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RANDOM PRESSURE EXCITATION OF SHELLS AND STATISTICAL DEPENDENCE EFFECT OF NORMAL MODE RESPONSE

by Robert E. Davis

Prepared under Contract No. NAS 1-3179 *by*
MCDONNELL AIRCRAFT CORPORATION
St. Louis, Mo.

for

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By Robert E. Davis
McDonnell Aircraft Corporation
St. Louis, Missouri

SUMMARY

The power spectrum of structural response to random excitation provides a valuable tool for evaluating structural integrity. Using a normal mode approach, equations are derived which give the response power spectrum of shell-type structures to random pressure excitation. Primary assumptions are that the exciting phenomenon is weakly ergodic, and that the structure is lightly damped and its motion is linear. The derivation is carried to a point such that direct solution of the final equations is possible, i.e., imaginary terms are eliminated and the equations are given as functions of positive frequency in cycles per second. The statistical dependence of the normal coordinate responses is accounted for in the equations, rather than being neglected as is usually done. Furthermore, the importance of these terms is investigated and a limiting case on their importance is obtained.

CONTENTS

	Page
SUMMARY	iii
SYMBOLS	vii
INTRODUCTION	1
DERIVATION OF RESPONSE EQUATIONS	1
General Aspects	1
Relation Between Cross-Spectrum and Fourier Transforms	2
Equation for Response Cross-Spectrum	6
Elimination of Imaginary Terms - Response Co-Spectrum	10
Final Equations in Terms of Positive Frequency in Cycles Per Second	12
CONTRIBUTION OF NORMAL COORDINATE CO-SPECTRA TO TOTAL RESPONSE	15
CONCLUDING REMARKS	21
APPENDIX A - UPPER LIMIT ON MAGNITUDE OF NORMAL COORDINATE CROSS-SPECTRUM	23
REFERENCES	27

SYMBOLS

A	Surface area
C	Co-spectrum, real part of S defined below
F	Generalized force
f	Frequency in cycles per second
G	Function of frequency defined in text
H	Function of frequency defined in text
i	$\sqrt{-1}$
M	Generalized mass
N	Number of normal modes
P	For a pair of modes, P is the percentage of response contributed by the normal coordinate co-spectrum compared to that from the sum of the normal coordinate power spectra and co-spectrum
p	Pressure
Q	Quad-spectrum, imaginary part of S defined below
q	Normal coordinate
R	For a pair of modes, R is the ratio of response contributed by the normal coordinate co-spectrum compared to that from the sum of the normal coordinate power spectra and co-spectrum
r	Radius of shell
S	Power spectrum or cross-spectrum depending upon: (1) alike or unlike superscripts, or (2) alike or unlike variables within parenthesis
T	Period of time over which spectral functions are determined, theoretically approaching infinity
t	Time
x	Coordinate defining distance along cone generator from apex to a point on shell

W	Deflection perpendicular to shell surface
Z	Reciprocal of complex frequency response function
α	Defined in text
β	Defined in text
ζ	Ratio of damping to critical damping
θ	Coordinate defining angular position on shell
ξ	Defined in text
τ	Time shift for auto-correlation or cross-correlation functions
ϕ	Modal deflection perpendicular to shell surface
ω	Frequency in radians per second

Subscripts and Superscripts

A	Indicates relationship to area
a,b	Indicate response points a and b, respectively
F	Indicates relationship to generalized force
I,J	Indicate modes I and J, respectively
k,l	Indicate sub-areas k and l, respectively
p,q,w	Indicates relationship to pressure, normal coordinate, and deflection, respectively

Matrix Notation

$[\ ]$	Square matrix
$\lfloor\ \rfloor$	Row matrix
$\lfloor\ \rfloor^T$	Transpose of row matrix
$\{\ \}$	Column matrix

NOTE: Double subscripts or superscripts appearing on symbols within a matrix imply all possible combinations of themselves as they vary from unity to their maximums. As the subscripts or superscripts vary, they give the row and column designation within the matrix; the first subscript or superscript gives the row, and the second gives the column. Thus, the variability of the subscripts or superscripts is limited by the number of rows and columns in the matrix.

Miscellaneous Notation

*	Denotes complex conjugate
( )	Parenthesis following a symbol indicates that the symbol is a function of those variables appearing within the parenthesis
$\langle\ \rangle$	Indicates a time average

Differentiation of a variable with respect to time is indicated by dots over the variable.

INTRODUCTION

In recent years, considerable interest has been shown in obtaining the response of continuous structures to random pressure excitation. This interest has resulted primarily from fatigue problems of panels and shell-type structures of spacecraft and aircraft subjected to aerodynamic buffet and/or jet or rocket noise.

Because of the random nature of the response to such excitation, it must be described in statistical terms. The most useful of these is the power spectrum for two reasons. First, the power spectrum gives the frequency distribution of response, and thus, indicates which frequency intervals contain the major energy contributions. Secondly, if it is assumed that the random response is Gaussian (normal), the power spectrum can be used to determine the probable time for fatigue failure to occur. This latter use is discussed by Powell (Reference 1) and is based on a derivation by Rice (Reference 2).

The purposes of this report are to derive equations for the deflection and acceleration response power spectra of shell-type structures to random pressure excitation, and to investigate the role played by the statistical dependence of the normal coordinate responses, denoted herein as the normal coordinate co-spectra. A conical shell is used initially as the structural representation to lend direction for the subsequent analytical development applicable to general shell-type structures. As implied above, a normal mode approach is used.

Matrix algebra is used in the derivation because the resulting equations are compact, the variables are conveniently separated, and numerical techniques of solution are simplified. The final equations include the normal coordinate co-spectra contributions and are given in a form which enables direct solution. The input forcing functions, pressure power spectra and co- and quad-spectra, are specifically defined.

The work relating to the effect of the normal coordinate co-spectra results in an equation giving the percentage of system response contributed by these terms compared to the total response. This work provides a general criteria for determining when these terms may be neglected; usually, they are neglected without true justification in order to simplify the calculations.

DERIVATION OF RESPONSE EQUATIONS

General Aspects

As previously stated, we wish to determine the response to random excitation in terms of power spectra. Since a power spectrum is a special case of a cross-spectrum (the cross-spectrum becomes a power spectrum when the functions concerned are identical), an equation will be derived for the response cross-spectrum. The derivation will be in terms of complex variables, with only the real part of the final equation retained. This real part is called the response co-spectrum, the imaginary part being the quad-spectrum.

The response cross-spectrum is a statistical function, and depends upon the statistical averages of the exciting phenomenon. Fundamentally, statistical averages imply the use of ensemble averages; however, when ensemble averages are not available, acceptable alternates, such as time averages, must be sought. When a random process is weakly ergodic, either time or ensemble averages give equivalent mean values, cross-correlation functions, and cross-spectral functions. A necessary condition, though not sufficient, for a process to be weakly ergodic is that it be stationary. If ensemble and time averages give equivalent results for all statistical properties, i.e., the complete probability structure may be determined from time averages, the processes are strongly ergodic. Sufficient conditions to insure strong ergodicity are that a random process be stationary, Gaussian, and have a continuous power spectrum. A more complete discussion of the above is given in Reference 3.

For the work herein, we assume weakly ergodic processes. We also assume processes having zero means, since we are dealing with linear systems and a non-zero mean can be handled as a separate problem. With these assumptions, a relationship can be derived between the cross-spectrum of a pair of functions and the Fourier transforms of these functions (Reference 4). Since the Fourier transform of the response may be obtained from the differential equations describing the system motion to forced excitation, the aforementioned relationship provides the "key" used to derive an equation for the response cross-spectrum.

Relation Between Cross-Spectrum and Fourier Transforms

Figure 1 shows the frustum of a conical shell. Assume that the response of this shell at points a and b to random pressure excitation is as shown in Figure 2. The cross-correlation function for $W(x_a,\theta_a,t)$ and $W(x_b,\theta_b,t)$ is then given by

$$\langle W(x_a,\theta_a,t+\tau)\, W(x_b,\theta_b,t)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} W(x_a,\theta_a,t+\tau)\, W(x_b,\theta_b,t)\,dt \qquad (1)$$

where τ is a time shift between the records. Assume that the infinite records for $W(x_a,\theta_a,t+\tau)$ and $W(x_b,\theta_b,t)$ are truncated such that they are zero outside the time interval $-T$ to T, where T is a very large time.

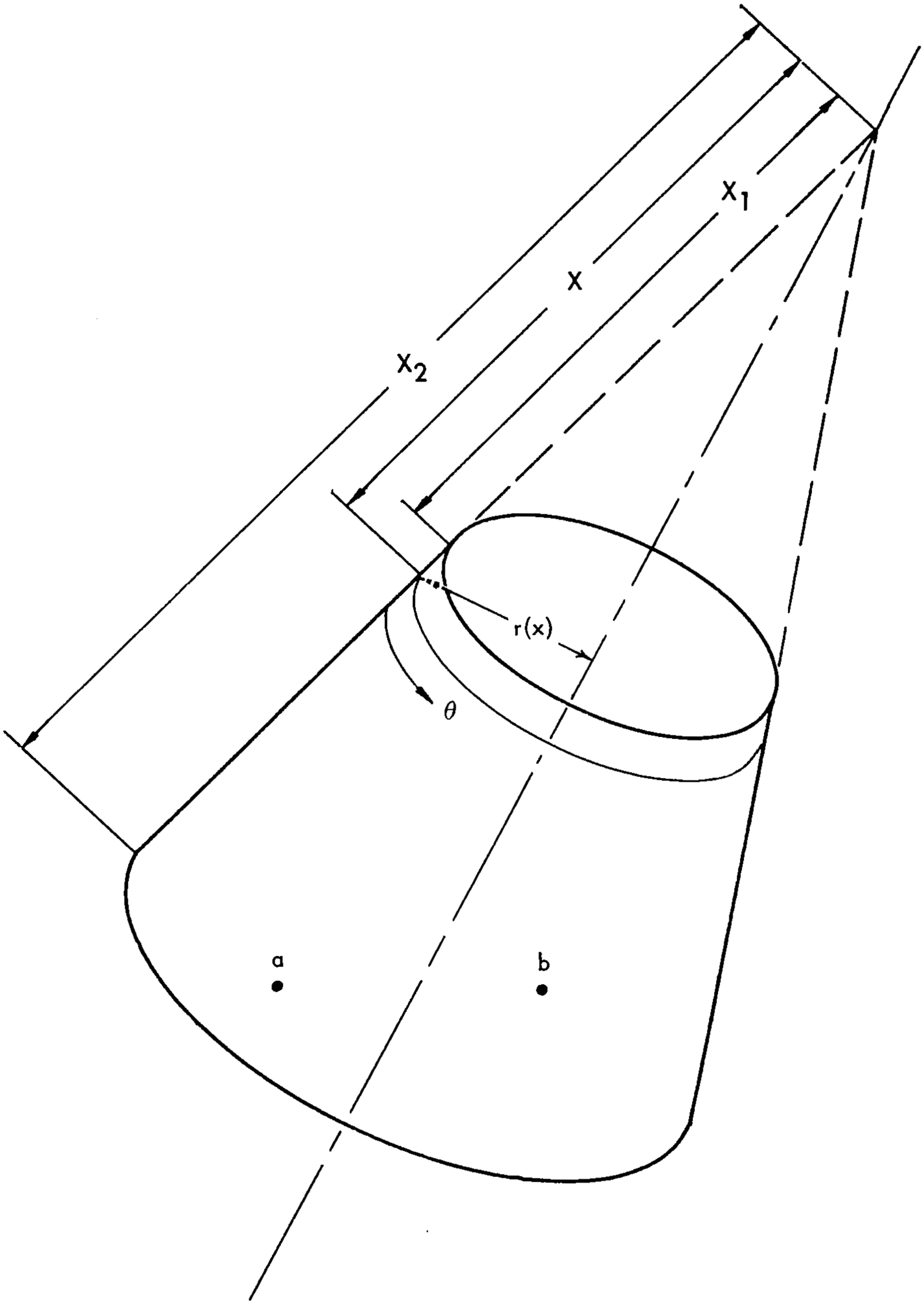

Figure 1 – Coordinate System for Frustum of Conical Shell

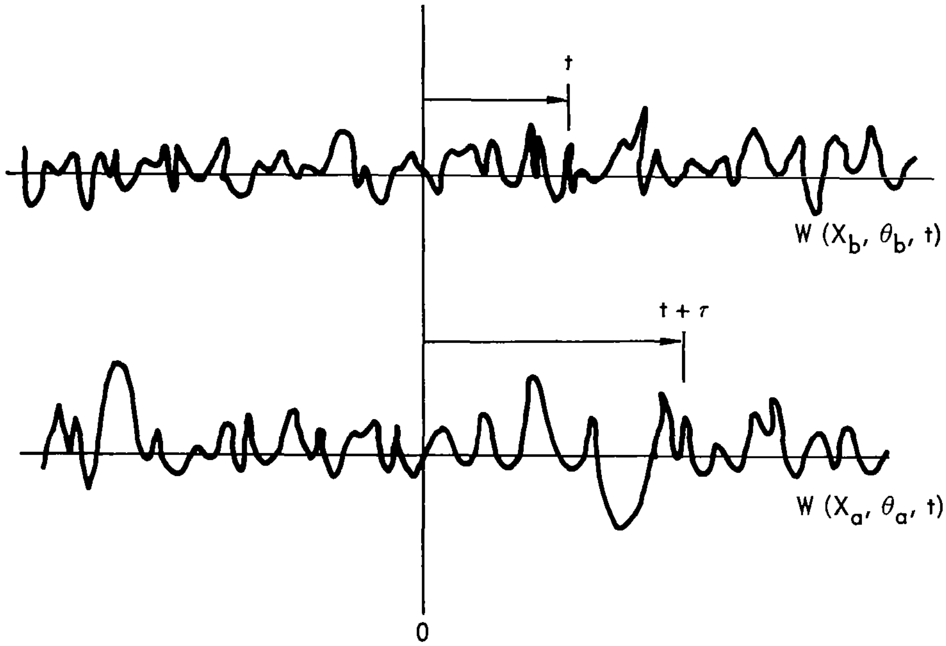

Figure 2 – Records of Random Functions of Time, $W(x_a, \theta_a, t)$ and $W(x_b, \theta_b, t)$

This assumes that the infinitely long time average can be approximated satisfactorily by a very long time average. Now Equation (1) can be written as

$$\langle W(x_a,\theta_a,t+\tau)\ W(x_b,\theta_b,t)\rangle = \frac{1}{2T} \int_{-\infty}^{\infty} W(x_a,\theta_a,t+\tau)\ W(x_b,\theta_b,t)dt \tag{2}$$

The cross-spectrum is defined as the Fourier transform of the cross-correlation function. Thus, we have

$$S_W^{ab}(\pm\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} \langle W(x_a,\theta_a,t+\tau)\ W(x_b,\theta_b,t)\rangle\, d\tau \tag{3}$$

The $\pm\omega$ is used to indicate both positive and negative frequencies. Later since physical meaning is limited to the positive frequency domain, we will limit certain results to only positive frequencies. Substituting Equation (2) into Equation (3), we obtain

$$S_W^{ab}(\pm\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} \frac{1}{2T} \int_{-\infty}^{\infty} W(x_a,\theta_a,t+\tau)\ W(x_b,\theta_b,t)dt\ d\tau$$

Introducing $e^{i\omega t}\ e^{-i\omega t} = 1$ and interchanging the order of integration results in

$$S_W^{ab}(\pm\omega) = \frac{1}{2T} \int_{-\infty}^{\infty} e^{-i\omega(t+\tau)}\ W(x_a,\theta_a,t+\tau)d\tau \int_{-\infty}^{\infty} e^{i\omega t}\ W(x_b,\theta_b,t)dt$$

Since t is a constant for the first integration, we can replace $d\tau$ by $d(t+\tau)$. Then letting $t+\tau = t'$ we get

$$S_W^{ab}(\pm\omega) = \frac{1}{2T} \int_{-\infty}^{\infty} e^{-i\omega t'}\ W(x_a,\theta_a,t')dt' \int_{-\infty}^{\infty} e^{i\omega t}\ W(x_b,\theta_b,t)dt \tag{4}$$

Using the fact that statistical properties of stationary processes are independent of the time origin, and assuming the existence of the Fourier transform of $W(x_a,\theta_a,t')$ and $W(x_b,\theta_b,t)$ - the transforms exist if the functions are piecewise continuous and the integrals,

$\int_{-\infty}^{\infty} |W(x_a,\theta_a,t')|dt'$ and $\int_{-\infty}^{\infty} |W(x_b,\theta_b,t)|dt$ are convergent (Reference 5) -

Equation (4) can be written as

$$S_w^{ab}(\pm\omega) = \frac{1}{2T} W(x_a,\theta_a,\pm\omega)\; W^*(x_b,\theta_b,\pm\omega) \tag{5}$$

This is the desired relationship between the cross-spectrum and the Fourier transforms. Note that when a = b, we have a power spectrum.

Equation for Response Cross-Spectrum

Proceeding to determine the response cross-spectrum, we must obtain $W(x_a,\theta_a,\pm\omega)$ and $W(x_b,\theta_b,\pm\omega)$ in terms of the excitation pressure and structural properties and then substitute into Equation (5). Assuming the frustum shown in Figure 1 is lightly damped with undamped normal modes $\phi_I(x,\theta)$, corresponding to the normal coordinates q_I, the equations of motion are given by

$$W(x_a,\theta_a,t) = \lfloor \phi_{aI} \rfloor \left\{ q_I(t) \right\} \tag{6}$$

$$\ddot{q}_I(t) + 2\zeta_I\, \omega_I\, \dot{q}_I(t) + \omega_I^{\,2}\, q_I(t) = \frac{1}{M_I} \int_{x_1}^{x_2} \int_0^{2\pi} \phi_I(x,\theta)p(x,\theta,t)r(x)\,d\theta\, dx \tag{7}$$

The assumption of light damping allows us to ignore cross damping terms in the above equation.

We obtain $W(x_a,\theta_a,\pm\omega)$ by taking the Fourier transform of Equation (6),

$$W(x_a,\theta_a,\pm\omega) = \lfloor \phi_{aI} \rfloor \left\{ q_I(\pm\omega) \right\} \tag{8}$$

A similar expression for $W^*(x_b,\theta_b,\pm\omega)$ will be needed and is obtained from the complex conjugate form of Equation (8) with a replaced by b,

$$W^*(x_b,\theta_b,\pm\omega) = \lfloor \phi_{bJ} \rfloor \left\{ q^*_J(\pm\omega) \right\} \tag{9}$$

In Equation (9), J is used rather than I for the purpose of distinguishing between different normal modes in the subsequent development. Substituting from Equations (8) and (9) into Equation (5), and obeying the rules for matrix multiplication, we obtain

$$\begin{aligned} S_w^{ab}(\pm\omega) &= \left\lfloor \phi_{aI} \right\rfloor \frac{1}{2T} \left\{ q_I(\pm\omega) \right\} \left\lfloor q^*_J(\pm\omega) \right\rfloor \left\lfloor \phi_{bJ} \right\rfloor^T \\ &= \left\lfloor \phi_{aI} \right\rfloor \left[S_q^{IJ}(\pm\omega) \right] \left\lfloor \phi_{bJ} \right\rfloor^T \end{aligned} \tag{10}$$

where the IJth element of $\left[S_q^{IJ}(\pm\omega)\right]$ is given by

$$S_q^{IJ}(\pm\omega) = \frac{1}{2T}\, q_I(\pm\omega)\ q^*{}_J(\pm\omega) \tag{11}$$

Equation (11) gives a normal coordinate power spectrum when I = J, and a normal coordinate cross-spectrum when I $\neq$ J. As first pointed out by Powell (Reference 1), the cross-spectra represent the statistical dependence between normal coordinate responses.

To obtain the elements of $\left[S_q^{IJ}(\pm\omega)\right]$ for substitution into Equation (10), we will use Equation (11), which means that $q_I(\pm\omega)$ and $q_J^*(\pm\omega)$ must be determined. We proceed to do this by taking the Fourier transform of Equation (7),

$$q_I(\pm\omega) = \frac{1}{Z_I(\pm\omega)} \int_{x_1}^{x_2} \int_0^{2\pi} \phi_I(x,\theta)p(x,\theta,\pm\omega)r(x)\,d\theta dx \tag{12}$$

where

$$Z_I(\pm\omega) = M_I\ \omega_I{}^2 \left(1 - \left(\frac{\omega}{\omega_I}\right)^2 + i\ 2\zeta_I \left(\frac{\omega}{\omega_I}\right)\right) \tag{13}$$

Similarly, for mode J we have

$$q_J^*(\pm\omega) = \frac{1}{Z^*{}_J(\pm\omega)} \int_{x_1}^{x_2} \int_0^{2\pi} \phi_J(x,\theta)p^*(x,\theta,\pm\omega)r(x)\,d\theta dx \tag{14}$$

$$Z_J^*(\pm\omega) = M_J\ \omega_J^2 \left(1 - \left(\frac{\omega}{\omega_J}\right)^2 - i\ 2\ \zeta_J \left(\frac{\omega}{\omega_J}\right)\right) \tag{15}$$

By substituting Equations (12) and (14) into Equation (11), we obtain

$$S_q^{IJ}(\pm\omega) = \frac{1}{Z_I(\pm\omega)\ Z^*{}_J(\pm\omega)} \int_{x_1}^{x_2} \int_0^{2\pi} \int_{x_1}^{x_2} \int_0^{2\pi} \Big(S_p(x,\theta,x',\ \theta',\pm\omega)\phi_I(x,\theta)\phi_J(x',\theta')\, r(x)r(x')\,d\theta dx d\theta' dx'\Big) \tag{16}$$

where

$$S_p(x,\theta,x',\theta',\pm\omega) = \frac{1}{2T}\, p(x,\theta,\pm\omega)\ p^*(x',\theta',\pm\omega) \tag{17}$$

and the primes indicate the order of integration. The function $S_p(x,\theta,x',\theta',\pm\omega)$, is the pressure cross-spectrum between any two points (x,θ) and (x',θ').

The solution of Equation (16) in its present form requires the analytical definition of $S_p(x,\theta,x',\theta',\pm\omega)$ over the complete shell surface. A rigorous definition would be extremely difficult, if not impossible, to obtain when the exciting pressure results from a highly turbulent flow over the shell surface. Yet this is precisely the most likely condition under which maximum excitation of spacecraft shell structures occurs. Thus, it is necessary to simplify Equation (16) before a solution is possible.

As a first step in this simplification, we divide the total shell surface into sub-areas. Now Equation (16) can be written as

$$S_q^{IJ}(\pm\omega) = \frac{1}{Z_I(\pm\omega)Z^*_J(\pm\omega)} \sum_k \sum_l \int_{A_l}\int_{A_k}\left(S_p(x,\theta,x',\theta',\pm\omega)\ \phi_I(x,\theta)\ \phi_J(x',\theta')\,dA_k\ dA'_l\right) \tag{18}$$

where A_k and A_l indicate the sub-areas and the indices k and l both vary from one to the total number of these areas. Suppose we assume that the pressure is identical for each point within any particular sub-area, but generally different between points not in the same sub-area. Then, for any two points, one in sub-area k and the other in sub-area l, the precise location within these sub-areas is no longer needed. That is, the precision implied by $S_p(x,\theta,x',\theta',\pm\omega)$ can now be replaced by the approximation $S_p^{kl}(\pm\omega)$. When $l = k$, $S_p^{kl}(\pm\omega)$ is a power spectrum of the pressure acting on area k. For $k \neq l$, $S_p^{kl}(\pm\omega)$ is the cross-spectrum between the pressure at the center of area k and the center of area l. We might call this the constant correlation assumption since we are assuming that the correlation (cross-spectrum) is constant between and two points lying in the same area, i.e., the cross-spectrum equals a power spectrum for points in the same area. This is a conservative assumption since the power spectrum is always positive (thus, its contribution is additive), whereas the cross-spectrum can be negative as well as positive. With the constant correlation assumption, Equation (18) can be written as

$$S_q^{IJ}(\pm\omega) = \frac{1}{Z_I(\pm\omega)\ Z^*_J(\pm\omega)} \sum_k \sum_l S_p^{kl}(\pm\omega)\left(\int_{A_k} \phi_I(x,\theta)\,dA_k \int_{A_l} \phi_J(x',\theta')\,dA'_l\right) \tag{19}$$

Now substituting

$$\phi^A_{Ik} = \int_{A_k} \phi_I(x,\theta)\,dA_k \qquad (20)$$

and

$$\phi^A_{Jl} = \int_{A_l} \phi_J(x',\theta')\,dA'_l \qquad (21)$$

into Equation (19) and rewriting in terms of matrix algebra gives

$$S^{IJ}_q(\pm\omega) = \frac{1}{Z_I(\pm\omega)\; Z^*_J(\pm\omega)} \left\lfloor \phi^A_{Ik} \right\rfloor \left[S^{kl}_p(\pm\omega) \right] \left\lfloor \phi^A_{Jl} \right\rfloor^T \qquad (22)$$

Substitution of this last equation into Equation (10) results in the deflection response cross-spectrum between points a and b.

To obtain the acceleration response cross-spectrum, the normal coordinate acceleration cross-spectra, $S^{IJ}_{\ddot{q}}(\pm\omega)$, must be determined and substituted into Equation (10) in place of $S^{IJ}_q(\pm\omega)$, i.e.,

$$S^{ab}_{\ddot{w}}(\pm\omega) = \left\lfloor \phi_{aI} \right\rfloor \left[S^{IJ}_{\ddot{q}}(\pm\omega) \right] \left\lfloor \phi_{bJ} \right\rfloor^T \qquad (23)$$

By using the properties of Fourier transforms, the transform of normal coordinate acceleration, $\ddot{q}_I(\pm\omega)$, can be related to $q_I(\pm\omega)$ as follows:

$$\ddot{q}_I(\pm\omega) = -\omega^2\, q_I(\pm\omega) \qquad (24)$$

With the above equation and a similar one for mode J, Equation (11) can be used to show that

$$S^{IJ}_{\ddot{q}}(\pm\omega) = \omega^4\, S^{IJ}_q(\pm\omega) \qquad (25)$$

Thus, the acceleration response cross-spectrum can be obtained by using Equations (22), (23) and (25).

Elimination of Imaginary Terms - Response Co-Spectrum

Equations (10) and (22) give the deflection response cross-spectrum at points a and b, but these equations contain imaginary terms. We will now limit ourselves to the real part of Equation (10), i.e., the co-spectrum of response at a and b. The only imaginary terms in Equation (10) are contained within $\left[S_q^{IJ}(\pm\omega)\right]$. These imaginary terms are the normal coordinate quad-spectra.

To eliminate the normal coordinate quad-spectra, we must separate $S_q^{IJ}(\pm\omega)$ into real and imaginary parts. This can be done by multiplying the numerator and denominator of Equation (22) by the complex conjugate of the denominator. After some manipulation involving Equations (13) and (15), we can write

$$\frac{1}{Z_I(\pm\omega)\ Z_J^*(\pm\omega)} = \frac{\left(1/M_I\ M_J\ \omega_I^2\ \omega_J^2\right)\left(G(\pm\omega) + i\ H(\pm\omega)\right)}{G^2(\pm\omega) + H^2(\pm\omega)} \tag{26}$$

where

$$G(\pm\omega) = \left(1 - \left(\frac{\omega}{\omega_I}\right)^2\right)\left(1 - \left(\frac{\omega}{\omega_J}\right)^2\right) + 4\ \zeta_I\ \zeta_J\ \frac{\omega^2}{\omega_I\ \omega_J} \tag{27}$$

$$H(\pm\omega) = 2\ \zeta_J\ \frac{\omega}{\omega_J}\left(1 - \left(\frac{\omega}{\omega_I}\right)^2\right) - 2\ \zeta_I\ \frac{\omega}{\omega_I}\left(1 - \left(\frac{\omega}{\omega_J}\right)^2\right) \tag{28}$$

The cross-spectrum of pressures acting at the centers of any two areas A_k and A_l can be written in terms of the co- and quad-spectrum as follows:

$$S_p^{kl}(\pm\omega) = C_p^{kl}(\pm\omega) - i\ Q_p^{kl}(\pm\omega) \tag{29}$$

where $C_p^{kl}(\pm\omega)$ and $Q_p^{kl}(\pm\omega)$ are given by

$$C_p^{kl}(\pm\omega) = \int_{-\infty}^{\infty} \cos\omega\tau \left\langle p_k(t+\tau)\ p_l(t)\right\rangle d\tau \tag{30}$$

$$Q_p^{kl}(\pm\omega) = \int_{-\infty}^{\infty} \sin\omega\tau \left\langle p_k(t+\tau)\ p_l(t)\right\rangle d\tau \tag{31}$$

These relations can be seen immediately if one writes an equation similar to Equation (3), but in terms of pressure rather than deflection.

Using Equation (29), we can write the matrix of pressure cross-spectra, $\left[S_p^{kl}(\pm\omega)\right]$, as

$$\left[S_p^{kl}(\pm\omega)\right] = \left[C_p^{kl}(\pm\omega)\right] - i\left[Q_p^{kl}(\pm\omega)\right] \tag{32}$$

The matrix $\left[C_p^{kl}(\pm\omega)\right]$ is symmetric and $\left[Q_p^{kl}(\pm\omega)\right]$ is skew-symmetric, i.e., $C_p^{lk}(\pm\omega) = C_p^{kl}(\pm\omega)$ and $Q_p^{lk}(\pm\omega) = -Q_p^{kl}(\pm\omega)$. This can be easily seen by rewriting Equation (5) with a and b interchanged. When this is done, we see that $S_w^{ba}(\pm\omega) = \left(S_w^{ab}(\pm\omega)\right)^*$ and thus, $C_w^{ba}(\pm\omega) = C_w^{ab}(\pm\omega)$ and $Q_w^{ba}(\pm\omega) = -Q_w^{ab}(\pm\omega)$. Similarly, it follows that $C_p^{lk}(\pm\omega) = C_p^{kl}(\pm\omega)$ and $Q_p^{lk}(\pm\omega) = -Q_p^{kl}(\pm\omega)$.

We now substitute Equations (26) and (32) into Equation (22) and retain only the real part, which gives

$$C_q^{IJ}(\pm\omega) = \frac{\left(1/M_I\,M_J\,\omega_I^2\,\omega_J^2\right) G(\pm\omega) \left\lfloor \phi_{Ik}^A \right\rfloor \left[C_p^{kl}(\pm\omega)\right] \left\lfloor \phi_{Jl}^A \right\rfloor^T}{G^2(\pm\omega) + H^2(\pm\omega)} + \frac{\left(1/M_I\,M_J\,\omega_I^2\,\omega_J^2\right) H(\pm\omega) \left\lfloor \phi_{Ik}^A \right\rfloor \left[Q_p^{kl}(\pm\omega)\right] \left\lfloor \phi_{Jl}^A \right\rfloor^T}{G^2(\pm\omega) + H^2(\pm\omega)} \tag{33}$$

The deflection response co-spectrum then follows from Equation (10) with $C_q^{IJ}(\pm\omega)$ replacing $S_q^{IJ}(\pm\omega)$, i.e.,

$$C_w^{ab}(\pm\omega) = \left\lfloor \phi_{aI} \right\rfloor \left[C_q^{IJ}(\pm\omega)\right] \left\lfloor \phi_{bJ} \right\rfloor^T \tag{34}$$

Similarly, the acceleration response co-spectrum is given by

$$C_{\ddot{q}}^{IJ}(\pm\omega) = \omega^4\, C_q^{IJ}(\pm\omega) \tag{35}$$

$$C_{\ddot{w}}^{ab}(\pm\omega) = \left\lfloor \phi_{aI} \right\rfloor \left[C_{\ddot{q}}^{IJ}(\pm\omega)\right] \left\lfloor \phi_{bJ} \right\rfloor^T \tag{36}$$

Final Equations in Terms of Positive Frequency in Cycles Per Second

When b equals a in Equation (34), one has the deflection power spectrum of response at a. In engineering applications, the power spectrum is usually defined only for the positive frequency domain, and in terms of frequency in cycles per second. In order to obtain the response equations as functions of positive frequency in cycles per second, we use as criteria the fact that the area under the power spectrum curve must equal the mean-square response.

The inverse Fourier transform of Equation (3) returns the cross-correlation function.

$$\langle W(x_a,\theta_a,t+\tau)\ W(x_b,\theta_b,t)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau}\ S_W^{ab}(\pm\omega)\ d\omega \tag{37}$$

Setting b = a and $\tau = 0$ in Equation (37) results in the mean-square response at a.

$$\langle W^2(x_a,\theta_a,t)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_W^{aa}(\pm\omega)\ d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_W^{aa}(\pm\omega)\ d\omega \tag{38}$$

In Equation (38), the fact has been used that $S_W^{aa}(\pm\omega)$ and $C_W^{aa}(\pm\omega)$ are identical, both being a power spectrum. We now change the variable in Equation (38) from ω to f.

$$\langle W^2(x_a,\theta_a,t)\rangle = \int_{-\infty}^{\infty} C_W^{aa}(\pm f)\ df \tag{39}$$

Since $C_W^{aa}(\pm f)$ is an even function, we write Equation (39) as

$$\langle W^2(x_a,\theta_a,t)\rangle = \int_{0}^{\infty} 2C_W^{aa}(\pm f)\ df = \int_{0}^{\infty} C_W^{aa}(f)\ df \tag{40}$$

where

$$C_W^{aa}(f) = 2C_W^{aa}(\pm f) \tag{41}$$

Thus, if we rewrite Equations (33) through (36) as functions of f, the associated mean-square responses will be given by the integrals of the resulting equations when b = a. Proceeding to write Equations (33) through (36) as functions of f, we obtain

$$C_q^{IJ}(f) = \frac{\left(1/M_I\ M_J\ \omega_I^2\ \omega_J^2\right) G(f) \left\lfloor \phi^A_{Ik} \right\rfloor \left[C_p^{kl}(f)\right] \left\lfloor \phi^A_{Jl} \right\rfloor^T}{G^2(f) + H^2(f)}$$

$$+ \frac{\left(1/M_I\ M_J\ \omega_I^2\ \omega_J^2\right) H(f) \left\lfloor \phi^A_{Ik} \right\rfloor \left[Q_p^{kl}(f)\right] \left\lfloor \phi^A_{Jl} \right\rfloor^T}{G^2(f) + H^2(f)} \qquad (42)$$

$$C_w^{ab}(f) = \left\lfloor \phi_{aI} \right\rfloor \left[C_q^{IJ}(f)\right] \left\lfloor \phi_{bJ} \right\rfloor^T \qquad (43)$$

$$C_{\ddot{q}}^{IJ}(f) = (2\pi)^4\ f^4\ C_q^{IJ}(f) \qquad (44)$$

$$C_{\ddot{w}}^{ab}(f) = \left\lfloor \phi_{aI} \right\rfloor\ C_{\ddot{q}}^{IJ}(f)\ \left\lfloor \phi_{bJ} \right\rfloor^T \qquad (45)$$

where

$$G(f) = \left(1 - \left(\frac{f}{f_I}\right)^2\right)\left(1 - \left(\frac{f}{f_J}\right)^2\right) + 4\ \zeta_I\ \zeta_J\ \frac{f^2}{f_I\ f_J} \qquad (46)$$

$$H(f) = 2\ \zeta_J\ \frac{f}{f_J}\left(1 - \left(\frac{f}{f_I}\right)^2\right) - 2\ \zeta_I\ \frac{f}{f_I}\left(1 - \left(\frac{f}{f_J}\right)^2\right) \qquad (47)$$

are from Equations (27) and (28). The associated mean-square responses are given by

$$\langle q_I^2(t) \rangle = \int_0^\infty C_q^{II}(f)\ df \qquad (48)$$

$$\langle W^2(x_a, \theta_a, t) \rangle = \int_0^\infty C_w^{aa}(f)\ df \qquad (49)$$

$$\langle \ddot{q}_I^2(t) \rangle = \int_0^\infty C_{\ddot{q}}^{II}(f)\, df \tag{50}$$

$$\langle \ddot{W}^2(x_a, \theta_a, t) \rangle = \int_0^\infty C_{\ddot{W}}^{aa}(f)\, df \tag{51}$$

Expressions for the elements of $\left[C_p^{kl}(f)\right]$ and $\left[Q_p^{kl}(f)\right]$ are obtained from Equations (30) and (31) by rewriting them in terms of $\pm f$, and then changing to f by using Equation (41). This yields,

$$C_p^{kl}(f) = 2 \int_{-\infty}^{\infty} \cos(2\pi f\tau) \; \langle p_k(t+\tau)\; p_l(t) \rangle \; d\tau \tag{52}$$

$$Q_p^{kl}(f) = 2 \int_{-\infty}^{\infty} \sin(2\pi f\tau) \; \langle p_k(t+\tau)\; p_l(t) \rangle \; d\tau \tag{53}$$

The "2" multiplying the integrals comes from the right-hand side of Equation (41).

Equations (42) - (51) are the desired relations for calculating structural response spectra to random pressure excitation. Their form allows direct numerical solution with the required forcing functions given by Equations (52) and (53).

It is of interest to note the simplification which results when the normal coordinate co-spectra are neglected. For this condition, Equation (42) becomes

$$C_q^{II}(f) = \frac{\left(1/M_I^2\, \omega_I^4\right) \left\lfloor \phi_{Ik}^A \right\rfloor \left[C_p^{kl}(f)\right] \left\lfloor \phi_{Il}^A \right\rfloor^T}{\left(1 - \left(\frac{f}{f_I}\right)^2\right)^2 + \left(2\, \zeta_I \left(\frac{f}{f_I}\right)\right)^2} \tag{54}$$

Thus, $\left[C_q^{IJ}(f)\right]$ is reduced to a diagonal matrix. Also, the pressure quad-spectra are no longer required since they do not appear in Equation (54). These simplifications greatly decrease the amount of calculation needed to determine the response spectra.

A quantitative estimate of these computational savings can be obtained by investigating the calculations required to determine the elements of $\left[C_q^{IJ}(f)\right]$. It is seen that the off-diagonal elements require approximately twice as much computation as the diagonal elements. Since $\left[C_q^{IJ}(f)\right]$ is symmetric, however, only one-half of the off-diagonal elements need be calculated. Thus, the amount of computation required to determine all elements of $\left[C_q^{IJ}(f)\right]$ is proportional to the order squared, where the order equals the total number of normal modes. If only the diagonal of $\left[C_q^{IJ}(f)\right]$ is to be determined, the computation is directly proportional to the order. Therefore, neglecting the normal coordinate co-spectra reduces the amount of calculation by a factor approximately equal to 1/N, where N is the number of normal modes.

CONTRIBUTION OF NORMAL COORDINATE CO-SPECTRA TO TOTAL RESPONSE

The actual solution of Equations (42) - (51) for a system having many normal modes and divided into many sub-areas requires a tremendous amount of computation. Unfortunately, this is exactly the situation we face in analyzing the response of shells to random pressure excitation. As previously discussed, neglecting the normal coordinate co-spectra reduces the amount of computation by the sizeable factor of 1/N and also eliminates the need for pressure quad-spectra. This leads to the need for investigating the effect of the normal coordinate co-spectra on the system response.

For convenience, we will limit the investigation to the effect on the deflection response power spectra. To determine this effect, we form the ratio

$$R(f) = \frac{C_w^{aa}(I,J,IJ,f) - C_w^{aa}(I,J,f)}{C_w^{aa}(I,J,IJ,f)} \tag{55}$$

where

$C_w^{aa}(I,J,IJ,f)$ - contribution to response power spectrum at point a due to normal coordinate power spectra and co-spectrum for modes I and J.

$C_w^{aa}(I,J,f)$ - contribution to response power spectrum at point a due to normal coordinate power spectra only for modes I and J.

The functional notation involving I, J and IJ is symbolic indicating dependence of $C_w^{aa}(f)$ on $C_q^{II}(f)$, $C_q^{JJ}(f)$, and $C_q^{IJ}(f)$, respectively.

The contributions, $C_w^{aa}(I,J,IJ,f)$ and $C_w^{aa}(I,J,f)$ can be determined from Equation (43) when b = a. Thus,

$$C_w^{aa}(I,J,IJ,f) = \phi_{aI}^2\, C_q^{II}(f) + \phi_{aJ}^2\, C_q^{JJ}(f) + 2\,\phi_{aI}\,\phi_{aJ}\, C_q^{IJ}(f) \tag{56}$$

$$C_w^{aa}(I,J,f) = \phi_{aI}^2\, C_q^{II}(f) + \phi_{aJ}^2\, C_q^{JJ}(f) \tag{57}$$

Substituting Equations (56) and (57) into Equation (55) and dividing top and bottom of the result by $\phi^2_{aI}\ C_q^{II}(f)$ gives

$$R(f) = \frac{2\left(\phi_{aJ}/\phi_{aI}\right)\left(C_q^{IJ}(f)/C_q^{II}(f)\right)}{1 + \left(\phi_{aJ}/\phi_{aI}\right)^2\left(C_q^{JJ}(f)/C_q^{II}(f)\right) + 2\left(\phi_{aJ}/\phi_{aI}\right)\left(C_q^{IJ}(f)/C_q^{II}(f)\right)} \tag{58}$$

If the normal coordinate co-spectra are neglected in a response calculation, it would appear that R(f) cannot be evaluated since $C_q^{IJ}(f)$ would be unavailable. Fortunately, we can obtain information about $C_q^{IJ}(f)$ without actually calculating it. We do this through an inequality given without proof in Reference 3.

$$\left(C_q^{IJ}(f)\right)^2 + \left(Q_q^{IJ}(f)\right)^2 \leq \left(C_q^{II}(f)\right)\left(C_q^{JJ}(f)\right) \tag{59}$$

Since this relationship is of prime importance to the subsequent development, it is verified in appendix A.

Through the use of Equation (59), we can determine the maximum possible $C_q^{IJ}(f)$. This is done by assuming that the left side equals the right side and that $Q_q^{IJ}(f)$ is zero. Thus, we can write

$$C_q^{IJ}(f)/C_q^{II}(f) = \pm\sqrt{C_q^{JJ}(f)/C_q^{II}(f)} \tag{60}$$

Substituting Equation (60) into Equation (58) results in

$$R(f) = \frac{\pm 2\left(\phi_{aJ}/\phi_{aI}\right)\sqrt{C_q^{JJ}(f)/C_q^{II}(f)}}{1 + \left(\phi_{aJ}/\phi_{aI}\right)^2\left(C_q^{JJ}(f)/C_q^{II}(f)\right) \pm 2\left(\phi_{aJ}/\phi_{aI}\right)\sqrt{C_q^{JJ}(f)/C_q^{II}(f)}} \tag{61}$$

If, from the $\pm$ sign, we choose such that $\left(\phi_{aJ}/\phi_{aI}\right)\sqrt{C_q^{JJ}(f)/C_q^{II}(f)}$ is negative, we will obtain a maximum absolute R(f) since the denominator will be minimized. On the other hand, this means that the effect of the normal coordinate co-spectra for modes I and J is negative, i.e., it subtracts from the total response. Thus, to neglect $C_q^{IJ}(f)$ would give a conservative answer. From a design standpoint, we need to investigate the effect of $C_q^{IJ}(f)$ when this effect is additive. Therefore, we must specify the positive value of $\left(\phi_{aJ}/\phi_{aI}\right)\sqrt{C_q^{JJ}(f)/C_q^{II}(f)}$ in Equation (61),

$$R(f) = \frac{2\left(\phi_{aJ}/\phi_{aI}\right)\sqrt{C_q^{JJ}(f)/C_q^{II}(f)}}{1 + \left(\phi_{aJ}/\phi_{aI}\right)^2\left(C_q^{JJ}(f)/C_q^{II}(f)\right) + 2\left(\phi_{aJ}/\phi_{aI}\right)\sqrt{C_q^{JJ}(f)/C_q^{II}(f)}} \tag{62}$$

The ratio R(f) can be evaluated at any number of frequencies. However, since the response close to the modal frequencies is by far the predominant response, evaluation of R(f) only at the modal frequency, f_I, should provide a satisfactory indication as to the effect of $C_q^{IJ}(f)$ on the response.

While Equation (62) provides a useful tool for determining the effect of $C_q^{IJ}(f)$ after a response calculation which neglects $C_q^{IJ}(f)$ has been made, it would be even more advantageous if this effect could be estimated before a calculation is made. It is possible to do this as will now be shown.

Restricting ourselves to R(f) evaluated at f_I for the reason discussed above, Equation (62) becomes

$$R(f_I) = \frac{2\left(\phi_{aJ}/\phi_{aI}\right)\sqrt{C_q^{JJ}(f_I)/C_q^{II}(f_I)}}{1 + \left(\phi_{aJ}/\phi_{aI}\right)^2\left(C_q^{JJ}(f_I)/C_q^{II}(f_I)\right) + 2\left(\phi_{aJ}/\phi_{aI}\right)\sqrt{C_q^{JJ}(f_I)/C_q^{II}(f_I)}} \tag{63}$$

Equation (54) can be used to determine $C_q^{II}(f_I)$ and $C_q^{JJ}(f_I)$ for substitution into Equation (63). Proceeding to do this and rearranging gives

$$R(f_I) = \frac{2\xi}{1 + \xi^2 + 2\xi} = \frac{2\xi}{(\xi + 1)^2} \tag{64}$$

where,

$$\xi = \left\{\frac{\phi_{aJ}^2\, C_F^{JJ}(f_I)}{M_J^2\, \omega_J^4\, (2\zeta_J)^2} \cdot \frac{M_I^2\, \omega_I^4\, (2\zeta_I)^2}{\phi_{aI}^2\, C_F^{II}(f_I)} \cdot \frac{(2\zeta_J)^2}{\left(1 - (f_I/f_J)^2\right)^2 + \left(2\zeta_J(f_I/f_J)\right)^2}\right\}^{1/2} \tag{65}$$

and,

$$C_F^{II}(f_I) = \lfloor \phi_{Ik}^A \rfloor \left[C_p^{kl}(f_I)\right] \lfloor \phi_{Il}^A \rfloor^T \tag{66}$$

$$C_F^{JJ}(f_I) = \lfloor \phi_{Jk}^A \rfloor \left[C_p^{kl}(f_I)\right] \lfloor \phi_{Jl}^A \rfloor^T \tag{67}$$

The functions $C_F^{II}(f_I)$ and $C_F^{JJ}(f_I)$ are the power spectra evaluated at f_I of the generalized forces for modes I and J, respectively. If the modal frequencies f_I and f_J are reasonably close (if they are not close, the normal coordinate co-spectra will certainly be negligible), we can assume that

$$C_F^{JJ}(f_I) \simeq C_F^{JJ}(f_J) \tag{68}$$

After substituting Equation (68) into Equation (65), and inspecting Equation (54), we see that

$$C_F^{JJ}(f_J)/(4\zeta_J^2 M_J^2 \omega_J^4) = C_q^{JJ}(f_J) \tag{69}$$

$$C_F^{II}(f_I)/(4\zeta_I^2 M_I^2 \omega_I^4) = C_q^{II}(f_I) \tag{70}$$

Use of Equations (69) and (70) to rewrite Equation (65) gives

$$\xi = \left\{ \frac{\phi_{aJ}^2 \, C_q^{JJ}(f_J)}{\phi_{aI}^2 \, C_q^{II}(f_I)} \cdot \frac{4\zeta_J^2}{\left(1 - (f_I/f_J)^2\right)^2 + \left(2\zeta_J(f_I/f_J)\right)^2} \right\}^{1/2} \tag{71}$$

Because we wish to determine $R(f_I)$ (and thus, ξ) before $C_q^{JJ}(f_J)$ and $C_q^{II}(f_I)$ are calculated, we must, if possible, eliminate them from Equation (71). It is possible to do this conservatively by investigating the effect of ξ on $R(f_I)$ in Equation (64). It can be shown by straightforward calculus that the maximum $R(f_I)$ occurs when $\xi = 1$. Furthermore, it can be shown that the maximum value of $4\zeta_J^2/\left\{\left(1 - (f_I/f_J)^2\right)^2 + \left(2\zeta_J\,(f_I/f_J)\right)^2\right\}$ equals $1/(1 - \zeta_J^2)$ and occurs when $f_I/f_J = (1 - 2\zeta_J^2)^{1/2}$. For light damping, this can, for practical purposes, be taken as

$$4\zeta_J^2/\left\{\left(1 - (f_I/f_J)^2\right)^2 + \left(2\zeta_J(f_I/f_J)\right)^2\right\} = 1$$

$$f_I/f_J = 1$$

Thus, for the condition that

$$4\zeta_J^2/\left\{\left(1 - (f_I/f_J)^2\right)^2 + \left(2\zeta_J\,(f_I/f_J)\right)^2\right\} = 1$$

$R(f_I)$ is maximized if

$$\phi_{aJ}\, C_q^{JJ}(f_J)/\phi_{aI}\, C_q^{II}(f_I) = 1$$

since this makes

$$\xi = 1$$

If

$$f_I/f_J \neq 1$$

then

$$4\zeta_J^2/\left\{\left(1 - (f_I/f_J)^2\right)^2 + \left(2\zeta_J(f_I/f_J)\right)^2\right\} < 1$$

Therefore, under this condition, ξ can only equal unity if

$$\phi^2_{aJ} \; C_q^{JJ}(f_J)/\phi_{aI} \; C_q^{II}(f_I) > 1$$

However, this latter situation can result only when mode J is stronger than mode I. This means that we are obtaining the effect of the normal coordinate co-spectrum at the frequency of the weaker mode. Since the stronger mode will probably be more damaging to the structure, it is reasonable to determine the importance of the normal coordinate co-spectrum from the value of $R(f_I)$ at the modal frequency of the stronger mode. Therefore, let us specify that the stronger mode is the I^{th} mode. With this stipulation, the ratio $\phi^2_{aJ} \; C_q^{JJ}(f_J)/\phi^2_{aI} \; C_q^{II}(f_I)$ is less than unity. Now we will conservatively assume that it equals unity (conservative in that ξ is made larger by this assumption). Writing Equation (71) with this assumption and simplifying gives

$$\xi = \left\{\frac{1}{(1/4\zeta_J^2)\left(1 - (f_I/f_J)^2\right)^2 + (f_I/f_J)^2}\right\}^{1/2} \tag{72}$$

Conversion of Equation (64) to a percentage gives the final result.

$$P(f_I) = 100 \; R(f_I) = \frac{200 \; \xi}{(\xi + 1)^2} \tag{73}$$

Equations (72) and (73) give a conservative estimate at the modal frequency of the stronger mode of the percentage error incurred by neglecting the normal coordinate co-spectrum in a response calculation. The results of Equation (73) are given in Figure 3 where $P(f_I)$ is plotted versus f_I/f_J for various damping ratios.

Two of the assumptions made in obtaining Equations (72) and (73) merit further discussion. It can be shown that the assumption used to

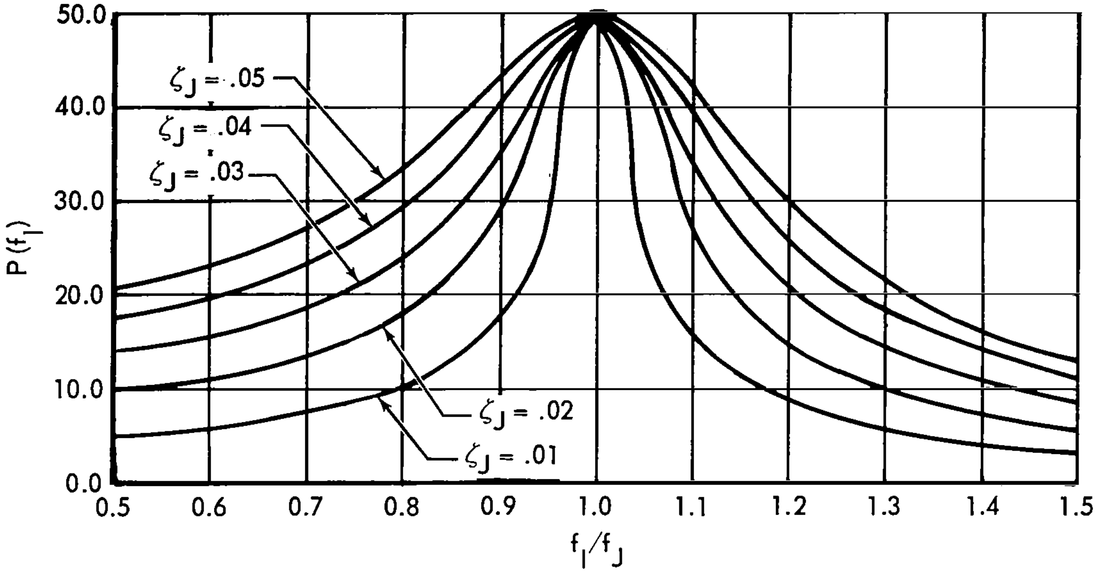

Figure 3 – Percent of Total Response from Modes I and J Contributed by Normal Coordinate Co-Spectrum

write Equation (60) requires the following relationships to exist between the power spectra and co- and quad-spectrum of the generalized forces.

$$Q_F^{IJ}(f)/C_F^{IJ}(f) = H(f)/G(f)$$

$$\left(C_F^{IJ}(f)\right)^2 + \left(Q_F^{IJ}(f)\right)^2 = S_F^{II}(f)\ S_F^{JJ}(f)$$

It is very improbable that either of the above relationships will be satisfied for an actual system subjected to random pressure excitation, and even much more improbable that both will be satisfied. Thus, we would expect that the left side of Equation (60) will be considerably less than the right side, which means the curves in Figure 3 are very conservative in that they represent a limiting case. This should be considered when one decides the percentage a normal coordinate co-spectrum must contribute, as determined from Figure 3, before it is no longer negligible.

The second assumption which merits discussion is that used to write Equation (68). We see that this assumption rests upon the modal frequencies being closely spaced and the excitation remaining fairly constant over the frequency separation interval. This later criterion may be less valid for high modal frequencies than for low ones, because a given frequency ratio indicates a much greater frequency separation interval at high frequencies. Thus, the results presented in Figure 3 must be viewed with increased caution at high modal frequencies.

CONCLUDING REMARKS

In terms of matrix algebra, equations have been derived (Equations (42) - (51)) for the response of shells to random pressure excitation. Primary assumptions in the derivation are: (1) the system may be represented by a superposition of normal modes and is lightly damped, and (2) the exciting phenomenon is ergodic.

Solution of the equations is made practical by the availability of high speed digital computers. However, for a system having many normal modes and sub-divided into many areas, computing time will be lengthy. By neglecting the normal coordinate co-spectra, the response equations are simplified and the amount of computation is reduced by a factor of approximately 1/N, where N is the number of normal modes.

Equations (72) and (73) or the curves of Figure 3 can be used to determine when the normal coordinate co-spectra may be neglected. It should be noted that the assumptions leading to these curves are conservative with the result that they provide the maximum possible normal coordinate co-spectra contribution. As one would expect, this contribution is lessened for increasing modal frequency separation and for decreasing damping.

APPENDIX A

UPPER LIMIT ON MAGNITUDE OF NORMAL COORDINATE CROSS-SPECTRUM

In order to verify Equation (59) of the main body of this report, we begin by writing an equation similar to Equation (5), but in terms of the normal coordinates instead of deflections and as a function of f rather than $\pm\omega$,

$$S_q^{IJ}(f) = \frac{1}{2T} \; q_I(f) \; q_J^*(f) \tag{A1}$$

By reviewing the derivation leading to Equation (3), it is seen that Equation (3), and thus Equation (A1), are approximations of the true answer which results when $T \rightarrow \infty$. Writing Equation (A1) as a limit gives

$$S_q^{IJ}(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \; q_I(f) \; q_J^*(f) \tag{A2}$$

Recalling that $q_I(f)$ and $q_J(f)$ are complex variables, we write them as

$$q_I(f) = \alpha_I(f) + i \; \beta_I(f) \tag{A3}$$

$$q_J(f) = \alpha_J(f) + i \; \beta_J(f) \tag{A4}$$

For simplicity, we drop the functional notation (f) until the derivation of Equation (59) is completed. With this in mind, we substitute from Equations (A3) and (A4) into Equation (A2), and write the result in terms of real and imaginary parts.

$$S_q^{IJ} = \lim_{T \rightarrow \infty} \frac{1}{2T} \; (\alpha_I \; \alpha_J + \beta_I \; \beta_J) - i \lim_{T \rightarrow \infty} \frac{1}{2T} \, (\alpha_I \; \beta_J - \alpha_J \; \beta_I) \tag{A5}$$

Since, for continuous functions, the limit of a sum equals the sum of the limits (Reference 6), Equation (A5) can be written as

$$S_q^{IJ} = (\langle \alpha_I \; \alpha_J \rangle + \langle \beta_I \; \beta_J \rangle) - i \; (\langle \alpha_I \; \beta_J \rangle - \langle \alpha_J \; \beta_I \rangle) \tag{A6}$$

where the bracket, $\langle \; \rangle$, indicates the time average which results when the limit is taken. If we now write an equation similar to Equation (29), but in terms of the normal coordinates, we obtain

$$S_q^{IJ} = C_q^{IJ} - i \; Q_q^{IJ} \tag{A7}$$

Equating real and imaginary parts of Equations (A6) and (A7) gives

$$C_q^{IJ} = (\langle \alpha_I \ \alpha_J \rangle + \langle \beta_I \ \beta_J \rangle) \tag{A8}$$

$$Q_q^{IJ} = (\langle \alpha_I \ \beta_J \rangle - \langle \alpha_J \ \beta_I \rangle) \tag{A9}$$

Using Equations (A8) and (A9) to form the left side of Equation (59) results in

$$(C_q^{IJ})^2 + (Q_q^{IJ})^2 = \langle \alpha_I \ \alpha_J \rangle^2 + \langle \beta_I \ \beta_J \rangle^2 + 2\langle \alpha_I \ \alpha_J \rangle \langle \beta_I \ \beta_J \rangle$$

$$+ \langle \alpha_I \ \beta_J \rangle^2 + \langle \alpha_J \ \beta_I \rangle^2 - 2 \langle \alpha_I \ \beta_J \rangle \langle \alpha_J \ \beta_I \rangle \tag{A10}$$

Since the limit of the product of two continuous functions equals the product of their limits (Reference 6), Equation (A10) reduces to

$$(C_q^{IJ})^2 + (Q_q^{IJ})^2 = \langle \alpha_I \ \alpha_J \rangle^2 + \langle \beta_I \ \beta_J \rangle^2 + \langle \alpha_I \ \beta_J \rangle^2 + \langle \alpha_J \ \beta_I \rangle^2 \tag{A11}$$

Now we form the right side of Equation (59). Writing Equation (A2) for S_q^{II} and S_q^{JJ} gives

$$S_q^{II} = \lim_{T \to \infty} \frac{1}{2T} \ q_I \ q_I^* \tag{A12}$$

$$S_q^{JJ} = \lim_{T \to \infty} \frac{1}{2T} \ q_J \ q_J^* \tag{A13}$$

Substituting Equations (A3) and (A4) into Equations (A12) and (A13), we obtain

$$S_q^{II} = \lim_{T \to \infty} \frac{1}{2T} \ (\alpha_I^2 + \beta_I^2) = (\langle \alpha_I^2 \rangle + \langle \beta_I^2 \rangle) \tag{A14}$$

$$S_q^{JJ} = \lim_{T \to \infty} \frac{1}{2T} \ (\alpha_J^2 + \beta_J^2) = (\langle \alpha_J^2 \rangle + \langle \beta_J^2 \rangle) \tag{A15}$$

Multiplication of Equations (A14) and (A15) gives

$$S_q^{II}\, S_q^{JJ} = \langle \alpha_I{}^2 \rangle \langle \alpha_J{}^2 \rangle + \langle \beta_I{}^2 \rangle \langle \beta_J{}^2 \rangle + \langle \alpha_I{}^2 \rangle \langle \beta_J{}^2 \rangle + \langle \alpha_J{}^2 \rangle \langle \beta_I{}^2 \rangle \qquad \text{(A16)}$$

Comparing Equations (A11) and (A16), we see that each term of Equation (A11) is $\leq$ the corresponding term of Equations (A16); i.e., $\langle \alpha_I\ \alpha_J \rangle^2 \leq \langle \alpha_I{}^2 \rangle \langle \alpha_J{}^2 \rangle$, etc. Therefore, $(C_q^{IJ})^2 + (Q_q^{IJ})^2 \leq S_q^{II}\ S_q^{JJ}$, which verifies Equation (59).

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