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CONTACT TRANSFORMATIONS AND THE THEORY OF OPTIMAL CONTROL

by R. S. DeZur and G. W. Haynes

Perpared under Contract No. NAS 2-2351 *by*
MARTIN COMPANY
Denver, Colo.

for

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Summary

The main results achieved for NAS 2-2351 are as follows:

1. The primary advantage in using contact transformations is that if the Hamilton-Jacobi partial differential equation can be reduced to a linear partial differential equation by a suitable contact transformation, then the number of ordinary differential equations required to generate a solution are reduced.

2. The Hamilton-Jacobi theory is inadequate, not because of the discontinuities introduced by the bounded controls, but because of the lack of sufficient conditions required to resolve the singular problem.

3. A new system of partial differential equations characterizing the control problem with an enlarged control set has been derived in conjunction with a new optimization procedure.

1. Discontinuities in the Hamilton-Jacobi Theory

The use of the Hamilton-Jacobi theory in the calculus of variations is well established;[1] the extension of these ideas and concepts to control theory[2] especially with regard to the determination of the feedback control, is quite recent. In these theories the Hamilton-Jacobi partial differential equation plays a central role, its complete integral determining in the classical sense, the solution of the canonical equations. One aspect of contact transformations in relationship to the Hamilton-Jacobi theory is that the form of the canonical equations is preserved under a contact transformation. The complete integral to the Hamilton-Jacobi equation can be veiwed as generating a contact transformation that reduces the problem to a point of equilibrium, thus yielding the solutions to the canonical equations. However, as pointed out by Kalman[2] in his extension of the Caratheodory[3] technique to control theory, the bounded controls create a lack of smoothness that makes the classical theory appear inadequate. It should be noted however that Kalman in his efforts to solve Bushaw's Problem[4] did not construct a complete integral to the Hamilton-Jacobi partial differential equation. In solving the same problem, the author transformed the Hamilton-Jacobi equation by using a Legendre contact transformation which is a transformation of both the independent and dependent variables. The resulting partial differential equation was linear, the only discontinuity being a forcing term. The solution to the partial differential equation was constructed by the method of characteristics. The main advantage of this entire treatment of the problem is that the number of ordinary differential equations to be solved are fewer in the linear case than in the non-linear case. The lack of adequate smoothness was not reflected in the treatment of the problem, and the reason for this is that the solution to the Hamilton-Jacobi equation was constructed along characteristics. The Hamilton-Jacobi equation is really a statement about the directional derivative of the cost functional; and the Hamiltonian of the system is smooth along the characteristics. As an indication of this smoothness, it is well known that when the Hamiltonian does not involve the time explicitly, it is a constant along the characteristics.

To illustrate the fact that the Hamilton-Jacobi equation for the problem of Bushaw has meaning everywhere, we shall evaluate the derivatives of the cost functional in the neighborhood of the switching curve. For the purposes of this demonstration we have reversed the strategy to that of transferring from the origin to any point (x_1, x_2) in minimum time. To facilitate the computation we shall consider the point $(1, 1+\delta)$; at this point the Hamilton-Jacobi equation becomes

$$\left.\frac{\partial V}{\partial x_1}\right|_{1,1+\delta}(1+\delta) - \left.\frac{\partial V}{\partial x_2}\right|_{1,1+\delta} + \left|\left.\frac{\partial V}{\partial x_2}\right|_{1,1+\delta}\right| - 1 = 0$$

The derivatives of the cost functional V (=t) are computed by determining the optimal times for the solutions of the system of differential equations,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + u \qquad |u| \leq 1$$

to achieve certain points.

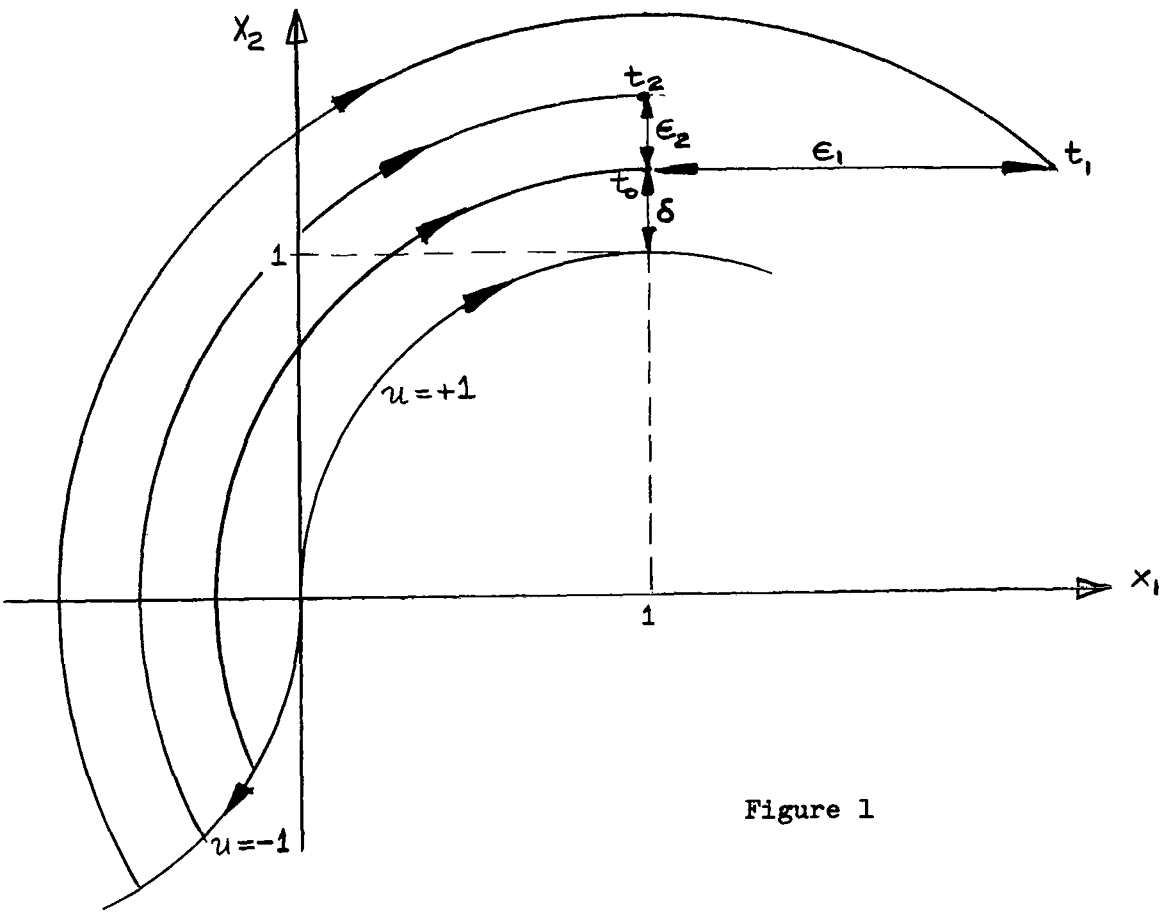

Figure 1

Denoting by t_o, t_1, and t_2 the optimal times to attain the points $(1, 1+\delta)$, $(1+\epsilon_1, 1+\delta)$ and $(1, 1+\delta+\epsilon_2)$ then

$$\left.\frac{\partial V}{\partial x_1}\right|_{1,1+\delta} = \lim_{\epsilon_1 \to 0} \frac{t_1 - t_o}{\epsilon_1} = \frac{1}{1+\delta}$$

$$\left.\frac{\partial V}{\partial x_2}\right|_{1,1+\delta} = \lim_{\epsilon_2 \to 0} \frac{t_2 - t_o}{\epsilon_1} = \frac{4 + 2\delta + \delta^2}{2(1+\delta)\sqrt{4\delta + \delta^2 + \delta^3 + \frac{\delta^4}{4}}}$$

It is readily seen that

$$\frac{\partial V}{\partial x_2}_{1,1+\delta} = O\left(\frac{1}{\sqrt{\delta}}\right)$$

and is undefined for $\delta = 0$; however, the Hamilton-Jacobi equation still holds, the singularities cancelling.

2. Inadequacies of the Hamilton-Jacobi Theory

One significant aspect of the Hamilton-Jacobi approach to control theory is that there are no singular problems, known by the authors, that are treated by this method. By singular, we refer specifically to those problems that are linear in the control, where the maximum principle fails to yield any information regarding the optimal control once the coefficient of the control in the Hamiltonian vanishes. The vanishing of the coefficient of the control is termed the singular condition, and sometimes is used to determine the singular control. One important question concerning these singular problems is how the singular condition should be interpreted with regard to the Hamilton-Jacobi equation.

In order to gain insight into this aspect of the Hamilton-Jacobi theory, consider the following example of a time optimal problem[5] for which an optimal singular arc exists.

Example 1. $\dot{x}_1 = x_1^2 - x_1 x_2 u$

$$\dot{x}_2 = - x_2 + u , \qquad |u| \leq 1 .$$

The problem is to transfer the state vector from $[1, 0]$ to $[2, 0]$ in minimum time. This problem can be treated by the Green's theorem approach[5], from which it can be determined that singular arc, as defined by

$$x_2 = 0$$

is the optimal strategy. The Hamilton-Jacobi equation for this problem is

$$\frac{\partial V}{\partial x_1} x_1^2 - \frac{\partial V}{\partial x_2} x_2 - \left| \frac{\partial V}{\partial x_2} - x_1^2 x_2 \frac{\partial V}{\partial x_1} \right| - 1 = 0$$

and to test its validity we shall compute the partial derivatives of the cost functional V(=t) at a representative point $P(x_1, x_2)$ on the singular arc. In evaluating these derivatives it should be observed that the composite trajectories used to determine these derivatives must satisfy the optimal strategy as indicated in Fig. 2.

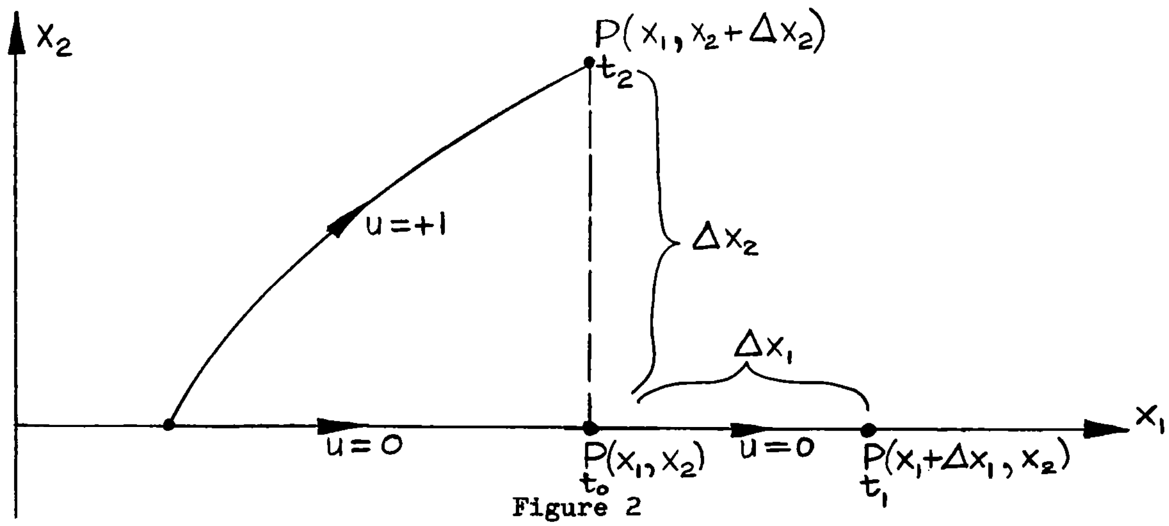

Figure 2

From the definition of the derivatives we have:

$$\frac{\partial V}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{t_1 - t_o}{\Delta x_1} \; ; \quad \frac{\partial V}{\partial x_2} = \lim_{\Delta x_2 \to 0} \frac{t_2 - t_o}{\Delta x_2}$$

where t_o, t_1 and t_2 are the optimal times to achieve the points $P(x_1,x_2)$, $P(x_1 + \Delta x_1 \;\; x_2)$ and $P(x_1, x_2 + \Delta x_2)$ respectively. Evaluating the derivatives for the point P(1,0) yields

$$\left.\frac{\partial V}{\partial x_1}\right|_{1,0} = 1 \qquad \left.\frac{\partial V}{\partial x_2}\right|_{1,0} = 0$$

which satisfies both the Hamilton-Jacobi equation and the singular condition, at the point P(1,0). This result is to be expected, since the optimal strategy was known in advance. In fact the optimal strategy is very crucial to the problem, and can give rise to a fallacy in the Hamilton-Jacobi theory as demonstrated in the next example[6].

Example 2.

This is an example[6] of a time optimal problem which possesses a non-optimal singular arc. The differential equations are

$$\dot{x}_1 = u$$

$$\dot{x}_2 = 1 + x_2 x_1^2 u \qquad |u| \leq 1$$

and the problem is to transfer the state vector $[x_1, x_2]$ from $[0,0]$ to $[0,\frac{1}{2}]$ in minimum time. This problem possesses a singular arc defined by $x_1 = 0$ yielding a singular control $u \equiv 0$; however, by the Green's theorem approach it can be shown that the optimal control is always bang-bang. Since in the Hamilton-Jacobi theory there is no knowledge known beforehand regarding the optimality of the singular arc, we shall assume the singular arc to be optimal, and seek a contradiction. The Hamilton-Jacobi equation for this problem is

$$\frac{\partial V}{\partial x_2} - \left| \frac{\partial V}{\partial x_1} + \frac{\partial V}{\partial x_2} x_2 x_1^2 \right| - 1 = 0 \ .$$

Evaluating the derivatives at a point $P(0,½)$ on the singular arc under the false assumption that the singular arc is optimum, yields

$$\left.\frac{\partial V}{\partial x_1}\right|_{0,½} = 0 \qquad \left.\frac{\partial V}{\partial x_2}\right|_{0,½} = 1$$

which satisfies the Hamilton-Jacobi equation at the point $P(0,½)$ and does not yield a contradiction. The fallacy is now obvious and indicates an inadequacy of the Hamilton-Jacobi theory for singular problems.

3. The Pfaffian Approach to Singular Problems

There remains to be resolved in the Hamilton-Jacobi approach such questions as the optimality of the singular arc and the role of the singular condition in the Hamilton-Jacobi theory. One feature that characterizes the singular problem is that the process of determination of the singular control does not involve the control bounds. This raises the conjecture that if the singular control is optimal, that is, the singular arc lies on the boundary of the reachable set, then it is better than any bang-bang control irrespective of the magnitude of the control bounds.

In order to test this conjecture then as the magnitude of the control gets very large it becomes necessary to include "delta functions" or "impulses" in the control set. It should be noted that this procedure of enlarging the control set has been treated by Kreindler[7] and Neustadt[8] in their investigations of linear systems. For the purposes of the elementary treatment given herein, it suffices to represent $u(t) = \frac{dy}{dt}$ where y is a function of bounded variation.

The feasibility of the conjecture is demonstrated for example 1, where the reachable sets obtainable in $0 \leq t \leq ½$ for the impulsive control, and the bounded control $\left|\frac{dy}{dt}\right| \leq 1$ are shown on Figure 3.

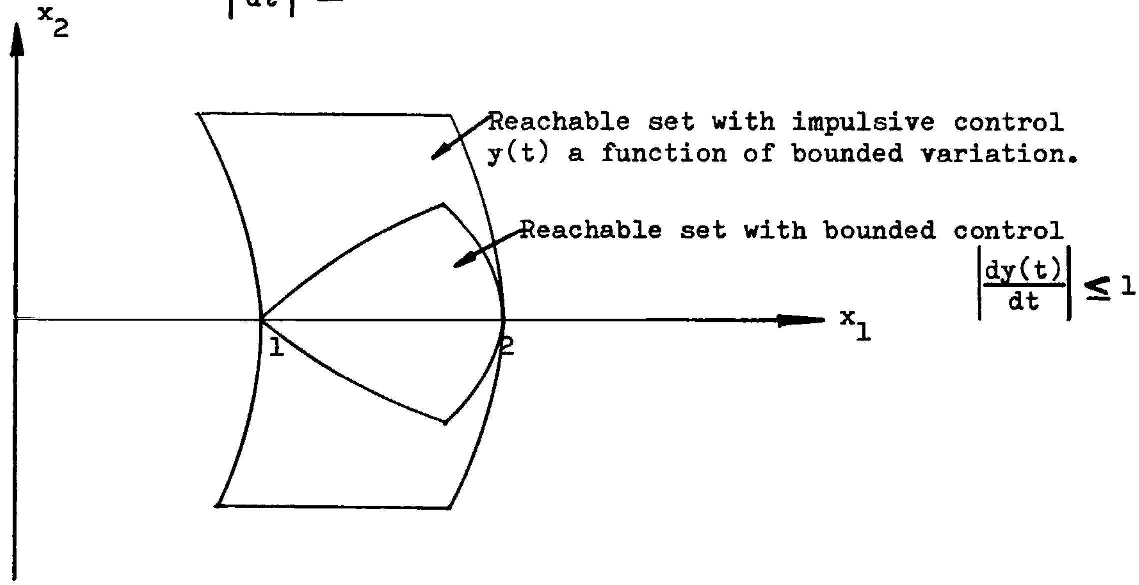

Figure 3

It is to be observed that the singular arc $x_2 = 0$ lies on the boundary of the reachable sets. The reachable set for the impulsive controls is larger, and this is due to the linearity of the control together with the consequence of impulsive controls that some points can be reached in zero time. Since the reachable set for the impulsive controls is more inclusive, then this is the set that should be inspected in order to ascertain the optimality of the singular arc. This fact was suggestive of the following pfaffian approach which is more general than the Green's theorem approach. Consider the following two dimensional system

$$\dot{x}_1 = A_1(x) + B_1(x)u; \quad \dot{x}_2 = A_2(x) + B_2(x)u \,. \tag{3.1}$$

The pfaffian associated with the differential equations (3.1) is

$$B_2(x)dx_1 - B_1(x)dx_2 = \left\{ B_2(x)\ A_1(x) - B_1(x)\ A_2(x) \right\} dt \,, \tag{3.2}$$

and this differential form holds independent of the control. It is assumed that this pfaffian is non-integrable[9]; otherwise all solutions of (3.1) will be contained in a surface independent of the control, so that the system would not be controllable. Two functions $W(x)$ and $\mu(x)$ which satisfy

$$\frac{\partial W(x)}{\partial x_1} = \mu(x)\ B_2(x) \quad \text{and} \quad \frac{\partial W(x)}{\partial x_2} = -\mu(x)\ B_1(x) \tag{3.3}$$

are determined. It should be noted that equations (3.3) do not uniquely determine $W(x)$ and $\mu(x)$; however, once $W(x)$ has been selected, then $\mu(x)$

is uniquely determined. If $W_1(x)$ and $\mu_1(x)$ are two functions that satisfy (3.3) and if $f(\cdot)$ is any C' function with derivative $f'(\cdot)$ then

$$
\begin{aligned}
W_2(x) &\overset{x}{\equiv} f(W_1(x)) \\
\mu_2(x) &\overset{x}{\equiv} \mu_1(x)\, f'(W_1(x))
\end{aligned}
\tag{3.4}
$$

also satisfy the relations (3.3). By virtue of (3.3) the pfaffian (3.2) transforms into

$$
dW = \mu(x) \left\{ B_2(x)\, A_1(x) - B_1(x) A_2(x) \right\} dt \tag{3.5}
$$

The procedure followed so far is the usual construction[9] used to determine the integrability conditions for the pfaffian (3.2); that is, if the right hand side of (3.5) can be expressed as a function of W alone, then the pfaffian (3.2) is integrable. There is, however, a different connotation to be inferred. A solution to the pfaffian (3.5) of the form W = constant, t = constant, represents an "impulsive" solution to the differential equations (3.1). By the inclusion of impulses in the control set, it does not cost any time for the state x to traverse a constant W line. In fact, W defines the wave front or zero cost line for the system (3.1), so the time optimal problem for the system (3.1) becomes that of determining the points on the constant W line where the time rate of change of W as given by (3.5) is extremized. The locus of all such points determines the singular arcs. Expressing (3.5) as

$$\frac{dW}{dt} = \frac{\partial W}{\partial x_1}(x)\ A_1(x) + \frac{\partial W}{\partial x_2}(x)\ A_2(x) \tag{3.6}$$

by using (3.3), then the values of x_1 and x_2 constrained to $W(x) = \text{constant}$, that make $\frac{dW}{dt}$ stationary are determined by

$$\frac{\partial^2 W(x)}{\partial x_1^2} A_1(x) + \frac{\partial W(x)}{\partial x_1} \frac{\partial A_1(x)}{\partial x_1} + \frac{\partial^2 W(x)}{\partial x_2 \partial x_1} A_2(x) + \frac{\partial W(x)}{\partial x_2} \frac{\partial A_2(x)}{\partial x_1} + \lambda \frac{\partial W(x)}{\partial x_1} = 0$$

$$\frac{\partial^2 W(x)}{\partial x_1 \partial x_2} A_1(x) + \frac{\partial W(x)}{\partial x_1} \frac{\partial A_1(x)}{\partial x_2} + \frac{\partial^2 W(x)}{\partial x_2^2} A_2(x) + \frac{\partial W(x)}{\partial x_2} \frac{\partial A_2(x)}{\partial x_2} + \lambda \frac{\partial W(x)}{\partial x_2} = 0$$

(3.7)

where λ is a lagrange multiplier. Since from (3.3), $W(x)$ is determined by

$$\frac{\partial W(x)}{\partial x_1} B_1(x) + \frac{\partial W(x)}{\partial x_2} B_2(x) \overset{x}{\equiv} 0 \tag{3.8}$$

then (3.7) can be reduced to

$$B_1(x)\left[\frac{\partial^2 W(x)}{\partial x_1^2} A_1(x) + \frac{\partial W(x)}{\partial x_1} \frac{\partial A_1(x)}{\partial x_1} + \frac{\partial^2 W(x)}{\partial x_2 \partial x_1} A_2(x) + \frac{\partial W(x)}{\partial x_2} \frac{\partial A_2(x)}{\partial x_1}\right]$$

$$+ B_2(x)\left[\frac{\partial^2 W(x)}{\partial x_1 \partial x_2} A_1(x) + \frac{\partial W(x)}{\partial x_1} \frac{\partial A_1(x)}{\partial x_2} + \frac{\partial^2 W(x)}{\partial x_2^2} A_2(x) + \frac{\partial W(x)}{\partial x_2} \frac{\partial A_2(x)}{\partial x_2}\right] = 0$$

(3.9)

which together with $W(x) = \text{constant}$, determine the values of x_1 and x_2 that make $\frac{dW}{dt}$ stationary. Differentiating (3.8), which is an identity in x, with respect to x_1 and also x_2, and using these expressions to eliminate the second partials of W from (3.9) and finally eliminating the first partials of W by (3.3), there results

$$\mu(x)\left[B_2(x)\left\{-\frac{\partial B_1(x)}{\partial x_1}A_1(x)+\frac{\partial A_1(x)}{\partial x_1}B_1(x)+\frac{\partial A_1(x)}{\partial x_2}B_2(x)-\frac{\partial B_1(x)}{\partial x_2}A_2(x)\right\}\right.$$
$$\left.-B_1(x)\left\{-\frac{\partial B_2(x)}{\partial x_1}A_1(x)-\frac{\partial B_2(x)}{\partial x_2}A_2(x)+\frac{\partial A_2(x)}{\partial x_1}B_1(x)+\frac{\partial A_2(x)}{\partial x_2}B_2(x)\right\}\right]=0 \qquad (3.10)$$

Defining a three vector X(x) having components in the t, x_1, and x_2 coordinates of $\left\{B_2(x)\ A_1(x)-B_1(x)\ A_2(x);\ -B_2(x);\ B_1(x)\right\}$, then (3.10) can be expressed in the succinct form

$$X(x)\cdot \operatorname{curl} X(x)=0 \qquad (3.11)$$

where $\mu(x)$, assumed not be be identically zero, has been deleted. This expression (3.11) determines the locus of the values of x along which $\frac{dW}{dt}$ is stationary. It should be observed that the integrability condition for the pfaffian (3.2) is

$$X(x)\cdot \operatorname{curl} X(x)\overset{x}{\equiv}0\,.$$

The optimality of the singular arc follows directly from whether $\frac{dW}{dt}$ is a maximum or a minimum and the change in the value of W required from the initial to the final points.

To illustrate this procedure consider example 1, where the system equations are

$$\dot{x}_1 = x_1^2 - x_1^2 x_2 u; \quad \dot{x}_2 = -x_2 + u$$

and the problem is to transfer the state from (1,0) to (2,0) in minimum time. The pfaffian is

$$dx_1 + x_1^2 x_2 \, dx_2 = (x_1^2 - x_1^2 x_2^2) dt \ .$$

A choice of W is $W = \frac{-1}{x_1} + \frac{x_2^2}{2}$, so the pfaffian becomes

$$dW = (1 - x_2^2) dt \tag{3.12}$$

The singular arc is given by $x_2 = 0$, which maximizes $\frac{dW}{dt}$, and since $W(1,0) = -1$, $W(2,0) = -\frac{1}{2}$, i.e., W increases, then the singular arc is optimal.

It should be noted for the above example that the cost (optimal time) to traverse from any permissible point to the final point (2,0), with an unbounded control set, is

$$V(x_1, x_2) = \frac{1}{x_1} - \frac{x_2^2}{2} - \frac{1}{2} \ .$$

This cost function does not satisfy the Hamilton-Jacobi equation with the singular condition imposed, but rather it satisfies the singular condition interpreted as a partial differential equation, which for example 1 is

$$x_1^2 x_2 \frac{\partial V}{\partial x_1} - \frac{\partial V}{\partial x_2} = 0 \ .$$

The resemblance between V and W should be noted, in fact

$$W + V + \frac{1}{2} = 0 .$$

From (3.12) with the singular condition imposed

$$\frac{dV}{dt} = -1$$

so that $V = t_{final} - t_{initial}$. The interesting aspect of the pfaffian approach described above is the emergence of a new partial differential equation for the cost functional, due in part to the enlarged control set.

4. A New Partial Differential Equation and Optimization Technique

To formally justify the new partial differential equation characterizing the singular problem, consider the problem of extremizing the integral

$$I = \int_1^2 \left[L(t,x)dt + A_1(t,x)dx_1 + \dots + A_n(t,x)dx_n, \right] \tag{4.1}$$

where t as usual represents an exceptional or evolutory axis. The curves considered as candidates for the extremal can possess ordinary discontinuities so that it suffices to consider a continuous parametrization of the extremal by $x = x(\sigma)$ and $t = t(\sigma)$ with the proviso that $t(\sigma)$ be monotone. One section of arc possible may be parametrized $t = \text{const}$, $x_1 = x_1(\sigma)$, $x_2 = x_2(\sigma)$, $x_3 = \text{const} \dots x_n = \text{const}$, where $x_1(\sigma)$ and $x_2(\sigma)$ are periodic in σ and project a Jordan curve C in the x_1, x_2 plane. The contribution to I along this section of arc becomes

$$\Delta I = \int_C \left[A_1(t,x)dx_1 + A_2(t,x)dx_2\right] = \oint_S \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) dS, \tag{4.2}$$

and since there is no limit to the number of times this curve may be traversed, it becomes possible to make I assume any value whatsoever. Hence if I is to possess an extremal value, ΔI must be zero; however, since the section of arc considered is quite arbitrary, then it follows that

$$\left(\frac{\partial A_2(t,x)}{\partial x_1} - \frac{\partial A_1(t,x)}{\partial x_2}\right)$$

must vanish identically. Similarly it follows for other possible sections of arc parametrized by $t = \text{constant}$, $x_1 = \text{const} \ldots$, $x_\alpha = x_\alpha(\sigma) \ldots x_\beta = x_\beta(\sigma) \ldots x_n = \text{constant}$ that

$$\frac{\partial A_\beta(t,x)}{\partial x_\alpha} - \frac{\partial A_\alpha(t,x)}{\partial x_\beta} \overset{t,x}{\equiv} 0 \tag{4.3}$$

Hence (4.3) is satisfied if the functions $A_\alpha(t,x)$ are the gradient components of some scalar function $V(t,x)$,

$$A_\alpha(t,x) \equiv \frac{\partial V(t,x)}{\partial x_\alpha} \tag{4.4}$$

The evolutory nature of the t axis prevents the construction of a closed curve in each of the t, x_α planes ($\alpha = 1 \ldots n$). For example, any parametrization of the form $t = t(\sigma)$: $x_1 = x_1(\sigma)$ $x_2 = \text{constant} \ldots x_n = \text{constant}$, will never project a closed curve in the t, x_1 plane if $t = t(\sigma)$ is a monotone function of σ. This prohibits the above technique from being performed on each of the t, x_α planes ($\alpha = 1 \ldots n$), so that $L(t,x)$ does not have to be a gradient component of the scalar function $V(t,x)$. The conclusion is that if I is to possess an extremal value under the assumption that the extremal arc can have ordinary discontinuities, then necessarily I must be of the form,

$$I = \int_1^2 \left[L(t,x)dt + \frac{\partial V(t,x)}{\partial x_1} dx_1 + \ldots + \frac{\partial V}{\partial x_n}(t,x)\, dx_n \right]. \qquad (4.5)$$

The extremal arc may be determined either by using Green's theorem as applied to each of the t, x_α planes ($\alpha = 1 \ldots n$), or to express (4.5) as

$$I = V(t,x)_2 - V(t,x)_1 + \int_1^2 \left\{ L(t,x) - \frac{\partial V(t,x)}{\partial t} \right\} dt$$

and to extremize the integrand $L(t,x) - \frac{\partial V(t,x)}{\partial t}$ pointwise. The analogous Euler-Lagrange equations are

$$\frac{\partial L(t,x(t))}{\partial x_\alpha} - \frac{\partial^2 V(t,x(t))}{\partial t\, \partial x_\alpha} = 0 \qquad (\alpha = 1 \ldots n) \qquad (4.6)$$

and they determine the extremal arcs $x = x(t)$. The Green's theorem approach would resolve the optimality of such arcs.

We shall develop the new partial differential equation for the problem of minimizing

$$I = \int_{(t,x)_{\text{initial}}}^{(t,x)_{\text{final}}} L(t,x)\, dt \qquad (4.7)$$

subject to the constraints

$$\frac{dx_\alpha}{dt} = A_\alpha(x) + B_\alpha(x)u \tag{4.8}$$

where the control u is to be selected from the enlarged control set and as noted previously is given the representation $u = \frac{dy}{dt}$. We essentially embed this problem into an equivalent integral form as (4.5) and perform this operation by a modification of the Kalman[2] artifice. Expressing (4.8) in pfaffian form, with the representation for the control included, as

$$dx_\alpha = A_\alpha(x)dt + B_\alpha(x)dy \tag{4.9}$$

which are adjoined to I by the introduction of a scalar function V(t,x,y) to yield

$$I = \int_{(t,x)_{initial}}^{(t,x)_{final}} \left\{ \left[L(t,x) - \frac{\partial V}{\partial x_\alpha}(t,x,y)A_\alpha(x) \right] dt + \frac{\partial V}{\partial x_\alpha}(t,x,y)dx_\alpha - \right.$$

$$\left. - \frac{\partial V}{\partial x_\alpha}(t,x,y)\ B_\alpha(x)dy \right\} \tag{4.10}$$

To obtain the equivalent integral form as (4.5) then V(t,x,y) is chosen so that

$$\frac{\partial V}{\partial x_\alpha}(t,x,y)\ B_\alpha(x) + \frac{\partial V}{\partial y}(t,x,y) \equiv 0 \tag{4.11}$$

and I reduces to

$$I = V(t,x,y)_{final} - V(t,x,y)_{initial} +$$

$$+ \int_{(t,x)_{initial}}^{(t,x)_{final}} \left[L(t,x) - \frac{\partial V}{\partial x_\alpha}(t,x,y)\, A_\alpha(x) - \frac{\partial V}{\partial t}(t,x,y) \right] dt \qquad (4.12)$$

The modification of the Kalman artifice employed is that along the extremal arc x(t) and y(t) which minimizes the integrand pointwise, the integrand is chosen to be zero.

$$L(t,x(t)) - \frac{\partial V}{\partial x_\alpha}(t,x\,(t),y(t))\, A_\alpha(x(t)) - \frac{\partial V}{\partial t}(t,x(t),y(t)) \overset{t}{\equiv} 0 \qquad (4.13)$$

Since the extremal arc x(t) y(t) is necessarily determined by

$$\frac{\partial L(t,x(t))}{\partial x_\gamma} - \frac{\partial^2 V(t,x(t),y(t))}{\partial x_\alpha\, \partial x_\gamma} A_\alpha(x(t)) - \frac{\partial V(t,x(t),y(t))}{\partial x_\alpha} \frac{\partial A_\alpha(x(t))}{\partial x_\gamma} -$$

$$- \frac{\partial^2 V(t,x(t),y(t))}{\partial t\, \partial x_\gamma} = 0 \qquad \alpha,\gamma = 1 \ldots n$$

and

$$\frac{\partial^2 V(t,x(t),y(t))}{\partial x_\alpha\, \partial y} A_\alpha(x(t)) + \frac{\partial^2 V}{\partial t\, \partial y}(t,x(t),y(t)) = 0 \qquad (4.14)$$

then the additional restriction on V(t,x,y) imposed by 4.13 can be expressed as

$$\frac{\partial L}{\partial t}(t,x(t)) - \frac{\partial^2 V(t,x(t),y(t))}{\partial x_\alpha\, \partial t} A_\alpha(x(t)) - \frac{\partial^2 V}{\partial t^2}(t,x(t),y(t)) = 0$$

$$(4.15)$$

With these conditions imposed then (4.12) becomes

$$I = V(t,x,y)_{final} - V(t,x,y)_{initial} \tag{4.16}$$

but since the cost I is independent of y, then V(t,x,y) must be independent of y, so that (4.11) reduces to the new partial differential equation for the cost.

This brief and formal derivation does not give any geometrical insight into the optimization process and the role of the singular arc. For the time optimal problem associated with the differential system (4.8) there exists a geometrical interpretation which is essentially an extension of the two dimensional optimization technique described previously. Since we are considering the time optimal problem associated with the differential system

$$\frac{dx_\alpha}{dt} = A_\alpha(x) + B_\alpha(x)\,\frac{dy}{dt} \qquad (\alpha = 1 \ldots n)$$

then the cost V will be a function of the state x only. A one-manifold W(x) in E^n is constructed satisfying

$$\frac{\partial W(x)}{\partial x_\alpha} B_\alpha(x) \overset{x}{\equiv} 0 \qquad (\alpha = 1 \ldots n) \tag{4.17}$$

It should be noted that the complete integral W to (4.17) will contain n arbitrary constants, (n-1) constants implicitly plus an additive constant. This manifold W(x) is a "zero cost" manifold since it costs no time to traverse the characteristics which are determined by

$$\frac{dx_\alpha}{dy} = B_\alpha(x) \tag{4.18}$$

and represents the impulsive solution to the differential system (4.8). The geometric interpretation of the singular arc for the time optimal problem is that it is the locus of those points of each manifold W(x) where $\frac{dW}{dt}$ is extremized. For solutions satisfying the differential system (4.8) we have

$$\frac{dW}{dt} = \frac{\partial W(x)}{\partial x_\alpha}\left[A_\alpha(x) + B_\alpha(x)\frac{dy}{dt}\right] \qquad (\alpha = 1 \dots n)$$

so that by virtue of (4.17)

$$\frac{dW}{dt} = \frac{\partial W(x)}{\partial x_\alpha} A_\alpha(x) \qquad (\alpha = 1 \dots n) \tag{4.19}$$

The points $x = \bar{x}$ confined to $W - W(x) = 0$ where $\frac{dW}{dt}$ is extremized is necessarily determined by

$$\frac{\partial^2 W}{\partial x_\alpha \partial x_\gamma}(\bar{x})\, A_\alpha(\bar{x}) + \frac{\partial W}{\partial x_\alpha}(\bar{x})\,\frac{\partial A_\alpha}{\partial x_\gamma}(\bar{x}) + \lambda\frac{\partial W(\bar{x})}{\partial x_\gamma} = 0 \qquad (\alpha, \gamma = 1 \dots n) \tag{4.20}$$

The n equations represented by (4.20) together with $W - W(\bar{x}) = 0$ can be solved, with suitable assumptions on $A(\bar{x})$ $B(\bar{x})$ and $W(\bar{x})$, for the $(n+1)$ quantities $\bar{x}$ and λ to yield

$$\begin{aligned} \lambda &= \lambda(W) \\ \bar{x} &= \bar{x}(W) \end{aligned} \tag{4.21}$$

Hence, substituting for $\bar{x}$ in (4.19) by (4.21) gives

$$\frac{dW}{dt} = \frac{\partial W}{\partial x_\alpha}(\bar{x}(W))\, A_\alpha(\bar{x}(W)) \tag{4.22}$$

which is solved for $W = \overline{W}(t)$, and thus yielding the solution of the singular arc

$$x(t) = \overline{x}(\overline{W}(t)) . \tag{4.23}$$

This then is the geometric interpretation of the singular arc, and has an analogy with the optimization procedure described previously if we define

$$V(x) = \overline{V}(W(x)) = \int^{W} \frac{dW}{\frac{\partial W}{\partial x_\alpha}(\overline{x}(W))\, A_\alpha(\overline{x}\,(W))} \tag{4.24}$$

Now, $V(x)$ satisfies

$$\frac{\partial V(x)}{\partial x_\alpha} B_\alpha(x) \overset{x}{\equiv} 0 \qquad (\alpha = 1 \ldots n)$$

furthermore from (4.22) and (4.24) we have

$$\frac{dV}{dt} = 1$$

so that for the time optimal problem $V(x)$ represents the cost, thus completing the analogy. To show that the arc (4.23) is singular we define

$$p_\alpha(t) = \frac{\partial W}{\partial x_\alpha}\left(\overline{x}(\overline{W}(t))\right) e^{\int^{t} \lambda(\overline{W}(\tau))\, d\tau} \qquad (\alpha = 1 \ldots n) \tag{4.25}$$

and observe that equations (4.20) become

$$\frac{d\, p_\alpha(t)}{dt} = -\, p_\gamma(t)\ \frac{\partial A_\gamma}{\partial x_\alpha}(x(t)) + \frac{\partial^2 W(x(t))}{\partial x_\alpha \partial x_\gamma} B_\gamma(x(t))\ u(t) e^{\int^{t} \lambda(\overline{W}(\tau)) d\tau} \tag{4.26}$$

Since (4.17) is an identity in x then

$$\frac{\partial^2 W(x)}{\partial x_\alpha \partial x_\gamma} B_\gamma(x) + \frac{\partial W(x)}{\partial x_\gamma} \frac{\partial B_\gamma(x)}{\partial x_\alpha} = 0$$

so that 4.26 becomes

$$\frac{dp_\alpha(t)}{dt} = - p_\gamma(t) \left[\frac{\partial A_\gamma(x(t))}{\partial x_\alpha} + \frac{\partial B_\gamma(x(t))}{\partial x_\alpha} u(t) \right]$$

$$(\alpha, \gamma = 1 \dots n)$$

thus showing that the Euler-Lagrange equations are satisfied. From (4.17) we have

$$p_\alpha(t) \, B_\alpha(x(t)) = 0$$

so that the singular condition is satisfied and hence the arc x(t) is singular.

5. Completeness and Integrability

For simplicity and to avoid at this stage a dimension problem, we shall consider a three-dimensional system with two controls (u,v) as follows

$$\begin{aligned} \dot{x}_1 &= A_1(x) + B_1(x)u + C_1(x)v \\ \dot{x}_2 &= A_2(x) + B_2(x)u + C_2(x)v \qquad (5.1) \\ \dot{x}_3 &= A_3(x) + B_3(x)u + C_3(x)v \end{aligned}$$

Assuming a time optimal problem then the singular condition for both controls u and v implies the existence of a function V satisfying

$$
\begin{aligned}
B_1 \frac{\partial V}{\partial x_1} + B_2 \frac{\partial V}{\partial x_2} + B_3 \frac{\partial V}{\partial x_3} &\equiv 0 \\
C_1 \frac{\partial V}{\partial x_1} + C_2 \frac{\partial V}{\partial x_2} + C_3 \frac{\partial V}{\partial x_3} &\equiv 0
\end{aligned}
\tag{5.2}
$$

The Poisson[10] operator applied to these two linear partial differential equations yields

$$
\left\{ C_\alpha \frac{\partial B_\gamma}{\partial x_\alpha} - B_\alpha \frac{\partial C_\gamma}{\partial x_\alpha} \right\} \frac{\partial V}{\partial x_\gamma} \equiv 0 \qquad (\alpha, \gamma = 1,2,3) \tag{5.3}
$$

If the linear partial differential equations (5.2) are complete, then equation (5.3) either vanishes identically or is a linear combination of equations (5.2), which implies the vanishing of the determinant

$$
D = \begin{vmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ \left\{ C_\alpha \frac{\partial B_1}{\partial x_\alpha} - B_\alpha \frac{\partial C_1}{\partial x_\alpha} \right\} & \left\{ C_\alpha \frac{\partial B_2}{\partial x_\alpha} - B_\alpha \frac{\partial C_2}{\partial x_\alpha} \right\} & \left\{ C_\alpha \frac{\partial B_3}{\partial x_\alpha} - B_\alpha \frac{\partial C_3}{\partial x_\alpha} \right\} \end{vmatrix} \tag{5.4}
$$

By eliminating the controls from equations (5.1) the system can be expressed as a single pfaffian

$$
\begin{aligned}
&(C_3B_2 - C_2B_3)dx_1 + (C_1B_3 - C_3B_1)dx_2 + (C_2B_1 - C_1B_2)dx_3 \\
&\quad + \left\{ C_1(A_3B_2 - A_2B_3) + C_2(A_1B_3 - A_3B_1) + C_3(A_2B_1 - A_1B_2) \right\} dt = 0
\end{aligned}
\tag{5.5}
$$

The completeness condition $D \equiv 0$ implies the integrability of the reduced pfaffian, ($t = \text{constant}$)

$$(C_3B_2 - C_2B_3)dx_1 + (C_1B_3 - C_3B_1)dx_2 + (C_2B_1 - C_1B_2)dx_3 = 0 \tag{5.6}$$

that is, it can be expressed, by a suitable choice of an integrating factor, as a total differential. By a theorem of Caratheodory[9], if the pfaffian (5.6) is integrable, then in any neighborhood of a given point there exists points which are not accessible from the given point along any path satisfying the pfaffian (5.6). Conversely, if the linear partial differential equations (5.2) are not complete then the pfaffian (5.6) would not be integrable and hence there would exist <u>some</u> neightborhood of a given point for which all points would be accessible from the given point by paths satisfying the pfaffian (5.6). From this can be inferred the result that if the system of linear partial differential equations is not complete, then the best time optimal strategy is impulsive, since the transfer from one state to another can be achieved in zero time. This is only a local result since the contra-positive of Caratheodory's theorem implies some neighborhood rather than any neighborhood. If however the system (5.2) of partial differential equations is complete, then the pfaffian (5.6) is integrable so that there exists a function $W(x)$ and an integrating factor $\mu(x)$ such that

$$\begin{aligned}
\frac{\partial W(x)}{\partial x_1} &\overset{x}{\equiv} \mu(x)\left[C_3(x)\ B_2(x) - C_2(x)\ B_3(x)\right] \\
\frac{\partial W(x)}{\partial x_2} &\overset{x}{\equiv} \mu(x)\left[C_1(x)\ B_3(x) - C_3(x)\ B_1(x)\right] \\
\frac{\partial W(x)}{\partial x_3} &\overset{x}{\equiv} \mu(x)\left[C_2(x)\ B_1(x) - C_1(x)\ B_2(x)\right]
\end{aligned} \tag{5.7}$$

Hence, the pfaffian (5.5) can be expressed as

$$dW + \mu\left[C_1(A_3B_2 - A_2B_3) + C_2(A_1B_3 - A_3B_1) + C_3(A_2B_1 - A_1B_2)\right] dt = 0 \tag{5.8}$$

and the techniques described previously can now be applied to determine the singular arc and its optimality. Therefore the integrability criterion for the reduced pfaffian (5.6) determines the exitence or non-existence of the singular arc.

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