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NOTES ON MAGNETOHYDRODYNAMICS – PART II

by

R. P. Kanwal

September 30, 1965

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IONOSPHERE RESEARCH LABORATORY

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Preface

This report is in continuation of the earlier one entitled "Notes on Magnetohydrodynamics - Part I". The first report is frequently referred to in the present part.

In the discussion of the chapter on the covariant formulation of magnetohydrodynamics we are compelled to make a slight change in notation. In the revised version of these notes this discrepancy will be eliminated.

Chapter VI

Simple Waves

1. One Dimensional Wave Propagation

In classical gas dynamics simple waves play a fundamental role in building up solutions in wave propagation. It is the result of the fact that only these waves can bound a region of constant flow if there are no shock waves. One of the most important examples of the generation of simple waves is by the withdrawal of a piston in a tube filled with gas at rest. This piston is withdrawn with increasing speed until a constant final velocity is attained. Simple waves in magnetohydrodynamics have essentially the same properties as those in gas dynamics. To analyze them we take all field and flow quantities to be function of time t and one space variable x. Furthermore let $\underline{H} = (H_1, H_2, H_3) = (H_x, H_y, H_z)$, and $\underline{v} = (v_1, v_2, v_3) = (u, v, w)$ wherein we impose no restriction on the presence or absence of the quantities (v, w) and (H_y, H_z). Under these circumstances the equation (4.1) implies that

$$H_x = \text{a constant}, \tag{1}$$

While the equation (4.2) to (4.5) reduce to

$$\frac{\partial H_y}{\partial t} + u \frac{\partial H_y}{\partial x} + H_y \frac{\partial u}{\partial x} - H_x \frac{\partial v}{\partial x} = 0, \tag{2}$$

$$\frac{\partial H_z}{t} + u \frac{\partial H_z}{\partial x} + H_z \frac{\partial u}{\partial x} - H_x \frac{\partial w}{\partial x} = 0. \tag{3}$$

$$\frac{\partial u}{\partial t} + \mu \frac{H_y}{\rho} \frac{\partial H_y}{\partial x} + \mu \frac{H_z}{\rho} \frac{\partial H_z}{\partial x} + u \frac{\partial u}{\partial x} + \frac{c^2}{\rho} \frac{\partial \rho}{\partial x} + B \frac{\partial S}{\partial x} = 0, \quad (4)$$

$$\frac{\partial v}{\partial t} - \mu \frac{H_x}{\rho} \frac{\partial H_y}{\partial x} + u \frac{\partial v}{\partial x} = 0 \quad (5)$$

$$\frac{\partial w}{\partial t} - \mu \frac{H_x}{\rho} \frac{\partial H_y}{\partial x} + u \frac{\partial w}{\partial x} = 0 \quad (6)$$

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} = 0 \quad (7)$$

$$\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} = 0, \quad (8)$$

where in equation (3), we have used the relation (4.21). These seven equations can be represented by a single vector equation

$$\frac{\partial \underline{U}}{\partial t} + \underline{\underline{A}} \frac{\partial \underline{U}}{\partial x} = 0, \quad (9)$$

where

$$\underline{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} B_y \\ B_z \\ u \\ v \\ w \\ \rho \\ S \end{bmatrix} \quad (10)$$

is a column vector and

$$\underline{\underline{A}} = \begin{bmatrix} u & o & H_y & -H_x & o & o & o \\ o & u & H_z & o & -H_x & o & o \\ \frac{\mu H_y}{\rho} & \frac{\mu H_z}{\rho} & u & o & o & \frac{c^2}{\rho} & B \\ \frac{-\mu H_x}{\rho} & o & o & u & o & o & o \\ o & \frac{-\mu H_x}{\rho} & o & o & u & o & o \\ o & o & \rho & o & o & u & o \\ o & o & o & o & o & o & u \end{bmatrix}, \quad (11)$$

is a (7 x 7) matrix:

In classical gas dynamics corresponding equations are discussed and simplified by the introduction of two invariants, the so called Riemann invariants. These are obtained by the integration of the characteristic system of the equation of motion. One of these invariants remains constant on one system of characteristic curves while the other on the second system of the characteristic curves. In terms of such an invariant J, the analysis leads to a relation

$$\frac{\partial J}{\partial t} + \frac{\lambda \partial J}{\partial x} = 0, \quad (12)$$

where λ is the slope of the corresponding characteristic curve. If a Riemann invariant J exists for the general n system of equations (6.9), we should expect a relation of the form (6.12). Moreover, as in gas dynamics, it should be accomplished by premultiplying the equation (6.9) by a row vector $\underline{D}$ with components d_1, d_2, d_3, ... d_n. This operation leads to the relation

$$\sum_{i=1}^{n} d_i \frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} a_{ij} d_i \frac{\partial u_j}{\partial x} = 0 , \qquad (13)$$

where a_{ij} are the elements of the matrix $\underline{\underline{A}}$ in equation (11). If we write the equation (12) in terms of the variables u_i, we have

$$\sum_{i=1}^{n} \frac{\partial J}{\partial u_i} \frac{\partial u_i}{\partial t} + \lambda \sum \frac{\partial J}{\partial u_j} \frac{\partial u_j}{\partial x} = 0 . \qquad (14)$$

By comparing (13) and (14) we obtain

$$d_i = \frac{\partial J}{\partial u_i} , \qquad i = 1, 2, \ldots n , \qquad (15)$$

and

$$\sum_{i=1}^{n} d_i a_{ij} = \lambda \frac{\partial J}{\partial u_j} \quad . \tag{16}$$

When we substitute for d_i from (15) into (16), there results the equation

$$\sum_{i=1}^{n} d_i a_{ij} = \lambda d_j \quad . \tag{17}$$

Therefore, we have two equations (15) and (17) for determining the vector d_i. From (15) we obtain

$$\frac{\partial d_i}{\partial u_j} = \frac{\partial^2 J}{\partial u_i \partial u_j} = \frac{\partial d_j}{\partial u_i} \quad . \tag{18}$$

This equation comprises a set of $\frac{n}{2}$ (n-1) conditions which must be satisfied by the variables d_i. This is the independent number of relations which survive after the omission of the indentities and the symmetries inherent in the above equation. The n homogeneous equations (17) give (n-1) independent equations. We, therefore, have $\frac{1}{2}$ (n-1)(n+2) conditions for determination of the n multiplier d_i. But only for n = 2, do we have the definitive system of equations; and that is precisely the case of classical gas dynamics. As such we cannot introduce Riemann invariants for the present problem.

Under somewhat weaker conditions than implied by the relation (12), Riemann invariants can be shown to exist. Indeed if we assume that in a simple wave region the quantities u_i are all functions of one of the variables u_i say u_1, then we can perform the differentiation with respect to this variable in equation (9) to obtain the expression

$$\frac{\partial \underline{U}}{\partial u_1} \frac{\partial u_1}{\partial t} + \underline{\underline{A}} \frac{\partial \underline{U}}{\partial u_1} \frac{\partial u_1}{\partial x}$$

$$= \left[\underline{\underline{I}} \frac{\partial u_1}{\partial t} + \underline{\underline{A}} \frac{\partial u_i}{\partial x} \right] \frac{\partial \underline{U}}{\partial u_1} = 0 . \tag{19}$$

Along a curve $u_1(x,t)$ = a constant, we have

$$\frac{dx}{dt} = - \frac{\partial u_1 / \partial t}{\partial u_1 / \partial x} = \lambda \tag{20}$$

Combining (19) and (20) we obtain

$$\left[\underline{\underline{A}} - \lambda \underline{\underline{I}} \right] d\underline{U} = 0 . \tag{21}$$

Again since the disturbance $d\underline{U} \neq 0$, implies the relation

$$\left[\underline{\underline{A}} - \lambda \underline{\underline{I}} \right] = 0 . \tag{22}$$

This equation gives us eigen values $\lambda^{(1)} \ldots \lambda^{(n)}$, leading to n simple wave solutions. Furthermore since all the dependent variables are functions of u_1, it follows that along a curve u_1 = constant, the quantities u_i and the elements of the matrix $\underline{\underline{A}} = \underline{\underline{A}}\ (\underline{U})$ are all constant. Therefore the corresponding eigen values as determined by (12) above are also constant, proving thereby that the line u_1 = constant is a straight line and hence lead to a simple wave solution. In fact, to each eigen value there corresponds a right eigen vector $\underline{r}$ of the matrix in the equation (22), i.e.

$$\left| \underline{\underline{A}} - \lambda^{(i)} \underline{\underline{I}} \right| \; r^{(i)} = 0, \qquad i = 1, \ldots n \; . \tag{23}$$

If we denote $r_1^{(i)}, r_2^{(i)}, \ldots r_n^{(i)}$ as the elements of the ith vector $\underline{r}$ then from (21) and (23) we have

$$\frac{du_1^{(i)}}{r_1^{(i)}} = \frac{du_2^{(i)}}{r_2^{(i)}} = \ldots = \frac{du_n^{(i)}}{r_n^{(i)}} = k \; , \tag{24}$$

where k is a constant. Thus for each $\lambda^{(i)}$, i = 1, 2 ... n, there is a system of first order differential equations of the form (24). Integration of this system leads to the required Riemann invariants.

2. Characteristics

From the column vector (10) we observe that $u_3 = u$, is x-component of the flow field; it is in the direction normal to the plane wave front. To derive the corresponding characteristic speeds we make the usual substitutions of Chapter IV in the system of equations (2-8), i.e.;

$$\frac{\partial F}{\partial t} \longrightarrow -\lambda\, dF \quad , \qquad \frac{\partial F}{\partial x} \longrightarrow dF, \tag{25}$$

and therefore

$$\frac{dF}{dt} \longrightarrow -UdF \quad , \qquad U = \lambda - u \quad . \tag{26}$$

When we take into account both the wave propagation in both positive as well as negative direction of the x-axis, the system of partial differential equations (2-8) yields:

$$\mp U\, dH_y + H_y du - H_x dv = 0 \tag{27}$$

$$\mp U\, dH_z + H_z du - H_x dw = 0 \tag{28}$$

$$\frac{\mu H_y}{\rho}\, dH_y + \frac{\mu H_z}{\rho}\, dH_z \mp U\, du + \frac{c^2}{\rho}\, d\rho = 0 \; . \tag{29}$$

$$\frac{-\mu H_x}{\rho}\, dH_y \mp U\, dv = 0 \; . \tag{30}$$

$$\frac{-\mu H_x}{\rho}\, dH_z \mp U\, dw = 0 \; , \tag{31}$$

$$\rho\, du \mp U\, d\rho = 0 \; , \tag{32}$$

$$\mp U\, dS = 0 \; . \tag{33}$$

For a nontrivial solution of this system of homogeneous equations, we should have

$$(\rho U)(\rho U^2 - \mu H_x^{\ 2})\left[\rho U^4 - \left\{\rho c^2 + \mu (H_x^{\ 2} + H_y^{\ 2} + H_z^{\ 2})\right\} + c^2 \mu H_x^{\ 2}\right] = 0. \tag{34}$$

We could have obtained this determinant directly from the relation (22). The equation (25) gives us the various wave speeds; the fast speed U_f, the slow speed U_s and the intermediate speed a, and the entropy wave with U = 0, as in Chapter IV. The fast and slow speeds are given by the factor

$$\rho U^4 - \left\{\rho c^2 + \mu (H_x^{\ 2} + H_y^{\ 2} + H_z^{\ 2})\right\} U^2 + c^2 \mu H_x^{\ 2} = 0. \tag{35}$$

Furthermore, as before,

$$U_s \leq c \leq U_f, \tag{36}$$

$$U_s \leq a \leq U_f, \tag{37}$$

where $a = \left(\frac{\mu H_x^{\ 2}}{\rho}\right)^{1/2}$, is the Alfven speed. From the relation (26) we observe that the values of λ for various characteristics, are given as

$$\lambda = \frac{dx}{dt} = u \pm U_f \text{ , etc.} \tag{38}$$

Corresponding to the values U_f, U_s, a and 0, we name the appropriate simple waves as fast, slow, transverse and entropy waves respectively. The set of equations (38) are the equations determining the characteristics in the x - t plane.

3. Derivation of Riemann invariants: the fast and slow waves

To derive the Riemann invariants we need the equations (24).

The value of the eigen vector $\underline{r}$ is determined from the equation (23), while the equation (23) is obtained from the given system of equations (21) by the change of $d\underline{U}$ to $\underline{r}$. The system of equations corresponding to (21) for our analysis is that given by the relations (27) to (33). Therefore the eigen vector $\underline{r}$ for fast waves (and similarly for other waves) is determined by the following set of algebraic equations:

$$- U_f r_1 + H_y r_3 - H_x r_4 = 0, \tag{39}$$

$$- U_f r_2 + H_z r_3 - H_x r_5 \; 0 , \tag{40}$$

$$\frac{\mu H_y}{\rho} r_1 + \frac{\mu H_z}{\rho} r_2 - U_f r_3 + \frac{c^2}{\rho} r_6 = 0 , \tag{41}$$

$$\frac{-\mu H_x}{\rho} r_1 - U_f r_4 = 0 , \tag{42}$$

$$\frac{-\mu H_x}{\rho} r_2 - U_f r_5 = 0 , \tag{43}$$

$$\rho r_3 - U_f r_6 = 0 , \tag{44}$$

$$- U_f r_7 = 0 . \tag{45}$$

These equations readily provide us with the solution for $\underline{r}$:

$$r_7 = 0 \tag{46}$$

$$r_6 = \frac{\rho r_3}{U_f} , \tag{47}$$

$$r_5 = \frac{-\mu H_x}{\rho U_f} r_2 \tag{48}$$

$$r_4 = \frac{-\mu H_x}{\rho U_f} r_1 \tag{49}$$

$$r_2 = \frac{U_f H_z}{(U_f^2 - a^2)} r_3 , \tag{50}$$

$$r_1 = \frac{U_f H_y}{(U_f^2 - a^2)} r_3 . \tag{51}$$

We chose the constant of proportionality in such a way that in the relation (47) we let $r_6 = \rho$, and then r_3 becomes equal to U_f. Thus

$$\underline{r}_f = \begin{bmatrix} \dfrac{U_f^2 H_y}{(U_f^2 - a^2)} \\[2ex] \dfrac{U_f^2 H_z}{(U_f^2 - a^2)} \\[2ex] U_f \\[2ex] \dfrac{-\mu H_x H_y U_f}{\rho(U_f^2 - a^2)} \\[2ex] \dfrac{-\mu H_x H_z U_f}{\rho(U_f^2 - a^2)} \\[2ex] \rho \\[2ex] 0 \end{bmatrix} \tag{52}$$

From the relations (24) and (52) we get the required differential equations:

$$\frac{(U_f^2 - a^2)\, dH_y}{U_f^2 H_y} = \frac{(U_f^2 - a^2)\, dH_z}{U_f^2 H_z} = \frac{du}{U_f}$$

$$= \frac{-\rho\, (U_f^2 - a^2)\, dv}{\mu H_x H_y U_f} = \frac{-\rho\, (U_f^2 - a^2)\, dw}{\mu H_x H_z U_f}$$

$$= \frac{d\rho}{\rho} = \frac{dS}{0} = k \;. \qquad (53)$$

These equations, in turn, yield Riemann invariants:

$$u - \int \frac{U_f}{\rho}\, d\rho = \text{constant}; \qquad (54)$$

$$S = \text{constant}, \qquad (55)$$

$$v + \int \frac{\mu H_x}{\rho k}\, dH_y = \text{constant}; \qquad (56)$$

$$v - \int \frac{\mu H_y}{H_z}\, dw = \text{constant}; \qquad (57)$$

$$\frac{H_y}{H_z} = \text{constant}; \qquad (58)$$

$$w + \int \frac{\mu H_x}{\rho k}\, d H_z = \text{constant} \;. \qquad (59)$$

The relation (58), i.e. $H_z = mH_y$, m = a constant, implies that the magnetic field vector can be written as $H = H_x \underline{i} + H_y (\underline{j} + m\underline{k})$, where $\underline{i}$, $\underline{j}$, $\underline{k}$, are the unit orthogonal vectors along the coordinate axes. We can therefore choose a new coordinate system such that $H_z = 0$. Furthermore the equations (30) and (31) give the relation

$$\frac{dw}{dv} = \frac{H_z}{H_y} = m \ , \tag{60}$$

which means that the flow and magnetic fields do not rotate across a simple wave. Hence we can refer to a coordinate system such that w and H_z are zero throughout the simple wave region. In such a system of coordinates the equations (27) to (32) become

$$\mp U \, dH_y + H_y \, du - H_x \, dv = 0 \ , \tag{61}$$

$$\mu \frac{H_y}{\rho} dH_y \mp U \, du + \frac{c^2}{\rho} d\rho = 0, \tag{62}$$

$$-\mu \frac{H_x}{\rho} dH_y \mp U \, dv = 0 \ , \tag{63}$$

$$\rho \, du \mp U \, d\rho = 0 \ , \tag{64}$$

$$S = \text{constant} \ , \tag{65}$$

while equation (35) reduces to

$$H_y^2 = \left(\frac{U^2}{c^2} - 1 \right) \left(\frac{\rho c^2}{\mu H_x^2} - \frac{c^2}{U^2} \right) H_x^2 \tag{66}$$

Let U stand for U_f or U_s. Define the quantities α and β such that

$$\alpha = \frac{U^2}{c^2} , \qquad \beta = \frac{\rho c^2}{\mu H_x^2} \tag{67}$$

The quantity $\alpha^{1/2}$ is the ratio of the wave speed to the sound speed, while $\beta^{1/2}$ is the ratio of sound speed to that of Alfven speed. Moreover the relation for α implies, when we put $c^2 = \frac{dp}{d\rho}$, that

$$U^2 \, d\rho = \alpha c^2 \, d\rho = \alpha \, dp \ . \tag{68}$$

If we take $c^2 = \frac{\gamma p}{\rho}$, i.e. take the equation of state as $p = A(S)\rho^\gamma$, the relation for β becomes

$$\beta = \frac{\gamma p}{\mu H_x^2} = p_0 p \ , \tag{69}$$

and since H_x is a constant, we find that β is proportional to p. The quantity $p_0 = \frac{\gamma}{\mu H_x^2}$.

Let us now try to seek the solution to our problem in terms of α and β. First of all we observe that equation (66) takes the form

$$H_y^2 = (\alpha - 1)\left(\beta - \frac{1}{\alpha}\right) H_x^2 \ . \tag{70}$$

Since H_x is a constant, H_y is determined in terms of α and β.

From the relations (62) and (64) we readily obtain the equation

$$(c^2 - U^2)\, d\rho + \frac{\mu}{2}\, d\,(H_y^2) = 0. \tag{71}$$

Moreover since $p = A(S)\,\rho^{\gamma}$, the quantity ρ can be given in terms of β with the help of the relation (69):

$$\rho = \rho_0\, \beta^{1/\gamma} \tag{72}$$

where ρ_0 is related to p_0 as

$$p_0 = A(S)\,\rho_0^{\gamma} \ . \tag{73}$$

Thus the value of $d\rho$ as required in the equation (71) can be given in terms of β:

$$d\rho = \frac{\rho_0}{\gamma}\ \beta^{\frac{1}{\gamma}-1}\ d\beta \ . \tag{74}$$

The other quantity in equation (71) which needs to be determined in terms of β is $d(H_y^2)$. For this we appeal to the equation (70) and get

$$d(H_y^2) = H_x^2\ d\left(\alpha\beta - 1 - \beta - \frac{1}{\alpha}\right) = \frac{\gamma p}{\mu\beta}\ d\left(\alpha\beta - 1 - \beta - \frac{1}{\alpha}\right), \tag{75}$$

where we have used the relation (69) and put H_x^2 as a function of β. Now observing the definition of α as given by the equation (67), and substituting from (74) and (75) into (71), the following equation results:

$$c^2\,(1-\alpha)\,\frac{\rho_0}{\gamma}\ \beta^{\frac{1}{\gamma}-1}\ d\beta + \frac{\gamma p}{2\beta}\ d\left(\alpha\beta - 1 - \beta - \frac{1}{\alpha}\right) = 0. \tag{76}$$

But $c^2 = \frac{\gamma p}{\rho}$ and hence (76) takes the simple form

$$\alpha^2(\alpha-1)\,d\beta = \hat{\gamma}\,(\alpha^2\beta - 1)\,ds, \tag{77}$$

where

$$\hat{\gamma} = \frac{\gamma}{2-\gamma} . \tag{78}$$

The equation (77) enables us to determine α in terms of β. This in turn, enables us to determine U in terms of β. Indeed the relation (67), and the definitions for p_0 and ρ_0, give

$$U^2 = c_0^{\,2} \; \alpha\beta^{1-\frac{1}{\gamma}} , \tag{79}$$

where $c_0^{\,2} = \gamma p_0/\rho_0$. Furthermore H_y (and hence also H_z in view of the relation (60))is also determined as a function of β as is obvious from the equation (70). Similarly the equations (69) and (72) give the values of p and ρ in terms of β. This leaves only the velocity vector (u, v, w) for whose determination we appeal to the equations (30), (31), and (32) and get

$$du = \pm\, c\alpha^{1/2} \rho^{-1} d\rho , \tag{80}$$

$$dv = \mp \frac{\mu}{c\,\alpha^{1/2}} H_x dH_y , \tag{81}$$

and

$$dw = \mp \frac{\mu}{c\,\alpha^{1/2}\rho} H_x dH_z . \tag{82}$$

Since the right hand sides of (80), (81), and (82) are known in terms of β, we can integrate these relations to evaluate u, v, and w. Thus, the evaluation of all the flow and field quantities depends on the relation

(77). Therefore we devote the next section to the discussion of this equation.

4. The solutions of the equation (77)

The integral curves of the equation (77) can be easily sketched graphically. The physically admissible portion of these curves can be completely separated into two branches. These branches will be denoted by α_f and α_s corresponding to the fast and slow waves respectively. As such α_f and α_s have to satisfy the inequalities:

$$\beta\alpha_f \geq 1 \ , \qquad \alpha_f \geq 1 \qquad (\beta\alpha_f^{\ 2} > 1), \tag{83}$$

$$\beta\alpha_s \leq 1 \ , \qquad \alpha_s \leq 1 \qquad (\beta\alpha_s^{\ 2} < 1), \tag{84}$$

where $\beta > 0$ and $\alpha_f \geq 0$. These inequalities follow from the relations (36), (37), (67), and (70). From the above two relations it follows that in the $(\alpha - \beta)$ plane the lines $\alpha\beta = 1$ and $\alpha = 1$, bound the fast wave region (f-region) from above and the slow wave region (s-region) from below. The point (1, 1) is a singular point (node). In the neighborhood of this point we set

$$\hat{\alpha} = \alpha - 1, \qquad \hat{\beta} = \beta - 1, \tag{85}$$

and the equation (77) takes the form

$$\hat{\alpha} \frac{d\hat{\beta}}{d\hat{\alpha}} = \hat{\gamma}\,\hat{\beta} + 2\hat{\gamma}\,\hat{\alpha} \ . \tag{86}$$

This equation is easily integrated;

$$\hat{\beta} = -\frac{\gamma}{\gamma-1}\hat{\alpha} + C\,\hat{\alpha}^{\hat{\gamma}}, \qquad C = \text{ a constant} \tag{87}$$

At the singular point (1, 1) all the integral curves have a common tangent with a negative slope for $\gamma > 1$:

$$\frac{d\beta}{d\alpha} = -\frac{\gamma}{\gamma-1}. \tag{88}$$

Since the derivative $d\beta/d\alpha$ vanishes on the curve $\alpha^2\beta = 1$, each integral curve has its tangent horizontal at the point of intersection with this curve. Furthermore from (77) and the inequalities (83) and (84) it follows that except at (1, 1), the value of $d\beta/d\alpha$ is always positive

$$\frac{d\beta}{d\alpha} > 0. \tag{89}$$

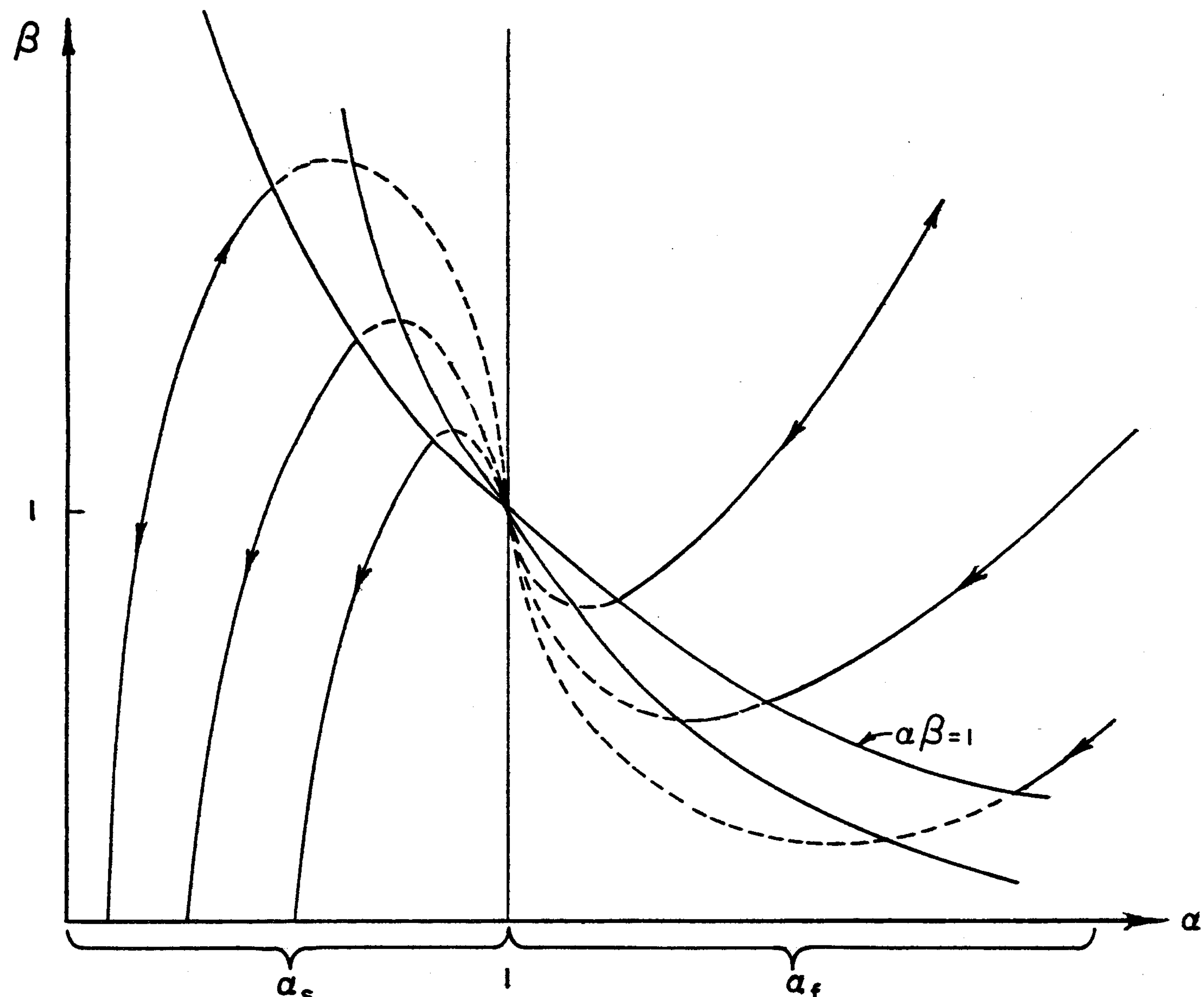
The integral curves are sketched in figure 1. The positions of the curves which are physically inadmissable are indicated by dashed lines. In fact from the relation (70) it follows that in these portions H_y becomes imaginary.

Riemann invariants can now be obtained directly from the integral of the equation (77). If we make the substitution (85) in the equation (77), it yields the solutions

$$R_f^m = \left|\alpha_f - 1\right|^{-\hat{\gamma}}\beta + \hat{\gamma}\int \alpha_f^{-2}\left|\alpha_f - 1\right|^{(-1+\hat{\gamma})} d\alpha_f, \tag{90}$$

and

$$R_s^m = \left|\alpha_s - 1\right|^{-\hat{\gamma}}\beta - \hat{\gamma}\int \alpha_s^{-2}\left|\alpha_s - 1\right|^{(-1+\hat{\gamma})} d\alpha_s. \tag{91}$$



THE INTEGRAL CURVES OF β VERSUS α

FIGURE I

The values of Riemann invariants R_f^m and R_s^m are determined by those in the constant state. Incidentally since the velocity vector does not enter these equations they are called magnetic Riemann invariants.

Let us now study the variation of the velocity field across the simple wave. In this connection we should substitute the value of $d\rho$ from the relation (74) in the equation for du, viz., (80):

$$du = \pm c\, \alpha^{1/2} \beta^{-1} \, d\beta , \tag{92}$$

where we have used the relation between ρ and ρ_0 as given by the equation (72). We can write c in terms of c_0 and β with the help of the relation (69) and (72):

$$c^2 = c_0^2 \;\; \beta^{1-\frac{1}{\gamma}} \; . \tag{93}$$

Substituting (93) into (92), we get

$$du = \epsilon \frac{c_0}{\gamma} \;\; \alpha^{1/2} \beta^{-1/2} \; (1+ \frac{1}{\gamma}) \; d\beta \, , \quad \epsilon = \pm 1 \; . \tag{94}$$

Therefore the behaviour of the jump in velocity in the x-direction in compression and rarefaction waves is the same as in classical gas dynamics.

From (53) it follows that the behaviour of dv and dw is the same. We need, therefore, to calculate dv only. To evaluate dv from (81) we should first calculate H_y. This follows from (70).

$$\frac{H_y}{H_x} = \operatorname{sgn}\left(\frac{H_{y1}}{H_x}\right) \sqrt{\frac{(\alpha - 1)\ (\alpha\beta - 1)}{\alpha}} \quad , \tag{95}$$

where $H_{y1}(\neq 0)$ is the value of H_y in the constant state. Furthermore H_y does not vanish anywhere except $\alpha\beta = 1$, and hence it does not change its sign across the slow and the fast waves. Therefore

$$\operatorname{sgn}\left(\frac{H_y}{H_x}\right) = \frac{H_{y1}}{H_x} \quad . \tag{96}$$

We now proceed to evaluate dv_y in precisely the same manner as for du and get

$$dv_y = -\epsilon \frac{a_0}{\gamma} \beta^{-1/2} (1+ \frac{1}{\gamma}) \frac{\sqrt{(\alpha - 1)\ (\alpha\beta - 1)}}{(\alpha\beta - 1)} \operatorname{sgn}(H_{y1}Hx)\ d\beta, \tag{97}$$

or

$$dv_y = \mp \epsilon \frac{a_0}{\gamma} \beta^{-1/2} (1+ \frac{1}{\gamma}) \sqrt{\frac{\alpha - 1}{\alpha\beta - 1}} \operatorname{sgn}(H_{y0}H_x)\ d\beta \ , \tag{98}$$

where the $\mp$ signs are for the fast wave and slow waves respectively.

The equations (94) and (98) can be integrated to give the velocity Riemann invariants:

$$R_x^v = v_x - \epsilon\ \gamma^{-1} c_0 \int \alpha^{1/2}\ \beta^{-1/2} (1+ \frac{1}{\gamma})\ d\beta \ , \tag{99}$$

$$R_y^{\,v} = v_y \pm \epsilon\, \gamma^{-1}\, c_0 \;\mathrm{sgn}\left(H_{y0} H_x\right) \int \beta^{-1/2\left(1+\frac{1}{\gamma}\right)} \sqrt{\frac{\alpha-1}{\alpha\beta-1}}\; d\beta \;. \tag{100}$$

By substituting α_f and α_s for α we derive the corresponding Riemann invariants for the fast and slow waves.

The above analysis suffices to determine the behaviour of the fast and slow simple waves. Let us recollect that β is proportional to both p and ρ. If β increases both p and ρ increase. Thus β increasing means that the wave is a compression wave propagating in the direction of β increasing. Similarly a rarefaction propagates in the direction of decreasing β. Furthermore when β increases α increases as is clear form the relation (89), and hence, since $U^2 = c^2$, U also increases. Let us consider a fast wave propagating in the positive direction along the x-axis into the region specified by ρ_1, p, $H_x > 0$ and $H_{y1} > 0$, and $v = w = 0$; i.e. specified by (α_1, β_1) in the (α, β) plane. Since we are considering the fast wave, the point (α_1, β_1) must be in the f-region of the figure 1. The behaviour of the wave is described in terms of a development of the parameter β. In addition to α and U increasing with β, we observe that because of the initial condition on H_y and the relation (70), H_y also increases with β.

From the relation (94), it further follows that u also increases with β. Thus the characteristic slope u + U increases with increasing β, i.e., fast wave has a tendency to become a shock. That is precisely what happens for the classical compression wave. However, the transverse velocity component v decreases with increasing β as can be read off from the relation (98). Thus the particle moves in the negative y-direction in the case of the fast compression wave. On the other hand the magnetic field increases in the positive y - direction. Similarly

U, H_y and u decrease with decreasing β, and the characteristic slope $u + U$ also decreases. If β continues to decrease so that the point on the integral curve approaches the critical curve $\beta\alpha = 1$, then H_y approaches zero, leading to the so called complete rarefaction. Since density remains finite in this case because β does, the cavitation does not occur in the fast rarefaction wave.

In the case of a slow wave propagating in the positive direction of the x-axis we assume the same initial conditions. However the initial point (α_1, β_1) now lies in the s-region of figure 1. Again, for the compression wave α_s and U increase, but H_y decreases as can be seen from the inequality (< only) (84) and the relation (70); H_y approaches zero as the point (α, β) approaches the line $\alpha\beta = 1$ and there the compression is complete. Since both u and U increase with β, the characteristic slope can tend to a slow shock. The transverse velocity v also increases. Finally in the slow rarefaction wave for which β decreases, α_s also decreases. In this case the integral curve can reach the line $\beta = 0$. Since ρ is proportional to β, ρ can also vanish, the cavitation can set in. The magnetic field H_y increases in this case. The value of u decreases. With U increasing and u decreasing, the characteristic slope becomes smoother. The value of v also decreases across a slow rarefaction wave.

5. Intermediate simple waves and material layers

In the equation (34), the factor $(\rho U^2 - \mu H_x^2)$ leads to the value of U:

$$U^2 = \frac{\mu H_x^2}{\rho} = a^2 \text{ Alfven speed.} \tag{101}$$

giving us the transverse simple wave. From Chapter IV equations (33), (43) and (44), it follows that

$$d\rho = du = dS = 0 , \tag{102}$$

$$d\underline{v}_T = \mp \frac{H_x}{|H_x|} \sqrt{\frac{\mu}{\rho}} \, d\underline{H}_T , \tag{103}$$

$$H_y \, dH_y + H_z \, dH_z = 0 , \tag{104}$$

where the $\mp$ signs correspond to the characteristic lines $dx/dt = u \pm a$, respectively. Thus ρ, u and S are constants. So is, of course, H_x. The jump in the magnitude of the magnetic field across this wave is zero. The tangential magnetic field, however, rotates:

$$H_y = H_{1T} \sin \psi , \tag{105}$$

$$H_z = H_{1T} \cos \psi , \tag{106}$$

where ψ is a parameter and H_{1T} is a constant. Similarly the tangential velocity field becomes

$$\underline{v}_T = \mp \quad (H_x) \, \underline{H}_T / \sqrt{\rho} + \underline{A} , \tag{107}$$

where $\underline{A}$ is a constant vector. Since the jumps take place only in the tangential components of the flow and field quantities, the transverse simple wave may be considered to be a shear flow. All particles on the same (y, z) - plane move in the same straight line.

There remains the factor $U = 0$ of the equation (34). This corresponds to the material layer (the contact layer). As explained in Chapter IV, if $H_x \neq 0$, over a contact layer then only the density and entropy change; all other flow and field quantities are constant. This excludes the existence of the shear material layer.

If $H_x = 0$, all other quantities may vary except the total pressure p^*, which remains constant.

Chapter VII

Covariant Formulation

Heretofore we have adopted only the quasi-equilibrium approximation to the electrodynamic equations by neglecting the displacement currents and charge accumulations. This approximation has merely allowed us the possibility that the velocities of fluid elements are comparable with the velocities of sound. In this chapter we plan to avoid that limitation by giving a covariant formulation to the equations of magnetohydrodynamics. Relativistic hydrodynamics has been formulated and put on firm foundations recently, while the equations of electrodynamics lend themselves readily to covariant description. We plan to couple these two disciplines. Specifically we shall consider a perfect and infinitely conducting fluid.

After formulating the appropriate equations we discuss the propagation of weak disturbances in the same manner as analyzed in Chapter IV. In fact we represent the wave front by a time-like hypersurface in an Einstein-Riemann space: By using the concept of singular surace across which all the magnetohydrodynamic quantities are continuous but the first derivatives of at least one of these quantities is discontinuous, we obtain various speeds of propagation of these surfaces. Then we take a suitable metric and give a space-time representation of the analysis. This enables us to compare the present discussion with the known results of non-relativistic magnetohydrodynamics as given in Chapter IV. Qualitatively the results are the same. Indeed, there are three kinds of waves--slow, intermediate and fast. The possibility of any of these waves exceeding the speed of light is excluded since we have taken the wave normals to be space-like.

The chapter ends with the discussion of the bicharacteristics and the surfaces of wave normals.

1. Let x^A be a general curvilinear coordinate system in the Einstein-Riemann 4-space (referred to hereafter as E-R space). The capital Roman indicies have the range 0, 1, 2, 3, and are subject to the summation convention over this range. Furthermore let g_{AB} denote the metric components of the E-R space and let W^A stand for the contravariant components of the unit time-like velocity 4-vector:

$$W^A = \frac{dx^A}{ds}, \quad s \text{ is the proper time;}$$

$$g_{AB}\, W^A\, W^B = 1 \tag{1}$$

The momentum energy tensor with components T_{AB} is composed of two parts $T^{(m)}_{AB}$ and $T^{(e)}_{AB}$. The material part $T^{(m)}_{AB}$ is given as

$$T^{(m)}_{AB} = (c^2\ \rho + p)\, W_A\, W_B - p g_{AB}, \tag{2}$$

where ρ is the proper density, p is the pressure and c is the velocity of light*. The components T_{AB} have the dimensions of energy. The electromagnetic part $T^{(e)}_{AB}$ is defined for infinitely conducting fluids in terms of the electromagnetic skewsymmatric tensor with components H_{AB} which in special relativity takes the form:

* In the previous chapters we have denoted the velocity of sound with the symbol c. However every effort is made to avoid confusion.

$$H_{AB} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \qquad (3)$$

The value of $T^{(e)}_{AB}$ is:

$$T^{(e)}_{AB} = \frac{1}{\mu}\left\{ \frac{1}{4} \; g_{AB} \, (H^{CD} \, H_{CD}) - H^{C}_{A} \; H_{CB} \right\}, \qquad (4)$$

where μ is the magnetic permeability.

The Maxwell equations can be written in covariant form in terms of the skewsymmetric tensor H_{AB} and the 4 vector j^B:

$$j^B = (Q, \; J^1, \; J^2, \; J^3), \qquad (5)$$

where Q is the scalar charge density while J^1, J^2, J^2 are the components of the electric current density. Another tensor which enters the Maxwell equations is the dual tensor H^*_{AB}.

$$H^{*\,AB} = \frac{1}{2} \; \eta^{ABCD} \; H_{CD}, \qquad (6)$$

where η^{ABCD} are the components of the customary permutation tensor. In fact the differential equation

$$H^{*\,AB}{}_{;\;A} = 0, \qquad (7)$$

where the semicolon denotes the covariant differentiation formed on g_{AB}, is equivalent to the set of the Maxwell equations

$$\text{curl}\,\underline{E} + \frac{\partial\,\underline{B}}{\partial\, t} = 0, \tag{8a}$$

$$\text{div}\,\underline{B} = 0. \tag{8b}$$

Similarly the equation

$$\frac{1}{\mu}\; H^{AB}_{\;\;;A} = j^{B}, \tag{9}$$

is equivalent to the other set of Maxwell equations:

$$\text{curl}\,\underline{H} = \frac{\partial\,\underline{D}}{\partial\, t} + \underline{J}, \tag{10a}$$

$$\text{div}\,\underline{D} = Q. \tag{10b}$$

Let us now introduce the magnetic 4-vector h_A and the electric 4-vector e_A:

$$h_A = \frac{1}{\mu}\; \overset{*}{H}_{AB}\, W^{B}, \tag{11}$$

$$e_A = H_{BA}\, W^{B}. \tag{12}$$

In view of the skew symmetry of H_{AB}, it readily follows that both the electromagnetic 4-vectors are orthogonal to $\underline{W}$:

$$h_A W^A = e_A W^A = 0. \tag{13}$$

Furthermore their components in the rest frame $W = 1$, $W^i = 0$, coincide with the magnetic and electric 3-vectors with components H_i and E_i. In these expressions as well as in the sequel, the Latin indices will have the range 1,2,3. For infinitely conducting fluids, it follows from OHM's law that $e_A = 0$. Therefore from (12) we have

$$H_{BA} W^B \equiv 0. \tag{14}$$

This means that the 4-vector $\underline{W}$ lies in the null domain of the skew-symmetric matrix $((H_{BA}))$. Hence the rank of this matrix is less than four. But an antisymmetric matrix has always an even rank. Thus the rank of this matrix is two (the rank zero is obviously excluded). From the definition of the dual tensor it follows that the matrix $((H^*_{AB}))$ has also rank two and that the 4-vector $\underline{W}$ lies in its non-null domain. Furthermore since $H^*_{AB}\, h^B \neq 0$, the vector h also lies in the non-null domain of this matrix. Since we have found two mutually orthogonal vectors lying in the non-null domain of the skew-symmetric matrix $((H^*_{AB}))$ of rank two, we can decompose it into the bivector form:

$$H^*_{AB} = \mu\,(W_A h_B - W_B h_A), \tag{15}$$

where the coefficient μ is accounted for by the relation (11).

Hence

$$H_{AB} = \mu\, \eta_{ABCD}\, W^C\, h^D. \tag{16}$$

By substituting the relation (15) into the set of Maxwell equations (7), we obtain an equation for the determination of the 4-vector $\underline{h}$,:

$$(h^A\, W^B - h^B\, W^A)_{;B} = 0. \tag{17}$$

Once we have found $\underline{h}$, we recover the quantities H_{AB} and $\underline{j}$ from the equations (16) and (9). Furthermore the value of $T^{(e)}_{AB}$ as given by the relation (4) can be evaluated in terms of the 4-vector $\underline{h}$. In fact we observe the simple relation:

$$H_{CD}\, H^{CD} = 2\,\mu^2\, h_A\, h^A = -\,2\,\mu^2\, |h|^2, \tag{18}$$

where we have used the fact that $\underline{h}$, being orthogonal to the time-like vector $\underline{W}$, is a space-like 4-vector.

Similarly

$$H^C_A\, H_{CB} = \mu^2 \left\{(g_{AB} - W_A W_B)\,|\,h|^2 + h_A\, h_B\right\}. \tag{19}$$

Substituting (18) and (19) into the equation (4) we get

$$T^{(e)}_{AB} = \mu \left\{(W_A W_B - \frac{1}{2}\, g_{AB})\,|\,h|^2 - \frac{1}{2}\, h_A h_B\right\}, \tag{20}$$

which is a symmetric tensor. Let us observe in passing that

$$W_A \, T^{(e)\;\;AB}_{\;\;;\,B} = 0 = h_A \, T^{(e)\;\;AB}_{\;\;;\,B} \, . \tag{21}$$

We are now in a position to write down the complete value of T_{AB} in terms of two unknown vectors $\underline{W}$ and $\underline{h}$:

$$T_{AB} = (c^2 \rho + p) \, W_A \, W_B - p \, g_{AB} + \mu \left\{ (W_A W_B - \tfrac{1}{2} g_{AB}) |h|^2 - h_A h_B \right\} . \tag{22}$$

The Einstein's field equations of general relativity imply that

$$T^{AB}_{\;\;;B} = 0. \tag{23}$$

When we substitute (22) in (23), we get the required partial differential equations of relativistic magneto-hydrodynamics. This completes the covariant formulation.

2. Definitive system of equations

Let us collect all the necessary equations and put them into the form needed for the subsequent analysis. We have ten variables: W_A, h_A, ρ and p. The second relation (1) provides us with the scalar equation which when differentiated yields

$$W^A W_{A;\,B} = 0. \tag{24}$$

The identity (13) when differentiated becomes

$$W^A h_{A;B} + W^A_{;B} \; h_A = 0. \tag{25}$$

The field equations (17) are

$$W^B \, h^A_{;B} + h^A \, W^B_{;B} - h^B \, W^A_{;B} - W^A h^B_{;B} = 0. \tag{26}$$

From the conservation law (23) we obtain

$$(c^2 \rho,_{B} + p_{,B}) \, W^A \, W^B + (c^2 \rho + p + \mu |h|^2) \, (W^A \, W^A_{;B} + W^B W^A_{;B})$$

$$- p_{,B} \, g^{AB} - 2 \mu \, (W^A W^B - \tfrac{1}{2} g^{AB}) \, h_c \, h^c_{;B}$$

$$- \mu \, (h^A_{;B} \, h_B + h^A \, h_{B;B}) = 0. \tag{27}$$

The thermodynamic relation, required for the present analysis, expresses the condition that

$$p_{,A} \, W^A - a^2 \, \rho_{,A} \, W^A = 0, \tag{28}$$

where

$$a^2 = \frac{\gamma p}{\rho} \,, \tag{29}$$

and γ is a material constant. Except for the identity (25), we have now ten equations in the ten unknown quantities mentioned above.

3. Sonic disturbances

We consider a weak disturbance whose position can be represented by a three dimensional time-like hypersurface $\Sigma(t)$, called a 3-wave, in the E-R space. Any time section t = constant, of Σ will be a two dimensional surface S(t), called the 2-wave, in the three dimensional Riemann space R(t). Let N_A be the components of the unit space-like 4-vector $\underline{N}$ which is normal to $\Sigma(t)$. Furthermore, as in the chapters IV and V, let the bracket $[F]$ stand for the jump of F across the sonic discontinuity. We assume that

$$[\rho] = [p] = [W_A] = [h_A] = [g_{AB}] = [g_{AB;C}] = [g_{AB;CD}] = 0. \tag{30}$$

As demonstrated in the chapter IV, when $[F] = 0$, but $[F_{;A}] \neq 0$, there results the relation

$$[F_{;A}] = N_A \;\; \delta F, \tag{31}$$

where δF is the strength of the discontinuity. Furthermore, since the components of the metric tensor and their first derivatives are continuous, it follows that the jump in the covariant derivatives of a quantity is equal to the jump in the ordinary derivatives. Also since the second derivatives of the components of the metric tensor are also continuous, it follows from the Einstein's field equations that the quantities T^{AB} are continuous across the wave front.

Applying the jump condition (31) to the system of equations given in the section 2 above, there result the relations:

$$W^A \, \delta W^A = 0, \tag{32}$$

$$W^A \, \delta h_A = - h_A \, \delta W^A, \tag{33}$$

$$L \, \delta h^A + h^A N_B \, \delta W^B - h_N \, \delta W^A - W^A N_B \, \delta h^B = 0, \tag{34}$$

$$(c^2 \, \delta\rho + \delta p) \, L \, W^A + (c^2 \rho + p + \mu |h|^2)(W^A N_B \delta W^B + L \delta W^A)$$

$$- \delta p \, N^A - \mu \, (2LW^A - N^A) \, h_c \, \delta h^c - \mu (h_N \delta h^A + N_B \, \delta h^B h^A) = 0, \tag{35}$$

and

$$\delta p = a^2 \, \delta\rho, \tag{36}$$

where

$$L = W^A N_A \text{ and } h_N = h^A \, N_A . \tag{37}$$

4. Alfven waves

Let T_A stand for the components of the tangent 4-vector $\underline{T}$ to the wave front. We shall take such a tangential direction $\underline{T}$ that it is orthogonal to the 3-base $\underline{N}$, $\underline{W}$, and $\underline{h}$. If we multiply (34) and (35) by T_A, we obtain the following two relations

$$L \, \delta h_T - h_N \, \delta W_T = 0, \tag{38}$$

$$(c^2 \rho + p + \mu \, |h|^2) \, L \, \delta W_T - \mu \, h_N \, \delta h_T = 0, \tag{39}$$

where $\delta h_T = \delta h^A \, T_A$ and $\delta \, W_T = \delta \, W^A T_A$. These two equations will give a non-trivial solution for $(\delta h_T, \delta W_T)$, if we have

$$L^2 (c^2 \rho + p + \mu |h|^2) - \mu h_N^2 = 0. \tag{40}$$

Let us now define

$$V = cL = c\ W^A N_A, \tag{41}$$

then (40) gives

$$V^2 = \frac{\mu h_N^2 c^2}{c^2 \rho + p + \mu |h|^2} \tag{42}$$

This relation, as we shall soon see, leads to the Alfven wave of nonrelativistic magnetohydrodynamics. It is independent of the thermodynamics in the above analysis. Moreover the same relation holds for incompressible as well as compressible fluids.

5. Fast and slow waves

We start with the field equation (34). Multiplying it by W_A and using (32) and (33), we derive:

$$L\, W_A\ \delta h^A = N_B\ \delta h^B = -\, L\, h_A\ \delta W^A. \tag{43}$$

Similarly if we multiply the equation (34) by h_A, it yields

$$h_A\ \delta h^A = \frac{1}{L}\ (|h|^2 N_B\ \delta W^B + h_N h_A\ \delta W^A) \tag{44}$$

Let us now eliminate δp from the equations (35) and (36). The scalar product of resulting equation with $\underline{h}$, with the help of some of the identities as derived above, simplifies into

$$h_A \, \delta W^A = \frac{1}{L} \quad \frac{a^2}{c^2 \rho + p} \, h_N \, \delta \rho \tag{45}$$

Similarly the scalar product of (35) with $\underline{W}$ and $\underline{N}$ leads to

$$N_B \quad \delta W^B = - L \, c^2 \, \frac{\delta \rho}{c^2 \rho + p} \quad , \tag{46}$$

and

$$\left\{ (c^2 + a^2) \, L^2 + a^2 \right\} \delta \rho + 2 \, (c^2 \rho \, + p) \, L \, N_B \, \delta W^B + a^2 \, \delta \rho -$$

$$\frac{\mu}{L} \quad h_N \, h_A \, \delta \, W^A = 0. \tag{47}$$

Substituting from (45) and (46) into (47), we readily obtain

$$\left[(c^2 - a^2) \, (c^2 \rho + p) L^4 - \left\{ \mu \, |h|^2 \, c^2 + a^2 (c^2 \rho + p) \right\} L^2 + \mu \, a^2 h_N^2 \right] \delta \rho = 0. \tag{48}$$

But $\delta \rho \neq 0$, and the above equation gives the required characteristic equation

$$(c^2 - a^2) \, (c^2 \rho + p) L^4 - \left\{ \mu \, | \, h|^2 \, c^2 + a^2 (c^2 \rho \, + \, p) \right\} L^2 + \mu \, a^2 \, h_N^2 = 0. \tag{49}$$

In terms of the quantity V as defined by the relation (41), the above equation (49) becomes

$$(1 - \frac{a^2}{c^2})(\rho + \frac{p}{c^2}) V^4 - \left\{ \mu |h|^2 + a^2(\rho + \frac{p}{c^2}) \right\} V^2 + \mu a^2 h_N^2 = 0. \tag{50}$$

When $h_A = 0$, this equation reduces to:

$$(1 - \frac{a^2}{c^2}) V^2 - a^2 = 0, \tag{51}$$

which gives the characteristic equation for relativistic hydrodynamics.

6. Space-time representation

A coordinate system can be introduced in the E-R space for which the square of the element of length ds^2, has the form

$$ds^2 = c^2 dt^2 - a_{ij} d_x^i d_x^i, \tag{52}$$

where a_{ij} are the coefficients of a positive definite quadratic form. These coefficients, in general, depend on the coordinate t, as well as the spatial coordinates $\underline{x}$.. Relative to this coordinate system the velocity 4-vector becomes

$$W_o = \frac{c}{\sqrt{1 - v^2/c^2}} \; ; \qquad W_i = \frac{-v_i}{\sqrt{1 - v^2/c^2}} \; ,$$

$$W^o = \frac{1}{c\sqrt{1 - v^2/c^2}} \; ; \qquad W^i = \frac{v^i}{c\sqrt{1 - v^2/c^2}}, \tag{53}$$

where v_i and v^i are the covariant and contravariant components of the velocity vector in the three-dimensional Riemann space R(t) whose metric is defined by the above quantities $a_{ij}(t, \underline{x})$, and v is the magnitude of the velocity.

The electromagnetic 4-vector $\underline{h}$ is now given as

$$h_o = \frac{(H_i v^i)}{\sqrt{1 - v^2/c^2}}, \quad h_i = -\frac{H_i}{\sqrt{1 - v^2/c^2}} - \frac{(\underline{E} \times \underline{V})_i}{\mu c \sqrt{1 - v^2/c^2}},$$

$$h^o = \frac{(H_i v^i)}{c^2\sqrt{1 - v^2/c^2}}, \quad h^i = \frac{H^i}{\sqrt{1 - v^2/c^2}} + \frac{(\underline{E} \times \underline{V})^i}{\mu c \sqrt{1 - v^2/c^2}} \tag{54}$$

Similarly the normal 4-vector $\underline{N}$ has the decomposition

$$N_o = \frac{G}{\sqrt{1 - G^2/c^2}}; \quad N_i = \frac{-n_i}{\sqrt{1 - G^2/c^2}},$$

$$N^o = \frac{G}{c^2\sqrt{1-G^2/c^2}}; \quad N^i = \frac{n^i}{\sqrt{1 - G^2/c^2}}, \tag{55}$$

where n_i and n^i are the covariant and contravariant components of the unit normal to S(t) in the space R(t) and G is the normal coordinate velocity of propagation of S(t) into R(t), as explained in Chapter IV. The direction of $\underline{N}$ is chosen, for definiteness, so that the associated vector $\underline{n}$ is directed into the region R(t) into which the surface S(t) is propagated; then the velocity G is positive.

The quantities L, V, h_N and $|h|^2$ become

$$L = N^A W_A = \frac{G - V_n}{c\sqrt{1-G^2/c^2}\ \sqrt{1-v^2/c^2}}\,, \tag{56}$$

$$V = cL = \frac{G - V_n}{\sqrt{1-G^2/c^2}\ \sqrt{1-v^2/c^2}}\,, \tag{57}$$

$$h_N = h_A N^A = \frac{(H_i\ V^i)G}{c^2\sqrt{1-G^2/c^2}} - \frac{H_n}{\sqrt{1-v^2/c^2}\ \sqrt{1-G^2/c^2}} - \frac{1}{\mu}\,\frac{(\underline{E}\times\underline{V})_i n^i}{c\sqrt{1-v^2/c^2}\ \sqrt{1-G^2/c^2}} \tag{58}$$

and

$$|h|^2 = -\,h^A h_A = -\,\frac{(H_i\ V^i)^2}{c^2(1-v^2/c^2)} + \frac{H^2}{\sqrt{1-v^2/c^2}} +$$

$$\frac{2\,H^i\,(\underline{E}\times\underline{V})_i}{\mu c(1-v^2/c^2)} + \frac{|(\underline{E}\times\underline{V})|^2}{\mu^2 c^2(1-v^2/c^2)} \tag{59}$$

From (57), (58) and (59) we observe the important fact that when

$$\frac{v}{c} << 1\,, \quad \frac{G}{c} << 1,$$

we have

$$V \sim G - V_n \overset{\text{def}}{=} U, \tag{60}$$

$$h_N \sim - H_n, \tag{61}$$

$$|h|^2 \sim H^2, \tag{62}$$

and a tends to the velocity of sound in non-relativistic uncharged fluids.

When we make these substitutions into the equations (42) and (50), they give the limits

$$U^2 = \mu H_n^2 \; / \rho, \tag{63a}$$

and

$$\rho \; U^4 - (\mu H^2 + a^2 \rho) U^2 + \mu \; a^2 H_N^2 = 0. \tag{63b}$$

These two relations agree completely with those found in Chapter IV, if we take into account the slight change in notation.

Qualitatively the results in the present case are similar to those in non-relativistic case. The velocities of the present waves are, however, comparable with the velocities of light, but none can exceed that velocity. This follows from the Darmois-Lichnerowicz formula for the speed u of wave propagation:

$$u^2 = \frac{c^2 \; W^A \; W_A}{W^A N_A - N^A N_A} \tag{64}$$

Since $N^A N_A = -1$, we get $u < c$.

7. Bicharacteristics

Let the hypersurface $\Sigma(t)$ have an equation:

$$f(x,\ t) = 0, \tag{65}$$

then the space-like 4-vector N_A is defined as

$$N_A = f_{,A} \Big/ \sqrt{-g^{PQ} f_{,P} f_{,Q}} \tag{66}$$

In terms of $f_{,A}$ we can write the characteristic relations (40) and (48) in the form

$$F(f_{,A}) = (c^2 \rho + p + \mu\,|h|^2)\,(W^A f_{,A})^2 - \mu\,(h^A f_{,A})^2, \tag{67}$$

$$F(f_{,A}) = (c^2 - a^2)\,(c^2 \rho + p)\,(W^A f_{,A})^4 + \left\{\mu\,|h|^2 c^2 + a^2(c^2\rho + p)\right\}(W^A f_{,A})^2\,(g^{PQ} f_{,\rho} f_{,Q})$$
$$- a^2 \mu\,(h^A f_{,A})^2\,(g^{PQ} f_{,\rho} f_{,Q}). \tag{68}$$

The equations (67) and (68) are first order partial differential equations for determining the integral surface f. This integral surface is spanned by characteristic rays. These rays are called the bicharacteristics for the original equations and are determined by the ordinary differential equations:

$$\frac{dx^A}{ds} = \frac{\partial F}{\partial (f_{,A})} \tag{69}$$

with s defined as a parameter along the bicharacteristics. Substituting from (67) and (68) into (69) leads to

$$\underset{(1)}{b}^A = \frac{1}{2}\frac{dx^A}{ds} = (c^2\rho + p + \mu|h|)\, W^P f_{,P}\, W^A - \mu\,(h^A f_{,A})\, h^A, \tag{70}$$

$$\begin{aligned}\underset{(2)}{b}^A = \frac{1}{2}\frac{dx^A}{ds} &= \left[2(c^2-a^2)(c^2\rho+p)\,(W^P f_{,P})^3 + \mu|h|^2 c^2 + a^2(c^2\rho+p)\right] W^A \\ &+ \left[\left\{\mu|h|^2 c^2 + a^2 (c^2\rho + p)\right\}(W^P f_{,P})^2 - a^2 \mu\,(h^P f_{,P})^2\right] g^{AB} f_{,B} \\ &- \left[a^2 \mu\,(h^P f_{,P})\,(g^{PQ} f_{,P} f_{,Q})\right] h^A. \end{aligned} \tag{71}$$

From (70) we observe that $\underset{(1)}{b}^A$ lies in the plane of the orthogonal 4-vectors W^A and h^A. The ray $\underset{(2)}{b}^A$ will also lie in the same plane if N^A does. We now set out to investigate that possibility.

We have $h^A / |h|$, as a unit space-like vector and W^A, as a unit time-like vector. Let us introduce two new mutually orthogonal unit 4-vectors, which are orthogonal to the 2-plane determined by W^A and $h^A/|h|$. Hence both Y^A and Z^A are space-like and span the null domain of H^{*AB} and therefore they span the non-null domain of H^{AB}. Thus we can decompose H^{AB}, in terms of Y^A and Z^A, giving (in view of the relations (15) and (16))

$$H^{AB} = \mu|h|\,(Y^A Z^B - Z^A Y^B). \tag{72}$$

The vectors W^A, $h^A/|h|$, Y^A and Z^A form a 4-tuple of mutually orthogonal unit vectors; W^A is time-like and $(h^A/|h|, Y^A, Z^A)$ are space-like. We can represent N_A as

$$N_A = A\,W_A + B\,h_A/|h| + c\,Y_A + D\,Z_A. \tag{73}$$

Since $N^A N_A = -1$, this gives

$$A^2 - B^2 - c^2 - D^2 = -1. \tag{74}$$

Also

$$L = A. \tag{75}$$

$$h_N = -B\,|h|. \tag{76}$$

The characteristic relations (40) and (48) become

$$\left\{c^2\rho + p + \mu|h|^2\right\} A^2 - \mu|h|^2 B^2 = 0, \tag{77}$$

$$(c^2-a^2)(c^2\rho+p)A^4 - \left\{\mu|h|^2 c^2 + a^2(c^2\rho+p)\right\} A^2 + a^2\mu|h|^2 B^2 = 0. \tag{78}$$

If the normal N_A is to lie in the 2-plane W_A and h_A, then from (71) and (72) it follows that

$$C = D = 0, \tag{79}$$

$$A^2 - B^2 = -1. \tag{80}$$

Thus A and B satisfy the equation for hyperbola as given by (80). The intersection of this hyperbola with the curve given by the equation (77) yields

$$A = \left\{\mu |h|^2 / (c^2 \rho + p)\right\}^{\frac{1}{2}} \; ; \; B = \pm \left\{1 + \frac{\mu |h|^2}{c^2 \rho + p}\right\}^{\frac{1}{2}} \tag{81}$$

Similarly the intersection of this hyperbola with the curve given by (78) leads to

$$A = \frac{a}{\sqrt{c^2 - a^2}} , \quad B = \pm \frac{c}{\sqrt{c^2 - a^2}} , \tag{82}$$

and the values as given by the relations (81). In fact when $B^2 = 1 + A^2$, the quadratic equation (77) becomes a factor of (78). It follows from (82) that there are two wave normals which lie in the plane (W_A, h_A) provided $c > a$.

Finally we observe from (74), (77) and (78) that $A = B = 0$; $C^2 + D^2 = 1$, always satisfy these equations. Thus there are ∞^1 characteristic hypersurfaces such that W_A and h_A lie in their local tangent hyperplanes.

8. Geometry of the wave fronts

Let us set

$$f_{,A} = X_A . \tag{83}$$

The equations (67) and (68) then give

$$F(X_A) = (c^2\rho+p+\mu|h|^2)(W^A X_A)^2 - \mu\,(h^A X_A)^2 = 0 \tag{84}$$

$$\begin{aligned} F(X_A) = {} & (c^2-a^2)\,(c^2\rho+p)\,(W^A X_A)^4 + \Big\{\mu\,|h|^2\ c^2 \\ & + a^2\,(c^2\rho + p)\Big\}\,(W^A X_A)^2\,(g^{CD} X_C X_D) \\ & - a^2\,\mu\,(h^A X_A)^2\,(g^{CD} X_C X_D) = 0. \end{aligned} \tag{85}$$

The equation $F(X_A) = 0$, represents, for fixed (x, t), a surface in X-space called the surface of wave normals. Let us take the Lorentz frame at the point $(\underline{x}, t)$ such that, $g^{oo} = 1$, $g^{ij} = -\delta^{ij}$. Let us also define a 4-tuple of mutually orthogonal vectors $\underline{e}_o$, $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$, at that point. For $\underline{e}_o$ and $\underline{e}_1$, we take $\underline{W}$ and $\underline{h}\,/\,|h|$ respectively. The equation (84), which gives the wave normal surface for Alfven waves, becomes

$$(c^2\,\rho + p + \mu\,|h|^2)\,X_0^2 - \mu\,X_1^2 = 0. \tag{86}$$

Factoring the left member of the equation (86) we obtain two planes in the E-R space.

The equation (85), similarly gives

$$(c^2 \rho + p + \mu |h|^2)\, c^2 X_0^4 - \left[\left\{a^2 (c^2 \rho + p) + \mu |h|^2 c^2\right\} (X_1^2 + X_2^2 + X_3^2) + a^2 \mu |h|^2 X_1^2\right]$$

$$+ a^2 \mu |h|^2 X_1^2 (X_1^2 + X_2^2 + X_3^2) = 0, \qquad (87)$$

which is the equation for a conoid. The intersection of this conoid with straight line shows that it is composed of two distinct nappes C_f (for fast wave) and C_s (for slow wave). The nappe C_f is interior and is convex. The light cone is interior to C_f. The two planes, which comprise the surface of the wave normals for the Alfven wave, always touch the conoid given by the equation (87). The geometrical shape of these wave normal surfaces is thus the same as in the case of non-relativistic magnetohydrodynamics. Since the diagram of wave fronts can be constructed from the surface of the wave normals, by a simple geometrical device, it follows that all the known geometrical facts about the magnetohydrodynamic wave fronts given in the Chapter IV are applicable to the present case.

Finally we observe that when $c^2 = a^2$, the equation (85) has a factor

$$g^{CD} X_C X_D = 0, \qquad (88)$$

which is the equation of the light cone. In this special case the light cone coincides with the nappe C_f of the fast wave.

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