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INVARIANT IMBEDDING AND PERTURBATION TECHNIQUES APPLIED TO DIFFUSE REFLECTION FROM SPHERICAL SHELLS

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The RAND Corporation
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PREFACE

This Memorandum was prepared as part of RAND's continuing study of Satellite Meteorology for the National Aeronautics and Space Administration under contract number NASr-21(07). The problem of diffuse reflection of light from a spherical shell is of central importance in meteorology and astrophysics. This Memorandum presents a method for assessing quantitatively the effects of sphericity on diffuse reflection patterns. This study is a necessary prerequisite for further work devoted to inverse problems for spherical shell atmosphere. This should aid in the planning and interpretation of meteorological satellite experiments.

SUMMARY

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This Memorandum presents a method for integrating numerically the invariant imbedding relation for the diffuse reflection coefficient of a spherical shell. The reflection coefficient of a shell is related to that of a slab via a perturbation technique.

The results of some test calculations are presented; they compare well with results obtained in an earlier Memorandum[1] using a different method. Interestingly enough, the perturbation technique proves useful for quite large values of the perturbation parameter.

Author

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I. INTRODUCTION

The equation for the scattering function of a spherical shell is a nonlinear partial differential integral equation. Little computational experience is available in the numerical solution of such complex functional equations. In an earlier paper,[1] we described an approach based upon the numerical estimation of partial derivatives. The results obtained seemed satisfactory from both the physical and computational viewpoints. Nevertheless, the importance of the equation and the newness of the calculation made it seem advisable to approach the problem from a second and independent point of view. We elected to use a perturbation technique based upon an expansion in inverse powers of the inner radius of the shell.

Results of this numerical experiment confirmed our earlier calculation to several significant figures. Remarkably enough, this was achieved using only the first and second terms in the perturbation expansion.

Let conical flux of net intensity π per unit area be uniformly incident upon a hollow shell of inner radius a and outer radius z. The intensity of diffusely reflected radiation in the direction arc cos v, due to incident flux with direction arc cos u, is

$$r(x,v,u) = \frac{S(z,v,u)}{4v} \quad , \tag{1}$$

where the S function satisfies the rather formidable functional equation

$$S_z + \frac{(1-v^2)}{zv} S_v + \frac{(1-u^2)}{zu} S_u + \left(\frac{1}{u} + \frac{1}{v}\right) S - \left(\frac{u^2+v^2}{zu^2v^2}\right) S$$

$$= \lambda \left[1 + \frac{1}{2} \int_0^1 S(z,v',u) \frac{dv'}{v'}\right] \left[1 + \frac{1}{2} \int_0^1 S(z,v,u') \frac{du'}{u'}\right] , \tag{2}$$

for $z \geq a$, with the initial condition $S(a,v,u) = 0$.[2,3] Here λ is the albedo for single scattering. We suppose that the core is a perfect absorber. The function S is symmetric in v and u, a fact which can be used either to test the accuracy of the numerical solution, or to reduce the size of the problem.

In a previous paper,[1] we obtained a numerical solution of this equation by means of an estimation of S_u as a linear combination of the values of S at suitably chosen points. This will be called Method I. In this note, we use a perturbation expansion in powers of $\frac{1}{a}$ and compare the results of the two calculations. The agreement is excellent.

II. PERTURBATION TECHNIQUE

Let us introduce the thickness of the shell, $x = z-a$, assume that $\frac{x}{a} \ll 1$, and substitute the formal expansion

$$S = S_o + \frac{S_1}{a} + \frac{S_2}{a^2} + \ldots \tag{3}$$

in Eq. (2), with the quantities S_o, S_1, S_2, ..., independent of a. S_o turns out to be the S-function for a plane parallel slab, a function which can be easily and accurately calculated.(1,3) Equating coefficients of powers of $\frac{1}{a}$, we obtain the following equations

$$(S_o)_x + \left(\frac{1}{u} + \frac{1}{v}\right) S_o = \lambda\left[1 + \frac{1}{2}\int_o^1 S_o(x,v', u)\,\frac{dv'}{v'}\right]\left[1 + \frac{1}{2}\int_o^1 S_o(x,v,u')\,\frac{du'}{u'}\right] ,$$

and

$$\begin{aligned} (S_1)_x &+ \left(\frac{1-v^2}{v}\right)(S_o)_v + \left(\frac{1-u^2}{u}\right)(S_o)_u + \left(\frac{1}{u} + \frac{1}{v}\right) S_1 - \frac{u^2+v^2}{u^2v^2}\, S_o = \\ &\lambda\left[1 + \frac{1}{2}\int_o^1 S_o(x,v,u')\,\frac{du'}{u'}\right]\left[\frac{1}{2}\int_o^1 S_1(x,v',u)\,\frac{dv'}{v'}\right] \\ &+ \lambda\left[1 + \frac{1}{2}\int_o^1 S_o(x,v',u)\,\frac{dv'}{v'}\right]\left[\frac{1}{2}\int_o^1 S_1(x,v,u')\,\frac{du'}{u'}\right] . \end{aligned} \tag{4}$$

Let us write $S_o^{ij}(x) = S_o(x,v_i,v_j)$, $S_1^{ij}(x) = S_1(x,v_i,v_j)$, $U^{ij}(x) = (S_o)_u$, $V^{ij}(x) = (S_o)_v$, where the v_i are the roots of the shifted Legendre polynomials. Using quadrature techniques as described in Refs. 4 and 5, we reduce Eq. (4) to a system of ordinary differential equations involving the functions introduced above. To solve these, we require the functions U^{ij} and V^{ij}. These partial derivatives are produced in two different ways. First of all, we can use the numerical estimation formula of Ref. 1. We shall refer to this as Method IIa. Secondly, we can turn to the equation for S_o and obtain, by means of partial differentiation, corresponding equations for $(S_o)_u$ and $(S_o)_v$. These are adjoined to Eq. (4) to provide a complete set of equations for the four functions S_o, $(S_o)_u$, $(S_o)_v$, S_1. Quadrature techniques applied to these produce the desired ordinary differential equations. This process involves the numerical solution of N(3N+1) simultaneous equations. We shall refer to this as Method IIb.

III. COMPUTATIONAL RESULTS

Calculation of the reflection function $r(x,v,u)$ was carried out with the two variations of the perturbation technique. Some checks consist of varying the order of the quadrature formula, varying the step length of integration, and increasing the inner radius a so as to approach the limiting value $a = \infty$. Other checks consist of comparisons among the three computational schemes (Methods I, IIa, and IIb). We found the results to be satisfactory.

In Fig. 1, we present three sets of curves for the reflection function r, for the case in which the albedo $\lambda = 1$, the inner radius $a = 50$, and the thickness $x = 3$. Each set corresponds to a different angle of incidence, arc cosine $u = 17.6$ degrees, 60.0 degrees, and 87.3 degrees. The abscissa is arc cosine v, the angle of reflection. Four curves are plotted for each incident direction, although these curves lie on top of one another in certain instances. These curves are labelled "Slab," "I," "IIa," and "IIb." The curves labelled "Slab" correspond to the r function in the limit as $a \to \infty$. Curves "I" are produced by a direct application of the estimation of derivative formula, the method described in Ref. 1. Curves "IIa" are produced by the perturbation method with the use of the derivative estimation formula, and curves "IIb" are produced by the perturbation method alone. The calculations are performed on an IBM 7044 machine with programs written in the FORTRAN IV source language. We used $N = 5$, an integration step size of 0.005, and an Adams-Moulton integration formula. Running times are comparable for Methods IIa and IIb.

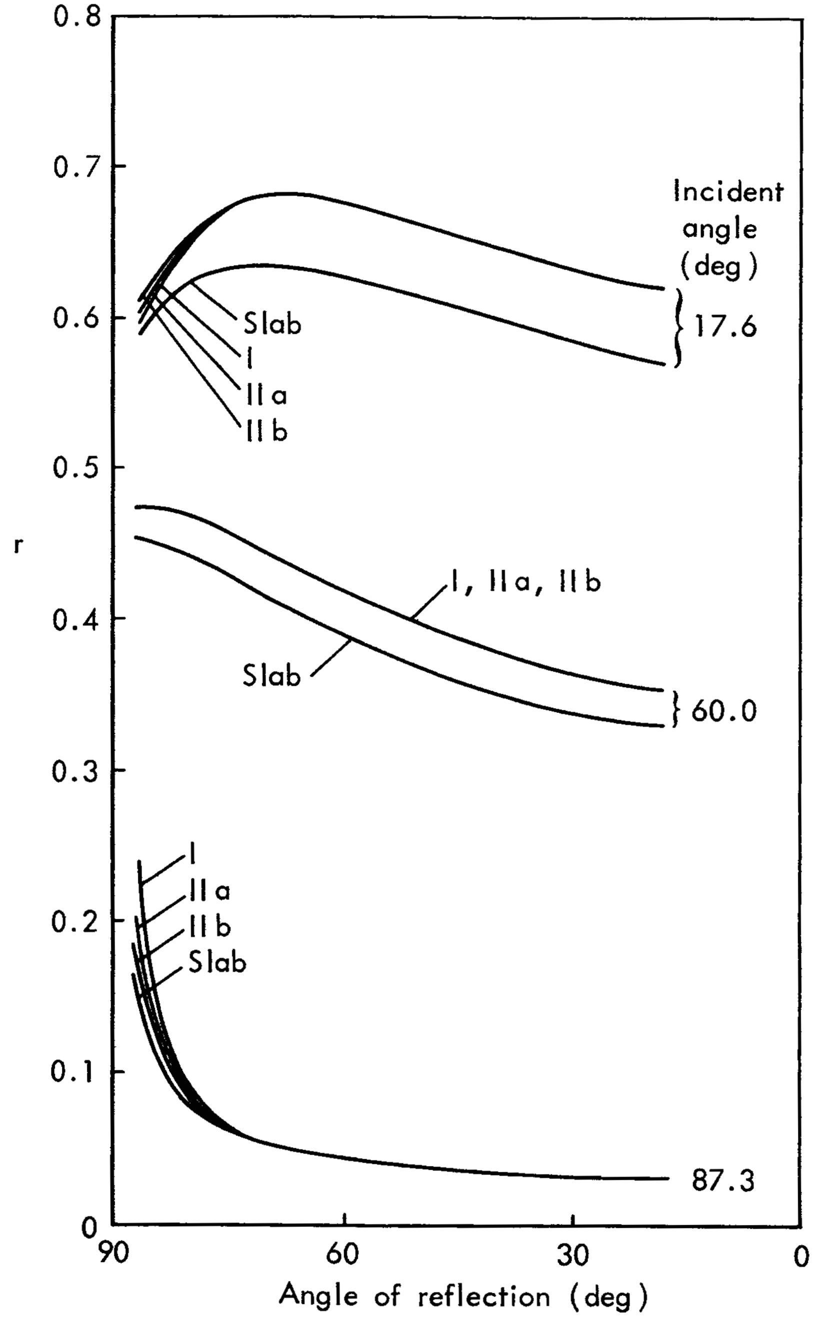

Fig. 1—Some reflected intensity patterns for a shell with albedo $\lambda = 1$, inner radius $a = 50$, and thickness $x = 3$, for various angles of incidence

No difficulties are encountered with the use of the perturbation technique. The comparison of results with the direct calculation of Ref. 1 is quite satisfactory. The present technique has the advantage of producing reflection functions for a variety of shell inner radii in a single calculation. Moreover, it appears to be stable, even when $\frac{x}{a}$ is fairly large. This ratio is $\frac{3}{50} = 0.06$ in the case discussed above. The technique is suitable for many other types of partial differential integral equations, and further results will be presented subsequently.

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