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VIBRATION OF A CIRCULAR CYLINDRICAL ELASTIC TANK, PARTIALLY FILLED WITH AN INCOMPRESSIBLE FLUID, UNDERGOING AN AXIAL ACCELERATION COMPOSED OF A UNIFORM AND A PERIODIC COMPONENT

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NORTHROP SPACE LABORATORIES

6025 TECHNOLOGY DRIVE, HUNTSVILLE, ALABAMA 35805
TELEPHONE 837-0580
P. O. BOX 1484

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Prepared by:

J. G. Goree
G. C. Kao
T. N. Lee
C. M. Pearson

Reviewed by:

M. A. Sloan, Jr.,
Branch Technical Supervisor

Approved:

Dr. Steve S. Hu, Director

RESEARCH & ANALYSIS SECTION
NORTHROP SPACE LABORATORIES
HUNTSVILLE DEPARTMENT
HUNTSVILLE, ALABAMA

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NOMENCLATURE

$X_1(t)$, $Y_1(t)$, $Z_1(t)$	Coordinates of a point relative to the fixed inertia reference frame
η, y, z	Cartesian coordinates of a point before deformation relative to the body reference frame
η, r, θ	Polar coordinates of a point before deformation relative to the body reference frame
$X(t)$, $\dot{X}(t)$, $\ddot{X}(t)$	Displacement, velocity, and acceleration respectively, of the body coordinate system with respect to the inertia reference frame
$u(\eta,\theta,t)$, $v(\eta,\theta,t)$, $w(\eta,\theta,t)$	Displacement components of the shell relative to the body reference frame as shown in Figure 1
a	Radius of middle surface of the shell
L	Length of the cylinder
h_s	Thickness of the shell
h_ℓ	Height of the liquid
ρ_s	Density of the shell
ρ_ℓ	Density of the liquid
M_s	Mass of the shell
M_ℓ	Mass of the liquid
$M = M_\ell + M_s$	Total combined mass
E	Modulus of Elasticity of the shell
ν	Poisson's ratio
$D = \dfrac{Eh_s^3}{12(1-\nu)^2}$	Modulus of rigidity of the shell

$\omega_{mn}^2 = \frac{g\lambda_{mn}}{a} \tanh (\lambda_{mn} \frac{h_\ell}{a})$	Natural frequency of free vibration of an incompressible liquid in a rigid circular tank with a free surface
Ω	Frequency of the periodic component of applied thrust
T_o	Magnitude of the constant component of applied thrust
T_Ω	Magnitude of the periodic component of applied thrust
γ	T_Ω/T_o
P_o	Constant over-ride pressure in cylinder
$P_s(\eta, \theta, t)$	Variable pressure on wall (assumed positive outward)
$\mathcal{S}(t)$	Unit step function $\mathcal{S}(t) = 0 \quad t \leq 0$; $= 1 \quad t > 0$
$\tau = t\omega^*$	Dimensionless time
ω^*	Arbitrary frequency used to non-dimensionalize time
$\xi = \frac{\eta}{L}$	Dimensionless axial coordinate
$\tilde{u}(\xi,\theta,\tau)$, $\tilde{v}(\xi,\theta,\tau)$, $\tilde{w}(\xi,\theta,\tau)$; $\tilde{U}_{mn}^{(i)}(\tau)$, $\tilde{V}_{mn}^{(i)}(\tau)$, $\tilde{W}_{mn}^{(i)}(\tau)$	Dimensionless displacements and time coefficients
$\sigma = \frac{h_s}{L}$; $\zeta = \frac{h_\ell}{L}$; $\alpha = \frac{h_\ell}{a}$; $\lambda = \frac{L}{a}$; $\omega = \frac{\Omega}{\omega^*}$; $\mu = \frac{\rho_s L^2 (\omega^*)^2}{E}$	Dimensionless constants

$$\Gamma^{(1)} = \frac{\rho_\ell g L}{E\sigma}$$

$$\Gamma^{(2)} = \frac{T_o \rho_\ell L}{ME\sigma}$$

$$\Gamma^{(3)} = \frac{\rho_\ell L^2 (\omega^*)^2}{E\sigma}$$

$$\Gamma^{(4)} = \frac{T_o \rho_s L}{ME}$$

$$\Gamma^{(5)} = \frac{\rho_s L g}{E}$$

$$\Gamma^{(6)} = \frac{g}{a(\omega^*)^2}$$

Dimensionless constants (applies to $\Gamma^{(1)}$ through $\Gamma^{(6)}$ above)

$$\tilde{P}_o = \frac{P_o}{E\sigma}$$

Dimensionless static pressure

$$\tilde{P}_s(\xi,\theta,\tau) = \frac{P_s(\eta,\theta,\tau)}{E\sigma}$$

Dimensionless dynamic pressure

$$I_{kn} = L\ \tilde{I}_{kn}$$

$$P^*_{kn} = L\ \tilde{P}^*_{kn}$$

$$P^{mj}_{kn} = L\ \tilde{P}^{mj}_{kn}$$

$$P^{oj}_{kn} = L\ \tilde{P}^{oj}_{kn}$$

$$P^{oj}_{on} = L\ (\omega^*)^2\ \tilde{P}^{oj}_{kn}$$

$$W^{(i)}_{mn}(t) = L\ \tilde{W}^{(i)}_{mn}(\tau)$$

Parameters defined in the text. The sub-scripts and super-scripts are both the usual summation indices. The symbol ~ denoted the dimensionless form of the parameter (applies to the six equations above)

$$\overline{F} = \mathcal{L}\,[F]$$

Laplace transform of the function F

LIST OF FIGURES

1.0 INTRODUCTION

The problem considered in this paper is the forced vibration of a circular cylindrical elastic shell that is partially filled with an incompressible liquid and is initially at rest in a uniform gravitational field. The strains in the shell are assumed to lie within the region of application of linear elasticity and the motion of the shell and the liquid is assumed to be such that it can be adequately described by small oscillation theory. While in the above state, the shell is subjected to an impulsive force composed of a constant and a periodically varying component resulting in the acceleration of the cylinder in the axial direction (see Figure 1). During this motion, it is assumed that the ends of the cylinder remain circular and that the centers of the two ends remain on a straight line that is always parallel to the X_1 coordinate axis. In addition, the ends of the shell are assumed to be freely supported as defined on page 6. These assumptions are not felt to place an undue restriction on either the vibratory motion of the cylinder or the stresses in the shell wall and the resulting "uncoupling" of the equations of motion is a distinct advantage in obtaining a tractable set of differential equations.

The assumed displacement functions for the shell are taken in a general enough form to allow for all motions of the cylinder within the restrictions imposed by the equations of motion and the boundary conditions.

In the computations involving the liquid it is assumed that the bottom of the cylinder is flat and rigid and has the same motion as the

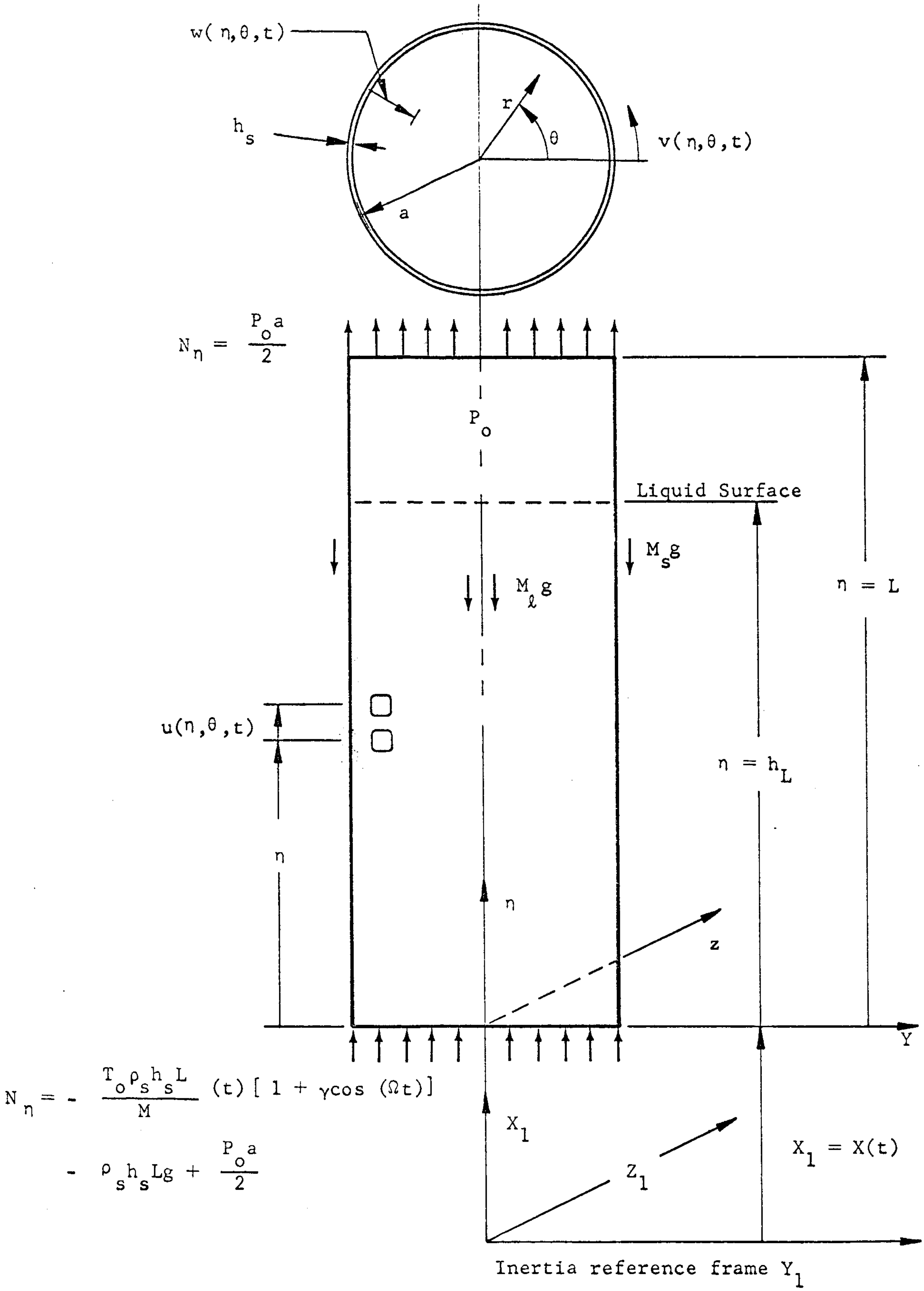

Figure 1 Coordinate System and the Displacement Functions of a Circular Cylinder

body coordinate system. The motion of the liquid is described in terms of a velocity potential. The pressure exerted by the liquid on the wall is given by Bernoulli's equation.

By letting the periodic component of the applied thrust vanish and the constant part be just sufficient to support the cylinder but give no acceleration, the problem of free vibration is obtained. In this case the initial conditions must be other than the static deflection state as pointed out in Section 6.0 of the paper.

Physically this problem may be interpreted as follows. Consider the cylinder to be the fuel tank of a liquid fueled rocket motor that is initially restrained at its base prior to launch. The usual launching procedure is to start the motor and allow it to reach approximately full thrust before releasing the restraining mechanism. Assume that during the hold-down time all of the thrust is absorbed by the supporting mechanism except for that constant portion, Mg, required to hold the rocket in a static vertical equilibrium position. When full thrust is reached the hold-down mechanism is released and the total thrust is applied to the motor.

2.0 EQUATIONS OF MOTION FOR THE SHELL

The static equilibrium equations are assumed to be those given by Donnell in reference 1, page 14. (Also see reference 2). By adding body forces, applied forces, and inertia forces to these equations the resulting dynamic equations of motion are

$$\frac{\partial N_\eta}{\partial \eta} + \frac{1}{a}\frac{\partial N_{\theta\eta}}{\partial \theta} - \rho_s h_s g = F(\eta) \tag{1}$$

$$\frac{1}{a}\frac{\partial N_\theta}{\partial \theta} + \frac{\partial N_{\eta\theta}}{\partial \eta} - \frac{Q_\theta}{a} = \rho_s h_s \frac{\partial^2 v}{\partial t^2} \tag{2}$$

$$\frac{\partial Q_\eta}{\partial \eta} + \frac{1}{a}\frac{\partial Q_\theta}{\partial \theta} + \frac{N_\theta}{a} = \rho_s h_s \frac{\partial^2 w}{\partial t^2} + P_s(\eta,\theta,t) \tag{3}$$

$$\frac{\partial M_{\eta\theta}}{\partial \eta} - \frac{1}{a}\frac{\partial M_\theta}{\partial \theta} + Q_\theta = 0 \tag{4}$$

$$\frac{1}{a}\frac{\partial M_{\theta\eta}}{\partial \theta} + \frac{\partial M_\eta}{\partial \eta} - Q_\eta = 0 \quad \text{and} \tag{5}$$

$$N_{\eta\theta} - N_{\theta\eta} = 0 \tag{6}$$

The bending moments and forces are related to the displacements as

$$M_\eta = -D\left[\frac{\partial^2 w}{\partial \eta^2} + \frac{\nu}{a^2}\frac{\partial^2 w}{\partial \theta^2}\right] \tag{7}$$

$$M_\theta = -D\left[\frac{1}{a^2}\frac{\partial^2 w}{\partial \theta^2} + \nu\frac{\partial^2 w}{\partial \eta^2}\right] \tag{8}$$

$$M_{\eta\theta} = - M_{\theta\eta} = \frac{D(1-\nu)}{a} \frac{\partial^2 w}{\partial\eta\partial\theta} \qquad (9)$$

$$N_\eta = \frac{Eh_s}{1-\nu^2} \left[\frac{\partial u}{\partial\eta} + \nu \left(\frac{1}{a} \frac{\partial v}{\partial\theta} - \frac{w}{a} \right) \right] \qquad (10)$$

$$N_\theta = \frac{Eh_s}{1-\nu^2} \left[\frac{1}{a} \frac{\partial v}{\partial\theta} - \frac{w}{a} + \nu \frac{\partial u}{\partial\eta} \right] \text{ and} \qquad (11)$$

$$N_{\eta\theta} = N_{\theta\eta} = \frac{Eh_s}{2(1+\nu)} \left[\frac{1}{a} \frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial\eta} \right] \qquad (12)$$

Solving for Q_θ and Q_η from equations (4) and (5) and substituting for the moments and forces from equations (7) through (12), the equations of motion in terms of the displacement components are

$$\frac{\partial^2 u}{\partial\eta^2} + \frac{(1-\nu)}{2a^2} \frac{\partial^2 u}{\partial\theta^2} + \frac{(1+\nu)}{2a} \frac{\partial^2 v}{\partial\eta\partial\theta} - \frac{\nu}{a} \frac{\partial w}{\partial\eta} = \frac{(1-\nu^2)}{Eh_s} \left[\rho_s h_s g + F(\eta) \right] \qquad (13)$$

$$\frac{(1+\nu)}{2a} \frac{\partial^2 u}{\partial\eta\partial\theta} + \frac{(1-\nu)}{2} \frac{\partial^2 v}{\partial\eta^2} + \frac{1}{a^2} \frac{\partial^2 v}{\partial\theta^2} - \frac{1}{a^2} \frac{\partial w}{\partial\theta} = \frac{(1-\nu^2)}{E} \rho_s \frac{\partial^2 v}{\partial t^2} \text{ and} \qquad (14)$$

$$\frac{\nu}{a} \frac{\partial u}{\partial\eta} + \frac{1}{a^2} \frac{\partial v}{\partial\theta} - \frac{w}{a^2} - \frac{h_s^2}{12} \nabla^4 w = \frac{(1-\nu^2)}{Eh_s} \left[\rho_s h_s \frac{\partial^2 w}{\partial t^2} + P_s(\eta,\theta,t) \right] \qquad (15)$$

In the above equations we have neglected the terms $\frac{Q_\theta}{a}$, $\frac{1}{a^2} \frac{\partial v}{\partial\theta}$, $\frac{1}{a} \frac{\partial v}{\partial\eta}$ and terms of the order $(h_s)^2$.

If a force given by $\Gamma(t)\, T_o \left[1 + \gamma \cos(\Omega t) \right] + Mg$ is applied to the cylinder which is initially at rest in a uniform gravitational field, the resulting acceleration of the center of mass of the cylinder will be

$$\ddot{X}_{cm}(t) = \Gamma(t)\,\frac{T_o}{M}\left[1 + \gamma\cos(\Omega t)\right] \tag{16}$$

The assumption is made that the motion of the body coordinate system located at the bottom of the cylinder will be approximately the same as the center of mass. With this assumption the force, F (η), is then

$$F(\eta) = \rho_s h_s \left\{ \frac{\partial^2 u}{\partial t^2} + \Gamma(t)\,\frac{T_o}{M}\left[1 + \gamma\cos(\Omega t)\right] \right\} \tag{17}$$

The pressure due to the liquid, $P_s(\eta,\theta,t)$, is obtained in Section 4.0 of this paper and is assumed positive outward.

The boundary conditions on the shell are that both edges are freely supported. This is taken to imply the following,

$$v = w = M_\eta = 0 \text{ at } \eta = 0 \text{ and } \eta = L \tag{18}$$

$$N_\eta = -\Gamma(t)\,\frac{T_o M_s}{2\pi a M}\left[1 + \gamma\cos(\Omega t)\right] - \frac{M_s g}{2\pi a} + \frac{P_o a}{2} \text{ at } \eta = 0 \tag{19}$$

$$N_\eta = \frac{P_o a}{2} \text{ at } \eta = L \tag{20}$$

Displacements satisfying these boundary conditions are assumed to be

$$u(\eta,\theta,t) = -(1-\nu^2)\left\{\Gamma(t)\,\frac{T_o \rho_s}{ME}\left[1+\gamma\cos(\Omega t)\right] + \frac{\rho_s g}{E}\right\}\left\{L\eta - \frac{\eta^2}{2}\right\}$$

$$+ \frac{(1-\nu^2)}{2Eh_s} P_o a\eta + \sum_{m=0}^{M}\sum_{n=0}^{N} U_{mn}^{(1)}(t)\cos\left(\frac{n\pi}{L}\eta\right)\cos(m\theta)$$

$$+ \sum_{m=1}^{M}\sum_{n=0}^{N} U_{mn}^{(2)}(t)\cos\left(\frac{n\pi}{L}\eta\right)\sin(m\theta) \tag{21}$$

$$v(\eta,\theta,t) = \sum_{m=1}^{M}\sum_{n=1}^{N} V_{mn}^{(1)}(t)\ \sin(\frac{n\pi}{L}\eta)\sin(m\theta) + \sum_{m=0}^{M}\sum_{n=1}^{N} V_{mn}^{(2)}(t)\sin(\frac{n\pi}{L}\eta)\cos(m\theta) \tag{22}$$

and

$$w(\eta,\theta,t) = \sum_{m=0}^{M}\sum_{n=1}^{N} W_{mn}^{(1)}(t)\sin(\frac{n\pi}{L}\eta)\cos(m\theta) + \sum_{m=1}^{M}\sum_{n=1}^{N} W_{mn}^{(2)}(t)\sin(\frac{n\pi}{L}\eta)\sin(m\theta) \tag{23}$$

where $U_{mn}^{(i)}(t)$, $V_{mn}^{(i)}(t)$, and $W_{mn}^{(i)}(t)$ are unknown functions to be determined.

If we now substitute the assumed displacements into the equations of motion and use Galerkin's method to eliminate the spatial dependence (see reference 3), we obtain the following set of differential equations in dimensionless time (τ).

$$\ddot{\tilde{U}}_{00}^{(1)}(\tau) = \frac{(1-\nu^2)}{3}\ \frac{d^2}{d\tau^2}\left\{\Gamma(\tau)\ \Gamma^{(4)}\left[1+\gamma\cos(\omega\tau)\right] + \Gamma^{(5)}\right\} \tag{24}$$

$$\ddot{\tilde{U}}_{0n}^{(1)}(\tau) + \frac{(n\pi)^2}{(1-\nu^2)\mu}\ \tilde{U}_{0n}^{(1)}(\tau) + \frac{n\pi\nu\lambda}{(1-\nu^2)\mu}\ \tilde{W}_{0n}^{(1)}(\tau)$$

$$= -\frac{2(1-\nu^2)}{(n\pi)^2}\ \frac{d^2}{d\tau^2}\left\{\Gamma(\tau)\ \Gamma^{(4)}\left[1+\gamma\cos(\omega\tau)\right] + \Gamma^{(5)}\right\} \tag{25}$$

$$\ddot{\tilde{U}}_{m0}^{(1)}(\tau) + \frac{(m\lambda)^2}{2(1+\nu)\mu}\ \tilde{U}_{m0}^{(1)}(\tau) = 0 \tag{26}$$

$$\ddot{\tilde{U}}_{m0}^{(2)}(\tau) + \frac{(m\lambda)^2}{2(1+\nu)\mu}\ \tilde{U}_{m0}^{(2)}(\tau) = 0 \tag{27}$$

$$\ddot{\tilde{V}}_{0n}^{(2)}(\tau) + \frac{(n\pi)^2}{2(1+\nu)\mu}\ \tilde{V}_{0n}^{(2)}(\tau) = 0 \tag{28}$$

$$\ddot{\tilde{W}}^{(1)}_{0s}(\tau) + \frac{1}{(1-\nu^2)\mu} \left[\lambda^2 + \frac{\sigma^2}{12}(s\pi)^4 \right] \tilde{W}^{(1)}_{0s}(\tau) + \frac{s\pi\nu\lambda}{(1-\nu^2)\mu} \tilde{U}^{(1)}_{0s}(\tau)$$

$$= -\frac{1}{\pi\mu} \int_0^1 \int_0^{2\pi} \tilde{P}_s(\xi,\theta,\tau)\ \sin(s\pi\xi)\ d\xi\, d\theta \qquad (29)$$

$$\frac{1}{(1-\nu^2)\mu} \left[(n\pi)^2 + \frac{(1-\nu)(m\lambda)^2}{2} \right] \tilde{U}^{(1)}_{mn}(\tau) + \ddot{\tilde{U}}^{(1)}_{mn}(\tau) + \frac{n\pi\nu\lambda}{(1-\nu^2)\mu} \tilde{W}^{(1)}_{mn}(\tau)$$

$$- \frac{mn\pi\lambda}{2(1-\nu)\mu} \tilde{V}^{(1)}_{mn}(\tau) = 0 \qquad (30)$$

$$\ddot{\tilde{V}}^{(1)}_{mn}(\tau) + \frac{1}{(1-\nu^2)\mu} \left[(m\lambda)^2 + \frac{(n\pi)^2(1-\nu)}{2} \right] \tilde{V}^{(1)}_{mn}(\tau) - \frac{m\lambda^2}{(1-\nu^2)\mu} \tilde{W}^{(1)}_{mn}(\tau)$$

$$- \frac{mn\pi\lambda}{2(1-\nu)\mu} \tilde{U}^{(1)}_{mn}(\tau) = 0 \qquad (31)$$

$$\frac{s\pi\nu\lambda}{(1-\nu^2)\mu} \tilde{U}^{(1)}_{ms}(\tau) - \frac{m\lambda^2}{(1-\nu^2)\mu} \tilde{V}^{(1)}_{ms}(\tau) + \frac{1}{(1-\nu^2)\mu} \left\{ \lambda^2 + \frac{\sigma^2}{12} \left[(s\pi)^2 + (m\lambda)^2 \right]^2 \right\} \tilde{W}^{(1)}_{ms}(\tau)$$

$$+ \ddot{\tilde{W}}^{(1)}_{ms}(\tau) = -\frac{2}{\pi\mu} \int_0^1 \int_0^{2\pi} \tilde{P}_s(\xi,\theta,\tau)\ \sin(s\pi\xi)\ \cos(m\theta)\ d\xi\, d\theta \qquad (32)$$

$$\frac{1}{(1-\nu^2)\mu} \left[(n\pi)^2 + \frac{(1-\nu)(m\lambda)^2}{2} \right] \tilde{U}^{(2)}_{mn}(\tau) + \frac{n\pi\nu\lambda}{(1-\nu^2)\mu} \tilde{W}^{(2)}_{mn}(\tau) + \ddot{\tilde{U}}^{(2)}_{mn}(\tau)$$

$$+ \frac{mn\pi\lambda}{2(1-\nu)\mu} \tilde{V}^{(2)}_{mn}(\tau) = 0 \qquad (33)$$

$$\ddot{\tilde{V}}^{(2)}_{mn}(\tau) + \frac{1}{(1-\nu^2)\mu} \left[(m\lambda)^2 + \frac{(n\pi)^2(1-\nu)}{2} \right] \tilde{V}^{(2)}_{mn}(\tau) + \frac{m\lambda^2}{(1-\nu^2)\mu} \tilde{W}^{(2)}_{mn}(\tau)$$

$$+ \frac{mn\pi\lambda}{2(1-\nu)\mu} \tilde{U}^{(2)}_{mn}(\tau) = 0 \quad \text{and} \qquad (34)$$

$$\frac{s\pi\nu\lambda}{(1-\nu^2)\mu} \tilde{U}_{ms}^{(2)}(\tau) + \frac{m\lambda^2}{(1-\nu^2)\mu} \tilde{V}_{ms}^{(2)}(\tau) + \frac{1}{(1-\nu^2)}\left\{\lambda^2 + \frac{\sigma^2}{12}\left[(s\pi)^2+(m\lambda)^2\right]^2\right\}\tilde{W}_{ms}^{(2)}(\tau)$$

$$+ \ddot{\tilde{W}}_{ms}^{(2)}(\tau) = - \frac{2}{\mu\pi}\int_0^1\int_0^{2\pi} \tilde{P}_s(\xi,\theta,\tau) \sin(n\pi\xi) \sin(m\theta)\, d\xi d\theta \qquad (35)$$

It is now necessary to determine $\tilde{P}_s(\xi,\theta,\tau)$ in terms of the unknown time coefficients of the displacement functions. With $\tilde{P}_s(\xi,\theta,\tau)$ known, the above equations can be solved for the unknown functions.

The "uncoupling" of the equations of motion mentioned in the introduction can be observed in equations (24) through (35). Due to the restrictions placed on the displacement functions by the boundary conditions, it is seen that in the application of Galerkin's method all the resulting equations are "uncoupled" except equations (29), (32), and (35). As $\tilde{P}_s(\xi,\theta,\tau)$ contains terms of the form $\sum_{n=1}^{N} \tilde{W}_{mn}^{(i)}(\tau) \sin\left(\frac{n\pi\xi}{\zeta}\right)$ and as $\sin\left(\frac{n\pi\xi}{\zeta}\right)$ is not orthogonal to $\sin(n\pi\xi)$ over the range $0<\xi<1$ unless $\zeta=1$, the right hand side of these equations will contain all N unknowns $\tilde{W}_{mn}^{(i)}(\tau)$ for a given m.

3.0 DESCRIPTION OF THE FLUID MOTION

If the motion of the fluid is irrotational, it follows that there exists a velocity potential which will be defined (ref. 4) such that

$$\vec{V} = \nabla\phi = \text{grad}\ \phi \tag{36}$$

In addition, if the fluid is assumed to be incompressible, it is known that the velocity potential must satisfy Laplace's equations

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial\eta^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \tag{37}$$

The velocity potential must also satisfy certain boundary and initial conditions which for the problem considered here are as follows:

$$\left.\frac{\partial\phi}{\partial\eta}\right|_{\eta=0} = \dot{X}(t) = \frac{T_o}{M}\left[t + \frac{\gamma}{\Omega}\sin(\Omega t)\right] \tag{38}$$

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=a} = -\frac{\partial w}{\partial t} = -\sum_{m=0}^{M}\sum_{n=0}^{N}\dot{W}_{mn}^{(1)}(t)\, f_n(\eta)\cos(m\theta)$$

$$-\sum_{m=1}^{M}\sum_{n=0}^{N}\dot{W}_{mn}^{(2)}(t)\, f_n(\eta)\sin(m\theta)\ \text{, where} \tag{39}$$

the inner radius of the cylinder is assumed to be approximately the same as the radius to the middle surface,

$$\left.\left[\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial\eta} + \dot{X}\ddot{X}\right]\right|_{\eta=h_\ell} = 0 \tag{40}$$

$$\left.\frac{\partial\phi}{\partial\eta}\right|_{t=0} = 0 \tag{41}$$

$$\left.\frac{\partial \phi}{\partial r}\right|_{t=0} = 0 \tag{42}$$

$$\left.\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right|_{t=0} = 0 \text{ and} \tag{43}$$

$$\left.\frac{\partial \phi}{\partial t}\right|_{t=0} = -\frac{P_o + \rho_\ell h_\ell g}{\rho_\ell} \tag{44}$$

Equation (40) is the free surface condition and follows from Bernoulli's equation if the magnitude of the velocity vector at a point on the surface is assumed to be

$$|\vec{V}| \approx \dot{X}(t) \quad \text{and} \tag{45}$$

the pressure above the fluid is constant. For these conditions Bernoulli's equation is

$$\frac{\partial \phi}{\partial t} + g\,X_1(t) + \frac{1}{2}(\dot{X})^2 + \frac{P}{\rho_\ell} = 0 \qquad \text{and as} \tag{46}$$

$$P(\eta,\theta,t)\Big|_{\eta=h_\ell} = P_o \qquad \text{then} \tag{47}$$

$$\left.\frac{dP}{dt}\right|_{\eta=h_\ell} = 0 \qquad \text{and} \tag{48}$$

$$\left[\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial \eta} + \dot{X}\ddot{X}\right]\Bigg|_{\eta=h_\ell} = 0 \qquad \text{where}$$

$$\frac{dX_1}{dt} = \frac{\partial \phi}{\partial \eta} \tag{49}$$

From equation (49)

$$\int_0^t \frac{dX_1(t)}{dt}\,dt = \int_0^t \frac{\partial \phi}{\partial \eta}\,dt \qquad \text{or} \tag{50}$$

$$X_1(t) - X_1(0) = \int_0^t \frac{\partial \phi}{\partial \eta} \, dt \tag{51}$$

and as the initial conditions are chosen such that $X_1(0) = \eta$ then

$$X_1(t) = \eta + \int_0^t \frac{\partial \phi}{\partial \eta} \, dt \tag{52}$$

and equation (46) can be written as

$$\frac{\partial \phi}{\partial t} + g\eta + g \int_0^t \frac{\partial \phi}{\partial \eta} \, dt + \frac{1}{2} (\dot{X})^2 + \frac{P}{\rho_\ell} = 0 \tag{53}$$

Equations (41), (42) and (43) specify an initially stationary fluid and equation (44) is the corresponding condition that

$$P(\eta, r, \theta, 0) = P_o + \rho_\ell \, g(h_\ell - \eta) \tag{54}$$

The solution to equation (37) is assumed to be of the form

$$\phi \, (\eta, r, \theta, t) = X(\eta) \; R(r) \; \Theta(\theta) \; T(t) \tag{55}$$

Substituting equation (55) into equation (37) and requiring that ϕ be periodic and single-valued in θ and bounded at $r = 0$, a solution to equation (37) is found to be

$$\phi(\eta, r, \theta, t) = A(t) + B(t)\eta + I_0 \left\{\frac{\pi}{2h_\ell} r\right\} C(t) \cos \left\{\frac{\pi}{2h_\ell} \eta\right\}$$

$$+ \sum_{j=1}^{\infty} J_0(\lambda_{0j} \frac{r}{a}) \, D_{0j}(t) \cosh (\lambda_{0j} \frac{\eta}{a})$$

$$+ \sum_{k=1}^{\infty} \frac{I_0(\frac{k\pi}{h_\ell} r)}{(\frac{k\pi}{h_\ell} a) \, I_0'(\frac{k\pi}{h_\ell} a)} \, E_{0k}(t) \cos(\frac{k\pi}{h_\ell} \eta)$$

$$+ \sum_{m=1}^{M} \left\{ F_m^{(1)}(t) r^m + \sum_{j=m}^{\infty} J_m(\lambda_{mj} \frac{r}{a}) \, D_{mj}^{(1)}(t) \cosh (\lambda_{mj} \frac{\eta}{a}) \right.$$

$$+ \sum_{k=1}^{\infty} \frac{I_m(\frac{k\pi}{h_\ell}r)}{(\frac{k\pi}{h_\ell}a)\, I_m'(\frac{k\pi}{h_\ell}a)} \; E_{mk}^{(1)}(t) \cos(\frac{k\pi}{h_\ell}\eta) \Bigg\} \quad \cos(m\theta)$$

$$+ \sum_{m=1}^{M} \Bigg\{ F_m^{(2)}(t) r^m + \sum_{j=m}^{\infty} J_m(\lambda_{mj}\frac{r}{a})\, D_{mj}^{(2)}(t) \cosh(\lambda_{mj}\frac{\eta}{a})$$

$$+ \sum_{k=1}^{\infty} \frac{I_m(\frac{k\pi}{h_\ell}r)}{(\frac{k\pi}{h_\ell}a)\, I_m'(\frac{k\pi}{h_\ell}a)} \; E_{mk}^{(2)}(t) \cos(\frac{k\pi}{h_\ell}\eta) \Bigg\} \quad \sin(m\theta) \qquad (56)$$

where $J_m(\lambda_{mj}\frac{r}{a})$ and $I_m(\frac{k\pi}{h_\ell}r)$ are Bessel functions of the first kind and the modified Bessel functions of the first kind respectively. The λ_{mj}'s are zeros of $J_m'(\lambda_{mj})$ and are ordered such that $\lambda_{00} = 0$; and $\lambda_{0j} \neq 0$, $j \geq 1$; and $\lambda_{mj} \neq 0$ for $j \geq m$ and $m \geq 1$. The root $\lambda_{00} = 0$ is retained as $J_0(0) = 1$.

The separation constants $\frac{k\pi}{h_\ell}$ are chosen with regard to the orthogonality of the cosine function over the half range $0<\eta<h_\ell$. The expressions containing the time functions A(t), B(t), C(t), $F_m^{(1)}(t)$ and $F_m^{(2)}(t)$ are obtained by taking particular values for the separation constants and are necessary in order to satisfy all of the boundary conditions. In satisfying the boundary and initial conditions it is necessary to use the orthogonality of the function $J_s(\lambda_{sp}\frac{r}{a})\cos(s\theta)$ with respect to a weight function of $\frac{r}{a}$ over the region $0<r<a$ and $0<\theta<2\pi$. This is as follows. (reference 5, page 195)

$$\int_0^a \int_0^{2\pi} \frac{r}{a} \; J_m(\lambda_{mj}\frac{r}{a}) \cos(m\theta)\, J_s(\lambda_{sp}\frac{r}{a}) \cos(s\theta) \frac{dr}{a}\, d\theta$$

$$= \delta_{sm}\ \delta_{\rho j}\ (1+\delta_{0s})\ \pi \left\{ \frac{\delta_{0p}}{2} + (1-\delta_{0p}) \left[\frac{(\lambda_{s\rho})^2-(s)^2}{2(\lambda_{s\rho})^2} \right] J_s^{\,2}(\lambda_{s\rho}) \right\}$$

$$j \geq m$$

$$\rho \geq s \qquad (57)$$

Requiring that the three boundary conditions, equations (38), (39) and (40) be satisfied, yields the following equations for the time functions. The complete derivation is shown in Appendix A.

$$A(t) = C_1 t + C_2 - \left(\frac{T_o}{M}\right)^2 \left\{ \frac{t^3}{6} + \frac{\gamma}{\Omega^3} \left[\sin(\Omega t) - (\Omega t)\cos(\Omega t)\right] - \frac{\gamma^2}{8\Omega^3}\sin(2\Omega t) \right\}$$

$$- hB(t) - g\left(\frac{T_o}{M}\right) \left\{ \frac{t^3}{6} - \frac{\gamma}{\Omega^3}\left[\sin(\Omega t) - \Omega t\right] \right\} + \sum_{n=o}^{N} \dot{W}_{0n}^{(1)}(t)\ P_{kn}^{\ *}$$

$$- \frac{2g}{a} \sum_{n=0}^{N} \int_0^t W_{0n}^{(1)}(\beta)\ d\beta\ I_{0n} \qquad (58)$$

$$B(t) = \frac{T_o}{M}\left[t + \frac{\gamma}{\Omega}\sin(\Omega t) \right] \qquad (59)$$

with the additional requirement that

$$\dot{B}(t) = \Gamma(t)\ \frac{T_o}{M}\left[1 + \gamma\cos(\Omega t)\right] \qquad (60)$$

$$C(t) = \frac{-1}{I_1\left(\frac{\pi}{2h_\ell}a\right)} \sum_{n=0}^{N} \dot{W}_{0n}^{(1)}(t)\ \ I_{0n} \qquad (61)$$

$$D_{0j}^{(1)}(t) = A_{0j}^{(1)}\cos(\omega_{0j}t) + B_{0j}^{(1)}\sin(\omega_{0j}t) + \sum_{n=0}^{N} \int_0^t \dddot{W}_{0n}^{(1)}(\beta)\ \frac{\sin\left[\omega_{0j}(t-\beta)\right]d\beta}{\omega_{0j}}\ P_{kn}^{0j}$$

$$+ \sum_{n=0}^{N} \int_0^t \dot{W}_{0n}^{(1)}(\beta)\ \frac{\sin\left[\omega_{0j}(t-\beta)\right]d\beta}{\omega_{0j}}\ P_{0n}^{0j} \qquad (62)$$

$$D_{mj}^{(i)}(t) = A_{mj}^{(i)} \cos(\omega_{mj} t) + B_{mj}^{(1)} \sin(\omega_{mj} t)$$

$$+ \sum_{n=0}^{N} \int_0^t \dddot{W}_{mn}^{(i)}(\beta) \frac{\sin\left[\omega_{mj}(t-\beta)\right]}{\omega_{mj}} d\beta \left[P_{kn}^{mj} + P_{0n}^{mj}\right] \tag{63}$$

$$E_{0k}(t) = - \frac{2a}{h_\ell} \sum_{n=0}^{N} \dot{W}_{0n}^{(1)}(t) \left[I_{kn} - \frac{(-1)^k I_{0n}}{1 - 4k^2} \right] \tag{64}$$

$$E_{mk}^{(i)}(t) = - \frac{2a}{h_\ell} \sum_{n=0}^{N} \dot{W}_{mn}^{(i)}(t) I_{kn} \tag{65}$$

$$F_m^{(i)}(t) = - \frac{1}{m h_\ell a^{m-1}} \sum_{n=0}^{N} \dot{W}_{mn}^{(i)}(t) I_{0n} \qquad i = 1, 2 \tag{66}$$

where the superposition integrals in equations (62) and (63) are obtained as particular solutions to the equations determining $D_{mj}^{(i)}(t)$, (see reference 6, page 444) and

$$I_{on} = \int_0^{h_\ell} f_n(\eta) \, d\eta \tag{67}$$

$$I_{kn} = \int_0^{h_\ell} f_n(\eta) \cos\left(\frac{k\pi}{h_\ell}\right) d\eta \tag{68}$$

$$H_{kj}^m = \int_0^a \frac{\frac{r}{a} I_m\left(\frac{k\pi}{h_\ell} a \frac{r}{a}\right)}{\frac{k\pi}{h_\ell} a \; I_m'\left(\frac{k\pi}{h_\ell} a\right)} J_m\left(\lambda_{mj} \frac{r}{a}\right) \frac{dr}{a} = \frac{1}{\frac{k\pi}{h_\ell} a \; I_m'\left(\frac{k\pi}{h_\ell} a\right) \left[\left(\frac{k\pi}{h_\ell} a\right)^2 + (\lambda_{mj})^2\right]}$$

$$\cdot \left\{ \frac{k\pi}{h_\ell} a \; J_m(\lambda_{mj}) \, I_{m+1}\left(\frac{k\pi}{h_\ell} a\right) + \lambda_{mj} I_m\left(\frac{k\pi}{h_\ell} a\right) J_{m+1}(\lambda_{mj}) \right\}$$

$$k \geq 1, \; j \geq m \tag{69}$$

$$H_{k0}^0 = \left(\frac{h_\ell}{k\pi a}\right)^2, \qquad k \geq 1 \tag{70}$$

$$P^{*}_{kn} = \sum_{k=1}^{\infty} \frac{4a}{h_\ell} \left(\frac{h_\ell}{k\pi a}\right)^2 (-1)^k \left[I_{kn} - \frac{(-1)^k I_{0n}}{1-4k^2} \right] \quad (71)$$

$$P^{mj}_{kn} = \frac{2(\lambda_{mj})^2}{\left[(\lambda_{mj})^2 - m^2\right] J_m^{\,2}(\lambda_{mj}) \cosh(\lambda_{mj} \frac{h_\ell}{a})} \sum_{k=1}^{\infty} H^m_{kj} (-1)^k \frac{2a}{h_\ell} I_{kn} \quad (72)$$

$$P^{0j}_{kn} = \frac{2}{J_0(\lambda_{0j}) \cosh(\lambda_{0j} \frac{h_\ell}{a})} \sum_{k=1}^{\infty} H^0_{kj} (-1)^k \frac{2a}{h_\ell} \left[I_{kn} - \frac{(-1)^k I_{0n}}{1 - 4k^2} \right] \quad (73)$$

$$P^{mj}_{0n} = \frac{2(\lambda_{mj})^2}{\left[(\lambda_{mj})^2 - m^2\right] J_m^{\,2}(\lambda_{mj}) \cosh(\lambda_{mj} \frac{h_\ell}{a})} \frac{a\, J_{m+1}(\lambda_{mj})\, I_{0n}}{m h_\ell \lambda_{mj}} \quad (74)$$

and

$$P^{0j}_{0n} = - \frac{2}{J_0(\lambda_{0j}) \cosh(\lambda_{0j} \frac{h_\ell}{a})} \frac{g a \left(\frac{\pi}{2h_\ell}\right)^2}{\left(\frac{\pi a}{2h_\ell}\right)^2 + (\lambda_{0j})^2} I_{0n} \quad (75)$$

The constants in the above equations are specified by the initial conditions on the fluid. Using equations (41), (42), and (43) and the necessary restriction that if $\vec{V}(0) = 0$ then $\dot{W}^{(i)}_{mn}(0) = 0$, the constants $A^{(1)}_{0j}$, $A^{(1)}_{mj}$, and $A^{(2)}_{mj}$ are found to be zero.

From equation (44) the following equations are obtained for the remaining constants except for C_2, which due to the Neumann type boundary and initial conditions, cannot be determined

$$C_1 + \frac{P_o + \rho_\ell g h_\ell}{\rho_\ell} + \left(\frac{T_o}{M}\right)^2 \frac{\gamma^2}{4\Omega^2} + \sum_{n=0}^{N} \ddot{W}^{(1)}_{0n}(0) \quad P^{*}_{kn}$$

$$- \frac{2g}{a} \sum_{n=0}^{N} W_{0n}^{(1)}(0)\, I_{0n}^{(1)} - \frac{4h_\ell}{\pi a} \cos(\frac{\pi}{2h_\ell}\eta) \sum_{n=0}^{N} \ddot{W}_{0n}^{(1)}(0)\, I_{0n}$$

$$- 2 \sum_{k=1}^{\infty} (\frac{h_\ell}{k\pi a})^2 \frac{2a}{h_\ell} \sum_{n=0}^{N} \ddot{W}_{0n}(0) \left[I_{kn} - \frac{(-1)^k I_{0n}}{1 - 4k^2} \right] \cos(\frac{k\pi}{h_\ell}\eta) = 0 \qquad (76)$$

$$B_{0j}^{(1)}\, \omega_{0j} \frac{J_0^{\,2}(\lambda_{0j})}{2(\lambda_{0j})^2} \cosh(\lambda_{0j} \frac{\eta}{a}) - \frac{\frac{\pi a}{2h_\ell} J_0(\lambda_{0j})}{(\frac{\pi a}{2h_\ell})^2 + (\lambda_{0j})^2} \cos(\frac{\pi}{2h_\ell}\eta) \sum_{n=0}^{N} \ddot{W}_{0n}^{(1)}(0)\, I_{0n}$$

$$- \sum_{k=1}^{\infty} H_{kj}^{0} \frac{2a}{h_\ell} \sum_{n=0}^{N} \ddot{W}_{0n}^{(1)}(0) \left[I_{kn} - \frac{(-1)^k I_{0n}}{1 - 4k^2} \right] \cos(\frac{k\pi}{h_\ell}\eta) = 0 \qquad (77)$$

$$\frac{B_{mj}^{(i)}\, \omega_{mj} \left[\lambda_{mj}^{\,2} - m^2 \right] J_m^{\,2}(\lambda_{mj}) \cosh(\lambda_{mj} \frac{\eta}{a})}{2(\lambda_{mj})^2} - \sum_{k=1}^{\infty} H_{kj}^{m} \frac{2a}{h_\ell} \sum_{n=0}^{N} \ddot{W}_{mn}^{(i)}(0)\, I_{kn} \cos(\frac{k\pi}{h_\ell})$$

$$- \frac{a}{mh_\ell \lambda_{mj}} J_{m+1}(\lambda_{mj}) \sum_{n=0}^{N} \ddot{W}_{mn}^{(i)}(0)\, I_{kn} = 0 \qquad (78)$$

It is seen that these equations cannot, in general, be satisfied for all η.

The constants can be determined approximately as follows. Writing equation (76) as

$$C_1 + G(\eta) = E_1 \qquad (79)$$

we will require that E_1 be a minimum in the sense of least squares. Therefore

$$\frac{\partial}{\partial C_1} \int_0^{h_\ell} (E_1)^2 \, d\eta = 0 \qquad (80)$$

Following the same procedure for the remaining equations, the constants are found to be

$$C_1 = \frac{2g}{a} \sum_{n=0}^{N} W_{0n}^{(1)}(0)\, I_{0n} - \left(\frac{T_o}{M}\right)^2 \frac{\gamma^2}{4\Omega^2} - \frac{P_o + \rho_\ell h_\ell g}{\rho_\ell}$$

$$- \sum_{n=0}^{N} \ddot{W}_{0n}^{(1)}(0) \left[P_{kn}^{*} - \frac{8h_\ell}{\pi^2 a} I_{0n}\right] \qquad (81)$$

$$B_{0j}^{(1)} = \frac{4(\lambda_{0j})^3 \sum_{n=0}^{N} \ddot{W}_{0n}^{(1)}(0)}{\omega_{0j} J_0^2(\lambda_{0j}) \left[\frac{1}{2} \sinh\left(2\lambda_{0j} \frac{h_\ell}{a}\right) + \left(\frac{\lambda_{0j}}{a}\right)\right] \left[\left(\frac{\pi}{2h_\ell}\right)^2 + \left(\frac{\lambda_{0j}}{a}\right)^2\right]}$$

$$\cdot \left\{ \frac{\left(\frac{\pi}{2h_\ell}\right)^2 J_0(\lambda_{0j}) \cosh\left(\lambda_{0j} \frac{h_\ell}{a}\right) I_{0n}}{\left(\frac{\pi a}{2h_\ell}\right)^2 + (\lambda_{0j})^2} + \sum_{k=1}^{\infty} H_{jk}^{0} \frac{2\lambda_{0j}}{ah_\ell} \sinh\left(\lambda_{0j} \frac{h_\ell}{a}\right) \right.$$

$$\left. \cdot \left[(-1)^k I_{kn} - \frac{I_{0n}}{1 - 4k^2}\right] \right\} \qquad (82)$$

and

$$B_{mj}^{(i)} = \frac{4(\lambda_{mj})^2 \sinh\left(\lambda_{mj} \frac{h_\ell}{a}\right) \sum_{n=0}^{N} \ddot{W}_{mn}^{(i)}(0)}{\omega_{mj} \left[(\lambda_{mj})^2 - m^2\right] J_m^2(\lambda_{mj}) \left[\frac{1}{2} \sinh\left(2\lambda_{mj} \frac{h_\ell}{a}\right) + \frac{\lambda_{mj}}{a}\right]}$$

$$\cdot \left\{ \frac{a J_{m+1}(\lambda_{mj}) I_{kn}}{mh_\ell (\lambda_{mj})^2} + \sum_{k=1}^{\infty} H_{kj}^{m} \frac{2\lambda_{mj}}{ah_\ell} \frac{(-1)^k I_{kn}}{\left(\frac{k\pi}{h_\ell}\right)^2 + \left(\frac{\lambda_{mj}}{a}\right)^2} \right\} \qquad i = 1,2 \qquad (83)$$

It should be noted that if the initial conditions on the shell are such that $\ddot{W}_{mn}^{(1)}(0) = \ddot{W}_{mn}^{(2)}(0) = 0$ the constants are then determined exactly as

$$C_1 = \frac{2g}{a} \sum_{n=0}^{N} W_{0n}^{(1)}(0)\, I_{0n} - \left(\frac{T_o}{M}\right)^2 \frac{\gamma^2}{4\Omega^2} - \frac{P_o + \rho_\ell g h_\ell}{\rho_\ell} \tag{84}$$

and

$$B_{0j}^{(1)} = B_{mj}^{(1)} = B_{mj}^{(2)} = 0 \tag{85}$$

The initial conditions necessary for the above are that the displacements of the wall are those displacements corresponding to the static forces of the fluid and of gravity. These conditions amount to omitting the transient part of the solution and therefore, if the free vibrations of the elastic shell are desired, the initial conditions must be other than these or there will of course be no vibratory motion.

4.0 PRESSURE EXERTED BY THE FLUID

The velocity potential is now known and the pressure exerted by the fluid on the wall is given by equation (53) with $r = a$ and $0<\eta<h_\ell$. The following integrals are necessary in evaluating the pressure:

$$k_1 = \int_0^t \left\{ \int_0^{t_1} \dot{W}_{mn}(\tau) \frac{\sin \omega_{mj}(t_1-\tau)\, d\tau}{\omega_{mj}} \right\} dt_1 \quad \text{and}$$

$$k_2 = \int_0^t \left\{ \int_0^{t_1} \dddot{W}_{mn}(\tau) \frac{\sin\omega_{mj}(t_1-\tau)\, d\tau}{\omega_{mj}} \right\} dt_1 .$$

Taking the Laplace transform

$$k_1 = \frac{1}{P} \mathcal{L}\left[\dot{W}_{mn}(\tau)\right] \frac{1}{P^2 + (\omega_{mj})^2} = \frac{\mathcal{L}\left[\dot{W}_{mn}(\tau)\right]}{(\omega_{mj})^2} \left[\frac{1}{P} - \frac{P}{P^2 + (\omega_{mj})^2}\right] \quad \text{and}$$

$$k_1 = \frac{1}{(\omega_{mj})^2} \left\{ W_{mn}(t) - W_{mn}(0) - \int_0^t \dot{W}_{mn}(\tau) \cos\omega_{mj}(t-\tau)\, d\tau \right\} \qquad (86)$$

Similarly

$$k_2 = \frac{1}{(\omega_{mj})^2} \left\{ \ddot{W}_{mn}(t) - \ddot{W}_{mn}(0) - \int_0^t \dddot{W}_{mn}(\tau) \cos \omega_{mj}(t-\tau)\, d\tau \right\} \qquad (87)$$

In order to include the pressure in the shell equations, it is found to be more convenient to expand the pressure in a series in terms of the orthogonal function $f_n(\eta)$. The pressure acting on the shell is

$$P_s(\eta,\theta,t) = \sum_{m=0}^{M} P^{(1)}(\eta,t)\cos(m\theta) + \sum_{m=1}^{M} P^{(2)}(\eta,t)\sin(m\theta), \quad 0 \le \eta \le h_\ell$$

$$= P_o, \qquad h_\ell \le \eta \le L \tag{88}$$

Writing $P_s(\eta,\theta,t)$ in a series we have

$$P_s(\eta,\theta,t) = \sum_{m=0}^{M}\sum_{s=0}^{\infty} P_{ms}^{(1)}(t)\, f_s(\eta)\cos(m\theta) + \sum_{m=1}^{M}\sum_{s=0}^{\infty} P_{ms}^{(2)}(t)\, f_s(\eta)\sin(m\theta) \tag{89}$$

where

$$P_{0s}^{(1)}(t) = \frac{2}{(1+\delta_{0s})L}\left\{\int_0^{h_\ell} P^{(1)}(\eta,t)\, f_s(\eta)d\eta + P_o\int_h^{L} f_s(\eta)d\eta\right\} \tag{90}$$

$$P_{ms}^{(1)}(t) = \frac{2}{(1+\delta_{0s})L}\int_0^{h_\ell} P^{(1)}(\eta,t)\, f_s(\eta)d\eta \qquad \text{and} \tag{91}$$

$$P_{ms}^{(2)}(t) = \frac{2}{(1+\delta_{0s})L}\int_0^{h_\ell} P^{(2)}(\eta,t)\, f_s(\eta)\, d\eta \tag{92}$$

Substituting equation (56) into equation (53) and then the results into equations (90), (91), and (92), the time coefficients may be determined. With these coefficients known in terms of $W_{mn}^{(1)}(t)$ and $W_{mn}^{(2)}(t)$, equation (88) may then be substituted into the shell equations and the functions $U_{mn}^{(1)}(t)$, $U_{mn}^{(2)}$, $V_{mn}^{(1)}(t)$, $V_{mn}^{(2)}(t)$, $W_{mn}^{(1)}(t)$, and $W_{mn}^{(2)}(t)$ evaluated. For the particular problem considered here, the function $f_n(\eta) = \sin(\frac{n\pi}{L}\eta)$ and equations (90), (91), and (92) give

$$P_{0s}^{(1)}(t) = \frac{2P_o}{s\pi}\left[1 - (-1)^s\right] + \frac{2\rho_\ell g h_\ell}{s\pi}\left[1 - \frac{L}{s\pi h_\ell}\sin\left(\frac{s\pi}{L}h_\ell\right)\right]$$

$$+ \frac{2\rho_\ell}{L}\left\langle -\Gamma(t)\frac{T_o}{M}\left[1 + \gamma\cos(\Omega t)\right]\left[\left(\frac{L}{s\pi}\right)^2 \sin\left(\frac{s\pi}{L}h_\ell\right) - \left(\frac{Lh_\ell}{s\pi}\right)\right]\right.$$

$$+ \frac{L}{s\pi}\left[1 - \cos\left(\frac{s\pi}{L}h_\ell\right)\right]\left\{\frac{2g}{a}\sum_{n=0}^{N} I_{0n}\left[W_{0n}^{(1)}(t) - W_{0n}^{(1)}(0)\right]\right.$$

$$\left. - \sum_{n=o}^{N}\ddot{W}_{0n}^{(1)}(t)\,P_{kn}^{*}\right\}$$

$$+ \sum_{n=o}^{N}\ddot{W}_{0n}(t)\left\{\frac{I_0\left(\frac{\pi}{2h_\ell}a\right) I_{0n}}{I_1\left(\frac{\pi}{2h_\ell}a\right)\left[\left(\frac{s\pi}{L}\right)^2 - \left(\frac{\pi}{2h_\ell}\right)^2\right]}\left[\frac{s\pi}{L} - \frac{\pi}{2h_\ell}\sin\left(\frac{s\pi}{L}h_\ell\right)\right]\right.$$

$$+ \sum_{k=1}^{\infty}\frac{I_0\left(\frac{k\pi}{h_\ell}a\right)\frac{2as\pi}{Lh_\ell}}{\left(\frac{k\pi}{h_\ell}a\right) I_1\left(\frac{k\pi}{h_\ell}a\right)\left[\left(\frac{s\pi}{L}\right)^2 - \left(\frac{k\pi}{h_\ell}\right)^2\right]}\left[1 - (-1)^k\cos\left(\frac{s\pi}{L}h_\ell\right)\right]$$

$$\left.\cdot\left[I_{kn} - \frac{(-1)^k I_{0n}}{1 - 4k^2}\right]\right\} - \sum_{j=1}^{\infty} J_0(\lambda_{0j})\frac{1}{\left(\frac{\lambda_{0j}}{a}\right)^2 + \left(\frac{s\pi}{L}\right)^2}$$

$$\cdot\left\{\frac{s\pi}{L} - \frac{s\pi}{L}\cos\left(\frac{s\pi}{L}h_\ell\right)\cosh\left(\lambda_{0j}\frac{h_\ell}{a}\right) + \frac{\lambda_{0j}}{a}\sin\left(\frac{s\pi}{L}h_\ell\right)\sinh\left(\lambda_{0j}\frac{h_\ell}{a}\right)\right\}$$

$$\cdot\left\{\sum_{n=0}^{N}\int_0^t \dddot{W}_{0n}^{(1)}(\beta)\cos\left[\omega_{0j}(t-\beta)\right]d\beta\,P_{kn}^{0j}\right.$$

$$\left.+ \sum_{n=0}^{N}\int_0^t \dot{W}_{0n}^{(1)}(\beta)\cos\left[\omega_{0j}(t-\beta)\right]d\beta\,P_{0n}^{0j}\right\}$$

$$+ \frac{I_0\left(\frac{\pi}{2h_\ell}a\right)\frac{s\pi^2}{2h_\ell L}g}{I_1\left(\frac{\pi}{2h_\ell}a\right)\left[\left(\frac{s\pi}{L}\right)^2 - \left(\frac{\pi}{2h_\ell}\right)^2\right]}\cos\left(\frac{s\pi}{L}h_\ell\right)\sum_{n=0}^{N}\left[W_{0n}^{(1)}(t) - W_{0n}^{(1)}(o)\right] I_{0n}$$

$$+ \sum_{j=1}^{\infty} J_0(\lambda_{0j}) \frac{g \frac{\lambda_{0j}}{a}}{(\frac{\lambda_{0j}}{a})^2 + (\frac{s\pi}{L})^2}$$

$$\cdot \left\{ \frac{s\pi}{L} \cos(\frac{s\pi}{L} h_\ell) \sinh(\lambda_{0j} \frac{h_\ell}{a}) - \frac{\lambda_{0j}}{a} \sin(\frac{s\pi}{L} h_\ell) \cosh(\lambda_{0j} \frac{h_\ell}{a}) \right\}$$

$$\cdot \left\{ \sum_{n=0}^{N} \frac{P_{0n}^{0j}}{(\omega_{0j})^2} \left[W_{0n}^{(1)}(t) - W_{0n}^{(1)}(0) - \int_0^t \dot{W}_{0n}^{(1)}(\beta) \cos\left[\omega_{0j}(t-\beta)\right] d\beta \right] \right.$$

$$\left. + \sum_{n=0}^{N} \frac{P_{kn}^{0j}}{(\omega_{0j})^2} \left[\ddot{W}_{0n}^{(1)}(t) - \int_0^t \ddot{W}_{0n}^{(1)}(\beta) \cos\left[\omega_{0j}(t-\beta)\right] d\beta \right] \right\}$$

$$- \frac{2ga}{h_\ell} \sum_{k=1}^{\infty} \frac{I_0(\frac{k\pi}{h_\ell} a) (\frac{k\pi}{h_\ell})^2}{(\frac{k\pi}{h_\ell} a) I_1(\frac{k\pi}{h_\ell} a) \left[(\frac{s\pi}{L})^2 - (\frac{k\pi}{h_\ell})^2 \right]} \sin(\frac{s\pi}{L} h_\ell)$$

$$\cdot \sum_{n=0}^{N} \left[W_{0n}^{(1)}(t) - W_{0n}^{(1)}(0) \right] \left[(-1)^k I_{kn} - \frac{I_{0n}}{1-4k^2} \right] \Big\rangle \qquad (93)$$

$s = 1, 2, \ldots,$ and

$$P_{ms}^{(i)}(t) = \frac{2\rho_\ell}{L} \Big\langle \frac{a}{mh_\ell} \sum_{n=0}^{N} \ddot{W}_{mn}^{(i)}(t) I_{0n} \frac{L}{s\pi} \left[1 - \cos(\frac{s\pi h_\ell}{L}) \right]$$

$$- \sum_{j=m}^{\infty} J_m(\lambda_{mj}) \frac{1}{(\frac{\lambda_{mj}}{a})^2 + (\frac{s\pi}{L})^2}$$

$$\cdot \left\{ \frac{s\pi}{L} - \frac{s\pi}{L} \cos(\frac{s\pi}{L} h_\ell) \cosh(\lambda_{mj} \frac{h_\ell}{a}) + \frac{\lambda_{mj}}{a} \sin(\frac{s\pi}{L} h_\ell) \sinh(\frac{\lambda_{mj}}{a} h_\ell) \right\}$$

$$\cdot \left\{ \sum_{n=0}^{N} \int_0^t \dddot{W}_{mn}^{(i)}(\beta) \cos\left[\omega_{mj}(t-\beta)\right] d\beta \left[P_{kn}^{mj} + P_{0n}^{mj} \right] \right\}$$

$$+ \sum_{k=1}^{\infty} \frac{I_m\left(\frac{k\pi}{h_\ell}a\right)}{\left(\frac{k\pi}{h_\ell}a\right) I_m'\left(\frac{k\pi}{h_\ell}a\right)} \sum_{n=0}^{\infty} \ddot{W}_{mn}^{(i)}(t)\, I_{kn} \frac{\frac{2s\pi a}{h_\ell L}\left[1 - (-1)^k \cos\left(\frac{s\pi}{L}h_\ell\right)\right]}{\left(\frac{s\pi}{L}\right)^2 - \left(\frac{k\pi}{L}\right)^2}$$

$$- \sum_{j=m}^{\infty} J_m(\lambda_{mj}) \frac{\frac{\lambda_{mj}g}{a}}{\left(\frac{\lambda_{mj}}{a}\right)^2 + \left(\frac{s\pi}{h_\ell}\right)^2}$$

$$\cdot \left\{ \frac{\lambda_{mj}}{a} \sin\left(\frac{s\pi}{L}h_\ell\right) \cosh\left(\lambda_{mj}\frac{h_\ell}{a}\right) - \frac{s\pi}{L} \cos\left(\frac{s\pi}{a}h_\ell\right) \sinh\left(\lambda_{mj}\frac{h_\ell}{a}\right) \right\}$$

$$\cdot \sum_{n=0}^{N} \frac{P_{kn}^{mj} + P_{0n}^{mj}}{(\omega_{mj})^2} \left\{ \ddot{W}_{mn}^{(i)}(t) - \int_0^t \dddot{W}_{mn}^{(i)}(\beta) \cos[\omega_{mj}(t-\beta)]\, d\beta \right.$$

$$\left. + \frac{2ga}{h_\ell} \sum_{k=1}^{\infty} \frac{I_m\left(\frac{k\pi}{h_\ell}a\right)(-1)^k}{\left(\frac{k\pi}{h_\ell}a\right) I_m'\left(\frac{k\pi}{h_\ell}a\right)} \frac{\left(\frac{k\pi}{h_\ell}\right)^2 \sin\left(\frac{s\pi}{L}h_\ell\right)}{\left(\frac{s\pi}{L}\right)^2 - \left(\frac{k\pi}{h_\ell}\right)^2} \sum_{n=0}^{N} \left[W_{mn}^{(i)}(t) - W_{mn}^{(i)}(0) \right] I_{kn} \right\rangle$$

$$i = 1, 2$$

$$s = 1, 2, \ldots, \qquad (94)$$

For $t = 0$ in the above equations

$$P_{0s}^{(1)}(0) = \frac{2P_o}{s\pi}\left[1 - (-1)^s\right] + \frac{2\rho_\ell g h}{s\pi}\left[1 - \frac{L}{s\pi h_\ell}\sin\left(\frac{s\pi}{L}h_\ell\right)\right] \qquad (95)$$

which is just the coefficient of the half range expansion of the static pressure in a series in terms of the orthogonal functions $f_n(\eta)$, and $P_{ms}^{(i)}(0) = 0, \quad i = 1,2$.

The expression obtained for $P_s(\eta,\theta,t)$ can now be substituted into the shell equations, equations (24) through (35) on pages 7 through 9. Making this substitution and performing the indicated integration in equations (29), (32), and (35), we obtain

$$\ddot{\tilde{W}}^{(1)}_{0s}(\tau) + \frac{1}{(1-\nu^2)\mu}\left[\lambda^2 + \frac{\sigma^2}{12}(s\pi)^4\right]\tilde{W}^{(1)}_{0s}(\tau) + \frac{s\pi\nu\lambda}{(1-\nu^2)\mu}\tilde{U}^{(1)}_{0s}(\tau) = -\frac{1}{\mu}\tilde{P}^{(1)}_{0s}(\tau) \tag{29)*}$$

$$\frac{s\pi\nu\lambda}{(1-\nu^2)\mu}\tilde{U}^{(1)}_{ms}(\tau) - \frac{m\lambda^2}{(1-\nu^2)\mu}\tilde{V}^{(1)}_{ms}(\tau) + \frac{1}{(1-\nu^2)\mu}\left\{\lambda^2 + \frac{\sigma^2}{12}\left[(s\pi)^2 + (m\lambda)^2\right]^2\right\}\tilde{W}^{(1)}_{ms}(\tau)$$

$$+ \ddot{\tilde{W}}^{(1)}_{ms}(\tau) = -\frac{1}{\mu}\tilde{P}^{(1)}_{ms}(\tau) \text{ and} \tag{32)*}$$

$$\frac{s\pi\nu\lambda}{(1-\nu^2)\mu}\tilde{U}^{(2)}_{ms}(\tau) + \frac{m\lambda^2}{(1-\nu^2)\mu}\tilde{V}^{(2)}_{ms}(\tau) + \frac{1}{(1-\nu^2)\mu}\left\{\lambda^2 + \frac{\sigma^2}{12}\left[(s\pi)^2 + (m\lambda)^2\right]^2\right\}\tilde{W}^{(2)}_{ms}(\tau)$$

$$+ \ddot{\tilde{W}}^{(2)}_{ms}(\tau) = -\frac{1}{\mu}\tilde{P}^{(2)}_{ms}(\tau) \tag{35)*}$$

Equations (93) and (94) in dimensionless form are

$$\tilde{P}^{(1)}_{0s}(\tau) = \frac{2}{s\pi}\tilde{P}_o\left[1-(-1)^s\right] + \frac{2\Gamma^{(1)}}{s\pi}\left[\zeta - \frac{1}{s\pi}\sin(s\pi\zeta)\right]$$

$$- 2\sqrt{\ }(\tau)\,\Gamma^2\left[1+\gamma\cos(\omega\tau)\right]\left[\left(\frac{1}{s\pi}\right)^2\sin(s\pi\zeta) - \frac{\zeta}{s\pi}\right]$$

$$+ \frac{2}{s\pi}\left[1-\cos(s\pi\zeta)\right]\left\{2\lambda\Gamma^{(1)}\sum_{n=0}^{N}\tilde{I}_{0n}\left[\tilde{W}^{(1)}_{0n}(\tau) - \tilde{W}^{(1)}_{0n}(0)\right]\right.$$

$$\left. - \sum_{n=0}^{N}\Gamma^{(3)}\ddot{\tilde{W}}^{(1)}_{0n}(\tau)\,\tilde{P}^{*}_{kn}\right\}$$

$$+ 2\Gamma^{(3)}\sum_{n=0}^{N}\ddot{\tilde{W}}^{(1)}_{0n}(\tau)\left\{\frac{I_0\left(\frac{\pi}{2\alpha}\right)\tilde{I}_{0n}}{I_1\left(\frac{\pi}{2\alpha}\right)\left[(s\pi)^2 - \left(\frac{k\pi}{\zeta}\right)^2\right]}\left[s\pi - \frac{\pi}{2\zeta}\sin(s\pi\zeta)\right]\right.$$

$$+ \frac{2}{\alpha} \sum_{k=1}^{\infty} \frac{s\pi I_0\left(\frac{k\pi}{\alpha}\right)}{\frac{k\pi}{\alpha} I_1\left(\frac{k\pi}{\alpha}\right) \left[(s\pi)^2 - \left(\frac{\pi}{2\zeta}\right)^2\right]} \left[1 - (-1)^k \cos(s\pi\zeta)\right] \left[\tilde{I}_{kn} - \frac{(-1)^k \tilde{I}_{0n}}{1 - 4k^2}\right] \Bigg\}$$

$$- 2\Gamma^{(3)} \sum_{j=1}^{\infty} \frac{J_0(\lambda_{0j})}{\lambda^2(\lambda_{0j})^2 + (s\pi)^2}$$

$$\cdot \left\{ s\pi - s\pi \cos(s\pi\zeta) \cosh(\lambda_{0j}\alpha) + \lambda(\lambda_{0j}) \sin(s\pi\zeta) \sinh(\lambda_{0j}\alpha) \right\}$$

$$\cdot \sum_{n=0}^{N} \left\{ \int_0^{\tau} \dddot{\tilde{W}}_{0n}^{(1)}(\tilde{\beta}) \cos\left[\tilde{\omega}_{0j}(\tau - \tilde{\beta})\right] d\tilde{\beta}\, \tilde{P}_{kn}^{0j} + \int_0^{\tau} \dot{\tilde{W}}_{0n}(\tilde{\beta}) \cos\left[\tilde{\omega}_{0j}(\tau - \tilde{\beta})\right] d\tilde{\beta}\, \tilde{P}_{0n}^{0j} \right\}$$

$$+ \frac{\Gamma^{(1)} I_0\left(\frac{\pi}{2\alpha}\right) s\pi^2 \cos(s\pi\zeta)}{\zeta\, I_1\left(\frac{\pi}{2\alpha}\right) \left[(s\pi)^2 - \left(\frac{\pi}{2\zeta}\right)^2\right]} \sum_{n=0}^{N} \left[\tilde{W}_{0n}^{(1)}(\tau) - \tilde{W}_{0n}^{(1)}(0)\right] \tilde{I}_{0n}$$

$$+ 2\lambda\Gamma^{(1)} \sum_{j=1}^{\infty} \frac{\lambda_{0j}\, J_0(\lambda_{0j})}{\lambda^2(\lambda_{0j})^2 + (s\pi)^2}$$

$$\cdot \left\{ s\pi \cos(s\pi\zeta) \sinh(\lambda_{0j}\alpha) - \lambda(\lambda_{0j}) \sin(s\pi\zeta) \cosh(\lambda_{0j}\alpha) \right\}$$

$$\cdot \sum_{n=0}^{N} \left\{ \frac{\tilde{P}_{0n}^{0j}}{(\tilde{\omega}_{0j})^2} \left[\tilde{W}_{0n}^{(1)}(\tau) - \tilde{W}_{0n}^{(1)}(0) - \int_0^{\tau} \dot{\tilde{W}}_{0n}^{(1)}(\beta) \cos\left[\tilde{\omega}_{0j}(\tau - \tilde{\beta})\right] d\tilde{\beta}\right] \right.$$

$$\left. + \frac{\tilde{P}_{kn}^{0j}}{(\tilde{\omega}_{0j})^2} \left[\ddot{\tilde{W}}_{0n}^{(1)}(\tau) - \int_0^{\tau} \dddot{\tilde{W}}_{0n}^{(1)}(\tilde{\beta}) \cos\left[\tilde{\omega}_{0j}(\tau - \tilde{\beta})\right] d\tilde{\beta} \right] \right\}$$

$$- \frac{4\Gamma^{(1)}}{\alpha} \sum_{k=1}^{\infty} \frac{I_0\left(\frac{k\pi}{\alpha}\right) \left(\frac{k\pi}{\zeta}\right)^2 \sin(s\pi\zeta)}{\frac{k\pi}{\alpha} I_1\left(\frac{k\pi}{\alpha}\right) \left[(s\pi)^2 - \left(\frac{k\pi}{\zeta}\right)^2\right]}$$

$$\cdot \sum_{n=0}^{N} \left[\tilde{W}_{0n}^{(1)}(\tau) - \tilde{W}_{0n}^{(1)}(0)\right] \left[(-1)^k \tilde{I}_{kn} - \frac{\tilde{I}_{0n}}{1 - 4k^2}\right] \quad \text{and} \qquad (93)^*$$

$$\tilde{P}_{ms}^{(i)}(\tau) = \frac{2\Gamma^{(3)}}{m\alpha} \sum_{n=0}^{N} \ddot{\tilde{W}}_{mn}^{(i)}(\tau) \frac{\tilde{I}_{0n}}{s\pi} \left[1 - \cos(s\pi\zeta)\right]$$

$$- 2\Gamma^{(3)} \sum_{j=m}^{\infty} \frac{J_m(\lambda_{mj})}{\lambda^2(\lambda_{mj})^2 + (s\pi)^2}$$

$$\cdot \left\{ s\pi - s\pi \cos(s\pi\zeta) \cosh(\lambda_{mj}\alpha) + \lambda\lambda_{mj} \sin(s\pi\zeta) \sinh(\lambda_{mj}\alpha) \right\}$$

$$\cdot \sum_{n=0}^{N} \int^{\tau} \dot{\tilde{W}}_{mn}^{(i)}(\tilde{\beta}) \cos\left[\tilde{\omega}_{mj}(\tau - \tilde{\beta})\right] d\tilde{\beta} \left[\tilde{P}_{kn}^{mj} - \tilde{P}_{0n}^{mj}\right]$$

$$+ \frac{2\Gamma^{(3)}}{\alpha} \sum_{k=1}^{\infty} \frac{I_m(\frac{k\pi}{\alpha})}{\frac{k\pi}{\alpha} I_m'(\frac{k\pi}{\alpha})} \sum_{n=0}^{N} \ddot{\tilde{W}}_{mn}^{(i)} \frac{\frac{2s\pi}{\alpha} \tilde{I}_{kn}}{(s\pi)^2 - (\frac{k\pi}{\zeta})^2} \left[1 - (-1)^k \cos(s\pi\zeta)\right]$$

$$- 2\lambda\Gamma^{(1)} \sum_{j=m}^{\infty} \frac{\lambda_{mj} J_m(\lambda_{mj})}{\lambda^2(\lambda_{mj})^2 + (s\pi)^2}$$

$$\cdot \left\{ \lambda\lambda_{mj} \sin(s\pi\zeta) \cosh(\lambda_{mj}\alpha) - s\pi \cos(s\pi\zeta) \sinh(\lambda_{mj}\alpha) \right\}$$

$$\cdot \sum_{n=0}^{N} \frac{\left[\tilde{P}_{kn}^{mj} + \tilde{P}_{0n}^{mj}\right]}{(\tilde{\omega}_{mj})^2} \left\{ \ddot{\tilde{W}}_{mn}^{(i)}(\tau) - \int_0^{\tau} \dddot{\tilde{W}}_{mn}^{(i)}(\tilde{\beta}) \cos\left[\tilde{\omega}_{mj}(\tau - \tilde{\beta})\right] d\tilde{\beta} \right\}$$

$$+ 4 \frac{\Gamma^{(1)}}{\alpha} \sum_{k=1}^{\infty} \frac{(-1)^k I_m(\frac{k\pi}{\alpha}) (\frac{k\pi}{\zeta})^2 \sin(s\pi\zeta)}{\frac{k\pi}{\alpha} I_m'(\frac{k\pi}{\alpha}) \left[(s\pi)^2 - (\frac{k\pi}{\zeta})^2\right]} \sum_{n=0}^{N} \left[\dot{\tilde{W}}_{mn}^{(i)}(\tau) - \dot{\tilde{W}}_{mn}^{(i)}(0)\right] \tilde{I}_{kn}$$

$$i = 1,2 \qquad (94)^*$$

5.0 METHOD OF SOLUTION TO THE SHELL EQUATIONS

In order to solve for the unknown displacement functions it is necessary to obtain a solution to equations (24), (26) through (28), and the three sets of equations: equations (25) and (29)*, equations (30), (31) and (32)*, equations (33), (34) and (35)*.

Equations (24), (26), (27) and (28) may be solved directly, and due to the statement of the problem, i.e. by the particular choice of initial conditions, it follows that all of the vibratory motion will be of a steady-state nature and therefore the solutions to these equations will be

$$\tilde{U}^{(1)}_{00}(\tau) = \frac{(1-\nu^2)}{3} \left\{ (\tau)\, \Gamma^{(4)} \left[1 + \lambda \cos(\omega\tau)\right] + \Gamma^{(5)} \right\} \tag{96}$$

$$\tilde{U}^{(1)}_{m0}(\tau) = \tilde{U}^{(2)}_{m0}(\tau) = \tilde{V}^{(2)}_{0n}(\tau) = 0 \tag{97}$$

Equation (96) follows if the requirement is made that the body reference frame remain with the body during the motion.

Solutions to the remaining equations are determined as follows: equations (25) and (29)* determine $\tilde{U}^{(1)}_{0s}(\tau)$ and $\tilde{W}^{(1)}_{0s}(\tau)$, equations (30), (31) and (32)* determine $\tilde{U}^{(1)}_{ms}(\tau)$, $\tilde{V}^{(1)}_{ms}(\tau)$ and $\tilde{W}^{(1)}_{ms}(\tau)$ and equations (33), (34) and (35)* determine $\tilde{U}^{(2)}_{ms}(\tau)$, $\tilde{V}^{(2)}_{ms}(\tau)$ and $\tilde{W}^{(2)}_{ms}(\tau)$. As $\tilde{P}^{(i)}_{ms}(\tau)$ contains the superposition integral, it seems that the most logical manner in which to obtain a solution to these three sets of equations is through the use of Laplace transforms. By taking the Laplace transform of the above three sets of equations and using the known initial conditions we can solve for $\tilde{U}^{(1)}_{0s}(\tau)$ in terms of $\tilde{W}^{(1)}_{0s}(\tau)$ from equation (25) and substitute the results into equation (29)*. Equation (29)* will now consist of N equations for the determination of the N unknowns $\tilde{W}_{0s}^{(1)}(\tau)$ in terms of the transform variable p. In a similar manner equations (30), (31), and (34), (35) may be solved for $\tilde{U}_{ms}^{(1)}(\tau)$, $\tilde{V}_{ms}^{(2)}(\tau)$ and $\tilde{U}_{ms}^{(2)}(\tau)$, $\tilde{V}_{ms}^{(2)}(\tau)$ in terms of $\tilde{W}_{ms}^{(1)}(\tau)$ and $\tilde{W}_{ms}^{(2)}(\tau)$ respectively and then equations (32)* and (35)* solved for $\tilde{W}_{ms}^{(1)}(\tau)$ and $\tilde{W}_{ms}^{(2)}(\tau)$. Equations (32)* and (35)* will each consists of M sets of equations and each set of equations will contain N equations in the N unknowns, $W_{mn}^{(i)}(\tau)$, n=1, 2, ..., N.

Methods for determining $\tilde{W}_{0s}^{(1)}(\tau)$, $\tilde{W}_{ms}^{(1)}(\tau)$ and $\tilde{W}_{ms}^{(2)}(\tau)$ are developed in Appendix B.

6.0 SPECIFICATION OF INITIAL CONDITIONS

At time t=0 the pressure is given by

$$P_s(\eta,\theta,0) = P_o + \rho_\ell g(h_\ell-\eta) \qquad 0\le\eta\le h_\ell \tag{98}$$

$$= P_o \qquad h_\ell\le\eta\le L \tag{99}$$

In view of the assumed form of the displacements, the pressure will be expanded in a Fourier sine series over the over the half range $0\le\xi\le1$. The coefficients are given by equations (95) as

$$\tilde{D}_s = \frac{2\tilde{P}_o}{s\pi}\left[1-(-1)^s\right] + \frac{2\Gamma^{(1)}}{s\pi}\left[1-\frac{1}{s\pi\zeta}\sin(s\pi\zeta)\right] \tag{100}$$

therefore

$$\tilde{P}_s(\xi,\theta,0) = \sum_{s=1}^{\infty} \tilde{D}_s \sin(s\pi\xi) \tag{101}$$

Due to the symmetry of loading the only non-zero initial displacements will be

$$\tilde{u}(\xi,\theta,0) = -(1-\nu^2)\,\Gamma^{(5)}\left[\xi-\frac{\xi^2}{2}-\frac{1}{3}\right] + \frac{(1-\nu^2)\,\tilde{P}_o\xi}{2\lambda} + \sum_{n=1}^{N}\tilde{U}_{0n}^{(1)}(0)\cos(n\pi\xi)$$

$\tilde{v}(\xi,\theta,0) = 0$ and

$$\tilde{w}(\xi,\theta,0) = \sum_{n=1}^{N}\tilde{W}_{0n}^{(1)}(0)\sin(n\pi\xi) \tag{102}$$

The equilibrium equations are

$$\frac{\partial^2\tilde{u}}{\partial\xi^2} + \frac{(1-\nu)\lambda^2}{2}\frac{\partial^2\tilde{u}}{\partial\theta^2} + \frac{(1+\nu)\lambda}{2}\frac{\partial^2\tilde{v}}{\partial\xi\partial\theta} - \nu\lambda\frac{\partial\tilde{w}}{\partial\xi} = (1-\nu^2)\,\Gamma^{(5)} \tag{103}$$

$$\frac{(1+\nu)\lambda}{2}\frac{\partial^2\tilde{u}}{\partial\xi\partial\theta} + \frac{(1-\nu)}{2}\frac{\partial^2\tilde{v}}{\partial\xi^2} + \lambda^2\frac{\partial^2\tilde{v}}{\partial\theta^2} - \lambda^2\frac{\partial\tilde{w}}{\partial\theta} = 0 \quad \text{and} \tag{104}$$

$$\nu\lambda \frac{\partial\tilde{u}}{\partial\xi} + \lambda^2 \frac{\partial\tilde{v}}{\partial\theta} - \lambda^2\tilde{w} - \frac{\sigma^2}{12} \tilde{\nabla}^4\tilde{w} = (1-\nu^2)\, \tilde{P}_s(\xi,\theta,0) \tag{105}$$

Substituting the displacements into the above equilibrium equations, we find that

$$\tilde{U}_{on}^{(1)}(0) = \frac{2\, \nu\lambda(1-\nu^2)}{(n\pi)^2 \left[\lambda^2(1-\nu^2) + \frac{\sigma^2}{12} (n\pi)^4\right]} \left\{ \tilde{P}_o \left[1 - (-1)^n\right] (1-\frac{\nu}{2}) + \Gamma^{(1)} \left[1 - \frac{\sin (n\pi\zeta)}{n\pi\zeta}\right] + \nu\lambda\Gamma^{(5)} \right\} \quad \text{and} \tag{106}$$

$$\tilde{W}_{on}^{(1)}(0) = - \frac{2(1-\nu^2)}{(n\pi) \left[\lambda^2(1-\nu^2) + \frac{\sigma^2}{12} (n\pi)^4\right]} \left\{ \tilde{P}_o \left[1 - (-1)^n\right] (1-\frac{\nu}{2}) + \Gamma^{(1)} \left[1 - \frac{\sin (n\pi\zeta)}{n\pi\zeta}\right] + \nu\lambda\Gamma^{(5)} \right\} \quad n = 1, 2, \ldots N \tag{107}$$

7.0 STABILITY CRITERIA OF THE SHELL-FLUID SYSTEM

It can be shown from Appendix B that the form for the solutions to the equations (B-35) and (B-69) is

$$\overline{W}_{mn}^{(i)}(p) = \frac{N(p)}{D(p)} \tag{108}$$

where N(p) and D(p) are polynomials in p.

The stability criteria is determined by the roots of the polynomial D(p), and in order to have a stable solution, the following conditions must be met (ref 7):

a) Every unrepeated root must be nonpositive

b) Every repeated real root is negative

c) Every pure imaginary root is unrepeated

d) Every general complex root has negative real part.

Any root which does not satisfy the above conditions will imply that the Shell-Fluid System is basically unstable. The unstable roots will provide the necessary information for future parametric study.

8.0 CONCLUSIONS AND RECOMMENDATIONS

The dynamic responses of a circular cylindrical tank partially filled with an incompressible fluid and subjected to a periodically varying longitudinal thrust have been analyzed.

The method of separation of variables has been used to determine the pressure exerted by the vibrating fluid on the wall of the shell. Assumptions on the small displacements both on the shell elements and the free surface of the liquid are emphasized, so that the linear theory can be applied.

By assuming the displacements to be general double series, and using Galerkin's method to solve the shell equations, a system of ordinary differential equations in the time functions $\tilde{U}_{mn}^{(i)}(\tau)$, and $\tilde{V}_{mn}^{(i)}(\tau)$, and $\tilde{W}_{mn}^{(i)}(\tau)$ was obtained. It is proposed to solve this system of equations using Laplace transforms and matrix theory. Because extensive numerical work is involved, the numerical solutions of the time functions, the natural frequencies of the Shell-Fluid System, and the instability criteria can only be obtained through the aid of an electronic computer.

For future study of the Shell-Fluid System, the following investigations are recommended:

(1) To develop a numerical solution for the time functions $\tilde{U}_{mn}^{(i)}(\tau)$, $\tilde{V}_{mn}^{(i)}(\tau)$, and $\tilde{W}_{mn}^{(i)}(\tau)$ thus completing the solutions of the displacement functions $\tilde{u}$, $\tilde{v}$, and $\tilde{w}$.

(2) To perform a parametric study on the stability of the Shell-Fluid System.

(3) To investigate the Shell-Fluid System subjected to a gimbaled, periodically-varying end thrust.

9.0 REFERENCES

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APPENDIX A

Derivations of the Time Functions: $A(t)$, $B(t)$, $C(t)$, $D_{0j}(t)$, $E_{0k}(t)$,

$D_{mj}^{(1)}(t)$, $D_{mj}^{(2)}(t)$, $F_m^{(1)}(t)$, $F_m^{(2)}(t)$,

$E_{mk}^{(1)}(t)$, $E_{mk}^{(2)}(t)$

A-I $B(t)$

Substitute eq. (56) into eq. (38), one obtains

$$B(t) = \frac{T_o}{M}\left[t + \frac{\gamma}{\Omega}\sin(\Omega t)\right] \tag{A-1}$$

A-II $C(t)$, $E_{0k}(t)$, $E_{mk}^{(1)}(t)$, $E_{mk}^{(2)}(t)$, $F_m^{(1)}(t)$, $F_m^{(2)}(t)$

Substitute eq. (56) into eq. (39), one has for m = 0

$$\left(\frac{\pi}{2h_\ell}\right) I_0'\left(\frac{\pi}{2h_\ell}a\right) C(t) \cos\left(\frac{\pi}{2h_\ell}\eta\right) + \frac{1}{a} E_{0k}(t)\cos\left(\frac{k\pi\eta}{h_\ell}\right) = -\sum_{n=0}^{N} \dot{W}_{0n}^{(1)}(t)\, f_n(\eta) \tag{A-2}$$

and for m>0

$$F_m^{(i)}(t)\, m\, a^{m-1} + \sum_{k=1}^{\infty} \frac{1}{a} E_{mk}^{(i)}(t)\cos\left(\frac{k\pi}{h_\ell}\eta\right) = -\sum_{n=0}^{N} W_{mn}^{(i)}(t)\, f_n(\eta)$$

$$i = 1,\ 2 \tag{A-3}$$

Multiply eq. (A-2) by $\cos(\frac{s\pi\eta}{h_\ell})$ and integrate over the range of η from 0 to h_ℓ as follows:

$$\left(\frac{\pi}{2h_\ell}\right) I_0'\left(\frac{\pi}{2h_\ell}a\right) C(t) \int_0^{h_\ell} \cos\left(\frac{\pi}{2h_\ell}\eta\right) \cos\left(\frac{s\pi\eta}{h_\ell}\right) d\eta$$

$$+\frac{1}{a}E_{0k}^{(1)}(t)\int_0^{h_\ell}\cos\left(\frac{k\pi}{h_\ell}\eta\right)\cos\left(\frac{s\pi\eta}{h_\ell}\right)d\eta = -\sum_{n=0}^{N}\dot{W}_{0n}^{(1)}(t)\int_0^{h_\ell} f_n(\eta)\cos\left(\frac{s\pi\eta}{h_\ell}\right)d\eta \qquad \text{(A-4)}$$

for $s = 0$,

$$C(t) = \frac{-\sum_{n=0}^{N}\dot{W}_{0n}^{(1)}(t)\, I_{0n}}{I_{0n}'\left(\frac{\pi a}{2h_\ell}\right)} \qquad \text{(A-5)}$$

where

$$I_{0n} = \int_0^{h_\ell} f_n(\eta)\, d\eta \qquad \text{(A-6)}$$

for $s>0$

$$E_{0k}(t) = \left(-\frac{2a}{h_\ell}\right)\sum_{n=0}^{N}\dot{W}_{0n}^{(1)}\left[I_{kn} - I_{0n}\frac{(-1)^k}{1-4k^2}\right] \qquad \text{(A-7)}$$

where

$$I_{kn} = \int_0^{h_\ell} f_n(\eta)\cos\left(\frac{k\pi\eta}{h_\ell}\right)d\eta \qquad \text{(A-8)}$$

Multiply eq. (A-3) by $\cos\left(\frac{s\pi}{h_\ell}\eta\right)$ and integrate over the interval of η from 0 to h_ℓ as follows:

$$\frac{1}{a}E_{mk}^{(i)}(t)\left(\frac{h_\ell}{2}\right) = -\sum_{n=0}^{N}\dot{W}_{mn}^{(1)}(t)\int_0^{h_\ell} f_n(\eta)\cos\left(\frac{k\pi}{h_\ell}\eta\right)d\eta \qquad \text{(A-9)}$$

then,

$$E_{mk}^{(i)}(t) = -\left(\frac{2a}{h_\ell}\right)\sum_{n=0}^{N}\dot{W}_{mn}^{(1)}(t)\; I_{kn} \qquad \text{(A-10)}$$

$$i = 1, 2$$

Integrate eq. (A-3) over the interval of η from 0 to h_ℓ, the coefficient $F_m^{(i)}(t)$ is found as

$$F_m^{(i)}(t) = \left(- \frac{1}{m\, a^{m-1}\, h_\ell}\right) \sum_{n=0}^{N} \dot{W}_{mn}^{(1)} \; I_{0n}$$

$$i = 1,\ 2 \qquad \text{(A-11)}$$

A-III $\underline{A(t),\ D_{0j}(t)}$

Substitute eq. (56) into eq. (40) for m = 0, one obtains eq. (A-12) as follows:

$$\ddot{A}(t) + \ddot{B}(t)\, h_\ell + \sum_{j=1}^{\infty} J_0(\lambda_{0j}\frac{r}{a})\, \ddot{D}_{0j}(t) \cosh(\lambda_{0j}\frac{h_\ell}{a})$$

$$+ \sum_{k=1}^{\infty} \frac{I_0(\frac{k\pi r}{h_\ell})}{\left(\frac{k\pi a}{h_\ell}\right) I_0'\left(\frac{k\pi a}{h_\ell}\right)} \ddot{E}_{0k}(t)\, (-1)^k$$

$$+ g\left\{ B(t) - \left(\frac{\pi}{2h_\ell}\right) I_0\left(\frac{\pi r}{2h_\ell}\right) C(t) \right.$$

$$\left. + \sum_{j=1}^{\infty} \left(\frac{\lambda_{0j}}{a}\right) J_0(\lambda_{0j}\frac{r}{a})\, D_{0j}(t) \sinh(\lambda_{0j}\frac{h_\ell}{a}) \right\} + \ddot{XX} = 0 \qquad \text{(A-12)}$$

Multiply eq. (A-12) by $(\lambda_{0j}\frac{r}{a})$ and integrate over the interval of r from 0 to a, the result is

$$\frac{1}{2}\left[\ddot{A}(t) + \ddot{B}(t)h_\ell\right] - \sum_{k=1}^{\infty} \frac{2a}{h_\ell}\left(\frac{h_\ell}{k\pi a}\right)^2 (-1)^k \sum_{n=0}^{N} \dot{W}_{0n}^{(1)}(t) \left[I_{kn} - \frac{(-1)^k I_{0n}}{1-4k^2}\right]$$

$$+ g\left[\frac{B(t)}{2} + \frac{1}{a}\sum_{n=0}^{N} \dot{W}_{0n}^{(1)} \; I_{0n}\right] + \frac{\dddot{XX}}{2} = 0 \qquad \text{(A-13)}$$

Integrate eq. (A-13) twice with respect to t, the coefficient A(t) is obtained as

$$A(t) = (C_1 t) + C_2 - h_\ell\, B(t) + \sum_{k=1}^{\infty} \frac{4a}{h} \left(\frac{h_\ell}{k\pi a}\right)^2 (-1)^k \sum_{n=0}^{N} \dot{W}_{0n}^{(1)}(t) \left[I_{kn} - \frac{(-1)^k\, I_{0n}}{1-4k^2}\right]$$

$$- g \left(\frac{T_o}{M}\right) \left\{\frac{t^3}{6} - \frac{\gamma}{\Omega^3}\left[\sin(\Omega t) - \Omega t\right]\right\}$$

$$- \left(\frac{2g}{a}\right) \sum_{n=0}^{N} I_{0n} \int_o^t \dot{W}_{0n}^{(1)}(\beta)\, d\beta$$

$$- \left(\frac{T_o}{M}\right)^2 \left\{ \frac{t^3}{6} + \frac{\gamma}{\Omega^3}\left[\sin(\Omega t) - (\Omega t)\cos(\Omega t)\right] - \frac{\gamma^2}{8\Omega^3} \sin(2\Omega t)\right\} \qquad \text{(A-14)}$$

where C_1 and C_2 are constants to be determined by the initial conditions.

Multiply eq. (A-12) by $(\lambda_{0s} \frac{r}{a})\, J_0(\lambda_{0s} \frac{r}{a})$ and integrate over the interval of r from 0 to a, the differential equation which contains the coefficient $D_{0j}(t)$ can be expressed as follows:

$$\ddot{D}_{0j}(t) + \omega_{0j}^2\, D_{0j}(t) - \sum_{n=0}^{N} \left\{ \frac{2\dddot{W}_{0n}^{(1)}(t)}{J_0'(\lambda_{0j}) \cosh(\lambda_{0j} \frac{h_\ell}{a})} \sum_{k=1}^{\infty} (-1)^k \left(\frac{2a}{h_\ell}\right) \left[I_{kn} - \frac{(-1)^k\, I_{on}}{1-4k^2}\right] \right.$$

$$\left. \cdot \frac{1}{(\lambda_{0j})^2 + \left(\frac{k\pi a}{h}\right)^2} + \dot{W}_{0n}^{(1)}(t) \frac{2 I_{0n} \left(\frac{\pi}{2h_\ell}\right)^2 (ga)}{J_0(\lambda_{0j}) \cosh(\lambda_{0j} \frac{h_\ell}{a}) \left[(\lambda_{0j})^2 + \left(\frac{\pi a}{h_\ell}\right)^2\right]} \right\} = 0$$

(A-15)

where

$$(\omega_{0j})^2 = \frac{g\lambda_{0j}}{a} \tanh\left(\lambda_{0j} \frac{h_\ell}{a}\right) \qquad \text{(A-16)}$$

The solution of eq. (A-15) can be found as

$$D_{0j}(t) = A_{0j}^{(1)} \cos(\omega_{0j} t) + B_{0j}^{(1)} \sin(\omega_{0j} t)$$

$$+ \sum_{n=0}^{N} \int_{o}^{t} \dddot{W}_{0n}^{(1)}(\beta) \frac{\sin\left[\omega_{0j}(t-\beta)\right]}{\omega_{0j}} P_{kn}^{0j} \, d\beta$$

$$+ \sum_{n=0}^{N} \int_{o}^{t} \dot{W}_{0n}^{(1)}(\beta) \frac{\sin\left[\omega_{0j}(t-\beta)\right]}{\omega_{0j}} P_{0n}^{0j} \, d\beta \qquad \text{(A-17)}$$

where $A_{0j}^{(1)}$ and $B_{0j}^{(1)}$ are constants to be determined by the initial conditions, and

$$P_{kn}^{0j} = \frac{2}{J_0(\lambda_{0j}) \cosh(\lambda_{0j} \frac{h_\ell}{a})} \sum_{k=1}^{\infty} H_{kj}^{0} (-1)^k \left(\frac{2a}{h_\ell}\right) \left[I_{kn} - \frac{(-1)^k I_{0n}}{1-4k^2} \right] \qquad \text{(A-18)}$$

$$P_{0n}^{0j} = \frac{2}{J_0(\lambda_{0j}) \cosh(\lambda_{0j} \frac{h_\ell}{a})} \frac{ga \left(\frac{\pi}{2h_\ell}\right)^2}{\left(\frac{\pi a}{2h_\ell}\right)^2 + (\lambda_{0j})^2} I_{0n} \qquad \text{(A-19)}$$

$$H_{kj}^{0} = \frac{1}{(\lambda_{0j})^2 + \left(\frac{k\pi a}{h_\ell}\right)^2} \qquad \text{(A-20)}$$

A-IV $\underline{D_{mj}^{(1)}(t),\ D_{mj}^{(2)}(t)}$

Substituting eq. (56) into eq. (40) and grouping terms for m>0 yields the following differential equation,

$$\ddot{F}_{m}^{(i)}(t)\, r^m + \sum_{j=m}^{\infty} J_m(\lambda_{mj} \frac{r}{a}) \ddot{D}_{mj}^{(i)}(t) \cos(\lambda_{mj} \frac{h_\ell}{a})$$

$$+ \sum_{k=1}^{\infty} \frac{I_m(\frac{k\pi r}{h_\ell})}{\left(\frac{k\pi a}{h_\ell}\right) I'_m\left(\frac{k\pi a}{h_\ell}\right)} \ddot{E}_{mk}^{(i)}(t) \, (-1)^k$$

$$+ g\left[\sum_{j=m}^{\infty} J_m(\lambda_{mj} \frac{r}{a}) \, D_{mj}^{(i)}(t) \, (\frac{\lambda_{mj}}{a}) \sinh(\lambda_{mj} \frac{h_\ell}{a})\right] = 0$$

$$i = 1, 2 \qquad \text{(A-21)}$$

Multiply eq. (A-21) by $(\lambda_{ms} \frac{r}{a}) \, J_m \, (\lambda_{ms} \frac{r}{a})$ and integrate over the interval of r from 0 to a, eq. (A-22) is obtained .

$$\ddot{D}_{mj}^{(i)}(t) + \omega_{mj}^2 \, D_{mj}^{(i)}(t) - \frac{2(\lambda_{mj})^2}{J_m^{\,2}(\lambda_{mj}) \left[(\lambda_{mj})^2 - m^2\right] \cosh(\lambda_{mj} \frac{h_\ell}{a})}$$

$$\cdot \sum_{n=1}^{N} \dddot{W}_{mn}^{(i)}(t) \left\{ \frac{(\lambda_{mj})^2 \, J_{m+1}(\lambda_{mj}) \, a}{m h_\ell \, (\lambda_{mj})} \, I_{0n} \right.$$

$$+ \sum_{k=1}^{\infty} (-1)^k \, (\frac{2a}{h_\ell}) \frac{I_{kn}}{\left(\frac{k\pi a}{h_\ell}\right) I'_m \left(\frac{k\pi a}{h_\ell}\right) \left[\left(\frac{\lambda_{mj}}{a}\right)^2 + \left(\frac{k\pi}{h_\ell}\right)^2\right]}$$

$$\left. \cdot \left[\left(\frac{k\pi a}{h_\ell}\right) J_m(\lambda_{mj}) \, I_{m+1}\left(\frac{k\pi a}{h_\ell}\right) + (\lambda_{mj}) \, I_m\left(\frac{k\pi a}{h_\ell}\right) J_{m+1}(\lambda_{mj})\right] \right\} = 0$$

$$i = 1, 2 \qquad \text{(A-22)}$$

The solution of eq. (A-22) can be found as

$$D_{mj}^{(i)}(t) = A_{mj}^{(i)} \cos(\omega_{mj} t) + B_{mj}^{(i)} \sin(\omega_{mj} t)$$

$$+ \sum_{n=0}^{N} \int_{o}^{t} \dddot{W}_{mn}^{(1)}(\beta) \, \frac{\sin\left[\omega_{mj}(t-\beta)\right]}{\omega_{mj}} \, d\beta \, \left[P_{kn}^{mj} + P_{0n}^{mj}\right]$$

$$i = 1, 2 \qquad \text{(A-23)}$$

Where $A_{mj}^{(i)}$ and $B_{mj}^{(i)}$ are constants to be determined by the initial conditions, and

$$(\omega_{mj})^2 = \frac{g\lambda_{mj}}{a} \tanh \left(\lambda_{mj} \frac{h_\ell}{a}\right) \tag{A-24}$$

$$P_{kn}^{mj} = \frac{2(\lambda_{mj})^2}{\left[(\lambda_{mj})^2 - m^2\right] J_m{}^2(\lambda_{mj}) \cosh\left(\lambda_{mj} \frac{h_\ell}{a}\right)} \sum_{k=1}^{\infty} H_{kj}^m (-1)^k \left(\frac{2a}{h_\ell}\right) I_{kn} \tag{A-25}$$

$$P_{0n}^{mj} = \frac{2(\lambda_{mj})^2}{\left[(\lambda_{mj})^2 - m^2\right] J_m{}^2(\lambda_{mj}) \cosh\left(\lambda_{mj} \frac{h_\ell}{a}\right)} \frac{a \; J_{m+1}(\lambda_{mj}) \, I_{0n}}{mh \; \lambda_{mj}} \tag{A-26}$$

APPENDIX B

Method of Solution to the Differential Equations (24) through (35)

B-I Solutions to Equations (24), (26), (27) and (28)

Integrating twice gives the solution to eq. (24)

$$\tilde{U}_{00}^{(1)}(\tau) = \frac{1-\nu^2}{3}\left\{ \_\!\!\lceil(\tau)\ \Gamma^{(4)}\ [1 + \gamma\cos(\omega\tau)] + \Gamma^{(5)} \right\} \quad \text{(B-1)}$$

Solving eqs. (26), (27) and (28) by Laplace gives

$$\tilde{U}_{m0}^{(1)}(\tau) = \tilde{U}_{m0}^{(1)}(0)\cos\left[\frac{m\lambda}{\sqrt{2(1+\nu)\mu}}\ \tau\right] + \dot{\tilde{U}}_{m0}^{(1)}(0)\ \sin\left[\frac{m\lambda}{\sqrt{2(1+\nu)\mu}}\ \tau\right] \quad \text{(B-2a)}$$

$$\tilde{U}_{m0}^{(2)}(\tau) = \tilde{U}_{m0}^{(1)}(0)\ \cos\left[\frac{m\lambda}{\sqrt{2(1+\nu)\mu}}\ \tau\right] + \dot{\tilde{U}}_{m0}^{(2)}(0)\ \sin\left[\frac{m\lambda}{\sqrt{2(1+\nu)\mu}}\ \tau\right] \quad \text{(B-2b)}$$

$$\tilde{V}_{0n}^{(2)}(\tau) = \tilde{V}_{0n}^{(2)}(0)\ \cos\left[\frac{n\pi}{\sqrt{2(1+\nu)\mu}}\right]\tau + \dot{\tilde{V}}_{0n}^{(2)}(0)\ \sin\left[\frac{n\pi}{\sqrt{2(1+\nu)\mu}}\ \tau\right] \quad \text{(B-2c)}$$

B-II Solutions to Equations (25) and (29)*

Let

$$\beta_{11} = \frac{(s\pi)^2}{(1-\nu^2)\mu}$$

$$\beta_{12} = \frac{s\pi\nu\lambda}{(1-\nu^2)\mu}$$

$$\beta_{13} = -\Gamma^{(4)}\ \frac{2(1-\nu^2)}{(s\pi)^2} \quad \text{(B-3)}$$

$$\beta_{22} = \frac{1}{(1-\nu^2)\mu}\quad \left[\lambda^2 + \frac{\sigma^2}{12}(s\pi)^4\right]$$

Then eqs. (25) and (29)* become

$$\ddot{\tilde{U}}_{0s}^{(1)}(\tau) + \beta_{11}\ \tilde{U}_{0s}^{(1)}(\tau) + \beta_{12}\tilde{W}_{0s}^{(1)}(\tau) = \beta_{13}\ \frac{d^2}{d\tau^2}\left\{ \_\!\!\lceil(\tau)\ [1 + \gamma\ \cos(\omega\tau)] \right\} \quad \text{(B-4)}$$

$$\ddot{\tilde{W}}_{0s}^{(1)}(\tau) + \beta_{12}\, \tilde{U}_{0s}^{(1)}(\tau) + \beta_{22}\ddot{\tilde{W}}_{0s}(\tau) = -\frac{1}{\mu}\ \tilde{P}_{0s} \qquad \text{(B-5)}$$

The Laplace transform of eq. (B-4) with respect to τ is

$$p^2\ \bar{U}_{0s}^{(1)}(p) - p\ \tilde{U}_{0s}^{(1)}(0) - \dot{\tilde{U}}_{0s}^{(1)}(0) + \beta_{11}\bar{U}_{0s}^{(1)}(p) + \beta_{12}\bar{W}_{0s}^{(1)} = \beta_{13}\left[p + \frac{rp^3}{p^2+\omega^2}\right]$$

or

$$\bar{U}_{0s}^{(1)}(p) = \frac{\left[\beta_{13}(1+\gamma)+\tilde{U}_{0s}^{(1)}(0)\right]p^3 + \dot{\tilde{U}}_{0s}^{(1)}(0)\ p^2 + \omega^2\left[\beta_{13}+\tilde{U}_{0s}^{(1)}(0)\right]p + \omega^2\dot{\tilde{U}}_{0s}^{(1)}(0)}{(p^2+\omega^2)(p^2+\beta_{11})}$$

$$- \frac{\beta_{12}\ \bar{W}_{0s}^{(1)}(p)}{p^2 + \beta_{11}} \qquad \text{(B-6)}$$

The Laplace transform of eq. (B-5) with respect to τ is

$$(p^2+\beta_{22})\bar{W}_{0s}^{(1)}(p) = -\frac{1}{\mu}\bar{P}_{0s}(p) + p\tilde{W}_{0s}^{(1)}(0) + \dot{\tilde{W}}_{0s}^{(1)}(0) - \beta_{12}\bar{U}_{0s}^{(1)}(p) \qquad \text{(B-7)}$$

where $\bar{P}_{0s}(p)$ is the Laplace transform $\tilde{P}_{0s}(\tau)$.

Substitution of eq. (B-6) into eq. (B-7) gives (The naming of the functions and coefficients of F_{02}, G_{31}, etc., is for the convenience of computer programming)

$$F_{02}\bar{W}_{0s}^{(1)}(p) = -\bar{P}_{0s}(p) + \mu\left[p\ \tilde{W}_{0s}^{(1)}(0) + \dot{\tilde{W}}_{0s}^{(1)}(0)\right] + F_{07} \qquad \text{(B-8)}$$

where

$$F_{02} = F_{02}\ (p,s) = \mu\left[p^2+\beta_{12} - \frac{\beta_{12}^2}{p^2+\beta_{11}}\right] \qquad \text{(B-9)}$$

$$F_{07} = F_{07}(p,s)$$

$$= -\mu\beta_{12}\frac{\left[\beta_{13}(1+\gamma)+\tilde{U}_{0s}^{(1)}(0)\right]p^3+\dot{\tilde{U}}_{0s}^{(1)}(0)\ p^2 + \omega^2\left[\beta_{13}+\tilde{U}_{0s}^{(1)}(0)\right]p+\omega^2\dot{\tilde{U}}_{0s}^{(1)}(0)}{(p^2+\omega^2)\ (p^2+\beta_{11})}$$

(B-10)

The Laplace transform of eq. (93)* with respect to τ is

$$\bar{P}_{0s}(p) = (G_{037}) \frac{1}{p} + (G_{038}) \frac{p}{p^2+\omega^2}$$

$$+ \sum_{n=1}^{N} \left[F_{03} \bar{W}_{0n}^{(1)}(p) + F_{04} \tilde{W}_{0n}^{(1)}(0) + F_{05} \dot{W}_{0n}^{(1)}(0) + F_{06} \ddot{W}_{0n}^{(1)}(0) \right] \tag{B-11}$$

where, for the convenience of computer programming, the first zero of $J_m'(\lambda_{mj})$ is assumed to be the first non-zero root of $J_m'(\lambda_{mj}) = 0$, therefore index j begins at i.

$$F_{03} = F_{03}(p, n, s, j) = (F_{03p2})\, p^2 + (F_{03p0})$$

$$+ \sum_{j=1}^{\infty} \frac{(F_{03sp4})\, p^4 + (F_{03sp2})\, p^2 + (F_{03sp0})}{p^2 + (\tilde{\omega}_{0j})^2} \tag{B-12}$$

$$F_{04} = F_{04}(p,n,s,j) = -(F_{03p2})\, p - (F_{03p0}) \frac{1}{p}$$

$$- \sum_{j=1}^{\infty} \frac{(F_{03sp4})\, p^3 + (F_{03sp2})\, p + (F_{03sp0})}{p^2 + (\tilde{\omega}_{0j})^2} \tag{B-13}$$

$$F_{05} = F_{05}(p,n,s,j)$$

$$= -(F_{03p2}) - \sum_{j=1}^{\infty} \frac{(F_{03sp4})\, p^2 + (F_{03sp0})}{p^2 + (\tilde{\omega}_{0j})^2} \tag{B-14}$$

$$F_{06} = F_{06}(p,n,s,j)$$

$$= \sum_{j=1}^{\infty} \left[\frac{G_{034j}\, \tilde{P}_{kn}^{0j}}{(\tilde{\omega}_{0j})^2} - F_{03sp4} \right] \frac{p}{p^2+(\tilde{\omega}_{0j})^2} \tag{B-15}$$

$$G_{037} = G_{037}(s) = \frac{2\tilde{P}_0}{s\pi}\left[1 - (-1)^s\right] + \frac{2\Gamma^{(1)}}{s\pi}\left[\zeta - \frac{1}{s\pi}\sin(s\pi\zeta)\right]$$

$$- 2\Gamma^{(2)}\left[\frac{1}{(s\pi)^2}\sin(s\pi\zeta) - \frac{\zeta}{s\pi}\right] \qquad \text{(B-16)}$$

$$G_{038} = G_{038}(s) = - 2\Gamma^{(2)}\gamma \left[\frac{1}{(s\pi)^2}\sin(s\pi\zeta) - \frac{\zeta}{s\pi}\right] \qquad \text{(B-17)}$$

$$F_{03p2} = F_{03p2}(n,s) = G_{032} - G_{036} \qquad \text{(B-18)}$$

$$F_{03p0} = F_{03p0}(n,s) = - G_{031} + G_{035} + G_{039} \qquad \text{(B-19)}$$

$$F_{03sp4} = F_{03sp4}(n,s,j) = G_{033j} \cdot \tilde{P}^{0j}_{kn} \qquad \text{(B-20)}$$

$$F_{03sp2} = F_{03sp2}(n,s,j) = G_{033j} \cdot \tilde{P}^{0j}_{0n} + G_{034j} \cdot \tilde{P}^{0j}_{kn} \qquad \text{(B-21)}$$

$$F_{03sp0} = F_{03sp0}(n,s,j) = G_{034j} \cdot \tilde{P}^{0j}_{0n} \qquad \text{(B-22)}$$

$$G_{031} = G_{031}(n,s) = - \frac{4\lambda}{s\pi}\Gamma^{(1)}\left[1 - \cos(s\pi\zeta)\right]\tilde{I}_{0n} \qquad \text{(B-23)}$$

$$G_{032} = G_{032}(n,s) = 2\Gamma^{(3)}\left\{\frac{I_0\left(\frac{\pi}{2\alpha}\right)\tilde{I}_{0n}}{I_1\left(\frac{\pi}{2\alpha}\right)\left[(s\pi)^2 - \left(\frac{\pi}{2\zeta}\right)^2\right]}\left[s\pi - \frac{\pi}{2\zeta}\sin(s\pi\zeta)\right]\right.$$

$$+ \frac{2}{\alpha}\sum_{k=1}^{\infty}\frac{s\pi\, I_0\left(\frac{k\pi}{\alpha}\right)}{\left(\frac{k\pi}{\alpha}\right)I_1\left(\frac{k\pi}{\alpha}\right)\left[(s\pi)^2 - \left(\frac{k\pi}{\zeta}\right)^2\right]}\left[1 - (-1)^k\cos(s\pi\zeta)\right]$$

$$\left.\left[\tilde{I}_{kn} - \frac{(-1)^k\,\tilde{I}_{0n}}{1 - 4k^2}\right]\right\} \qquad \text{(B-24)}$$

$$G_{035} = G_{035}(n,s)$$

$$= - \frac{4\Gamma^{(1)}}{\alpha}\sum_{k=1}^{\infty}\frac{I_0\left(\frac{k\pi}{\alpha}\right)\left(\frac{k\pi}{\zeta}\right)^2\sin(s\pi\zeta)}{\frac{k\pi}{\alpha}\,I_1\left(\frac{k\pi}{\alpha}\right)\left[(s\pi)^2 - \left(\frac{k\pi}{\zeta}\right)^2\right]}\left[(-1)^k\,\tilde{I}_{kn} - \frac{\tilde{I}_{0n}}{1-4k^2}\right] \qquad \text{(B-25)}$$

$$G_{036} = G_{036}(n,s) = \frac{2}{s\pi} \left[1 - \cos(s\pi\zeta)\right] \cdot \Gamma^{(3)} \tilde{P}^{*}_{kn} \tag{B-26}$$

$$G_{039} = G_{039}(n,s) = \Gamma^{(1)} \frac{I_0 \frac{\pi}{2\alpha} (s\pi)^2 \cos(s\pi\zeta)}{\zeta I_1 \left(\frac{\pi}{2\alpha}\right) \left[(s\pi)^2 - \left(\frac{\pi}{2\zeta}\right)^2\right]} \tilde{I}_{0n} \tag{B-27}$$

$$G_{033j} = G_{033j}(n,s,j) = -2\,\Gamma^{(3)} \frac{J_0(\lambda_{0j})}{(\lambda\lambda_{0j})^2 + (s\pi)^2}$$

$$\cdot \left[s\pi - s\pi \cos(s\pi\zeta) \cosh(\lambda_{0j}\alpha) + \lambda\lambda_{0j} \sin(s\pi\zeta) \sinh(\lambda_{0j}\alpha)\right] \tag{B-28}$$

$$G_{034j} = G_{034j}(n,s,j) = 2\,\lambda\Gamma^{(1)} \frac{\lambda_{0j} J_0(\lambda_{0j})}{(\lambda\lambda_{0j})^2 + (s\pi)^2}$$

$$\cdot \left[s\pi \cos(s\pi\zeta) \sinh(\lambda_{0j}\alpha) - \lambda\lambda_{0j} \sin(s\pi\zeta) \cosh(\lambda_{0j}\alpha)\right] \tag{B-29}$$

The coefficients $\tilde{I}_{0n}$, $\tilde{I}_{kn}$, $\tilde{P}^{*}_{kn}$, $\tilde{P}^{0j}_{kn}$, and $\tilde{P}^{0j}_{0n}$ are the dimensionless forms of I_{0n}, I_{kn}, P^{0j}_{kn} and P^{0j}_{0n} respectively.

When $f_n(\eta)$ is taken to be $\sin(\frac{n\pi}{L}\eta)$, eq. (67) gives

$$\tilde{I}_{0n} = \frac{1 - \cos(n\pi\zeta)}{n\pi} \tag{B-30}$$

Eq. (68) gives

$$\tilde{I}_{kn} = \frac{n}{\left[n^2 - \left(\frac{k}{\zeta}\right)^2\right]\pi} \left[1 - (-1)^k \cos(n\pi\zeta)\right] \tag{B-31}$$

Eq. (71) gives

$$\tilde{P}^{*}_{kn} = \frac{4\alpha}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \left[(-1)^k \tilde{I}_{kn} - \frac{\tilde{I}_{0n}}{1 - 4k^2}\right] \tag{B-32}$$

Eq. (73) gives

$$\tilde{P}^{0j}_{kn} = \frac{4\alpha}{\pi^2 J_0(\lambda_{0j}) \cosh(\lambda_{0j}\alpha)} \sum_{k=1}^{\infty} \frac{1}{k^2} \left[(-1)^k \tilde{I}_{kn} - \frac{\tilde{I}_{0n}}{1 - 4k^2}\right] \tag{B-33}$$

Eq. (75) gives

$$\tilde{P}_{0n}^{0j} = \frac{2\Gamma^{(6)}}{J_0(\lambda_{0j}) \cosh (\lambda_{0j}\alpha)} \cdot \frac{\left(\frac{\pi}{2\alpha}\right)^2}{\left(\frac{\pi}{2\alpha}\right)^2 + \left(\lambda_{0j}\right)^2} \tilde{I}_{0n} \qquad \text{(B-34)}$$

Substitution of eq. (B-11) into eq. (B-8) gives

$$\sum_{n=1}^{N} \left[F_{03} + \delta_{ns} F_{02} \right] \bar{W}_{0n}^{(1)}(p)$$

$$= - \sum_{n=1}^{N} \left[F_{04} \tilde{W}_{0n}^{(1)}(0) + F_{05} \dot{\tilde{W}}_{0n}^{(1)}(0) + F_{06} \ddot{\tilde{W}}_{0n}^{(1)}(0) \right]$$

$$+ \mu \left[p \tilde{W}_{0s}^{(1)}(0) + \dot{\tilde{W}}_{0s}^{(1)}(0) \right] + F_{07} - G_{037} \frac{1}{p} - G_{038} \frac{p}{p^2 + \omega^2}$$

$$s = 1, 2, \ldots N \qquad \text{(B-35)}$$

This is a system of N equations to be solved for the N unknowns $\bar{W}_{0n}^{(1)}(p)$, n = 1, 2, ... N. Taking the inverse transform of $\bar{U}_{0n}^{(1)}(p)$ and $\bar{W}_{0n}^{(1)}(p)$ gives $\tilde{U}_{0n}^{(1)}(\tau)$ and $\tilde{W}_{0n}^{(1)}(\tau)$.

B-III <u>Solution of Equations (30), (31), (32)*, and (33), (34), (35)*</u>

Equations (30), (31), (32)* and (33), (34), (35)* are of the following forms

$$\ddot{\tilde{U}}^{(i)}_{ms}(\tau) + \alpha_{11}\tilde{U}^{(i)}_{ms}(\tau) + \alpha_{12}\tilde{V}^{(i)}_{ms}(\tau) + \alpha_{13}\tilde{W}^{(i)}_{ms}(\tau) = 0 \qquad \text{(B-36)}$$

$$\ddot{\tilde{V}}^{(i)}_{ms}(\tau) + \alpha_{12}\tilde{U}^{(i)}_{ms}(\tau) + \alpha_{22}\tilde{V}^{(i)}_{ms}(\tau) + \alpha_{23}\tilde{W}^{(i)}_{ms}(\tau) = 0 \qquad \text{(B-37)}$$

$$\ddot{\tilde{W}}^{(i)}_{ms}(\tau) + \alpha_{13}\tilde{U}^{(i)}_{ms}(\tau) + \alpha_{23}\tilde{V}^{(i)}_{ms}(\tau) + \alpha_{33}\tilde{W}^{(i)}_{ms}(\tau) = -\frac{1}{\mu}\tilde{P}_{ms} \qquad \text{(B-38)}$$

where for eqs. (30), (31), (32)*

$$i = 1$$

$$\alpha_{11} = \frac{1}{(1-\nu^2)\mu}\left[(s\pi)^2 + \frac{(m\lambda)^2(1-\nu)}{2}\right]$$

$$\alpha_{12} = -\frac{ms\pi\lambda}{2(1-\nu)\mu}$$

$$\alpha_{13} = \frac{s\pi\nu\lambda}{(1-\nu^2)\mu} \qquad \text{(B-39)}$$

$$\alpha_{22} = \frac{1}{(1-\nu^2)\mu}\left[(m\lambda)^2 + \frac{(s\lambda)^2(1-\nu)}{2}\right]$$

$$\alpha_{23} = -\frac{m\lambda^2}{(1-\nu^2)\mu}$$

$$\alpha_{33} = \frac{1}{(1-\nu^2)\mu}\left\{\lambda^2 + \frac{\sigma^2}{12}\left[(s\pi)^2 + (m\lambda)^2\right]^2\right\}$$

For equations (33), (34), (35)*

$$i = 2$$

$$\alpha_{11} = \frac{1}{(1-\nu^2)\mu} \left[(s\pi)^2 + \frac{(m\lambda)^2(1-\nu)}{2} \right]$$

$$\alpha_{12} = \frac{ms\pi\lambda}{2(1-\nu)\mu}$$

$$\alpha_{13} = \frac{s\pi\nu\lambda}{(1-\nu^2)\mu}$$

$$\alpha_{22} = \frac{1}{(1-\nu^2)\mu} \left[(m\lambda)^2 + \frac{(s\pi)^2(1-\nu)}{2} \right] \qquad \text{(B-40)}$$

$$\alpha_{23} = \frac{m\lambda^2}{(1-\nu^2)\mu}$$

$$\alpha_{33} = \frac{1}{(1-\nu^2)\mu} \left\{ \lambda^2 + \frac{\sigma^2}{12} \left[(s\pi)^2 + (m\lambda)^2 \right]^2 \right\}$$

The Laplace transforms of eqs. (B-36), (B-37) and (B-38) with respect to τ are:

$$(p^2 + \alpha_{11}) \, \bar{U}_{ms}^{(i)}(p) + \alpha_{12} \bar{V}_{ms}^{(i)}(p) = - \alpha_{13} \, \bar{W}_{ms}^{(i)}(p) + p \, \tilde{U}_{ms}^{(i)}(0) + \dot{\tilde{U}}_{ms}^{(i)}(0) \qquad \text{(B-41)}$$

$$\alpha_{12} \bar{U}_{ms}^{(i)}(p) + (p^2 + \alpha_{22}) \, \bar{V}_{ms}^{(i)}(p) = - \alpha_{23} \bar{W}_{ms}^{(i)}(p) + p \tilde{V}_{ms}^{(i)}(0) + \dot{\tilde{V}}_{ms}^{(i)}(0) \qquad \text{(B-42)}$$

$$(p^2 + \alpha_{33}) \, \bar{W}_{ms}^{(i)}(p) + \alpha_{13} \bar{U}_{ms}^{(i)}(p) + \alpha_{23} \bar{V}_{ms}^{(i)}(p)$$

$$= - \frac{1}{\mu} \, \bar{P}_{ms}^{(i)}(p) + p \, \tilde{W}_{ms}^{(i)}(0) + \dot{\tilde{W}}_{ms}^{(i)}(0) \qquad \text{(B-43)}$$

where $\bar{P}_{ms}^{(i)}(p)$ is the Laplace transform of $\tilde{P}_{ms}^{(i)}(\tau)$.

Solving eqs. (B-41) and (B-42) simultaneously for $\bar{U}_{ms}^{(i)}(p)$ and $\bar{V}_{ms}^{(i)}(p)$ gives

$$\bar{U}^{(i)}_{ms}(p) = \frac{N_5(p) + N_6(p)\,\bar{W}^{(i)}_{ms}(p)}{D_3(p)} \tag{B-44}$$

$$\bar{V}^{(i)}_{ms}(p) = \frac{N_7(p) + N_8(p)\,\bar{W}^{(i)}_{ms}(p)}{D_3(p)} \tag{B-45}$$

where

$$N_5(p) = \tilde{U}^{(i)}_{ms}(0)\,p^3 + \dot{\tilde{U}}^{(i)}_{ms}(0)\,p^2 + \left[\alpha_{22}\tilde{U}^{(i)}_{ms}(0) - \alpha_{12}\tilde{V}^{(i)}_{ms}(0)\right] \tag{B-46}$$

$$N_6(p) = -\,\alpha_{13}\,p^2 + \alpha_{12}\alpha_{23} - \alpha_{22}\alpha_{13} \tag{B-47}$$

$$N_7(p) = \tilde{V}^{(i)}_{ms}(0)\,p^3 + \dot{\tilde{V}}^{(i)}_{ms}(0)\,p^2 + \left[\alpha_{11}\tilde{V}^{(i)}_{ms}(0) - \alpha_{12}\tilde{U}^{(i)}_{ms}(0)\right] \tag{B-48}$$

$$N_8(p) = -\,\alpha_{23}\,p^2 + \alpha_{12}\alpha_{13} - \alpha_{11}\alpha_{23} \tag{B-49}$$

$$D_3(p) = p^4 + (\alpha_{11}+\alpha_{22})\,p^2 + (\alpha_{11}\alpha_{22} - \alpha_{12}^2) \tag{B-50}$$

Substitution of eqs. (B-44) and (B-45) into eq. (B-43) gives

$$F_2(p)\,\bar{W}^{(i)}_{ms}(p) = -\,\bar{P}^{(i)}_{ms} + \mu\left[p\tilde{W}^{(i)}_{ms}(0) + \dot{\tilde{W}}^{(i)}_{ms}(0)\right] + F_7(p) \tag{B-51}$$

where

$$F_2(p) = \mu\left[\frac{\alpha_{13}\,N_6(p) + \alpha_{23}\,N_8(p)}{D_3(p)} + p^2 + \alpha_{23}\right] \tag{B-52}$$

$$F_7(p) = -\,\mu\,\frac{\alpha_{13}\,N_5(p) + \alpha_{23}\,N_7(p)}{D_3(p)} \tag{B-53}$$

The Laplace transform of eq. (94)* with respect to τ is

$$\bar{P}^{(i)}_{ms}(p) = \sum_{n=1}^{N}\left[F_3\bar{W}^{(i)}_{mn}(p) + F_4\tilde{W}^{(i)}_{mn}(0) + F_5\dot{\tilde{W}}^{(i)}_{mn}(0) + F_6\ddot{\tilde{W}}^{(i)}_{mn}(0)\right] \tag{B-54}$$

where

$$F_3 = F_3(p,m,n,s,j) = (G_{31}+G_{32})p^2 + G_{35}$$

$$+ p^4 \sum_{j=1}^{\infty} \frac{G_{33j}}{p^2 + (\tilde{\omega}_{mj})^2} + p^2 \sum_{j=1}^{\infty} \frac{G_{34j}(\tilde{\omega}_{mj})^2}{p^2 + (\tilde{\omega}_{mj})^2} \quad \text{(B-55)}$$

$$F_4 = F_4(p,m,n,s,j) = -(G_{31}+G_{32})p - G_{35}$$

$$- p^3 \sum_{j=1}^{\infty} \frac{G_{33j}}{p^2 + (\tilde{\omega}_{mj})^2} + p \sum_{j=1}^{\infty} \frac{G_{34j}}{p^2 + (\tilde{\omega}_{mj})^2} \quad \text{(B-56)}$$

$$F_5 = F_5(p,m,n,s,j) = -(G_{31}+G_{32})$$

$$- p^2 \sum_{j=1}^{\infty} \frac{G_{33j}}{p^2 + (\tilde{\omega}_{mj})^2} - \sum_{j=1}^{\infty} \frac{G_{34j}}{p^2 + (\tilde{\omega}_{mj})^2} \quad \text{(B-57)}$$

$$F_6 = F_6(p,m,n,s,j)$$

$$= p \sum_{j=1}^{\infty} \left\{ - \frac{G_{33j}}{p^2 + (\tilde{\omega}_{mj})^2} + \frac{G_{34j}}{\left[p^2 + (\tilde{\omega}_{mj})^2\right] (\tilde{\omega}_{mj})^2} \right\} \quad \text{(B-58)}$$

$$G_{31} = G_{31}(m,n,s) = \frac{2\Gamma^{(3)}}{m\alpha s\pi} \left[1 - \cos(s\pi\zeta)\right] \tilde{I}_{0n} \quad \text{(B-59)}$$

$$G_{32} = G_{32}(m,n,s)$$

$$= 4\pi s\Gamma^{(3)} \sum_{k=1}^{\infty} \frac{I_m\left(\frac{k\pi}{\alpha}\right)}{\frac{k\pi}{\alpha} I_m'\left(\frac{k\pi}{\alpha}\right)} \frac{\left[1 - (-1)^k \cos(s\pi\zeta)\right]}{(s\pi)^2 - \left(\frac{k\pi}{\zeta}\right)^2} \tilde{I}_{kn} \quad \text{(B-60)}$$

$$G_{35} = G_{35}(m,n,s)$$

$$= -\frac{4\Gamma^{(1)}}{\alpha} \sum_{k=1}^{\infty} (-1)^k \frac{I_m\left(\frac{k\pi}{\alpha}\right) \cdot \left(\frac{k\pi}{\zeta}\right)^2 \sin(s\pi\zeta)}{\frac{k\pi}{\alpha} I_m'\left(\frac{k\pi}{\alpha}\right)\left[(s\pi)^2 - \left(\frac{k\pi}{\zeta}\right)^2\right]} \tilde{I}_{kn} \quad \text{(B-61)}$$

$$G_{33j} = G_{33j}(m,n,s,j) = -\,2\Gamma^{(3)}\,\frac{J_m(\lambda_{mj})}{(\lambda\lambda_{mj})^2+(s\pi)^2}\left[\tilde{P}^{mj}_{kn} + \tilde{P}^{mj}_{0n}\right]$$

$$\cdot[s\pi - s\pi\cos(s\pi\zeta)\cosh(\lambda_{mj}\alpha) + \lambda\lambda_{mj}\sin(s\pi\zeta)\sinh(\lambda_{mj}\alpha)] \qquad \text{(B-62)}$$

$$G_{34j} = G_{34j}(m,n,s,j) = -\,2\lambda\Gamma^{(1)}\,\frac{\lambda_{mj}\,J_m(\lambda_{mj})}{(\lambda\lambda_{mj})^2+(s\pi)^2}\left[\tilde{P}^{mj}_{kn} + \tilde{P}^{mj}_{0n}\right]$$

$$\cdot\,[\lambda\lambda_{mj}\sin(s\pi\zeta)\cosh(\lambda_{mj}\alpha) - s\pi\cos(s\pi\zeta)\sinh(\lambda_{mj}\alpha)] \qquad \text{(B-63)}$$

The coefficients $\tilde{I}_{kn}$, $\tilde{H}^{m}_{jk}$, $\tilde{P}^{mj}_{kn}$ and $\tilde{P}^{mj}_{0n}$ are the dimensionless forms of I_{kn}, H^{m}_{jk}, P^{mj}_{kn} and P^{mj}_{0n} respectively:

When $f_n(\eta) = \sin\left(\frac{n}{L}\eta\right)$, eq. (68) gives

$$\tilde{I}_{kn} = \frac{n}{\left[n^2-\left(\frac{k}{3}\right)^2\right]\pi}\;[1 - (-1)^k\cos(n\pi\zeta)] \qquad \text{(B-64)}$$

Eq. (69) gives

$$\tilde{H}^{m}_{jk} = \frac{1}{\frac{k\pi}{\alpha}\,I_m\left(\frac{k\pi}{\alpha}\right)\left[\left(\frac{k\pi}{\alpha}\right)^2 + (\lambda_{mj})^2\right]}$$

$$\cdot\left[\frac{k\pi}{\alpha}J_m(\lambda_{mj})\,I_{m+1}\left(\frac{k\pi}{\alpha}\right) + \lambda_{mj}I_m\left(\frac{k\pi}{\alpha}\right)J_{m+1}(\lambda_{mj})\right] \qquad \text{(B-65)}$$

Eq. (72) gives

$$\tilde{P}^{mj}_{kn} = \frac{4n(\lambda_{mj})^2}{\pi\alpha\left[(\lambda_{mj})^2 - m^2\right]J^2_m(\lambda_{mj})\cosh(\lambda_{mj}\alpha)}$$

$$\cdot\sum_{k=1}^{\infty}\frac{\tilde{H}^{m}_{kj}}{\left[n^2 - \left(\frac{k}{\zeta}\right)^2\right]}\left[(-1)^k - \cos(n\pi\zeta)\right] \qquad \text{(B-66)}$$

Eq. (74) gives

$$\tilde{P}_{On}^{mj} = \frac{2(\lambda_{mj})\ J_{m+1}\ (\lambda_{mj})}{\pi mn\alpha \left[(\lambda_{mj})^2 - m^2\right]\ J_m^2(\lambda_{mj})\ \cosh(\lambda_{mj}\alpha)} \left[1 - \cos(n\pi\zeta)\right] \tag{B-67}$$

Substitution of eq. (B-54) into eq. (B-51) gives

$$F_2\ \overline{W}_{ms}^{(i)}(p) = -\sum_{n=1}^{N} \left[F_3\ \overline{W}_{mn}^{(i)}(p) + F_4\tilde{W}_{mn}^{(i)}(0) + F_5\dot{\tilde{W}}_{mn}^{(i)}(0) + F_6\ddot{\tilde{W}}_{mn}^{(i)}(0)\right]$$

$$+\ \mu\left[p\ \tilde{W}_{ms}^{(i)}(0) + \dot{\tilde{W}}_{ms}^{(i)}(0)\right] + F_7 \tag{B-68}$$

or

$$\sum_{n=1}^{N} \left[F_3 + \delta_{ns}\ F_2\right]\ \overline{W}_{mn}^{(i)}(p)$$

$$= -\sum_{n=1}^{N} \left[F_4\tilde{W}_{mn}^{(i)}(0) + F_5\tilde{W}_{mn}^{(i)}(0) + F_6\ddot{\tilde{W}}_{mn}^{(i)}(0)\right.$$

$$+\mu\left[p\tilde{W}_{ms}^{(i)}(0) + \dot{\tilde{W}}_{ms}^{(i)}(0)\right] + F_7$$

$$m = 1,2,\ \ldots\ M$$

$$s = 1,2,\ \ldots\ N \tag{B-69}$$

For each value of m and i, eq. (B-69) represents a system of N equations to be solved for the N unknowns $\overline{W}_{mn}^{(i)}$, n = 1, 2, ... N. Substitution of $W_{mn}^{(i)}(p)$ into eqs. (B-44) and (B-45) gives $\overline{U}_{mn}^{(i)}(p)$ and $\overline{V}_{mn}^{(i)}(p)$. Taking the inverse transforms of $\overline{U}_{mn}^{(i)}(p)$, $\overline{V}_{mn}^{(i)}(p)$ and $\overline{W}_{mn}^{(i)}(p)$ gives $\tilde{U}_{mn}^{(i)}(\tau)$, $\tilde{V}_{mn}^{(i)}(\tau)$ and $\tilde{W}_{mn}^{(i)}(\tau)$.

B-IV Final Comments on the Numerical Approach to Eqs. (B-35) and (B-69)

Eqs. (B-35) and (B-69) represent a total of 2M+1 sets of equations each of which contains N equations. To solve each set of these equations, we have to compute the coefficients of the functions F's in p corresponding to different indexes m, n, s, and j. Since the coefficients of each system of equations are functions of the transform variable p, numerical procedures in addition to the matrix inversion method must be used in the computer programs. After $\bar{U}_{mn}^{(i)}(p)$, $\bar{V}_{mn}^{(i)}(p)$ and $\bar{W}_{mn}^{(i)}(p)$ have been solved in terms of p, they have to be separated into partial fractions before the inverse transform can be taken to obtain the time functions $\tilde{U}_{mn}^{(i)}(\tau)$, $\tilde{V}_{mn}^{(i)}(\tau)$ and $\tilde{W}_{mn}^{(i)}(\tau)$. This involves a tremendous amount of work in computer programming as well as analytical calculation.