ETAI,ACC)                           SHELL042
      ARB= -(1.0/(BETA**2))*FJ2(ALPHA,ONOB,ETAI,ACC)                      SHELL043
      TE = FJ5(ALPHA,ONOB,TWOB,FK2,ETAI,ACC)                              SHELL044
      SLA = FK2*(FJ7(ALPHA,ONOB,TWOB,2.*(ALPHA-1.),FK2,EPS,ETAI,ACC) -    SHELL045
     1   FJ7(ALPHA,ONOB,TWOB,3.*ALPHA-2.,FK2,EPS,ETAI,ACC)/BETA +          SHELL046
     2   TE/ALPHA)                                                        SHELL047
      SLB = -(FK2/(BETA**2))*(FJ6(ALPHA,ONOB,ALPHA,TWOB,FK2,ETAI,ACC) +   SHELL048
     1   BETA*TE)                                                         SHELL049
      TA = (PO/SIGO) * ( ARA- AR*SLA/SL ) / SL                            SHELL050
      TB = (PO/SIGO) * (ARB- AR * SLB/ SL) /SL                            SHELL051
   24 WRITE OUTPUT TAPE 6, 25, ALPHA, BETA, AR, SL, RR, T, TA, TB         SHELL052
   25 FORMAT ( 1H0, 5X, 8F12.6)                                           SHELL053
```

```
C
C   TO CALCULATE V, W, AND THEIR DERIVATIVES
C
      SU= FJ4(ALPHA,ONOB,ONOB,FK2,ETAI,ACC)
      TE = FK2*FJ5(ALPHA,ONOB,THOB,FK2,ETAI,ACC)
      SUA= TE/ALPHA - FJ7(ALPHA,ONOB, ONOB+2.   ,ALPHA,FK2,EPS,ETAI,ACC)
     1  /BETA + FK2*FJ7(ALPHA,ONOB, THOB+1.0, 2.*(ALPHA-1.),FK2,EPS,
     2   ETAI,ACC) - FK2*(TWOB-1.)*FJ7(ALPHA,ONOB,THOB,3.*ALPHA-2.,
     3   FK2,EPS,ETAI,ACC)
      SUB= -(1./(BETA**2))*(FJ6(ALPHA,ONOB,1.,ONOB+2.,FK2,ETAI,ACC) +
     1   2.*FK2*FJ6(ALPHA,ONOB,ALPHA,THOB,FK2,ETAI,ACC) + BETA*TE)
      V = FJ1(ALPHA,TWOB,ACC)
      VA = -(TWOB)*FJ3(ALPHA,TWOB,EPS,ETAI,ACC)
      VB = -(2.0/(BETA**2))*FJ2(ALPHA,TWOB,ETAI,ACC)
      W = SU *T
      WA = SU*TA + T*SUA
      WB = SU*TB + T*SUB
C
C   TO FIND THE VALUES OF MU AND LAMDA
C
      CALL FMULAM ( FP, FQ, FM, FN, VMIN, WMAX, V, W, FMU, FLAM, NC )
C
      WRITE OUTPUT TAPE 6, 26, V, VA, VB, W, WA, WB, FMU, FLAM
   26 FORMAT (10X,  8F11.6 )
      IF (FMU) 50, 30, 30
   30 IF (FLAM) 50, 31, 31
C
C   TO CALCULATE FALPHA AND FBETA (THAT IS, ATIL AND BTIL)
C
   31 ATIL = FLAM * VA/ VMIN - FMU * WA/ WMAX
      BTIL = FLAM * VB/ VMIN - FMU * WB/ WMAX
      ATILS = ( ATIL*VA + BTIL*VB ) /VMIN
      BTILS = ( ATIL*WA + BTIL*WB ) / WMAX
      VO = V/VMIN
      WO = W/WMAX
C
C   PROGRAM TERMINATES WHEN QQ LESS THAN QEPS
C
      QQ = SQRT (ATIL**2 + BTIL**2)
      IF ( QQ-QEPS ) 48, 48, 35
C
C   TO DETERMINE STEP SIZE, CONTROL ON FKAP
C
   35 FKAPS = ETAO / SQRT (ATILS**2 + BTILS**2)
      IF (FKAPS - FKV) 36, 36, 37
   36 FKAP = FKAPS
      GO TO 40
   37 FKAP = FKV
   40 DA = FKAP * ATIL
      DB = FKAP * BTIL
      WRITE OUTPUT TAPE 6, 42, ALPHA, BETA, DA, DB, VO, WO, FKAP, ATIL,
     1   BTIL, ATILS, BTILS
   42 FORMAT (6F9.5, 5E12.5)
      ALPHA = ALPHA+ DA
      BETA = BETA + DB
      IF ( NCONT - 50  ) 14, 50, 50
   48 WRITE OUTPUT TAPE 6, 49, ALPHA, BETA, VO, WO, ATIL, BTIL, ATILS,
     1   BTILS
   49 FORMAT ( 1H-, 8F13.5 /1H2 )
   50 CONTINUE
   69 CONTINUE
      CALL SYSTEM
      END
```

```
$       FORTRAN IBM
$       PUNCH OBJECT
      SUBROUTINE FMULAM(P,Q,FM,FN,VMIN,WMAX,V,W,FMU,FLAM,NC)            SUBFML01
    1 VV = V/VMIN                                                        SUBFML02
    2 WW = W/WMAX                                                        SUBFML03
    3 IF (VV - 1.0) 4, 4, 10                                             SUBFML04
    4 IF (WW - 1.0) 10, 5, 5                                             SUBFML05
    5 WRITE OUTPUT TAPE 6, 6, VMIN, WMAX, VV, WW                         SUBFML06
    6 FORMAT(7H VMIN = ,E12.6,8H WMAX = ,E12.6,6H  V = ,E14.8,6H  W = , SUBFML07
     1E14.8,// 5X, 43HTHE PRIMARY REQUIREMENTS CANNOT BE REACHED )       SUBFML08
    7 FMU = -1.0                                                         SUBFML09
    8 FLAM = -1.0                                                        SUBFML10
    9 RETURN                                                             SUBFML11
   10 IF (V-1.0) 11, 5, 5                                                SUBFML12
   11 IF (VV - 1.0) 12, 12, 15                                           SUBFML13
   12 FMU = 0.0                                                          SUBFML14
   13 FLAM = 1.0                                                         SUBFML15
   14 IF (WW) 5, 5, 9                                                    SUBFML16
   15 IF (WW) 5, 5, 16                                                   SUBFML17
   16 IF (WW - 1.0) 20, 17, 17                                           SUBFML18
   17 FLAM = 0.0                                                         SUBFML19
   18 FMU = 1.0                                                          SUBFML20
   19 RETURN                                                             SUBFML21
   20 FLAM = (( 1./VMIN - VV)**P)/ ((VV-1. )**Q)                         SUBFML22
   21 FMU = (WW**FM)/((1.0 - WW)**FN)                                    SUBFML23
   22 GO TO (23,26), NC                                                  SUBFML24
   23 TEMP = SQRT(FLAM**2 + FMU**2)                                      SUBFML25
   24 FLAM = FLAM/TEMP                                                   SUBFML26
   25 FMU  = FMU/TEMP                                                    SUBFML27
   26 RETURN                                                             SUBFML28
      END                                                                SUBFML29
```

```
$      FORTRAN IBM
$PUNCH OBJECT
      FUNCTION FJ1(P,Q,ACC)                                              FJ1 01
    1 ODD = 0.0                                                          FJ1 02
    2 INT = 1                                                            FJ1 03
    3 V = 1.0                                                            FJ1 04
    4 EVEN = 0.0                                                         FJ1 05
      AREA1 = 0.0                                                        FJ1 06
      IF (Q) 19, 5, 6                                                    FJ1 07
    5 ENDS = 2.0                                                         FJ1 08
      GO TO 7                                                            FJ1 09
    6 ENDS = 1.0                                                         FJ1 10
    7 H = 1.0/V                                                          FJ1 11
    8 ODD = EVEN + ODD                                                   FJ1 12
    9 X = H/2.                                                           FJ1 13
   10 EVEN = 0.0                                                         FJ1 14
   11 DO 13 I = 1, INT                                                   FJ1 15
   12 EVEN = EVEN + ((1.0 - X**P)**Q)                                    FJ1 16
   13 X = X + H                                                          FJ1 17
   14 AREA = (ENDS + 4.0*EVEN + 2.0*ODD)*H/6.0                           FJ1 18
   15 R = ABSF(AREA1/AREA - 1.0) - ACC                                   FJ1 19
   16 IF (R) 25, 25, 17                                                  FJ1 20
   17 IF (INT - 16384) 21, 19, 19                                        FJ1 21
   18 FORMAT (23H J1(P,Q) NOT CONVERGENT)                                FJ1 22
   19 WRITE OUTPUT TAPE 6, 18                                            FJ1 23
   20 CALL SYSERR                                                        FJ1 24
   21 AREA1 = AREA                                                       FJ1 25
   22 INT = 2*INT                                                        FJ1 26
   23 V = 2.0*V                                                          FJ1 27
   24 GO TO 7                                                            FJ1 28
   25 FJ1 = AREA                                                         FJ1 29
   26 RETURN                                                             FJ1 30
      END                                                                FJ1 31
```

```
$     FORTRAN IBM
$PUNCH OBJECT
      FUNCTION FJ2(P,Q,ETAI, ACC)
    1 DP = P
      E=  1.-(1.0-ETAI)**P
      OME = 1.0 -ETAI
    3 DEL = 1.0 - 1.0/DP
    5 F0 = 1.0
    6 F1 = (1.0 - 0.5*E)**(-DEL)
    7 F2 = (1.0 - E)**(-DEL)
    8 DQ = Q
    9 A = 4.0*F1 - F2 - 3.0*F0
   10 B = 2.0*F2 - 4.0*F1 + 2.0*F0
   11 T1 = DQ + 1.0
   12 T2 = DQ + 2.0
   13 T3 = DQ + 3.0
   140EN = (ELOG (E)*(F0 + T1*(A/T2 + B/T3)) - (F0 + T1*T1*(A/(T2*T2)
     1  + B/(T3*T3)))/T1)*(E**T1)/(T1*DP)
   15 ODD = 0.0
   16 INT = 1
   17 V = 1.0
   18 EVEN = 0.0
      AREA1 = 0.0
   19 ENDS = ((1.0 - OME**DP)**DQ)*ELOG (1.0 - OME**DP)
   20 H = OME/V
   21 ODD = EVEN + ODD
   22 X = H/2.0
   23 EVEN = 0.0
   24 DO 26 I = 1, INT
   25 EVEN = EVEN + ((1.0 - X**DP)**DQ)*ELOG (1.0 - X**DP)
   26 X = X + H
   27 AREA = (ENDS + 4.0*EVEN + 2.0*ODD)*H/6.0
   28 R = ABSF(AREA1/AREA - 1.0) - ACC
   29 IF (R) 38, 38, 30
   30 IF (INT - 16384) 34, 31, 31
   31 WRITE OUTPUT TAPE 6, 32
   32 FORMAT (23H J2(P,Q) NOT CONVERGENT)
   33 CALL SYSERR
   34 AREA1 = AREA
   35 INT = 2*INT
   36 V = 2.0*V
   37 GO TO 20
   38 FJ2 = AREA + EN
   39 RETURN
      END
```

```
$     FORTRAN IBM
$PUNCH OBJECT
      FUNCTION FJ3(P,Q,EPS, ETAI, ACC)                                  FJ3 01
    1 DP = P                                                            FJ3 02
    2 DQ = Q - 1.0                                                      FJ3 03
    3 T1 = DP + 1.0                                                     FJ3 04
    4 T2 = T1/(DP + 2.0)                                                FJ3 05
    5 T3 = T1/(DP + 3.0)                                                FJ3 06
    6 F0 = 1.0                                                          FJ3 07
    7 E = EPS                                                           FJ3 08
    8 F1 = (1.0 - (0.5*E)**DP)**DQ                                      FJ3 09
    9 F2 = (1.0 - E**DP)**DQ                                            FJ3 10
   10 A = 4.0*F1 - F2 - 3.0*F0                                          FJ3 11
   11 B = 2.0*F2 - 4.0*F1 + 2.0*F0                                      FJ3 12
   120EN = (ELOG (E)*(F0 + T2*A + T3*B) - (F0 + T2*T2*A + T3*T3*B)/T1)*(FJ3 13
     1E**T1)/T1                                                         FJ3 14
   13 DEL = 1.0/DP                                                      FJ3 15
      OME = 1.0-ETAI                                                    FJ3 16
      E= 1.0- OME**DP                                                   FJ3 17
   15 T1 = DQ + 1.0                                                     FJ3 18
   16 T2 = T1/(DQ + 2.0)                                                FJ3 19
   17 T3 = T1/(DQ + 3.0)                                                FJ3 20
   18 F0 = 0.0                                                          FJ3 21
   19 F1 = ((1.0 - 0.5*E)**DEL)*ELOG (1.0 - 0.5*E)                      FJ3 22
   20 F2 = ((1.0 - E)**DEL)*ELOG (1.0 - E)                              FJ3 23
   21 A = 4.0*F1 - F2 - 3.0*F0                                          FJ3 24
   22 B = 2.0*F2 - 4.0*F1 + 2.0*F0                                      FJ3 25
   23 EN = EN + (F0 + T2*A + T3*B)*(E**T1)/(T1*DP*DP)                   FJ3 26
      E = EPS                                                           FJ3 27
   240ENDS = ((1.0 - E**DP)**DQ)*(E**DP)*ELOG (E)                       FJ3 28
     1 + ((1.0 - OME**DP)**DQ)*(OME**DP)*ELOG (OME)                     FJ3 29
   25 ODD = 0.0                                                         FJ3 30
   26 INT = 1                                                           FJ3 31
   27 V = 1.0                                                           FJ3 32
   28 EVEN = 0.0                                                        FJ3 33
   29 AREA1 = 0.0                                                       FJ3 34
   30 H = (OME - E)/V                                                   FJ3 35
   31 ODD = EVEN + ODD                                                  FJ3 36
   32 X = E + H/2.                                                      FJ3 37
   33 EVEN = 0.0                                                        FJ3 38
   34 DO 36 I = 1, INT                                                  FJ3 39
   35 EVEN = EVEN + ((1.0 - X**DP)**DQ)*(X**DP)*ELOG (X)                FJ3 40
   36 X = X + H                                                         FJ3 41
   37 AREA = (ENDS + 4.0*EVEN + 2.0*ODD)*H/6.0                          FJ3 42
   38 R = ABSF(AREA1/AREA - 1.0) - ACC                                  FJ3 43
   39 IF (R) 48, 48, 40                                                 FJ3 44
   40 IF (INT - 16384) 44, 42, 42                                       FJ3 45
   41 FORMAT (23H J3(P,Q) NOT CONVERGENT)                               FJ3 46
   42 WRITE OUTPUT TAPE 6, 41                                           FJ3 47
   43 CALL SYSERR                                                       FJ3 48
   44 AREA1 = AREA                                                      FJ3 49
   45 INT = 2*INT                                                       FJ3 50
   46 V = 2.0*V                                                         FJ3 51
   47 GO TO 30                                                          FJ3 52
   48 FJ3 = AREA + EN                                                   FJ3 53
   49 RETURN                                                            FJ3 54
      END                                                               FJ3 55
```

```
$      FORTRAN IBM
$PUNCH OBJECT
      FUNCTION FJ4(P,Q,S,FK2, ETAI, ACC)
    1 DG = S + Q - 1.0
    2 T1 = DG + 1.0
    3 T2 = T1/(DG + 2.0)
    4 T3 = T1/(DG + 3.0)
      OME = 1.0 - ETAI
      E = 1.0- OME**P
    6 DP = P
    8 DEL = -2.0*(1.0 - 1.0/DP)
      DQ = 2.0*(1.0 -  Q)
      IF (Q - 1.0) 11, 10, 36
   10 F0 = SQRT (1.0/FK2 + 1.0)
      GO TO 12
   11 F0 = 1.0
   12 DF = FK2
   13 F1 = SQRT (((1.0 - 0.5*E)**DEL)*((0.5*E)**DQ)/FK2 + 1.0)
   14 F2 = SQRT (((1.0 - E)**DEL)*(E**DQ)/FK2 + 1.0)
   15 A = 4.0*F1 - F2 - 3.0*F0
   16 B = 2.0*F2 - 4.0*F1 + 2.0*F0
   17 EN = (F0 + T2*A + T3*B)*(E**T1)*SQRT (DF)/(T1*DP)
   18 ODD = 0.0
   19 INT = 1
   20 V = 1.0
   21 EVEN = 0.0
      AREA1 = 0.0
      EE = 1.0
      IF (P - 1.0) 36, 22, 23
   22 EE = SQRT (1.0 + FK2)
   230ENDS = EE   + ((1.0 - OME**DP)**DG)*SQRT ((1.0 - OME**DP)**DQ + DF*
     1(OME**(2.0*DP - 2.0)))
   24 DDP = 2.0*(DP - 1.0)
   25 H = OME/V
   26 ODD = EVEN + ODD
   27 X = H/2.0
   28 EVEN = 0.0
   29 DO 31 I = 1, INT
   300EVEN = EVEN + ((1.0 - X**DP)**DG)*SQRT ((1.0 - X**DP)**DQ + DF*(X*
     1*DDP))
   31 X = X + H
   32 AREA = (ENDS + 4.0*EVEN + 2.0*ODD)*H/6.0
   33 R = ABSF(AREA1/AREA - 1.0) - ACC
   34 IF (R) 43, 43, 35
   35 IF (INT - 16384) 39, 36, 36
   36 WRITE OUTPUT TAPE 6, 37
   37 FORMAT (25H J4(P,Q,S) NOT CONVERGENT)
   38 CALL SYSERR
   39 AREA1 = AREA
   40 INT = 2*INT
   41 V = 2.0*V
   42 GO TO 25
   43 FJ4 = AREA + EN
   44 RETURN
      END
```

```
$      FORTRAN IBM
$PUNCH OBJECT
       FUNCTION FJ5(P,Q,S,FK2, ETAI, ACC)                               FJ5 01
       OME = 1.0 - ETAI                                                 FJ5 02
     2 G = S - Q - 1.0                                                  FJ5 03
     3 QQ = 2.0*(1.0 - Q)                                               FJ5 04
     4 DEL = 1.0 - 1.0/P                                                FJ5 05
       TDEL = 2.0*DEL                                                   FJ5 06
       FK = FK2                                                         FJ5 07
       IF (Q - 1.0) 6, 5, 35                                            FJ5 08
     5 FK = 1.0 + FK                                                    FJ5 09
     6 F0 = 1.0/SQRT (FK)                                               FJ5 10
       EPS = 1.0 -OME**P                                                FJ5 11
     7 TPS = 0.5*EPS                                                    FJ5 12
     8 F1 = ((1.0 - TPS)**DEL)/SQRT (TPS**QQ + FK2*((1.0 - TPS)**TDEL)) FJ5 13
     9 F2 = ((1.0 - EPS)**DEL)/SQRT (EPS**QQ + FK2*((1.0 - EPS)**TDEL)) FJ5 14
    10 A = 4.0*F1 - F2 - 3.0*F0                                         FJ5 15
    11 B = 2.0*F2 - 4.0*F1 + 2.0*F0                                     FJ5 16
    12 T1 = G + 1.0                                                     FJ5 17
    13 T2 = T1/(G + 2.0)                                                FJ5 18
    14 T3 = T1/(G + 3.0)                                                FJ5 19
    15 EN = (F0 + T2*A + T3*B)*(EPS**T1)/(T1*P)                         FJ5 20
    16 ODD = 0.0                                                        FJ5 21
    17 INT = 1                                                          FJ5 22
    18 V = 1.0                                                          FJ5 23
    19 EVEN = 0.0                                                       FJ5 24
       AREA1 = 0.0                                                      FJ5 25
       EE = 0.0                                                         FJ5 26
       IF (P - 1.0) 35, 20, 21                                          FJ5 27
    20 EE = 1.0/SQRT (1.0 + FK2)                                        FJ5 28
    21 ENDS = SQRT ((1.0 - OME**P)**QQ + FK2*(OME**(2.0*P - 2.0)))      FJ5 29
    22 PP = 2.0*(P - 1.0)                                               FJ5 30
    23 ENDS = (OME**PP)*((1.0 - OME**P)**G)/ENDS + EE                   FJ5 31
    24 H = OME/V                                                        FJ5 32
    25 ODD = EVEN + ODD                                                 FJ5 33
    26 X = H/2.0                                                        FJ5 34
    27 EVEN = 0.0                                                       FJ5 35
    28 DO 30 I = 1, INT                                                 FJ5 36
    290EVEN = EVEN + (X**PP)*((1.0 - X**P)**G)/SQRT ((1.0 - X**P)**QQ + FJ5 37
     1 FK2*(X**PP))                                                     FJ5 38
    30 X = X + H                                                        FJ5 39
    31 AREA = (ENDS + 4.0*EVEN + 2.0*ODD)*H/6.0                         FJ5 40
    32 R = ABSF(AREA1/AREA - 1.0) - ACC                                 FJ5 41
    33 IF (R) 42, 42, 34                                                FJ5 42
    34 IF (INT - 16384) 38, 35, 35                                      FJ5 43
    35 WRITE OUTPUT TAPE 6, 36                                          FJ5 44
    36 FORMAT (25H J5(P,Q,S) NOT CONVERGENT)                            FJ5 45
    37 CALL SYSERR                                                      FJ5 46
    38 AREA1 = AREA                                                     FJ5 47
    39 INT = 2*INT                                                      FJ5 48
    40 V = 2.0*V                                                        FJ5 49
    41 GO TO 24                                                         FJ5 50
    42 FJ5 = AREA + EN                                                  FJ5 51
    43 RETURN                                                           FJ5 52
       END                                                              FJ5 53
```

```
$     FORTRAN IBM
$PUNCH OBJECT
      FUNCTION FJ6(P,Q,R,S,FK2, ETAI, ACC)
    1 TQ = 2.0*(1.0 - Q)
    2 TP = 2.0*(P - 1.0)
    3 TR = 2.0*(R - 1.0)
    4 G = S - Q - 1.0
    5 TD = 2.0*(1.0 - 1.0/P)
      EPS = 1.0- (1.-ETAI)**P
    6 TPS = 0.5*EPS
    7 TT = 2.0*R/P - 1.0 - 1.0/P
    8 T1 = G + 1.0
    9 T2 = T1/(G + 2.0)
      T3 = T1/(G + 3.0)
      FK = FK2
      IF (Q - 1.0) 11, 10, 39
   10 FK = 1.0 + FK
   11 F0 = 1.0/SQRT (FK)
   12 OMW = 1.0 - TPS
   13 F1 = (OMW**TT)/SQRT (TPS**TQ + FK2*(OMW**TD))
   14 OMW = 1.0 - EPS
   15 F2 = (OMW**TT)/SQRT (EPS**TQ + FK2*(OMW**TD))
      OME = 1.0 - ETAI
   17 A = 4.0*F1 - F2 - 3.0*F0
   18 B = 2.0*F2 - 4.0*F1 + 2.0*F0
   190EN = (ELOG (EPS)*(F0 + T2*A + T3*B) - (F0 + T2*T2*A + T3*T3*B)/T1)
     1*(EPS**T1)/(T1*P)
   20 ODD = 0.0
   21 INT = 1
   22 V = 1.0
   23 EVEN = 0.0
   24 AREA1 = 0.0
   25 OMW = 1.0 - OME**P
   260ENDS =((OME**TR)*(OMW**G)/SQRT (OMW**TQ + FK2*(OME**TP)))*ELOG (OM
     1W)
   27 H = OME/V
   28 ODD = EVEN + ODD
   29 X = H/2.0
   30 EVEN = 0.0
   31 DO 34 I = 1, INT
   32 OMW = 1.0 - X**P
   330EVEN = EVEN +((X**TR)*(OMW**G)/SQRT (OMW**TQ + FK2*(X**TP)))*ELOG
     1(OMW)
   34 X = X + H
   35 AREA = (ENDS + 4.0*EVEN + 2.0*ODD)*H/6.0
   36 RR = ABSF(AREA1/AREA - 1.0) - ACC
   37 IF (RR) 46, 46, 38
   38 IF (INT - 16384) 42, 39, 39
   39 WRITE OUTPUT TAPE 6, 40
   40 FORMAT (27H J6(P,Q,R,S) NOT CONVERGENT)
   41 CALL SYSERR
   42 AREA1 = AREA
   43 INT = 2*INT
   44 V = 2.0*V
   45 GO TO 27
   46 FJ6 = AREA + EN
   47 RETURN
      END
```

```
$     FORTRAN IBM
$PUNCH OBJECT
      FUNCTION FJ7(P,Q,S,T,FK2,EPS, ETAI, ACC)
    1 TQ = 2.0*(1.0 - Q)
    2 TP = 2.0*(P - 1.0)
    3 G = S - Q - 2.0
    4 T1 = T + 1.0
    5 T2 = T1/(T + 2.0)
      T3 = T1/(T + 3.0)
      IF (P - 1.0) 53, 6, 7
    6 F0 = 1.0/SQRT (1.0 + FK2)
      GO TO 8
    7 F0 = 1.0
    8 TPS = 0.5*EPS
    9 OMW = 1.0 - TPS**P
   10 F1 = (OMW**G)/SQRT (OMW**TQ + FK2*(TPS**TP))
   11 OMW = 1.0 - EPS**P
   12 F2 = (OMW**G)/SQRT (OMW**TQ + FK2*(EPS**TP))
   13 A = 4.0*F1 - F2 - 3.0*F0
   14 B = 2.0*F2 - 4.0*F1 + 2.0*F0
   15 TT = F0 + T2*A + T3*B
   16 TU = F0 + T2*T2*A + T3*T3*B
   17 EN = (ELOG (EPS)*TT - TU/T1)*(EPS**T1)/T1
      OME = 1.0 - ETAI
   19 T4 = 2.0*(1.0 - 1.0/P)
   20 T5 = T/P - 1.0 + 1.0/P
   21 T1 = G + 2.0
   22 T2 = T1/(G + 3.0)
      T3 = T1/(G + 4.0)
      FK = FK2
      IF (Q - 1.0) 24, 23, 53
   23 FK = 1.0 + FK
   24 F0 = -1.0/SQRT (FK)
      E = 1.0- OME**P
      TPS = 0.5 * E
      OMW = 1.- TPS
   27 F1 =  (OMW**T5)/SQRT (TPS**TQ + FK2*(OMW**T4)) * ELOG(OMW) /TPS
   28 OMW = 1.0 - E
      F2 =  (OMW**T5)/SQRT ( E **TQ + FK2*(OMW**T4)) * ELOG(OMW) /E
   30 A = 4.0*F1 - F2 - 3.0*F0
   31 B = 2.0*F2 - 4.0*F1 + 2.0*F0
   32 EN = EN + (F0 + T2*A + T3*B)*( E **T1)/(T1*P*P)
   33 ODD = 0.0
   34 INT = 1
   35 V = 1.0
   36 EVEN = 0.0
   37 AREA1 = 0.0
      OMW = 1.0 - EPS**P
   38 ENDS = (EPS**T)*(OMW**G)*ELOG (EPS)/SQRT (OMW**TQ + FK2*(EPS**TP))
   39 O = 1.0 - OME**P
   40 ENDS=ENDS+(OME**T)*(O**G)*ELOG (OME)/SQRT (O**TQ+FK2*(OME**TP))
   41 H = (OME - EPS)/V
   42 ODD = EVEN + ODD
   43 X = EPS + H/2.0
   44 EVEN = 0.0
   45 DO 48 I = 1, INT
   46 O = 1.0 - X**P
   47 EVEN = EVEN + (X**T)*(O**G)*ELOG (X)/SQRT (O**TQ + FK2*(X**TP))
   48 X = X + H
   49 AREA = (ENDS + 4.0*EVEN + 2.0*ODD)*H/6.0
   50 R = ABSF(AREA1/AREA - 1.0) - ACC
   51 IF (R) 60, 60, 52
   52 IF (INT - 16384) 56, 53, 53
   53 WRITE OUTPUT TAPE 6, 54
   54 FORMAT (27H J7(P,Q,S,T) NOT CONVERGENT)
   55 CALL SYSERR
   56 AREA1 = AREA
   57 INT = 2*INT
   58 V = 2.0*V
   59 GO TO 41
   60 FJ7 = AREA + EN
   61 RETURN
      END
```

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TABLE I ITERATION RESULTS FOR $\lambda=\lambda_1$, $\mu=\mu_1$, $b/a=1$, $\alpha_o=1.5$, $\beta_o=1.5$

Step	α	β	$\frac{V}{2\pi ab^2}$	$\frac{W(\frac{\sigma_o}{p_o})}{4\pi ab^2 \rho g}$	$\frac{t}{b}(\frac{\sigma_o}{p_o})$	F_α	F_β	$\Delta\alpha$	$\Delta\beta$
1	1.5000	1.5000	.53743	.43810	.5000	.29499	.24269	.0885	.0728
2	1.5885	1.5728	.56271	.44979	.5000	.27202	.22630	.0816	.0679
3	1.6701	1.6407	.58448	.46001	.5000	.25255	.21206	.0758	.0636
4	1.7459	1.7043	.60343	.46903	.5000	.26893	.22758	.2228	.1885
5	1.9686	1.8928	.65276	.49306	.5000	.83353	.72064	.2501	.2162
6	2.2187	2.1090	.69862	.52328	.5069	-.28840	-.28654	-.0865	-.0860
7	2.1322	2.0231	.68274	.51165	.5035	-.05475	-.08112	-.0164	-.0243
8	2.1189	1.9915	.67861	.50850	.5025	.01648	-.01889	.0049	-.0057
9	2.1288	1.9802	.67860	.50846	.5024	.01672	-.01885	.0050	-.0057
10	2.1389	1.9689	.67859	.50841	.5024	.01689	-.01889	.0051	-.0057
11	2.1490	1.9575	.67859	.50837	.5023	.01705	-.01896	.0051	-.0057
12	2.1593	1.9461	.67858	.50832	.5023	.01721	-.01899	.0052	-.0057
13	2.1696	1.9347	.67856	.50827	.5022	.01756	-.01886	.0053	-.0057
14	2.1801	1.9233	.67856	.50823	.5022	.01758	-.01905	.0053	-.0057
15	2.1998	1.9107	.67935	.50875	.5023	.00421	-.03187	.0021	-.0159
16	2.2151	1.8891	.67884	.50830	.5021	.01305	-.02400	.0065	-.0120
17	2.2323	1.8690	.67865	.50807	.5020	.01655	-.02106	.0083	-.0105
18	2.2505	1.8494	.67857	.50793	.5019	.01806	-.01998	.0090	-.0100
19	2.2692	1.8299	.67853	.50783	.5017	.01883	-.01960	.0094	-.0098
20	2.2883	1.8105	.67851	.50774	.5016	.01927	-.01957	.0096	-.0098
21	2.3078	1.7911	.67849	.50764	.5015	.01971	-.01951	.0099	-.0098
22	2.3277	1.7716	.67847	.50754	.5014	.02008	-.01952	.0100	-.0098
23	2.3478	1.7521	.67846	.50745	.5013	.02044	-.01955	.0102	-.0098
24	2.3684	1.7326	.67844	.50735	.5012	.02080	-.01959	.0104	-.0098
25	2.3893	1.7130	.67843	.50725	.5011	.02116	-.01963	.0106	-.0098
26	2.4105	1.6933	.67842	.50715	.5010	.02153	-.01966	.0108	-.0098
27	2.4321	1.6737	.67840	.50704	.5008	.02189	-.01968	.0110	-.0098
28	2.4541	1.6540	.67838	.50693	.5007	.02227	-.01970	.0111	-.0099
29	2.4765	1.6343	.67837	.50681	.5005	.02276	-.01955	.0114	-.0098
30	2.4992	1.6146	.67836	.50670	.5004	.02310	-.01967	.0116	-.0098
31	2.5224	1.5948	.67834	.50659	.5003	.02345	-.01974	.0117	-.0099

TABLE II ITERATION RESULTS FOR $\lambda=\lambda_1$, $\mu=\mu_1$, $b/a=1$, $\alpha_o=2.0$, $\beta_o=3.0$

Step	α	β	$\frac{V}{2\pi ab^2}$	$\frac{W(\frac{\sigma_o}{p_o})}{4\pi ab^2 \rho g}$	$\frac{t}{b}(\frac{\sigma_o}{p_o})$	F_α	F_β	$\Delta\alpha$	$\Delta\beta$
1	2.0000	3.0000	.73915	.55381	.5151	-.99584	-.64496	-.2988	-.1935
2	1.7013	2.8065	.69653	.52338	.5075	-.32702	-.21005	-.0981	-.0630
3	1.6031	2.7435	.68018	.51171	.5042	-.02135	-.03251	-.0064	-.0098
4	1.5967	2.7337	.67866	.51062	.5040	.01177	-.01391	.0035	-.0042
5	1.6003	2.7296	.67881	.51071	.5039	.00845	-.01582	.0025	-.0048
6	1.6028	2.7248	.67879	.51069	.5039	.00887	-.01562	.0027	-.0047
7	1.6055	2.7201	.67879	.51068	.5039	.00886	-.01566	.0027	-.0047
8	1.6108	2.7107	.67879	.51066	.5039	.00893	-.01570	.0027	-.0047
9	1.6162	2.7013	.67879	.51064	.5039	.00900	-.01574	.0027	-.0047
10	1.6270	2.6824	.67879	.51060	.5039	.00914	-.01582	.0027	-.0047
11	1.6381	2.6634	.67876	.51055	.5039	.00955	-.01574	.0029	-.0047
12	1.6493	2.6443	.67877	.51051	.5039	.00943	-.01598	.0028	-.0048
13	1.6607	2.6251	.67877	.51047	.5039	.00958	-.01606	.0029	-.0048
14	1.6723	2.6058	.67876	.51042	.5039	.00973	-.01615	.0029	-.0048
15	1.6840	2.5864	.67876	.51038	.5038	.00989	-.01623	.0030	-.0049
16	1.6960	2.5668	.67876	.51033	.5038	.01005	-.01632	.0030	-.0049
17	1.7081	2.5472	.67875	.51028	.5038	.01022	-.01640	.0031	-.0049
18	1.7204	2.5275	.67875	.51023	.5038	.01039	-.01649	.0031	-.0050
19	1.7330	2.5077	.67875	.51018	.5038	.01056	-.01658	.0032	-.0050
20	1.7457	2.4877	.67874	.51012	.5037	.01074	-.01667	.0032	-.0050
21	1.7587	2.4677	.67874	.51007	.5037	.01093	-.01676	.0033	-.0050
22	1.7719	2.4475	.67874	.51001	.5037	.01112	-.01685	.0033	-.0051
23	1.7853	2.4273	.67873	.50996	.5036	.01131	-.01694	.0034	-.0051
24	1.7989	2.4071	.67873	.50990	.5036	.01147	-.01704	.0034	-.0051
25	1.8128	2.3866	.67872	.50984	.5036	.01171	-.01711	.0035	-.0051
26	1.8269	2.3660	.67872	.50978	.5035	.01192	-.01721	.0036	-.0052
27	1.8412	2.3453	.67870	.50972	.5035	.01205	-.01717	.0036	-.0052
28	1.8558	2.3247	.67870	.50966	.5035	.01251	-.01728	.0038	-.0052
29	1.8708	2.3038	.67870	.50958	.5034	.01277	-.01734	.0038	-.0052
30	1.8860	2.2828	.67871	.50952	.5034	.01280	-.01758	.0038	-.0053

TABLE III ITERATION RESULTS FOR $\lambda=\lambda_1$, $\mu=\mu_1$, b/a=2, α_o=1.5, β_o=1.5

Step	α	β	$\frac{V}{2\pi ab^2}$	$\frac{W(\frac{\sigma_o}{p_o})}{4\pi ab^2 \rho g}$	$\frac{t}{b}(\frac{\sigma_o}{p_o})$	F_α	F_β	$\Delta\alpha$	$\Delta\beta$
1	1.5000	1.5000	.53743	.62673	.5000	.29499	.24269	.0885	.0728
2	1.5885	1.5728	.56271	.63813	.5000	.27202	.22630	.0816	.0679
3	1.6701	1.6407	.58448	.64838	.5000	.25255	.21206	.0758	.0636
4	1.7459	1.7043	.60343	.65763	.5000	26.8590	22.7980	.2312	.1963
5	1.9771	1.9006	.65449	.68397	.5000	.72402	.70662	.2172	.2120
6	2.1944	2.1126	.69661	.70714	.5000	-.06577	.03817	-.0197	.0115
7	2.1303	2.1692	.69507	.70564	.5000	-.04389	.05289	-.0132	.0159
8	2.0792	2.2325	.69515	.70509	.5000	-.04058	.05212	-.0122	.0156
9	2.0317	2.2945	.69528	.70462	.5000	-.03797	.05084	-.0114	.0153
10	1.9872	2.3549	.69540	.70419	.5000	-.03561	.04950	-.0107	.0149
11	1.9455	2.4137	.69553	.70380	.5000	-.03342	.04814	-.0100	.0144
12	1.9063	2.4709	.69566	.70343	.5000	-.03135	.04680	-.0094	.0140
13	1.8696	2.5264	.69579	.70310	.5000	-.02940	.04543	-.0088	.0136
14	1.8012	2.6200	.69601	.70259	.5000	-.02680	.04309	-.0161	.0259
15	1.7494	2.7211	.69626	.70210	.5000	-.02346	.04044	-.0141	.0243
16	1.6956	2.8159	.69650	.70169	.5000	-.02089	.03796	-.0125	.0228
17	1.6475	2.9049	.69673	.70135	.5000	-.01870	.03561	-.0112	.0214
18	1.6043	2.9883	.69695	.70107	.5000	-.01680	.03339	-.0101	.0200
19	1.5656	3.0666	.69716	.70084	.5000	-.01515	.03132	-.0091	.0188
20	1.5305	3.1400	.69736	.70065	.5000	-.01370	.02938	-.0082	.0176
21	1.5065	3.1920	.69751	.70053	.5000	-.01273	.02801	-.0076	.0168
22	1.4770	3.2577	.69769	.70040	.5000	-.01159	.02628	-.0070	.0158
23	1.4502	3.3195	.69789	.70030	.5000	-.01091	.02453	-.0065	.0147
24	1.4256	3.3773	.69803	.70020	.5000	-.00966	.02317	-.0058	.0139
25	1.4032	3.4316	.69816	.70013	.5000	-.00884	.02178	-.0053	.0131
26	1.3826	3.4827	.69835	.70007	.5000	-.00817	.02043	-.0049	.0123
27	1.3636	3.5306	.69849	.70002	.5000	-.00747	.01923	-.0045	.0115
28	1.3463	3.5757	.69862	.69999	.5000	-.00688	.01809	-.0041	.0109
29	1.3302	3.6182	.69875	.69997	.5000	-.00641	.01702	-.0038	.0102
30	1.3154	3.6582	.69887	.69996	.5000	-.00590	.01601	-.0035	.0096
31	1.3017	3.6957	.69898	.69995	.5000	-.00544	.01509	-.0033	.0091

TABLE IV ITERATION RESULTS FOR $\lambda=\lambda_1$, $\mu=\mu_1$, $b/a=2$, $\alpha_0=2.0$, $\beta_0=3.0$

Step	α	β	$\frac{V}{2\pi ab^2}$	$\frac{W(\frac{\sigma_o}{p_o})}{4\pi ab^2 \rho g}$	$\frac{t}{b}(\frac{\sigma_o}{p_o})$	F_α	F_β	$\Delta\alpha$	$\Delta\beta$
1	2.0000	3.0000	.73915	.72761	.5000	-.46258	-.21858	-.1388	-.0656
2	1.8612	2.9344	.72206	.71693	.5000	-.31011	-.12583	-.0930	-.0378
3	1.7682	2.8967	.70998	.70953	.5000	-.18587	-.05414	-.0558	-.0162
4	1.7124	2.8804	.70272	.70513	.5000	-.10019	-.00669	-.0301	-.0020
5	1.6824	2.8784	.69912	.70294	.5000	-.05330	.01847	-.0160	.0055
6	1.6664	2.8840	.69761	.70199	.5000	-.03224	.02937	-.0097	.0088
7	1.6567	2.8928	.69704	.70160	.5000	-.02379	.03344	-.0071	.0100
8	1.6496	2.9028	.69685	.70143	.5000	-.02050	.03476	-.0062	.0104
9	1.6434	2.9132	.69680	.70135	.5000	-.01915	.03506	-.0058	.0105
10	1.6377	2.9238	.69680	.70130	.5000	-.01851	.03498	-.0056	.0105
11	1.6161	2.9653	.69689	.70115	.5000	-.01735	.03398	-.0052	.0102
12	1.5957	3.0056	.69700	.70102	.5000	-.01646	.03292	-.0049	.0099
13	1.5763	3.0446	.69711	.70090	.5000	-.01563	.03188	-.0047	.0096
14	1.5579	3.0825	.69721	.70080	.5000	-.01485	.03088	-.0045	.0093
15	1.5361	3.1281	.69733	.70068	.5000	-.01392	.02969	-.0084	.0178
16	1.5039	3.1977	.69752	.70052	.5000	-.01263	.02786	-.0076	.0167
17	1.4746	3.2630	.69770	.70039	.5000	-.01148	.02615	-.0069	.0157
18	1.4480	3.3243	.69790	.70028	.5000	-.01048	.02455	-.0063	.0147
19	1.4237	3.3818	.69804	.70019	.5000	-.00958	.02306	-.0058	.0138
20	1.4014	3.4359	.69820	.70012	.5000	-.00878	.02167	-.0053	.0130
21	1.3809	3.4867	.69840	.70007	.5000	-.00811	.02033	-.0049	.0122

TABLE V ITERATION RESULTS FOR $\lambda=\lambda_2$, $\mu=\mu_2$, b/a=1

Step	α	β	$\frac{V}{2\pi ab^2}$	$\frac{W(\frac{\sigma_o}{p_o})}{4\pi ab^2 \rho g}$	$\frac{t}{b}(\frac{\sigma_o}{p_o})$	F_α	F_β	$\Delta\alpha$	$\Delta\beta$
1	3.0000	3.0000	.80611	.60146	.5238	-.05923	-.05224	-.1777	-.1567
2	2.8223	2.8433	.78954	.58952	.5219	-.06529	-.05730	-.1959	-.1719
3	2.6265	2.6714	.76878	.57453	.5192	-.07086	-.06187	-.2126	-.1856
4	2.4139	2.4858	.74268	.55566	.5153	-.07227	-.06293	-.2168	-.1888
5	2.1971	2.2970	.71143	.53294	.5097	-.05642	-.04991	-.1693	-.1497
6	2.0278	2.1473	.68281	.51206	.5039	-.00851	-.01119	-.0255	-.0336
7	2.0023	2.1137	.67719	.50794	.5026	.00605	.00053	.0182	.0016
8	2.0205	2.1153	.67928	.50944	.5030	.00046	-.00402	.0014	-.0121
1	2.5000	2.5000	.75000	.56092	.5164	-.07211	-.06410	-.0721	-.0641
2	2.4279	2.4359	.74052	.55404	.5149	-.07125	-.06341	-.0713	-.0634
3	2.3566	2.3725	.73062	.54684	.5132	-.06854	-.06117	-.0685	-.0612
4	2.2881	2.3113	.72056	.53951	.5114	-.06330	-.05683	-.0633	-.0568
5	2.2248	2.2545	.71074	.53233	.5095	-.05507	-.05002	-.0551	-.0500
6	2.1697	2.2045	.70172	.52577	.5077	-.04418	-.04101	-.0442	-.0410
7	2.1256	2.1635	.69414	.52022	.5062	-.03206	-.03100	-.0321	-.0310
8	2.0935	2.1325	.68835	.51599	.5049	-.02082	-.02172	-.0208	-.0217
9	2.0727	2.1108	.68437	.51307	.5041	-.01202	-.01445	-.0120	-.0145
10	2.0607	2.0963	.68188	.51123	.5035	-.00605	-.00952	-.0061	-.0095
11	2.0546	2.0868	.68041	.51015	.5032	-.00239	-.00650	-.0024	-.0065
1	4.0000	2.0000	.80000	.59625	.5212	-.04804	-.07411	-.0480	-.0741
2	3.9520	1.9259	.79348	.59140	.5203	-.04939	-.07776	-.0494	-.0778
3	3.9026	1.8481	.78631	.58605	.5192	-.05070	-.08166	-.0507	-.0817
4	3.8519	1.7665	.77837	.58011	.5178	-.05189	-.08576	-.0519	-.0858
5	3.8000	1.6807	.76958	.57350	.5163	-.05282	-.08990	-.0528	-.0899
6	3.7472	1.5908	.75982	.56610	.5144	-.05324	-.09376	-.0532	-.0938
7	3.6939	1.4971	.74898	.55782	.5122	-.05273	-.09669	-.0527	-.0967
8	3.6412	1.4004	.73704	.54864	.5096	-.05063	-.09744	-.0506	-.0974
9	3.5906	1.3029	.72414	.53862	.5065	-.04591	-.09385	-.0459	-.0939
10	3.5447	1.2091	.71082	.52814	.5030	-.03744	-.08295	-.0374	-.0830
11	3.5072	1.1261	.69826	.51874	.4994	-.00550	-.00138	-.0055	-.0014

TABLE VI ITERATION RESULTS FOR $\lambda=\lambda_3$, $\mu=\mu_3$, b/a=1

Step	α	β	$\frac{V}{2\pi ab^2}$	$\frac{W(\frac{\sigma_o}{p_o})}{4\pi ab^2 \rho g}$	$\frac{t}{b}(\frac{\sigma_o}{p_o})$	F_α	F_β	$\Delta\alpha$	$\Delta\beta$
1	1.5000	1.5000	.53743	.43810	.5000	.29499	.24269	.1838	.1512
2	1.6838	1.6512	.58787	.46162	.5000	.24949	.20988	.2121	.1784
3	1.8959	1.8296	.63745	.48553	.5000	.10529	.09039	.2106	.1808
4	2.1064	2.0104	.67902	.50889	.5026	.03332	.02410	.0667	.0482
5	2.1731	2.0586	.68997	.51692	.5050	.02556	.01793	.0511	.0359
6	2.2242	2.0944	.69793	.52274	.5067	.02035	.01376	.0407	.0275
7	2.3259	2.1606	.71246	.53335	.5100	.01173	.00680	.0235	.0136
8	2.3868	2.1937	.72019	.53898	.5110	.00761	.00342	.0152	.0069
9	2.4274	2.2100	.72473	.54228	.5118	.00532	.00153	.0107	.0031
10	2.4380	2.2131	.72584	.54306	.5120	.00479	.00109	.0096	.0022
1	2.5000	3.0000	.77759	.58111	.5206	-.01458	-.01210	-.0292	-.0242
2	2.4709	2.9758	.77447	.57887	.5202	-.01373	-.01149	-.0275	-.0230
3	2.4434	2.9528	.77145	.57670	.5198	-.01286	-.01088	-.0257	-.0218
4	2.4177	2.9311	.76856	.57463	.5194	-.01199	-.01027	-.0240	-.0205
5	2.3937	2.9105	.76581	.57265	.5190	-.01112	-.00966	-.0222	-.0193
6	2.3715	2.8912	.76320	.57077	.5187	-.01025	-.00906	-.0205	-.0181
7	2.3150	2.8403	.75626	.56578	.5177	-.00778	-.00736	-.0156	-.0147
8	2.2728	2.7992	.75070	.56178	.5168	-.00562	-.00590	-.0112	-.0118
9	2.2428	2.7663	.74642	.55869	.5162	-.00385	-.00471	-.0077	-.0094
10	2.2284	2.7482	.74419	.55708	.5158	-.00288	-.00406	-.0058	-.0081
1	3.0000	2.0000	.75000	.56001	.5151	-.00521	-.00949	-.0104	-.0190
2	2.9896	1.9810	.74796	.55847	.5148	-.00455	-.00885	-.0091	-.0177
3	2.9805	1.9633	.74606	.55707	.5144	-.00393	-.00825	-.0079	-.0165
4	2.9726	1.9468	.74430	.55576	.5141	-.00334	-.00767	-.0067	-.0153
5	2.9603	1.9172	.74123	.55346	.5135	-.00229	-.00663	-.0046	-.0133
6	2.9521	1.8916	.73869	.55154	.5130	-.00140	-.00575	-.0028	-.0115
7	2.9473	1.8694	.73660	.54997	.5126	-.00066	-.00500	-.0013	-.0100
8	2.9460	1.8594	.73570	.54930	.5124	-.00034	-.00468	-.0007	-.0094

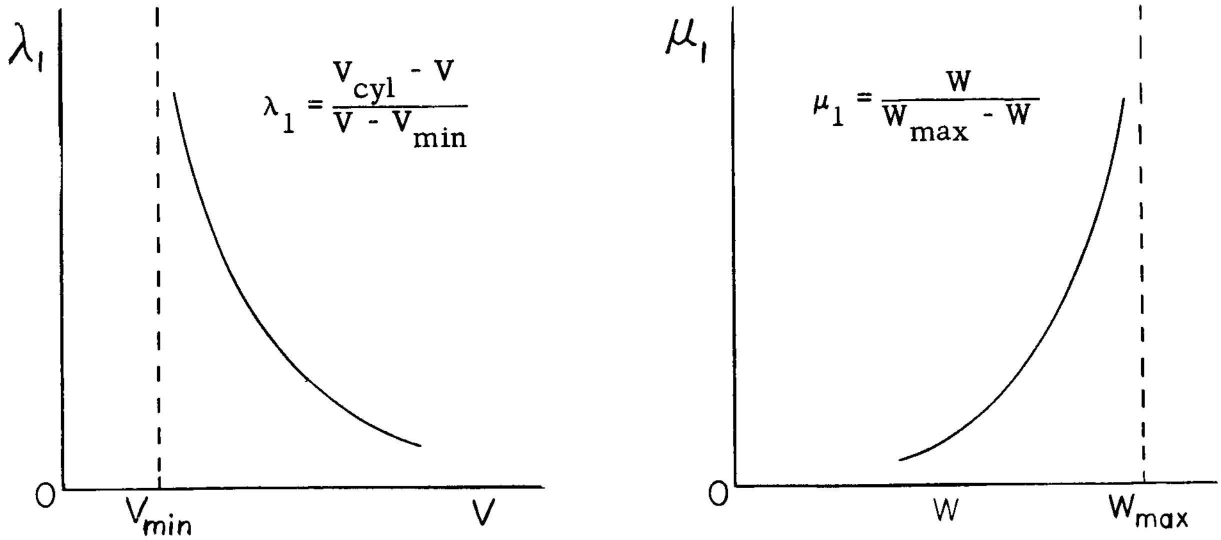

Fig. 1 Weighting Functions λ_1 and μ_1

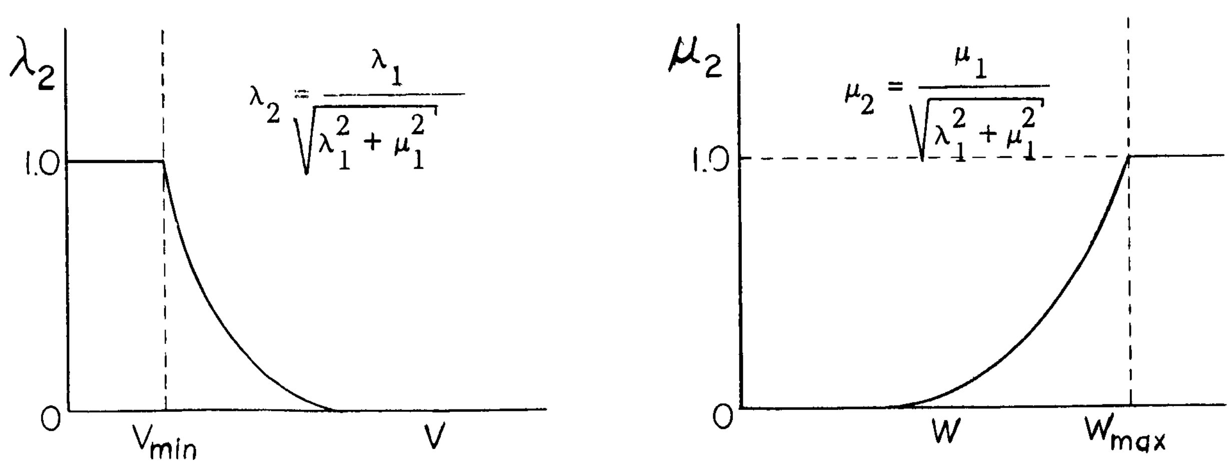

Fig. 2 Weighting Functions λ_2 and μ_2

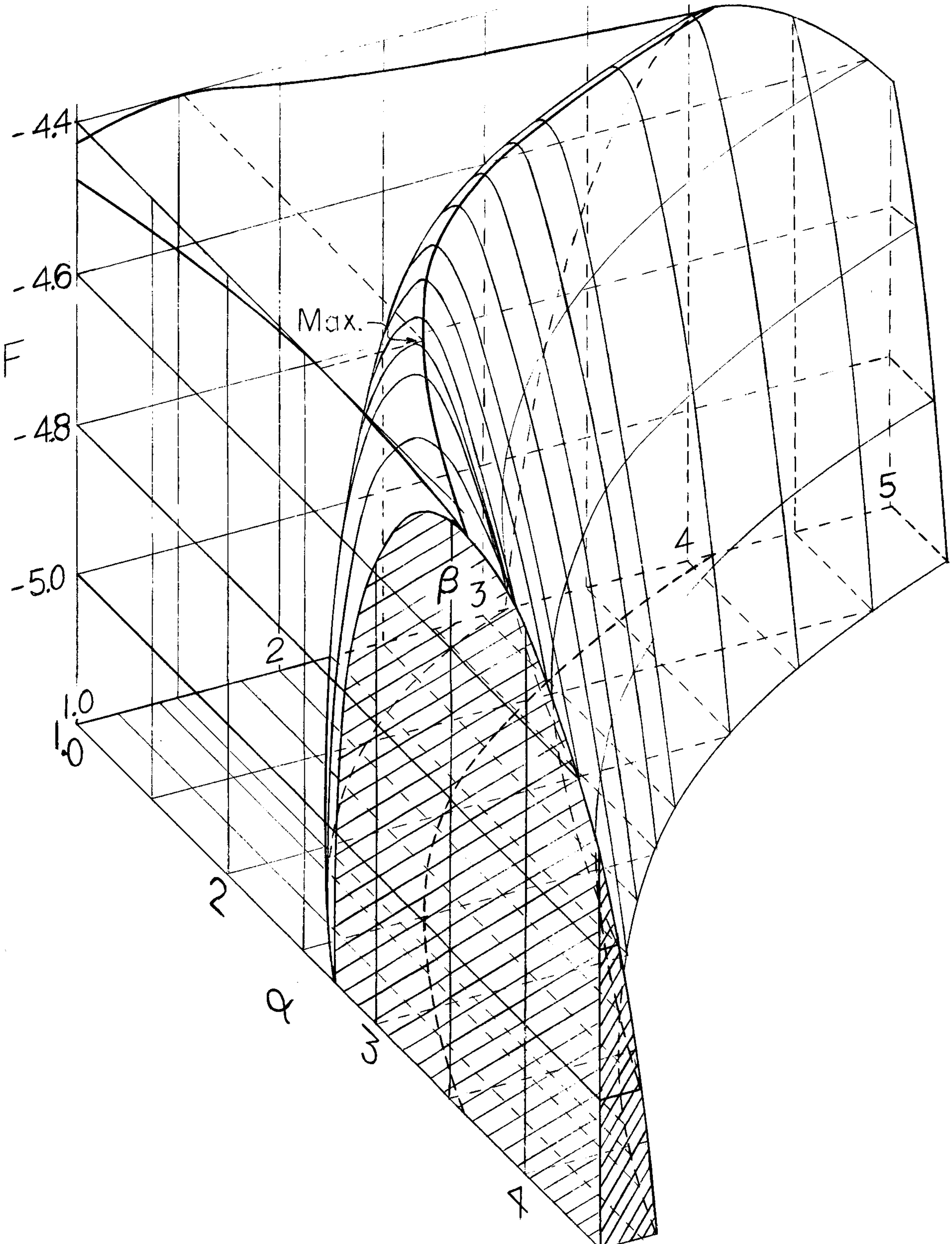

Fig. 3 Variation of the Function F with α and β

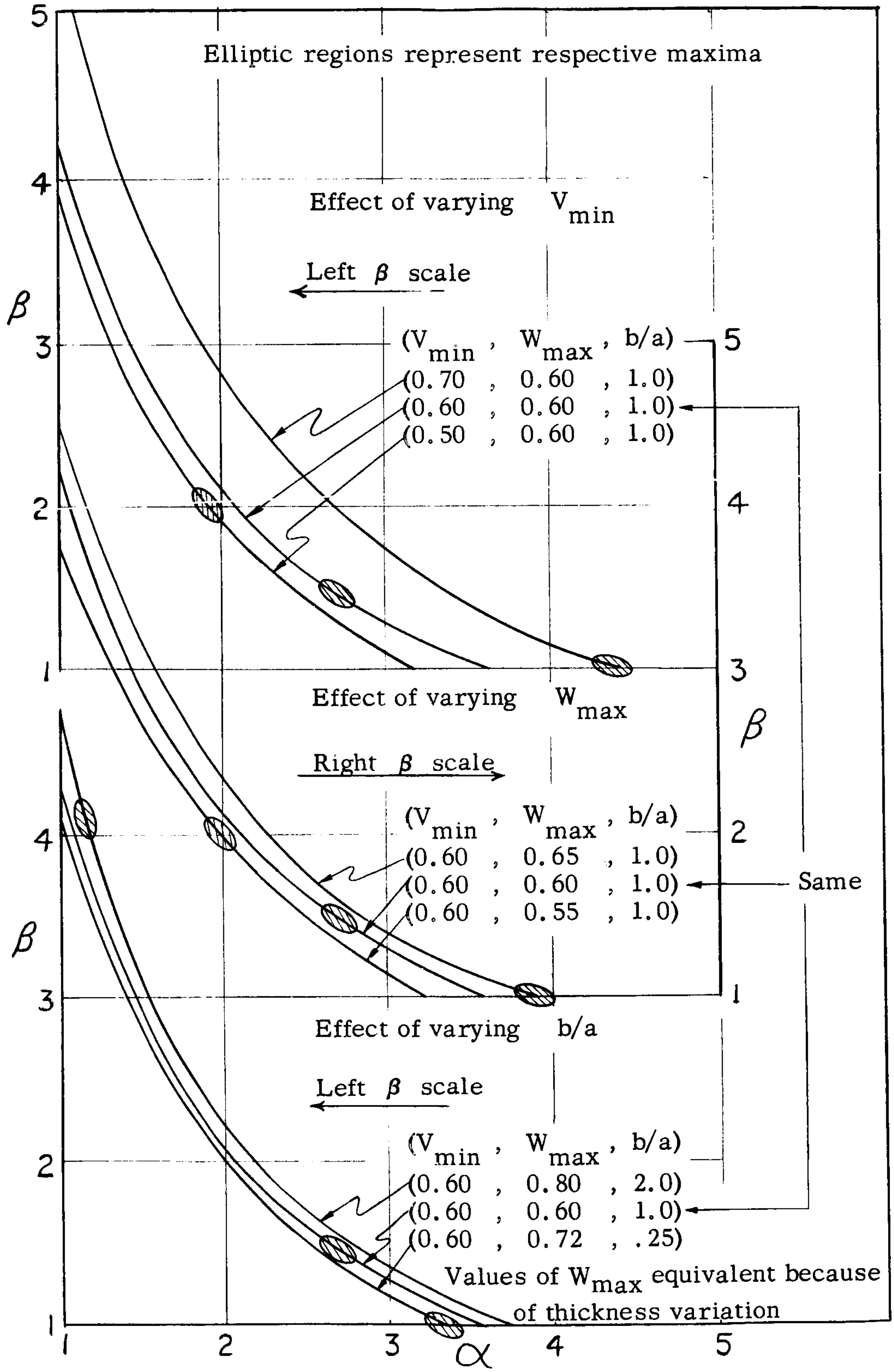

Fig. 4 Variation in Projections of Ridges on α-β Plane

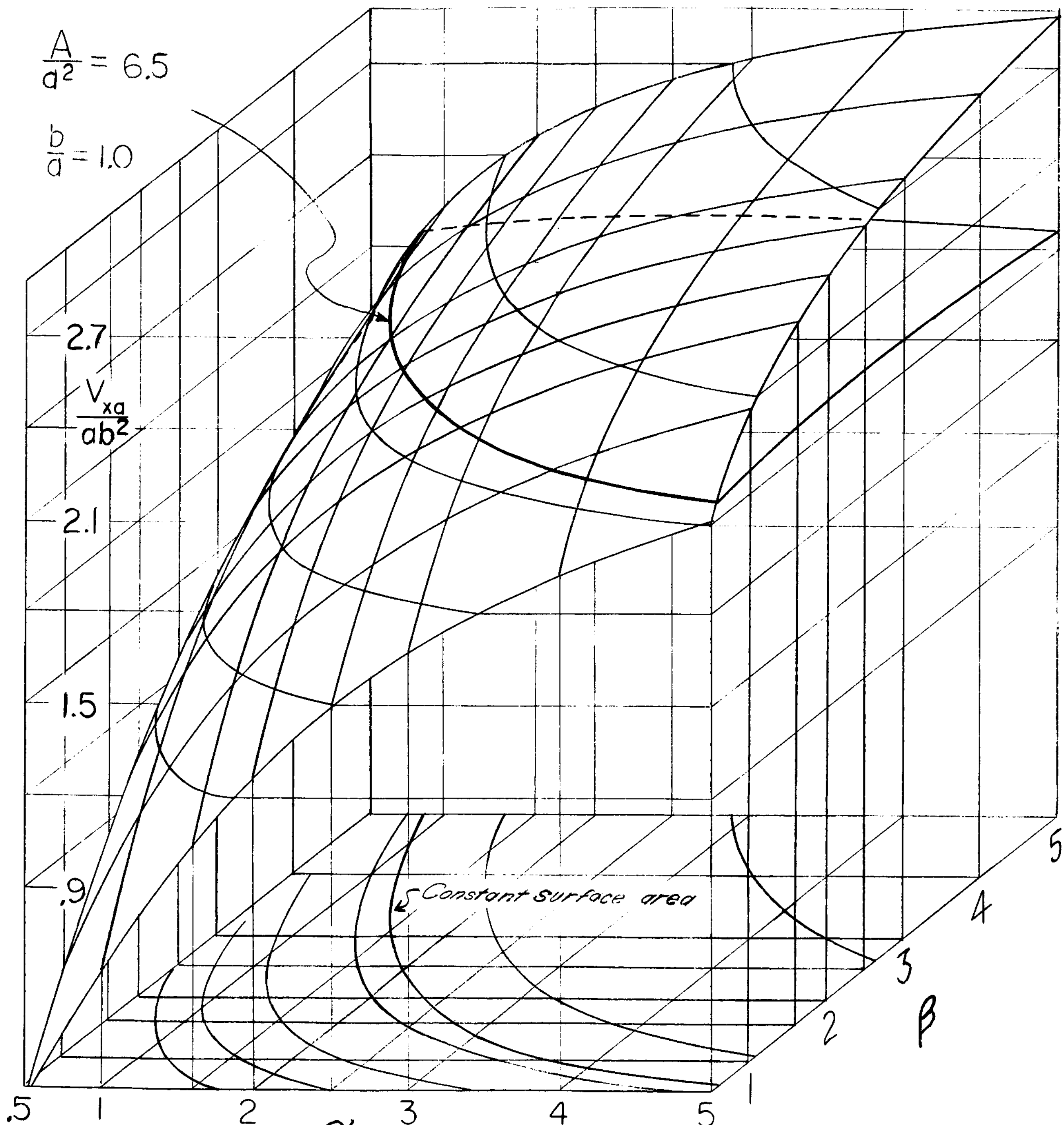

Fig.5 Variation in V_{xa}/ab^2 with α and β

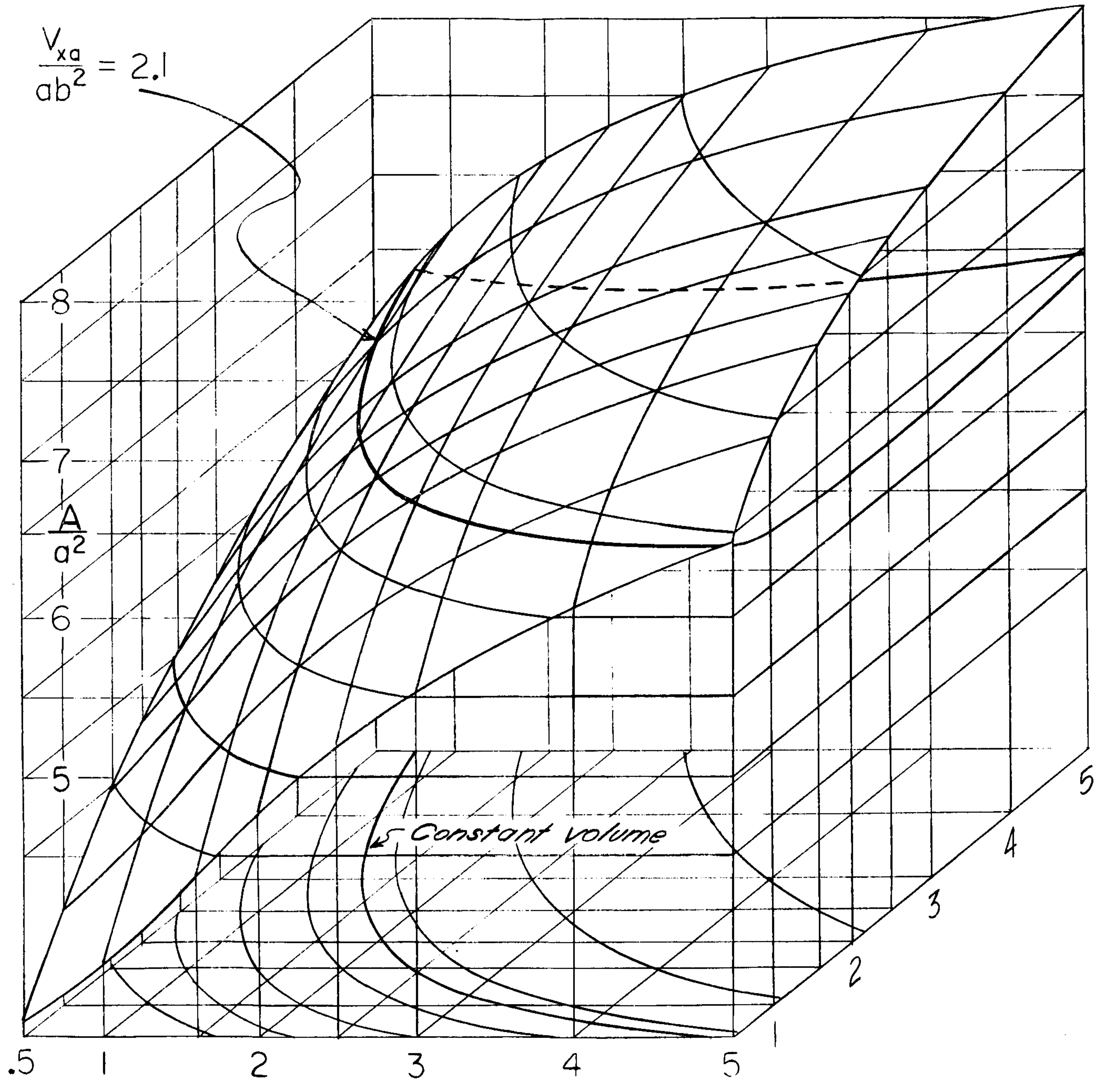

Fig.6 Variation in A/a^2 with α and β for $b/a = 1.0$

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