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THE THERMAL EFFECT ON CONICAL SHELLS OF LINEARLY VARYING THICKNESS

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ABSTRACT

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A study of an isotropic conical shell of linearly varying thickness under a surface temperature is made. The thermal effect on the shell is represented by an equivalent load. Asymptotic particular solutions due to the thermal load are obtained. One may combine these solutions with the complimentary solutions of the shell obtained in a previous report [1] [3] to constitute a set of complete solutions. A numerical example of a semi-circular cone frustum subjected to temperature functions that are constant along the meridians and have a sinusoidal distribution in the circumferential direction is given.

Author

Introduction

Analytical solutions of conical shells including thermal effect so far are not available. In this report an attempt is made to get an asymptotic solution of an isotropic conical shell with linearly varying thickness by

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[1] This work was sponsored by George C. Marshall Space Flight Center, NASA, under Contract NAS 8-11155 with the University of Alabama.

[2] Associate Professor of Engineering Mechanics, University of Alabama.

[3] Numbers in brackets designate references at end of report.

following the method developed in a previous report [1] in which the basic equations and complimentary solutions of these equations have been given. These solutions can be applied here without any alteration. Thus the particular solution of the system of equations due to a thermal effect on such shells will be sought.

The thermal effect may be represented by an equivalent load which will be referred to as a thermal load. The thermal load will be derived in the next section. The derivation is started by considering a shell of revolution which in general has two principal curvatures in two respective membrane directions. Letting one of the two curvatures vanish and specifying the other, the thermal load for a conical shell is obtained. The so-derived thermal load has components in all three directions of the reference coordinates used.

The temperature distribution considered is assumed to be a linear function of the normal coordinate and an arbitrary function of the two membrane coordinates. Such temperature distribution is commonly used in shell theory. For instance this was the case assumed for cylindrical shells in [2].

It is shown in this study that, for asymptotic solutions, the temperature variation in the normal direction is negligible. The asymptotic solutions are discussed. The solutions of a particular case of constant temperature distribution along the meridians of the conical shell and of sinusoidal distribution in the circumferential direction are presented. Combining these solutions with the complimentary solutions of the shell obtained in the previous report, a numerical example of a semi-circular cone frustum is given. The displacements, stress resultants and stress couples of the cone frustum are presented graphically. It is found that the effect of such a thermal load is quite similar to a lateral normal load.

The Thermal Loads

Let ϕ and θ be a set of orthogonal curvilinear coordinates describing the middle surface of a shell of revolution with a set of principal radii r_ϕ and r_θ. When the classical Duhamel-Newmann law of thermoelasticity [3] is used, the stresses, strains and temperature assume the following relations

$$\sigma_\phi = \frac{E}{1-\nu^2}\left[\epsilon_\phi + \nu\epsilon_\theta - (1+\nu)\beta T\right]$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[\epsilon_\theta + \nu\epsilon_\phi - (1+\nu)\beta T\right]$$

$$\sigma_{\theta\phi} = \frac{E}{2(1+\nu)}\epsilon_{\theta\phi} \qquad (1)$$

where σ_ϕ and σ_θ are normal stresses, ϵ_ϕ and ϵ_θ, normal strains in the ϕ and θ directions, $\sigma_{\theta\phi}$ and $\epsilon_{\theta\phi}$ are shearing stress and strain respectively, T is a temperature function and β is the coefficient of linear expansion.[4]

In what follows the relations for the part of the additional stresses associated with the temperature function T only will be concerned because those for the other part are supposedly known. The additional stresses may be expressed in the forms

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[4] Symbols other than those defined in this report follow those given in [1].

$$\sigma_{\phi}^{T} = -\frac{E\beta}{1-\nu} T(\phi, \Theta, z)$$

$$\sigma_{\Theta}^{T} = -\frac{E\beta}{1-\nu} T(\phi, \Theta, z) \tag{2}$$

and assume

$$T(\phi, \Theta, z) = T_0(\phi, \Theta) + z\, T_1(\phi, \Theta) \tag{3}$$

where the coordinate z is in the normal direction of the middle surface positive outward. The corresponding membrane stress resultants N_{ϕ}^{T}, N_{Θ}^{T} and stress couples M_{ϕ}^{T}, M_{Θ}^{T} per unit length due to the stresses (2) defined by

$$N_{\phi}^{T} = \int_{-t/2}^{t/2} \sigma_{\phi}^{T} \left(1 + \frac{z}{r_{\Theta}}\right) dz\,, \qquad N_{\Theta}^{T} = \int_{-t/2}^{t/2} \sigma_{\Theta}^{T} \left(1 + \frac{z}{r_{\phi}}\right) dz$$

$$M_{\phi}^{T} = -\int_{-t/2}^{t/2} \sigma_{\phi}^{T} \left(1 + \frac{z}{r_{\Theta}}\right) z dz, \quad M_{\Theta}^{T} = -\int_{-t/2}^{t/2} \sigma_{\Theta}^{T} \left(1 + \frac{z}{r_{\phi}}\right) dz \tag{4}$$

may be expressed in the following forms

$$N_{\phi}^{T} = -\frac{Et}{1-\nu}\beta\left[T_0 + \frac{T_1}{r_{\Theta}} \frac{t^2}{12}\right]$$

$$N_{\Theta}^{T} = -\frac{Et}{1-\nu}\beta\left[T_0 + \frac{T_1}{r_{\phi}} \frac{t^2}{12}\right]$$

$$M_{\phi}^{T} = \frac{E\beta}{1-\nu}\left[T_1 + \frac{T_0}{r_{\phi}}\right]\frac{t^2}{12} \tag{5}$$

$$M_{\phi}^{T} = \frac{E\beta}{1-\nu}\left[T_1 + \frac{T_0}{r_{\Theta}}\right]\frac{t^2}{12}$$

The foregoing expressions may be converted into those for conical shells; simply let

$$r_\theta = \infty, \quad \theta = \alpha, \quad r_\theta = s \cot\alpha \tag{6}$$

and the results are

$$N_s^T = -\frac{Et}{1-\nu}\beta\left[T_o + \frac{t^2}{12}\frac{T_1}{s}\tan\alpha\right]$$

$$N_\theta^T = -\frac{Et}{1-\nu}\beta\left[T_o\right]$$

$$M_s^T = \frac{E\beta}{1-\nu}\frac{t^3}{12}\left[\frac{T_o}{s}\tan\alpha + T_1\right] \tag{7}$$

$$M_\theta^T = \frac{E\beta}{1-\nu}\frac{t^3}{12}\left[T_1\right]$$

For conical shells with linearly varying thickness using $t = \delta s$, expressions (7) become

$$N_s^T = -\frac{E\beta\delta}{1-\nu}\left[T_o s + kT_1 s^2 \tan\alpha\right]$$

$$N_\theta^T = -\frac{E\beta\delta}{1-\nu}\left[T_o s\right]$$

$$M_s^T = \frac{E\beta\delta}{1-\nu}\, k\left[T_o s^2 \tan\alpha + T_1 s^3\right] \tag{8}$$

$$M_\theta^T = \frac{E\beta\delta}{1-\nu}\, k\left[T_1 s^3\right]$$

where $k = \frac{\delta^2}{12}$. When these stress resultants and couples are substituted into equilibrium equations (4) of reference [1] and denoting the additional terms by P_s^T, P_r^T and P_Θ^T one has

$$P_s^T = (sN_s^T)^\cdot - N_\Theta^T$$

$$= -\frac{E\beta\delta}{1-\nu}\left[T_0^\cdot s^2 + T_0 s + 3kT_1 s^2 \tan\alpha + kT_1^\cdot s^3 \tan\alpha \right]$$

$$P_r^T = sN_\Theta^T \tan\alpha + 2s(M_s^T)^\cdot + s^2(M_s^T)^{\cdot\cdot}$$

$$= -\frac{E\beta\delta}{1-\nu}\left\{ T_0 s^2 \tan\alpha - k\left[(s^5\ddot{T}_1 + 8s^4\dot{T}_1 + 12s^3 T_1) \right.\right.$$

$$\left.\left. + (s^4\ddot{T}_0 + 6s^3\dot{T}_0 + 6s^2 T_0)\tan\alpha \right] \right\} \tag{9}$$

$$P_\Theta^T = s(N_\Theta^T)' \sec\alpha - (M_\Theta^T)' \tan\alpha \sec\alpha$$

$$= -\frac{E\beta\delta}{1-\nu}\left[(s^2 T_0') + k\,T_1'\, s^3 \tan\alpha \right] \sec\alpha$$

The above three expressions may be considered as the three components of the thermal load in the respective directions.

Asymptotic Solutions

It has been discussed in [1] that for thin shells the asymptotic solutions are pertinent for practical purpose. In what follows asymptotic particular solutions of the shell due to the thermal load will be sought.

Retaining the terms of the lowest order of k the thermal loads (9) are simplified to the following form:

$$P_s^T = -\frac{E\beta\delta}{1-\nu}\left[\dot{T}_0 s^2 + T_0 s\right]$$

$$P_r^T = -\frac{E\beta\delta}{1-\nu}\left[T_0 s^2 \tan\alpha\right]$$

$$P_\theta^T = -\frac{E\beta\delta}{1-\nu}\left[s^2 T_0' \sec\alpha\right] \tag{10}$$

Note that the temperature function T_1 is not involved in these expressions.

For asymptotic solutions, the set of membrane equations may be used. Using the dimensionless variable y as independent variable to replace the s, the three equilibrium equations of membrane theory including the thermal loads (10) referring to equations (42) of reference [1] may be given in the following forms

$$\frac{1-\nu}{8}\left[y^2 u,_{yy} + 3yu,_y - 8u\right] + \frac{1+\nu}{4}\, yu,_{\theta y}\sec\alpha + u,_{\theta\theta}\sec^2\alpha$$

$$+ (2-\nu)\, v,_\theta \sec\alpha + w,_\theta \sec\alpha\tan\alpha = (1+\nu)\beta L\, y^2 T_{0,\theta} \sec\alpha$$

(11)

$$\frac{1+\nu}{8}\, yu,_{\theta y}\sec\alpha - \frac{3}{2}(1-\nu)u,_\theta \sec\alpha + \frac{1}{4}y^2 v,_{yy} + \frac{3}{4}y\, v,_y$$

$$+\frac{1-\nu}{2} v,_{\theta\theta}\sec^2\alpha - (1-\nu)v + \frac{1}{2}\nu\, y\, w,_y \tan\alpha - (1-\nu)\, w\tan\alpha$$

$$= (1+\nu)\beta L y^2\left[\frac{1}{2}\, y\, T_{0,y} + T_0\right]$$

$$u,_\theta \sec\alpha + \frac{1}{2}\nu\, y\, v,_y + v + w\tan\alpha = (1+\nu)\beta\, L\, y^2\, T_0 \tan\alpha$$

Let

$$T_o = Q_n y^{\omega_n} \begin{matrix} \cos \\ \sin \end{matrix} \frac{n\pi\theta}{\theta_1} \tag{12}$$

where Q_n and ω_n are prescribed constants presumably real and finite. The particular solutions of equations (11) may be assumed in the following fashions.

$$\begin{aligned} U^T &= A_n y^{\lambda_n^* - 1} \begin{matrix} \sin \\ \cos \end{matrix} \frac{n\pi\theta}{\theta_1} \\ V^T &= B_n y^{\lambda_n^* - 1} \begin{matrix} \cos \\ \sin \end{matrix} \frac{n\pi\theta}{\theta_1} \\ W^T &= C_n y^{\lambda_n^* - 1} \begin{matrix} \cos \\ \sin \end{matrix} \frac{n\pi\theta}{\theta_1} \end{aligned} \tag{13}$$

in which coefficients A_n, B_n and C_n are to be determined. On substituting expressions (12) and (13) for equations (11), and on factoring out the sinusoidal functions then setting

$$\lambda_n^* = \omega_n + 3 \tag{14}$$

one will have three linear algebraic equations for three unknowns A_n, B_n and C_n, and the equations can be solved by the Cramer's rule provided that λ_n^* does not make the determinant of the equations vanish. When $\omega_n = 0$, $\lambda_n^* = 3$ is one of the roots which will make the determinant vanish as has been shown in [1]. And physically this represents the case in which the temperature is constant along meridians. In such case the solutions which are obtained by the same method as for the lateral normal uniform load given in [1] are shown below.

Let

$$U^T = \left[d_1 + d_2 \ln y \right] y^2 \begin{matrix} \sin \\ \cos \end{matrix} \frac{n\pi\Theta}{\Theta_1}$$

$$V^T = b\, y^2 \begin{matrix} \cos \\ \sin \end{matrix} \frac{n\pi\Theta}{\Theta_1}$$

$$W^T = c(1 + \ln y)\, y^2 \begin{matrix} \cos \\ \sin \end{matrix} \frac{n\pi\Theta}{\Theta_1} \tag{14}$$

in which d_1 d_2 b and c are constants to be determined. When the assumed solutions (14) combined with (12) are substituted into equations (11) one can show that

$$d_1 = \mp \frac{1}{2} \frac{m\beta L Q_n}{1-\nu} \left\{ (1+3\nu) - (5-\nu)\tan\alpha - \left[\frac{2}{3} m^2 - \frac{1}{m^2}(1+\nu) \right] (1-\tan\alpha) \right\}$$

$$d_2 = \frac{1}{3} \frac{m\beta L Q_n}{1-\nu} \left\{ (1+4\nu-m^2) - \tan\alpha\,(7-2\nu-m^2) \right\}$$

$$b = \frac{1}{6} \frac{m^2\beta L Q_n}{1-\nu} \left\{ 1 - \tan\alpha + \frac{3}{m^2}\left[\tan\alpha\,(1-2\nu) + 1 \right] \right\}$$

$$c = \frac{m}{\tan\alpha}\, d_2 \tag{15}$$

The corresponding stress resultants due to the thermal loads are readily obtained by use of the elastic law. The results are given as follows.

$$N_s^{\ T} = \frac{E\delta}{1-\nu^2} \left[(1+\nu)\, b - m\nu\,(d_1 - d_2) \right] y^2 \begin{matrix} \cos \\ \sin \end{matrix} \frac{n\pi\Theta}{\Theta_1}$$

$$N_\Theta^{\ T} = \frac{E\delta}{1-\nu^2} \left[\nu(1+\nu)\, b - m(d_1 - d_2) \right] y^2 \begin{matrix} \cos \\ \sin \end{matrix} \frac{n\pi\Theta}{\Theta_1}$$

$$T_s^{\ T} = \frac{E\delta}{4(1+\nu)} \left[d_2 + 2mb \right] y^2 \begin{matrix} \sin \\ \cos \end{matrix} \frac{n\pi\Theta}{\Theta_1} \tag{16}$$

The stress couples induced by such thermal loads are of higher order, thus negligible. Combining the solutions (14) and (16) with the complementary solutions obtained in [1] , one has the complete solutions for this case.

Numerical Example

Take the semi-circular truncated cone which has two generators simply supported with the smaller circular end fixed and the other end free as discussed in [1] as an example. The same parameters are assumed that

$$\alpha = 75^{\circ}, \quad \nu = \frac{1}{3} \quad \text{and} \sqrt{\frac{L_1}{L}} = 0.90 \tag{17}$$

Numerical results for $n = 1$ and 2 are computed so that one can use them for the types of symmetrical and asymmetrical distribution of temperature similar to those of wind loads discussed in reference [1] . The results are given in the form

$$F_n(y, \Theta) = f_n(y) \begin{matrix} \sin \\ \cos \end{matrix} \frac{n\pi\Theta}{\Theta_1} \qquad n = 1 \text{ and } 2 \tag{18}$$

in which the function $f_n(y)$ are presented in Figs. 1 to 10 for $\frac{t}{R} = 0.004$, 0.006 and 0.008.

Acknowledgement

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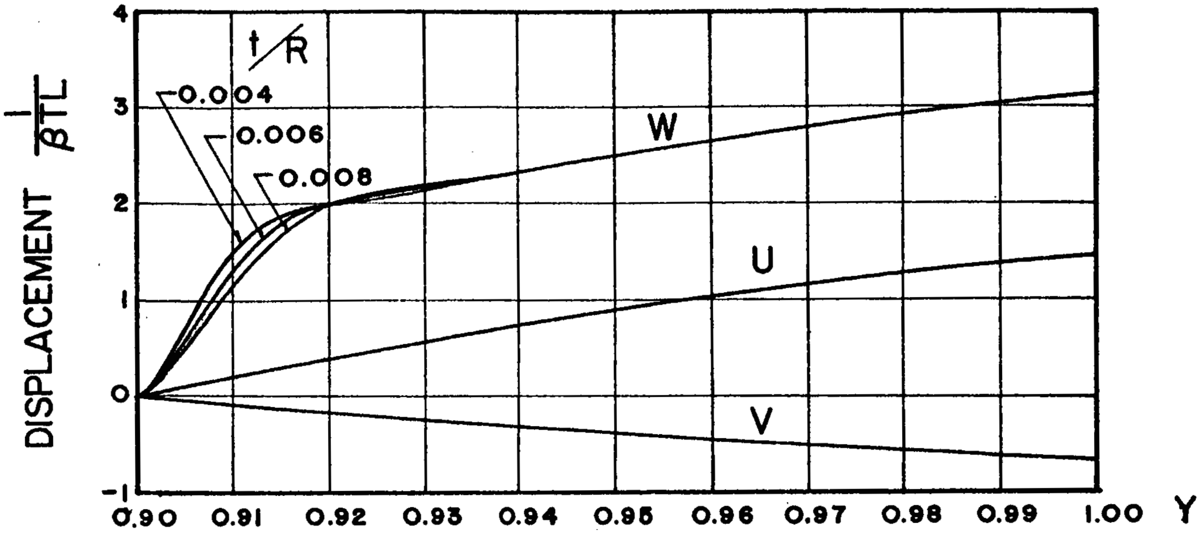

FIG. 1 DISPLACEMENTS W, V AND U
(N = 1)

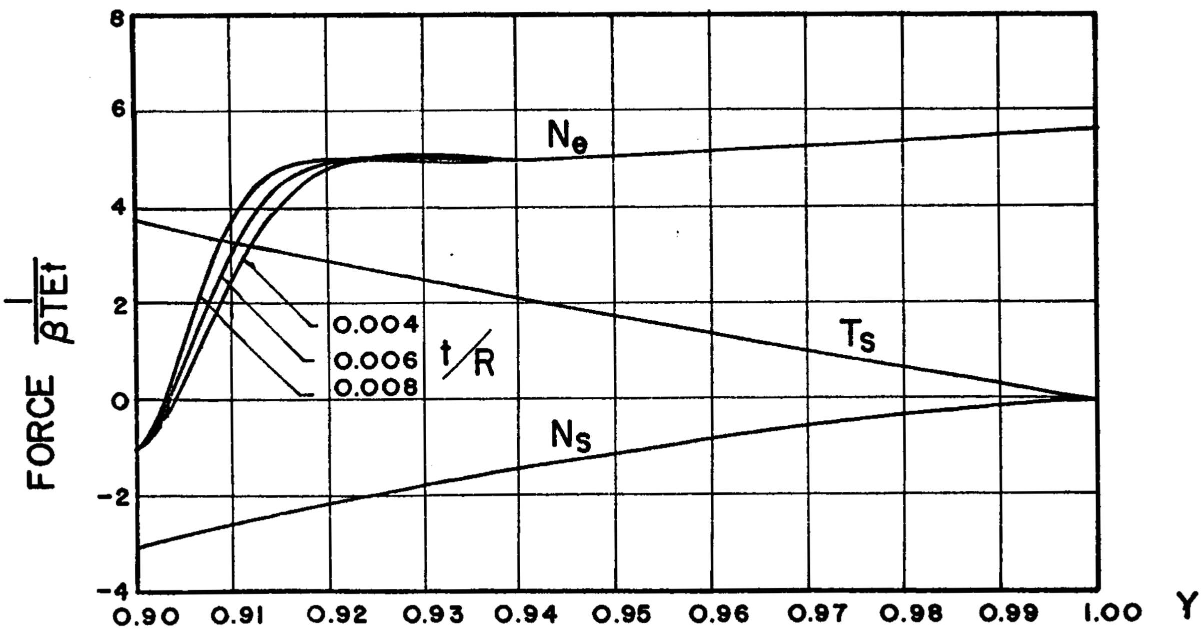

FIG. 2 MEMBRANE FORCES N_θ T_S AND N_S
(N = 1)

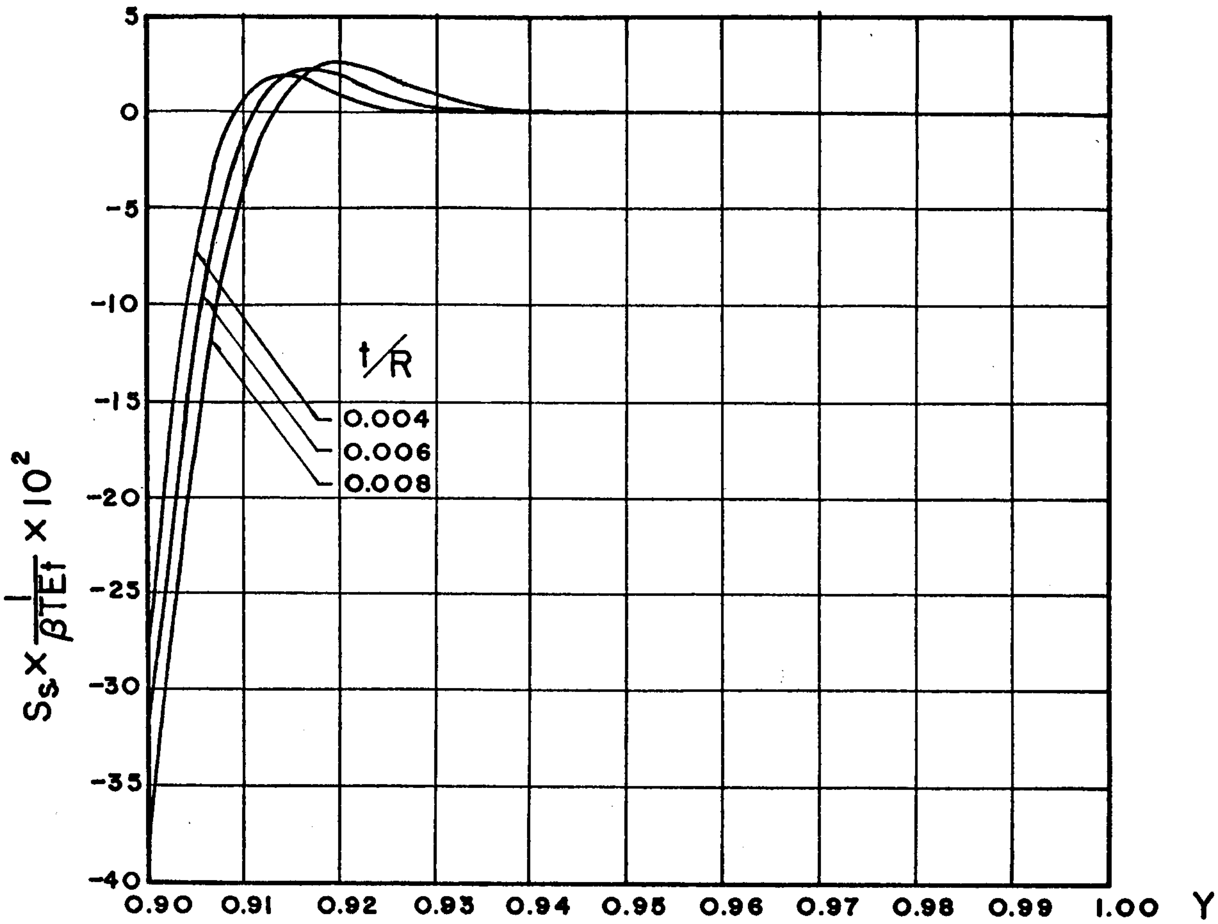

FIG. 3 TRANSVERSE SHEARING FORCE S_s (N=1)

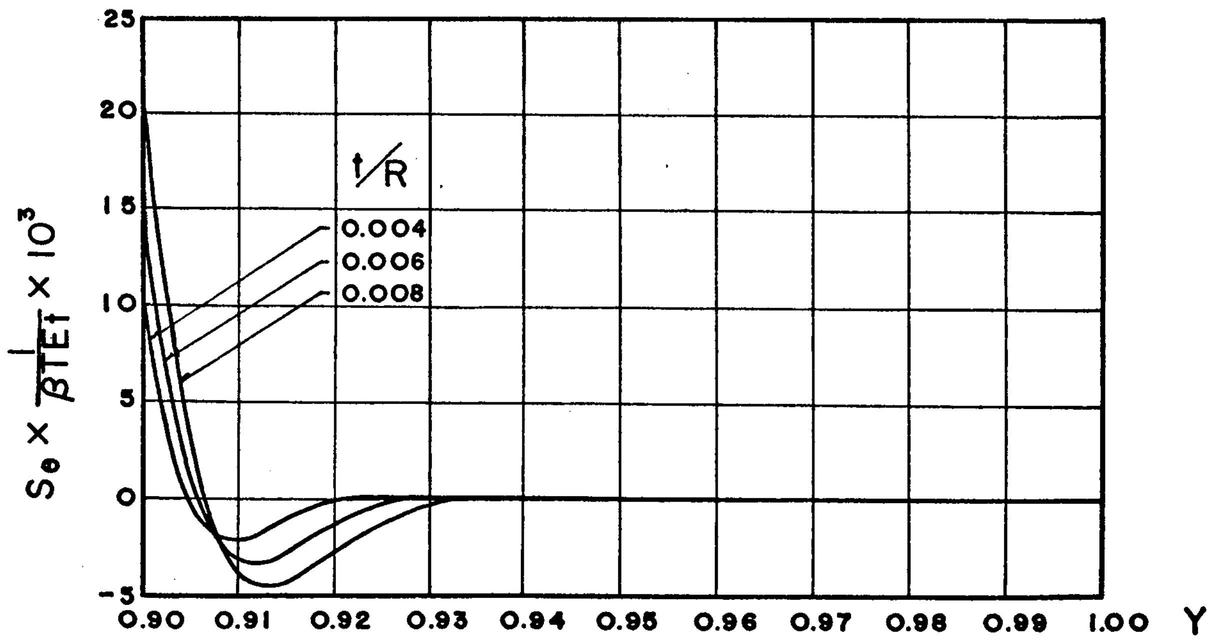

FIG. 4 TRANSVERSE SHEARING FORCE S_θ (N=1)

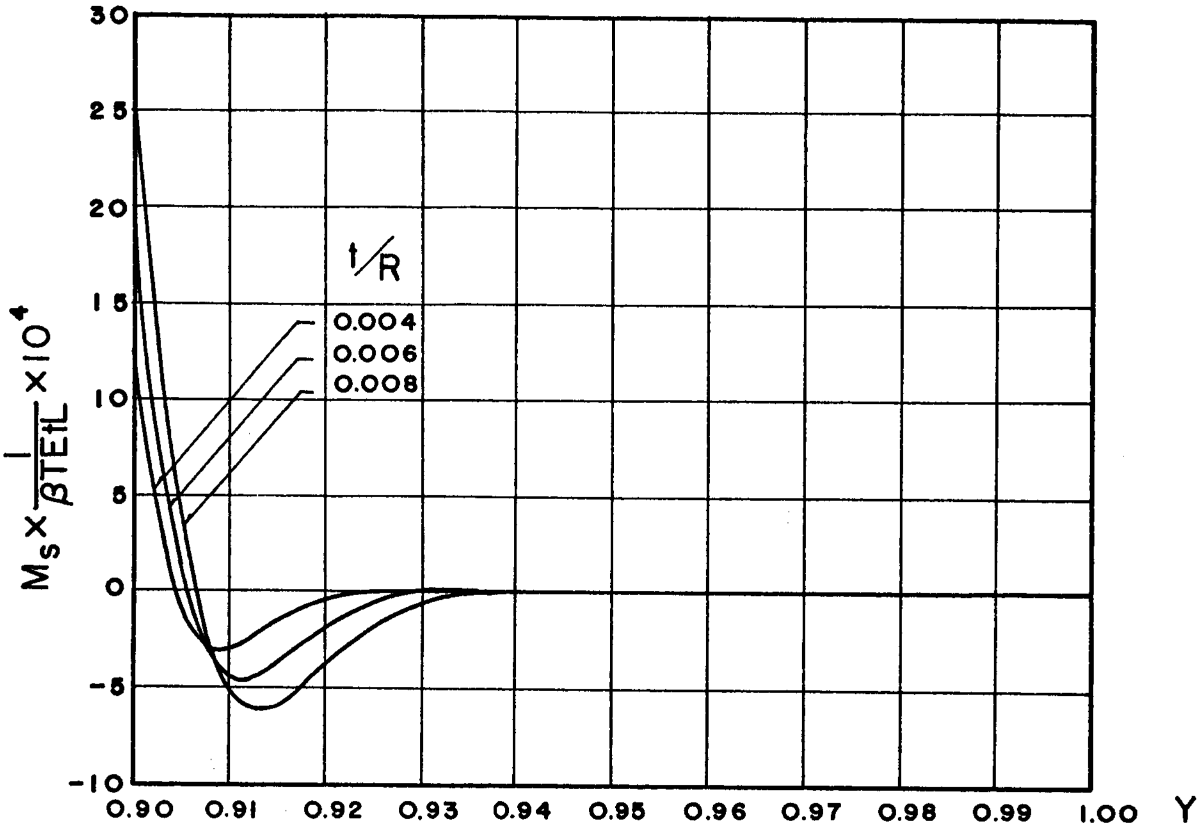

FIG. 5 NORMAL MOMENT M_S (N=1)

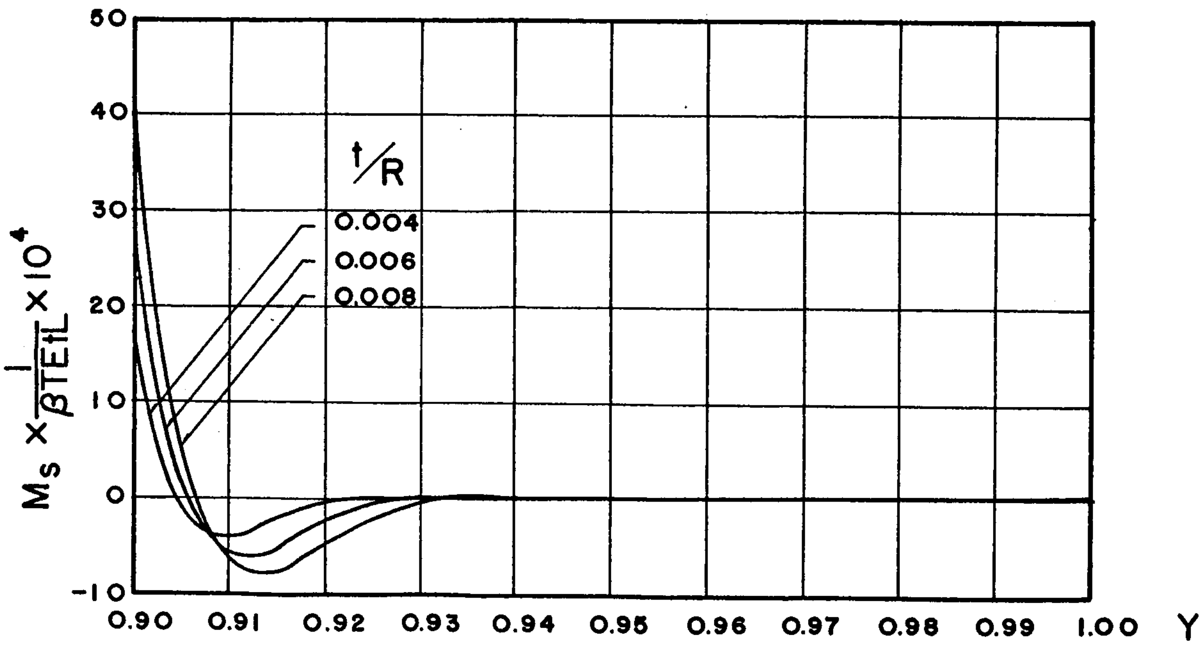

FIG. 6 NORMAL MOMENT M_S (N=2)

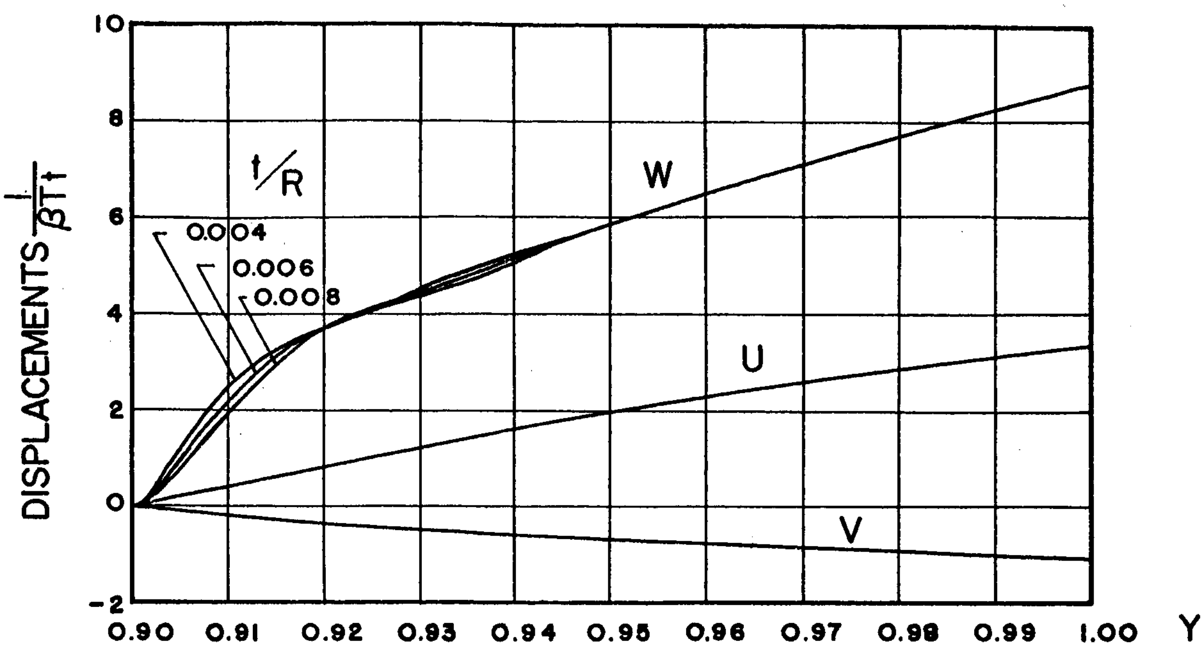

FIG. 7 DISPLACEMENTS W, V AND U (N=2)

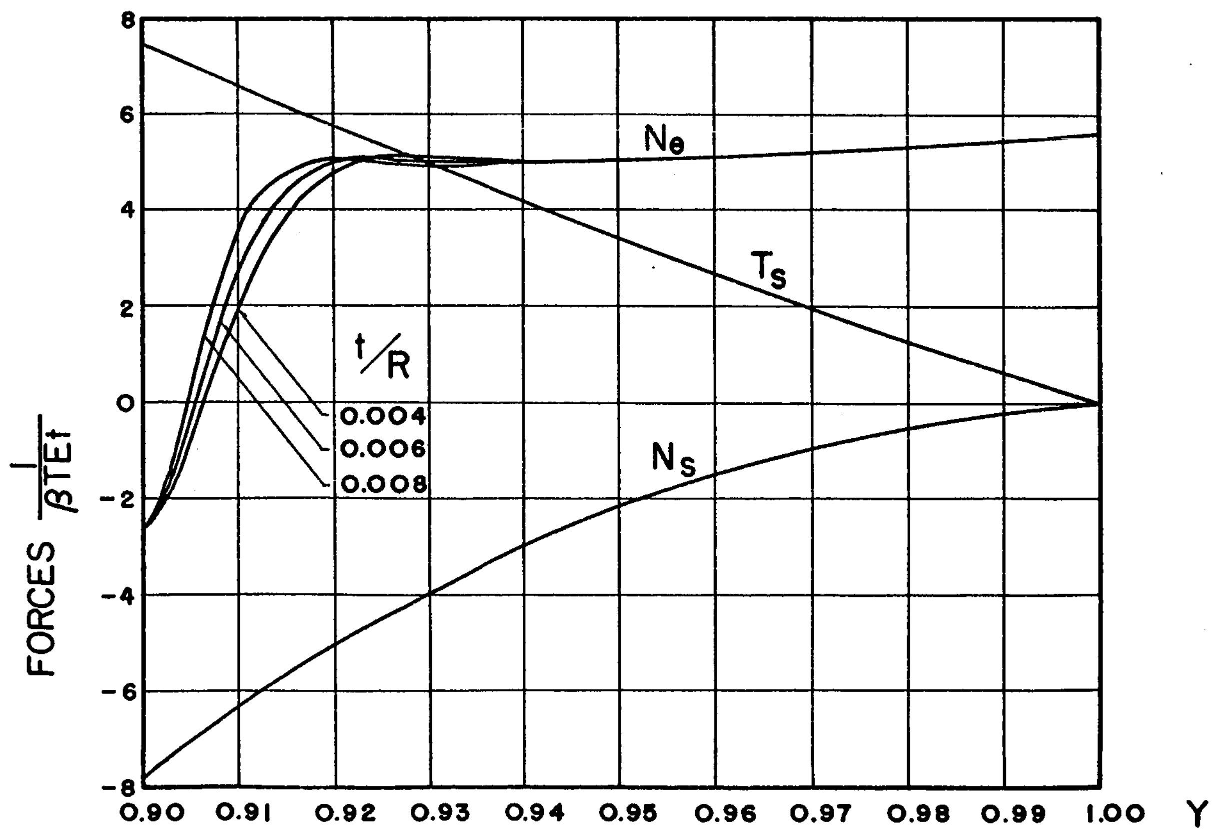

FIG. 8 MEMBRANE FORCES N_θ, T_s AND N_s (N=2)

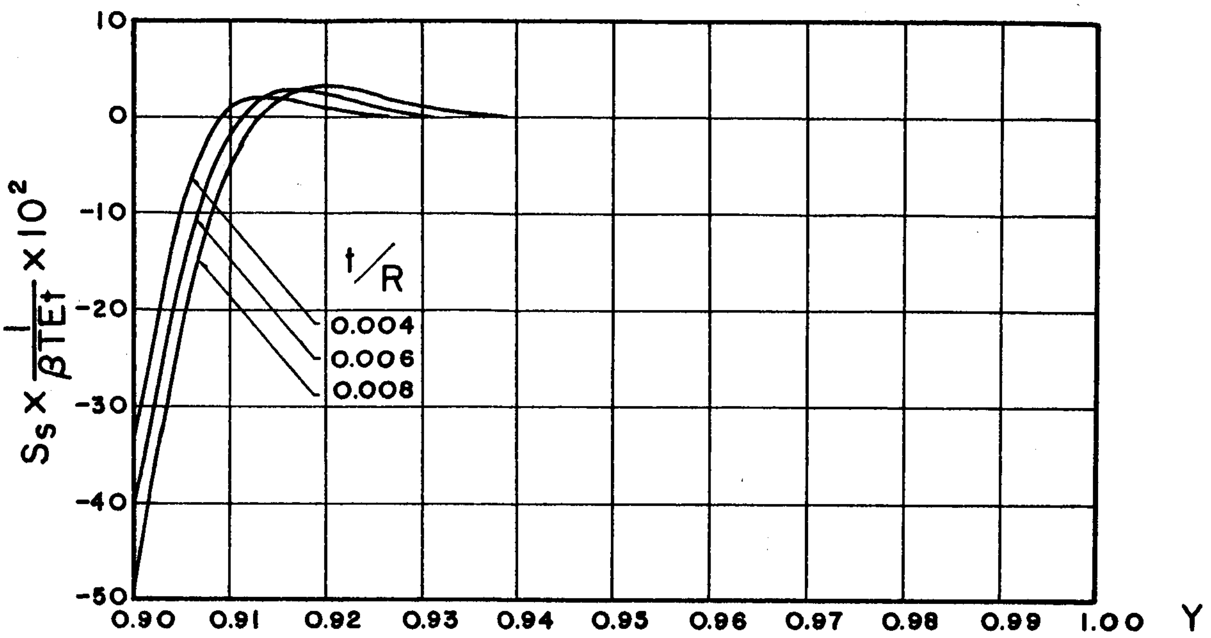

FIG. 9 TRANSVERSE SHEARING FORCE S_S (N=2)

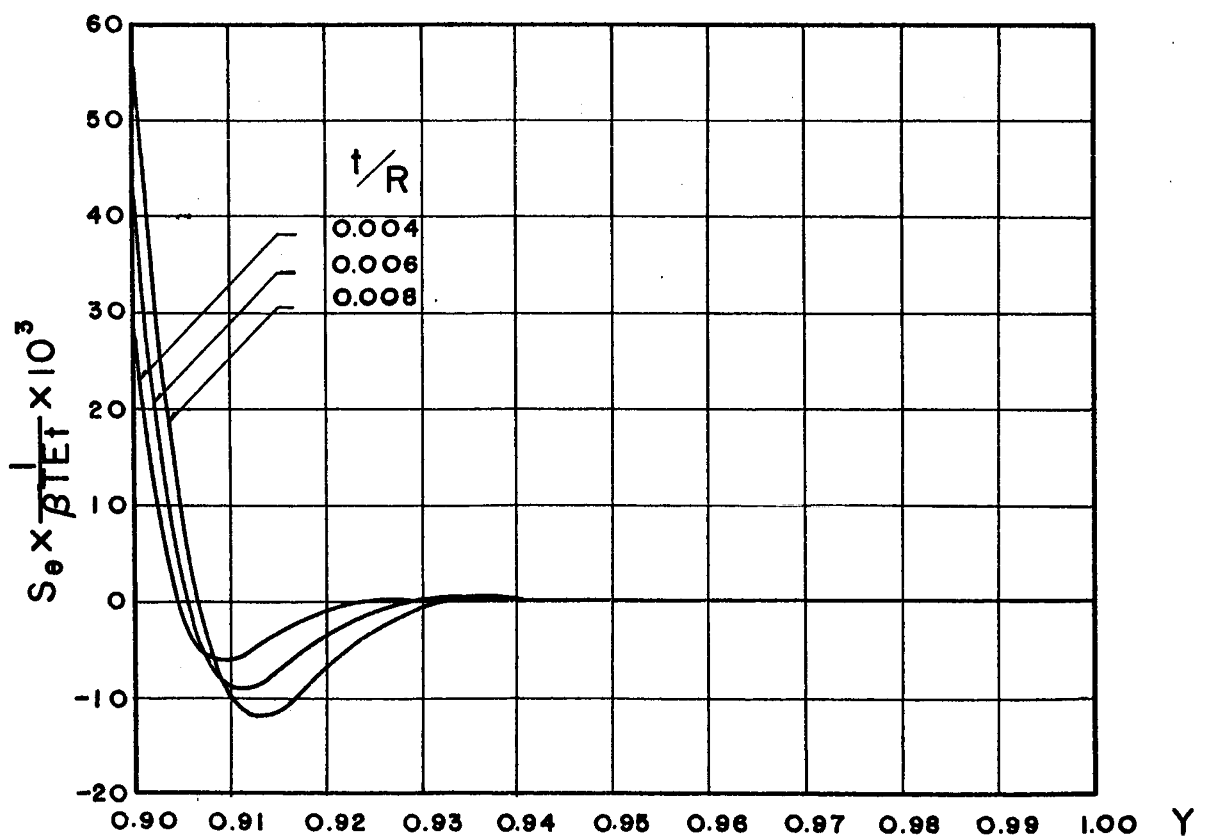

FIG. 10 TRANSVERSE SHEARING FORCE S_θ (N=2)