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NASA CR 59233

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Princeton, N.J.

UNPUBLISHED PRELIMINARY DATA

EQUILIBRIUM CONFIGURATIONS BETWEEN DENSITY AND TOPOGRAPHIC SURFACE IRREGULARITIES IN A PURELY ELASTIC EARTH MODEL

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FACILITY FORM 602

N65-35309 (ACCESSION NUMBER)	(THRU)
12 (PAGES)	1 (CODE)
CR 59233 (NASA CR OR TMX OR AD NUMBER)	13 (CATEGORY)

GPO PRICE $ ______

CSFTI PRICE(S) $ ______

Hard copy (HC) 1.00

Microfiche (MF) .50

ff 653 July 65

October 1, 1964

Technical Report No. 2, Prepared under NASA Research Grant NsG-556 to Princeton University by National Aeronautics and Space Administration

Equilibrium configurations between density and topographic surface irregularities in a purely elastic earth model.

Abstract

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A two dimensional purely elastic earth model is used to study elastic equilibrium configurations between density and topographic surface irregularities (mountains). Given a density deficiency $\delta\rho(x,y)$ the shape of the mountain in equilibrium with $\delta\rho(x,y)$ is determined by specifying that the vertical displacements due both to the mountain and $\delta\rho(x,y)$ vanish at the surface. The compression due to density irregularities is evaluated and numerical examples are given. Author

Introduction

In a previous paper (Durney, Stresses induced in a purely elastic earth model under various tectonic loads, to be published, hereafter referred to as I) we calculated the stresses in the mantle and crust of the earth by density variations or by topographic surface irregularities. A two dimensional model was adopted, all quantities being constant in a direction perpendicular to the x-y plane. The x axis was chosen horizontally from left to right and the y axis vertically up. The equations for the stresses were solved for arbitrary density distributions and surface loads, which were shown to be equivalent to mountains if their height was small compared to their width. Consideration of equilibrium situations provides a good method for treating elastic problems in the earth where body forces and surface loads (mountains) are important (this was done in (I) in the case of the isostatic adjustment of a continent and a density deficiency beneath.)

In the present paper our aim is to study more in detail these equilibrium configurations between surface loads and density irregularities. Consider the case of figure (1) where the density deficiency $\delta\rho(x,y)$ is supposed to be given. To perform the calculations we shall take $\delta\rho(x,y) = \rho_1(0)\,\eta(\ell,d)\,e^{-(x/D)^2}$

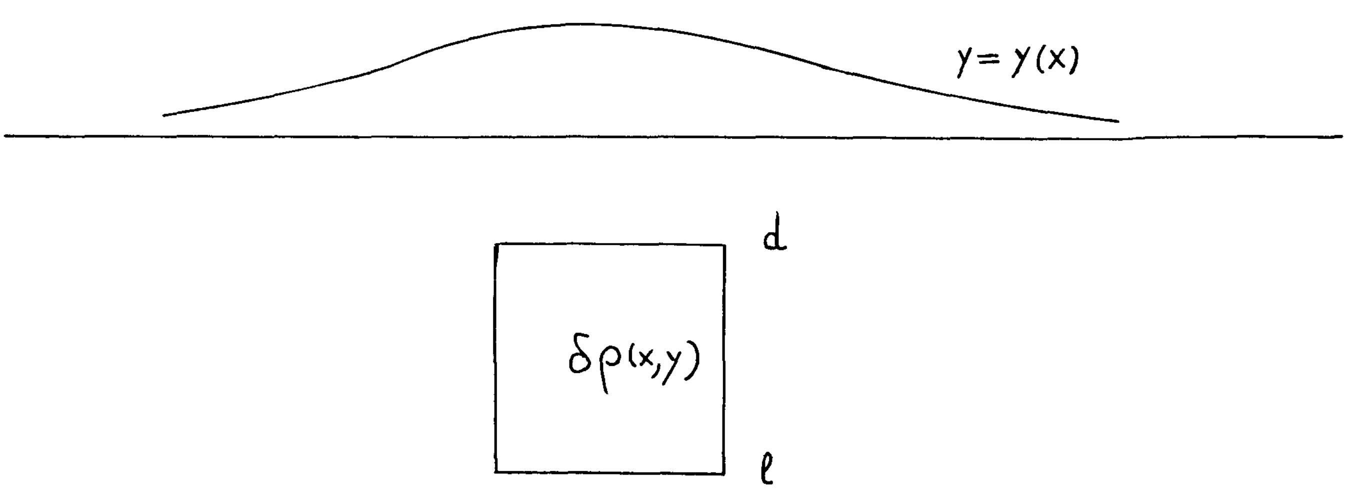

Figure 1

where $\eta(\ell,d)=1$ for $d \geq y \geq \ell$ and zero otherwise. We shall determine the mountain profile $y=y(X)$ such that the displacements due to both the surface load $-g\rho_0(0)y(X)$ and the density deficiency $\delta\rho(x,y)$ vanish at $y=0$. This is an equilibrium situation that could have arisen, for example, due to plastic flow. The curve $y=y(X)$ does not give the elastic displacements due to $\delta\rho(x,y)$; these are infinite in a plane geometry. In (I) we saw that this is due to the fact that the elastic displacements are mainly determined by the very long wavelengths of the Fourier spectrum of $e^{-(X/D)^2}$ ($\delta\rho(x,y)=\rho_1(0)\eta(\ell,d)e^{-(X/D)^2}$) It will be seen further on that $y=y(X)$, on the other hand, does not depend strongly on these long wavelengths as long as the depth of $\delta\rho(x,y)$ is not too large; the use of a spherical geometry would not alter significantly the shape of the curve $y=y(X)$.

We shall proceed to solve the elastic equations for $\omega(X,y)=u_{y,x}(X,y)-u_{x,y}(X,y)$ ($u_{y,x}(x,y)=\frac{\partial u_y(x,y)}{\partial x}$, $u_{x,y}(x,y)=\frac{\partial u_x(x,y)}{\partial y}$, $\omega(x,y)$ is the only non vanishing component of the vector $\underset{\sim}{\omega}(x,y)=\nabla\wedge\underset{\sim}{u}(x,y)$ in the two dimensional case). It will be shown that the Fourier transforms of $u_y(x,0)$ can be very simply expressed in terms of the Fourier transform of $\omega(x,0)$. $u_x(x,y)$ and $u_y(x,y)$ are the horizontal and vertical components of the elastic displacements.

Solutions of the elastic equations

The equations for the displacements can be written in any of the following forms (Sokolnikoff 1956)

$$\mu\nabla^2\underset{\sim}{u}+(\lambda+\mu)\nabla\vartheta+\underset{\sim}{F}=0 \tag{1a}$$

$$-\mu\,\nabla\wedge\underset{\sim}{\omega}+(\lambda+2\mu)\nabla\vartheta+\underset{\sim}{F}=0 \tag{1b}$$

$$(\lambda+2\mu)\nabla^2\underset{\sim}{u}+(\lambda+\mu)\nabla\wedge\underset{\sim}{\omega}+\underset{\sim}{F}=0 \tag{1c}$$

For the body force we take $\underset{\sim}{F}=(0,-g\,\delta\rho(x,y))$ and we shall restrict ourselves to $\delta\rho(x,y)=\rho_1(y)\cos kx$. Fourier integrals will then be used in the case of more general density distributions. Taking the curl of (1b) we obtain

$$\mu\nabla^2\underset{\sim}{\omega}+\nabla\wedge\underset{\sim}{F}=0 \tag{2}$$

We shall again restrict ourselves to the two dimensional case (all quantities being independent of z) and solve the elastic equations with boundary conditions expressing that the forces at the plane $y=0$ are equivalent to a pure vertical load: $f_x(x)=0$ and $f_y(x)=-g\,\rho_0(0)y(x)$, $f_y(x)$ is the weight of a mountain of profile $y=y(x)$ and density $\rho_0(0)$.

It is easily seen that these boundary conditions give rise to the following relations for $y=0$:

$$u_{y,x}(x,0)=\frac{1}{2}\,\omega(x,0) \tag{3a}$$

$$u_{y,y}(x,0)=\frac{1}{2\mu}\left(f_y(x)-\lambda\,\vartheta(x,0)\right) \tag{3b}$$

If $\omega(x,y)=\omega(y,k)\sin kx$ is substituted into (2) a differential equation is obtained for $\omega(y,k)$. Writing that $\omega(y,k)$ vanishes for large depths it is found that

$$\begin{aligned}\omega(y,k)&=-\frac{kg}{\mu}e^{ky}\int_0^y e^{-2ky}\,dy\int_{-\infty}^y e^{ky}\rho_1(y)\,dy+\omega(0,k)e^{ky}\\&=\omega_1(y,k)+\omega(0,k)e^{ky}\end{aligned} \tag{4}$$

$\dot{\omega}(0,k)$ is the value of $\dot{\omega}(y,k)$ at $y=0$ and must be determined from the boundary conditions (3). The equation (1b) gives for $\vartheta(y,k)$ $(\vartheta(y,x)=\vartheta(y,k)\cos kx)$

$$\vartheta(y,k) = -\frac{\mu}{k(\lambda+2\mu)}\,\omega'(y,k) \tag{5}$$

with $\omega'(y,k) = \dfrac{\partial\omega(y,k)}{\partial y}$

If we define $u_x(x,y)=u(y,k)\sin kx$, $u_y(x,y)=v(y,k)\cos kx$ $f_y(x) = -g\rho_0(0)\,y(k)\cos kx$ then the boundary conditions (3) can be written:

$$v(0,k) = -\frac{1}{2k}\,\omega(0,k) \tag{6a}$$

$$v'(0,k) = -\frac{g\rho_0(0)\,y(k)}{2\mu} + \frac{\omega'(0,k)\lambda}{2k(\lambda+2\mu)} \tag{6b}$$

To determine $\omega(0,k)$ we proceed in the following way: equation (1c) for $v(y,k)$ is solved with the boundary condition (6a) (and, of course no displacements at infinity); relation (6b) determines then $\omega(0,k)$. Performing the calculations it is found that

$$\omega(0,k) = 2\int_{-\infty}^{0} k\,\omega_1(y,k)e^{ky}dy + g\frac{(\lambda+2\mu)}{\mu(\lambda+\mu)}\Big[\int_{-\infty}^{0}\rho_1(y)\,e^{ky}dy + y(k)\rho_0(0)\Big] \tag{7}$$

We shall specialize the above formulas for the case $f_y(x)=0$ (the plane $y=0$ is a free surface) and a density excess of the form $\delta\rho(x,y)=\rho_1(0)\eta(\ell,d)e^{-(\frac{x}{D})^2}$ $\eta(\ell,d)=1$ for $d\geq y\geq \ell$ and zero otherwise). The problem being linear it is enough to write the expressions for the case $\delta\rho(x,y,\ell)=\rho_1(0)\theta(y-\ell)e^{-(x/D)^2}$

Where $\theta(y-\ell)=1$ for $y\geq \ell$ and zero for $y<\ell$. After some calculations it is found that

$$\omega(x,y)=\frac{D}{\sqrt{\pi}}\int_0^\infty \omega(y,k)\sin kx\, e^{-(kD/2)^2}dk \tag{8a}$$

with $\omega(y,k)=\omega_1(y,k)+\omega(0,k)e^{ky}$ where

$$\omega_1(y,k)=\frac{g\rho_1(0)}{k\mu}\Big[\Big(1-e^{ky}-\frac{e^{k\ell}}{2}\left(e^{-ky}-e^{ky}\right)\Big)\theta(y-\ell)$$

$$-\left(1-\frac{e^{k\ell}+e^{-k\ell}}{2}\right)e^{ky}\,\theta(\ell-y)\Big) \tag{8b}$$

and

$$\omega(0,k)=\frac{g\rho_1(0)}{k\mu}\Big[1+e^{k\ell}(k\ell-1)+\frac{\lambda+2\mu}{\lambda+\mu}\left(1-e^{k\ell}\right)\Big] \tag{8c}$$

Identically $\vartheta(x,y)$ is given by

$$\vartheta(x,y)=\frac{D}{\sqrt{\pi}}\int_0^\infty \vartheta(y,k)\cos kx\, e^{-(kD/2)^2} \tag{9}$$

where $\vartheta(y,k)$ can be obtained from (5) by using expressions (8b) and (8c).

It is now easy to calculate the displacements for $y=0$; with the help of (6a) we obtain

$$u_y(x,0)=-\frac{D}{2\sqrt{\pi}}\int_0^\infty \frac{\omega(0,k)}{k}\cos kx\, e^{-(kD/2)^2}dk \tag{10}$$

Expression (10) gives thus the elastic displacements at $y=0$ due to a density deficiency of the form

$$\delta\rho(x,y,\ell) = \rho_1(0)\,\theta(y-\ell)\,e^{-(x/D)^2}$$

From the above formulas it is also easy to calculate the displacements due to a mountain of profile $y = y(x) = \int_0^\infty \cos kx \; y(k)\, dk$ and density $\rho_0(0)$ They are given by

$$u_y(x,0) = -\frac{g\rho_0(0)(\lambda+2\mu)}{2\mu(\lambda+\mu)} \int_0^\infty \frac{\cos kx}{k}\, y(k)\, dk \qquad (11)$$

We calculate now $y(k)$ and thus $y(x)$ such that the displacements due to both the mountain of profile $y=y(x)$ and the density deficiency $\delta\rho(x,y) = \rho_1(0)\eta(\ell,d)\,e^{-(x/D)^2}$ vanish for $y=0$. For simplicity we take a square density deficiency with $2D=L=d-\ell$, the vertical and horizontal dimensions of $\delta\rho(x,y)$ are, in this case, approximately equal. For $y=y(x)$ it is then obtained:

$$y(x) = -\frac{L\rho_1(0)}{2\sqrt{\pi}\,\rho_0(0)} \int_0^\infty e^{-(t/4)^2 + td/L} \cos(tx/L) \,*$$
$$\left\{\left[\frac{(1-td/L)(1-e^{-t})}{t} - e^{-t}\right]\frac{\lambda+\mu}{\lambda+2\mu} + \frac{1-e^{-t}}{t}\right\} dt \qquad (12)$$

This expression gives thus the profile of the mountain of density $\rho_0(0)$ in elastic equilibrium with a density deficiency $\delta\rho(x,y) = \rho_1(0)\,\eta(\ell,d)\,e^{-(x/D)^2}$ $(2D = d-\ell)$ It is seen that as long as d/L is not very large, small (i.e. long wavelengths) do not play a predominant role; in this case the plane model is a good approximation to the spherical geometry. On the other hand if $d/L \gg 1$ it cannot be expected that the plane model will give valid results because the horizontal dimension of $y(x)$ becomes very large in this case.

Another simple way of obtaining expression (12) is noticing from (3a) that if $u_y(x,0)=0$ then also $\omega(x,0)=0$ We can thus solve the elastic equations with the boundary conditions $\omega(x,0)=0$ and $F_x(x,0)=0$ (no shear stresses). The vertical load $F_y(x,0)$ cannot, now, be fixed arbitrarily but it is determined from the above boundary conditions. The result obtained in this way gives a force equal to $-g\rho_0(0)y(x)$ with $y(x)$ given by expression (12)

Geophysical Applications

In figure (2) we have plotted expression (12); the profile of the mountain in elastic equilibrium with a square density deficiency of horizontal and vertical dimensions equal to 200 km (L = 200 km.) for different depths of $\delta\rho(x,y)$ ($d=0$, 100 and 200 km.). The rapid flattening of the mountain with increasing d is to be noticed. The mass of the mountain in each case remains nevertheless constant and equal to the mass due to the density deficiency. The mass of the mountain is found by (12) to be $M(y(x))=\sqrt{\pi}\,L^2|\rho_1(0)|/2$ On the other hand the total mass due to the density deficiency is given by $M(\delta\rho)=\sqrt{\pi}\,L^2\rho_1(0)/2$ The cancelation of $M(y(x))$ and $M(\delta\rho)$ is a consequence of the fact that the elastic displacements due to the mountain and $\delta\rho(x,y)$ are finite. In a plane geometry finite elastic displacements imply zero total mass (I).

We have written the expression of $y(x)$ for the case of a square density deficiency; another case of interest arises when $d=0$ (c.f. Fig. 1) and $D \gg |\ell|$ In this limit it is easily shown that $y(x)$ is given by $y(x) \sim \frac{\ell\rho_1(0)}{\rho_0(0)} e^{-(x/D)^2}$ In (I) we arrived at this result by writing that the stresses due to the mountain and the density deficiency compensate for $|y| > |\ell|$ that is, that the mountain and the density deficiency are in isostatic equilibrium.

In figure (3) we have plotted contours of constant compression and dilation for a density irregularity of the form $\delta\rho(x,y)=\rho_1(0)\left[\eta(-200\,km, 0)+\eta(-400\,km, -200\,km)\right]e^{-(x/D)^2}$ with $D=100\,km$ and $\rho_1(0)=-3.5\,gr/cm^3$; $\delta\rho(x,y)$

consists thus of a square density deficiency of 200 km. on each side, the top coinciding with the surface $y=0$ and an identical square density excess at a depth of 200 km. (Fig. 4)

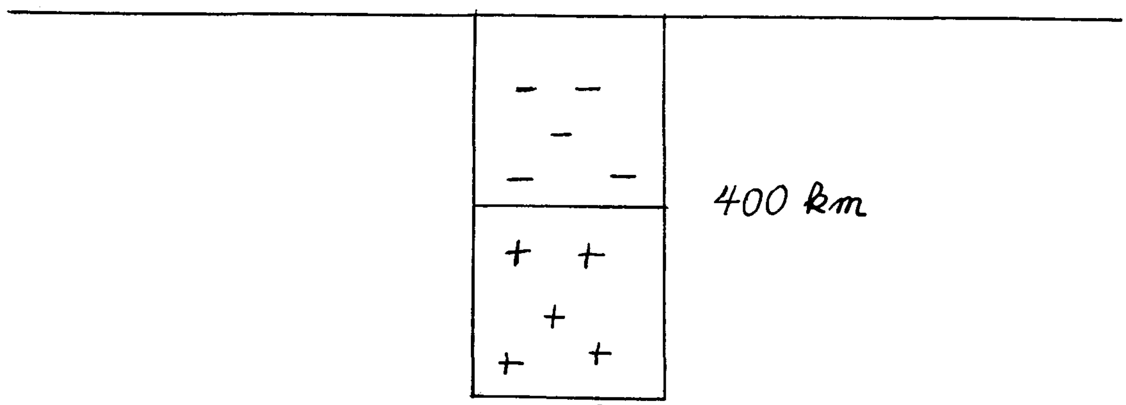

Figure 4

In the region near the surface there is dilation at the center and compression at a distance of about 200 km. from the center. If we take a density irregularity consisting only of the density excess of figure (4) (the lower part) we find also a compression $(\tau_{xx}(0,0)+\tau_{yy}(0,0))$ of about 185 bars at the center; as $\tau_{yy}(0,0)$ is zero this compression is due to the large value of the horizontal stress $\tau_x(0,0)$.

Acknowledgments

I would like to thank Professor W.M. Elsasser for suggesting this problem to me and for many helpful discussions.

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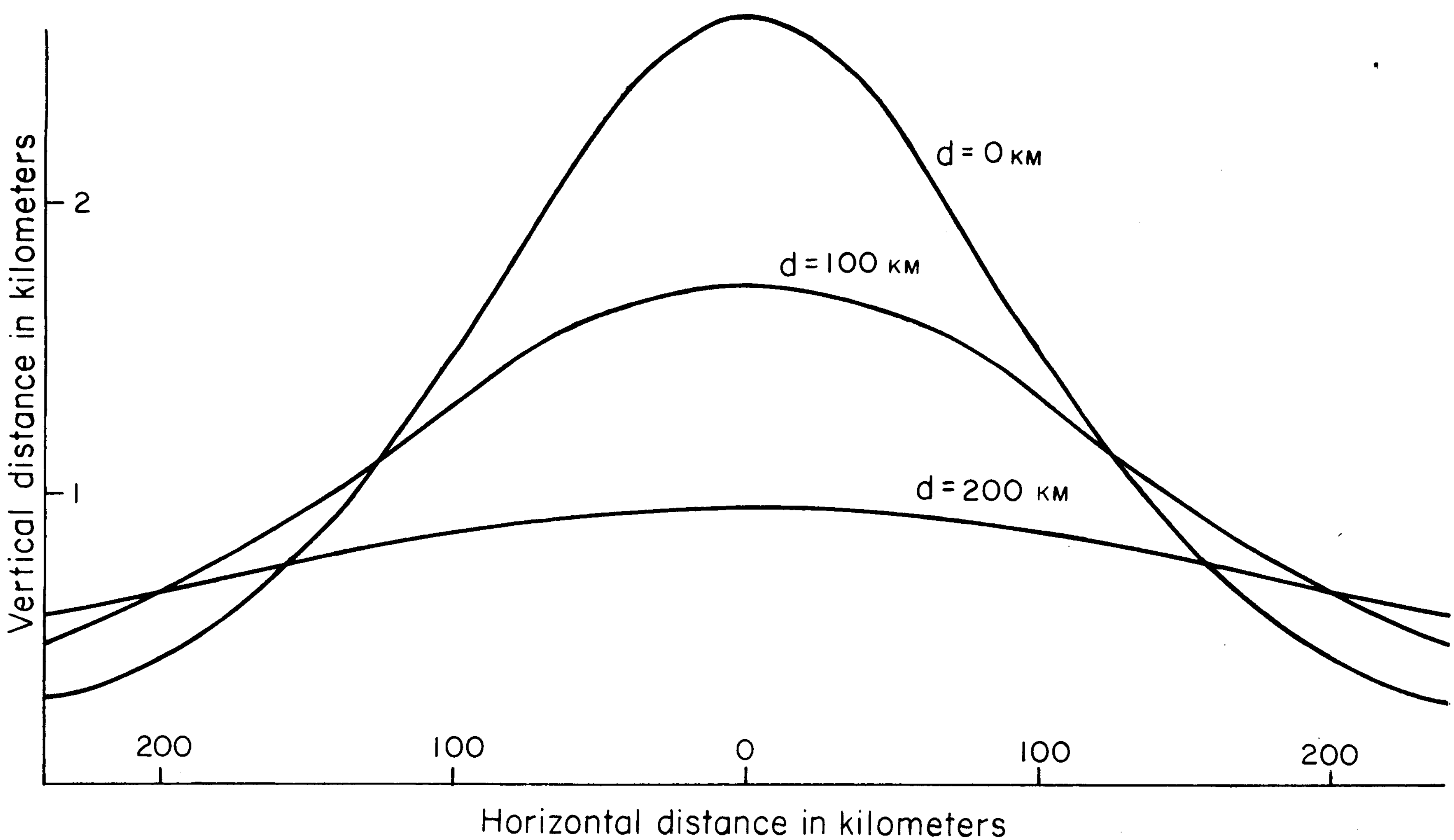

Fig. 2. Profile of the mountains in elastic equilibrium with a density deficiency $\delta\rho$ of 1% for different depths of $\delta\rho$ as shown in figure 1. The horizontal and vertical dimensions of $\delta\rho$ are equal to 200 km.

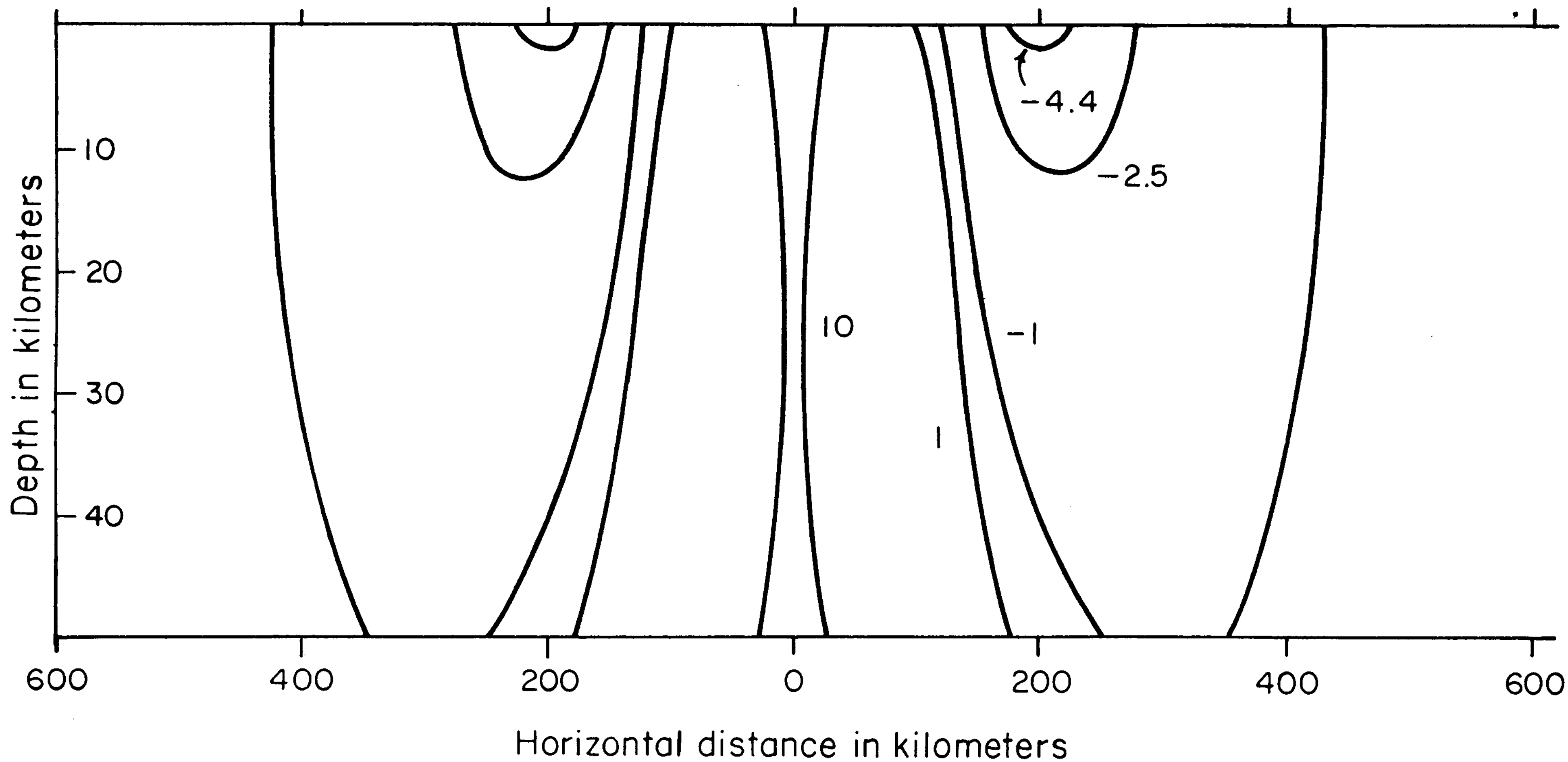

Fig. 3. Contours of equal compression and dilation ($2\operatorname{div}\underset{\sim}{u}\,10^5$) induced by the density irregularity of figure 4 ($|\delta\rho| = 3.5\ 10^{-2}$ gr./cm^3)