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ON CONICAL SHELLS OF LINEARLY VARYING THICKNESS SUBJECTED TO LATERAL NORMAL LOADS

Prepared By

Chin Hao Chang

Submitted to

George C. Marshall Space Flight Center

National Aeronautics and Space Administration

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FORWARD

This is a progress report for part of works performed for contract No. NAS8-11155 under the phase of analytic study of a cantilever conical shell subjected to wind and thermal loads.

Most of the results given in this report were given in a Technical Report* submitted previously. However, the results have been checked further by a somewhat different approach from the one used in that report. Some corrections are made. Thus this report may be regarded as the final one for the phase of the analysis of a cantilever conical shell subjected to lateral normal loads. For completeness the entire problem and basic equations are briefed.

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*"An Asymptotic Solution For Conical Shells of Linearly Varying Thickness", by C. H. Chang, Technical Report C submitted to George C. Marshall Space Flight Center, NASA for Contract No. NAS8-5168 by Bureau of Engineering Research, University of Alabama, March 1964.

Progress Report for NASA Contract NAS8-11155

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ABSTRACT

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The basic equations of conical shells of linearly varying thickness and an approach to the homogeneous solutions were given in "Stresses in Shells" by W. Flügge. The homogeneous solutions were hinged on an eighth degree characteristic equation. In this report, along the line of the theory, the characteristic equation is given in a different form, and a method of solving the equation is also presented. When let the ratio of the end thickness to total length approach zero asymptotically, it is found that the solution consists of two parts: membrance and bending. The two parts are coupled by the lateral displacement. The particular solutions due to lateral normal loads are also given along with a numerical example of a truncated semi-circular cone. Author

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Introduction

The theory of conical shell of linearly varying thickness in the framework of generalized plane stresses of linear theory of elasticity along with a general approach of solving the basic equations has been given in Reference [1][1]. The three homogeneous equilibrium equations in terms of three displacement components were solved by the classic method of separation of variables; in turn the solutions were hinged on an eighth degree characteristic equation.

The basic equations may be regarded as the results of series expansion of the stresses and displacements in a parameter k which depends on the ratio of the thickness to length, and only the terms of zero and first order of k are retained. Along this line, in this paper the characteristic equation is presented in a different form and is solved by a method being approximate but consistent with the theory.

Of the eight roots of the equation, four are real and the other four are complex. When let the parameter k approach zero asymptotically it is found that the solution of the real roots is of membrane theory while that of the complex roots is of bending effect. A general asymptotical solution is given including eight undetermined constants.

Generally there would be no difficulties in obtaining the particular solutions of the system due to lateral normal loads. When the load is uniformly distributed along meridians, the solution,

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[1]Numbers in brackets designate referencès at end of paper

however, is near a singularity of the system. It is at a singularity for the asymptotical solution. The particular solution of a such case is given.

As an example for illustration, a semi-circular truncated cone which has two generators simply supported with the smaller circular end fixed and the other end free is given. It is shown that the bending effects confine in the neighborhood of the clamped edge as it would be expected.

Basic Equations

Let θ, s be circumferential and meridional coordinates of the middle surface of an isotropic conical cone and u, v, w, be circumferential, meridional and normal displacement components respectively. Outward w is positive. When the thickness of shell h is proportional to s and independent to θ, one has

$$h = \delta s \tag{1}$$

where δ is a constant which for thin shells is very small. The elastic law assumes the following relationships between the stress resultants and displacement components.[2]

$$
\begin{aligned}
N_s &= \mathcal{D}[sv^{\cdot} + \nu (u' \sec \alpha + v + w \tan \alpha) - k\, s^2 w^{\cdot\cdot} \tan \alpha] \\
N_\theta &= \mathcal{D}[u' \sec \alpha + v + w \tan \alpha + \nu s v^{\cdot} \\
&\quad + k(v \tan \alpha + w \tan^2 \alpha + w'' \sec^2 \alpha + sw^{\cdot}) \tan \alpha] \\
N_{s\theta} &= \mathcal{D}\frac{1-\nu}{2} [su^{\cdot} - u + v' \sec \alpha \\
&\quad + k (su^{\cdot} - u - \frac{sw^{\cdot\prime}}{\sin \alpha} + \frac{w'}{\sin \alpha}) \tan^2 \alpha]
\end{aligned} \tag{2}
$$

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[2] Further details see Reference [1]

$$N_{\theta s} = \mathcal{D}\frac{1-\nu}{2}\Big[s\dot{u} - u + v' \sec\alpha$$

$$+ k\left(v' \sec\alpha + \frac{s\dot{w}'}{\sin\alpha} - \frac{w'}{\sin\alpha}\right)\tan^2\alpha\Big]$$

$$M_s = \mathcal{D}ks[s^2\ddot{w} - s\dot{v} \tan\alpha + \nu(w'' \sec^2\alpha + s\dot{w} - u' \sec\alpha \tan\alpha)]$$

$$M_\theta = \mathcal{D}ks[w'' \sec^2\alpha + s\dot{w} + w \tan^2\alpha + v \tan\alpha + \nu s^2\ddot{w}]$$

$$M_{s\theta} = \mathcal{D}k(1-\nu)\, s[(s\dot{w}' - w') \sec\alpha - (s\dot{u} - u)\tan\alpha]$$

$$M_{\theta s} = \mathcal{D}k(1-\nu)\, s[(s\dot{w}' - w' + \tfrac{1}{2} v' \tan\alpha)\sec\alpha - \tfrac{1}{2}(s\dot{u} - u)\tan\alpha]$$

in which N_s ..., $M_{\theta s}$ are stress resultants and stress moments per unit length. The dots indicate partial differentiation with respect to s and primes with respect to θ; α is the complement of the half central angle of the cone,

$$\mathcal{D} = \frac{E\delta}{1-\nu^2} \quad \text{and} \quad k = \frac{\delta^2}{12} \tag{3}$$

where E is Young's modulas of elasticity and ν Poisson's ratio.

The six equations of equilibrium may be given in the following forms:

$$\begin{aligned}
(sN_s)^{\cdot} + N'_{\theta s} \sec\alpha - N_\theta &= -P_s s \\
(sN_{s\theta})^{\cdot} + N'_\theta \sec\alpha + N_{\theta s} - Q_\theta \tan\alpha &= -P_\theta s \\
N_\theta \tan\alpha + Q'_\theta \sec\alpha + (sQ_s)^{\cdot} &= P_r s \\
(sM_s)^{\cdot} + M'_{\theta s} \sec\alpha - M_\theta &= sQ_s \\
(sM_{s\theta})^{\cdot} + M'_\theta \sec\alpha + M_{\theta s} &= sQ_\theta \\
s(N_{\theta s} - N_{s\theta}) &= M_{\theta s} \tan\alpha
\end{aligned} \tag{4}$$

where Q_s and Q_θ are the transverse shear forces per unit length and acting on sections perpendicular to the s and θ

directions; P_r, P_s and P_θ are surface loads per unit area in normal, meridional and circumferential directions respectively.

Dropping the last one of equations (4) which is an identity and making use of the fourth and fifth of equations (4) to eliminate the transverse shearing forces Q_s and Q_θ in the other three equations, one has the three equations of equilibrium of the forms:

$$\begin{aligned} & s(sN_{s\theta})^{\cdot} + sN_\theta{}' \sec\alpha + s\,N_{\theta s} - (sM_{s\theta})^{\cdot} \tan\alpha \\ & \quad - M_{\theta s} \tan\alpha - M'_\theta \tan\alpha \sec\alpha = - P_\theta s^2 \\ & (sN_s)^{\cdot} + N_{\theta s}{}' \sec\alpha - N_\theta = - P_s\, s \\ & sN_\theta \tan\alpha + s(sM_s)^{\cdot\cdot} + (sM'_{s\theta})^{\cdot} \sec\alpha + (sM'_{\theta s})^{\cdot} \sec\alpha \\ & \quad + M_\theta'' \sec^2\alpha - sM_\theta{}^{\cdot} = P_r s^2 \end{aligned} \tag{5}$$

Substitution of elastic law (2) for equations (5) results in the following equations of equilibrium in terms of the displacements:

$$\begin{aligned} & \frac{1-\nu}{2}s^2u^{\cdot\cdot} + u''\sec^2\alpha + (1-\nu)su^{\cdot} - (1-\nu)u + \frac{1+\nu}{2}sv'^{\cdot}\sec\alpha \\ & \quad + (2-\nu)v'\sec\alpha + w'\tan\alpha\sec\alpha + k[\frac{3}{2}(1-\nu)s^2u^{\cdot\cdot}\tan\alpha \\ & \quad + 3(1-\nu)su^{\cdot}\tan\alpha - 3(1-\nu)u\tan\alpha - \frac{3-\nu}{2}s^2w'^{\cdot\cdot}\sec\alpha \\ & \quad - 3(1-\nu)sw'^{\cdot}\sec\alpha + 3(1-\nu)w'\sec\alpha]\tan\alpha = -\frac{P_\theta s}{\mathcal{D}} \\ & \frac{1+\nu}{2}su'^{\cdot}\sec\alpha - \frac{3}{2}(1-\nu)u'\sec\alpha + s^2v^{\cdot\cdot} + \frac{1-\nu}{2}v''\sec^2\alpha \\ & \quad + 2sv^{\cdot} - (1-\nu)v + \nu sw^{\cdot}\tan\alpha - (1-\nu)w\tan\alpha \\ & \quad + k[\frac{1-\nu}{2}v''\tan\alpha\sec^2\alpha - v\tan\alpha - s^3w^{\cdot\cdot\cdot} + \frac{1-\nu}{2}sw''^{\cdot}\sec^2\alpha \\ & \quad - 3s^2w^{\cdot\cdot} - \frac{3-\nu}{2}w''\sec^2\alpha - sw^{\cdot} - w\tan^2\alpha]\tan\alpha = -\frac{P_s s}{\mathcal{D}} \end{aligned} \tag{6}$$

$$[u'\sec\alpha + \nu s v^{\cdot} + v + w\tan\alpha]\tan\alpha + k[-\frac{3-\nu}{2}s^2u'^{\cdot\cdot}\sec\alpha$$

$$- (3+\nu)su'^{\cdot}\sec\alpha + (3-5\nu)u'\sec\alpha - s^3v^{\cdot\cdot\cdot} + \frac{1-\nu}{2}sv'^{\cdot\cdot}\sec^2\alpha$$

$$- 6s^2v^{\cdot\cdot} + (2-\nu)v''\sec^2\alpha - 7sv^{\cdot} - v(1-\tan^2\alpha)]\tan\alpha$$

$$+ k[s^4w^{\cdot\cdot\cdot\cdot} + 2s^2w''^{\cdot\cdot}\sec^2\alpha + w^{IV}\sec^4\alpha + 8s^3w^{\cdot\cdot\cdot} + 4sw''^{\cdot}\sec^2\alpha$$

$$+ (11+3\nu)s^2w^{\cdot\cdot} + 2w''\tan^2\alpha\sec^2\alpha - (5-6\nu)w''\sec^2\alpha$$

$$- 2(1-3\nu)sw^{\cdot} - w(1-\tan^2\alpha)\tan^2\alpha] = \frac{P_r s}{\mathcal{D}}$$

Consider a segment of cone being bounded by $\theta = 0$ and θ_1 and $s = L_1$ and L, $L_1 < L$. For convenience, a nondimensional variable y is introduced such that

$$y = \sqrt{\frac{s}{L}} \tag{7}$$

On the observations over equations (6), the displacement fuctions may be assumed in the forms:

$$\begin{aligned} u &= A_n y^{\lambda_n - 1} \begin{matrix}\sin\\ \cos\end{matrix} \frac{n\pi\theta}{\theta_1} \\ v &= B_n y^{\lambda_n - 1} \begin{matrix}\cos\\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1} \\ w &= C_n y^{\lambda_n - 1} \begin{matrix}\cos\\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1} \end{aligned} \tag{8}$$

in which A_n, B_n, C_n and λ_n are constants to be determined. Physically speaking the upper set of the sinusoidal functions in (8) is for a complete cone ($\theta_1 = 2\pi$) while the lower one is for a segment of cone with two generator edges simply supported so that along $\theta = 0$ and θ_1 ($<2\pi$)

$$w = 0, \; v = 0, \; N_\theta = 0, \text{ and } M_\theta = 0 \tag{9}$$

The reactions along the two generator edges are given by

$$S_\theta = Q_\theta + M^{\cdot}_{\theta s} \qquad \text{at } \theta = 0 \text{ and } \theta_1 \tag{10}$$

The S_θ is transverse shearing force at a section perpendicular to the θ direction. The shearing force Q_θ may be obtained from equations (4). In what follows the case in which only the lateral normal load appears is considered.[3] Thus

$$P_\theta = P_s = 0$$

and let

$$P_r = P_{rn}(y) \begin{matrix}\cos \\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1} \tag{11}$$

Substitution of the assumed displacements and loading functions into equations (6) yields

$$\begin{aligned} d_{11}A_n + d_{12}B_n + d_{13}C_n &= 0 \\ d_{21}A_n + d_{22}B_n + d_{23}C_n &= 0 \\ d_{31}A_n + d_{32}B_n + d_{33}C_n &= \frac{L}{\mathcal{D}}P_{rn}(y)y^{3-\lambda n} \end{aligned} \tag{12}$$

where

$$\begin{aligned} d_{11} &= \frac{1-\nu}{8}(1 + 3k\tan^2\alpha)(9 - \lambda_n^2) + m^2 \\ d_{12} &= \pm\frac{1}{4}[(7 - 5\nu) + (1 + \nu)\lambda_n]m \\ d_{13} &= \pm[1 + \frac{k}{8}(3(9 - 11\nu) + 8\nu\lambda - (3 - \nu)\lambda_n^2)]m\tan\alpha \\ d_{22} &= \frac{1}{4}(1 - \lambda_n^2) + (1 - \nu)(1 + \frac{1}{2}m^2) + k\tan^2\alpha(1 + \frac{1-\nu}{2}m^2) \\ d_{23} &= \frac{1}{2}\tan\alpha[(2 - \nu) - \nu\lambda_n] \\ &\quad - \frac{1}{8}k\tan\alpha[(1 - 8\tan^2\alpha + 2(7 - 3\nu)m^2) \\ &\quad - (3 + 2(1 - \nu)m^2)\lambda_n + 3\lambda_n^2 - \lambda_n^3] \\ d_{33} &= \tan^2\alpha + \frac{1}{16}k[(13 - 12\nu) - 16(1 - \tan^2\alpha)\tan^2\alpha \\ &\quad + 8(11 - 12\nu - 4\tan^2\alpha)m^2 + 16m^4 \\ &\quad - 2(7-6\nu + 4m^2)\lambda_n^2 + \lambda_n^4] \end{aligned} \tag{13}$$

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[3]When the other loads exist, one may follow a similar procedure and by superposition to get the appropriate solution.

and in which

$$m = \frac{n\pi}{\theta_1} \sec\alpha \tag{14}$$

The expressions for d_{21}, d_{31}, d_{32} are obtained by replacing λ_n with $-\lambda_n$ in d_{12}, d_{13}, d_{23}, respectively. The plus and minus signs which appear in front of one term correspond to the upper and lower set of sinusoidal functions henceforth.

In order to have non-trivial homogeneous solutions of the system of equations (12), the determinant of the coefficients must vanish. This results in an eight degree characteristic equation for λ_n. Neglecting the terms of second and higher power of k as it has been done in the derivation of elastic law (2) yields the characteristic equation in the following form:

$$G[\lambda_n^4 - 10\lambda_n^2 + 9] + k\,[\lambda_n^8 - g_6\lambda_n^6 + g_4\lambda_n^4 - g_2\lambda_n^2 + g_0] = 0 \tag{15}$$

in which

$$\begin{aligned}
G &= 16(1 - \nu^2)\tan^2\alpha \\
g_6 &= 4(7 - 4\nu) - 8\nu\tan^2\alpha + 16m^2 \\
g_4 &= 2[127 - 136\nu + 24\nu^2 \\
&\quad - 4(8 + 3\nu)\tan^2\alpha + 8(4 - 3\nu^2)\tan^4\alpha] \\
&\quad + 16[(17 - 12\nu) - 6\tan^2\alpha]\, m^2 - 96m^4 \\
g_2 &= 4[203 - 316\nu + 120\nu^2 \\
&\quad - 2(80 - 61\nu)\tan^2\alpha + 40(4 - 3\nu^2)\tan^4\alpha] \\
&\quad + 16[(71 - 72\nu) - 4(13 - 10\nu)\tan^2\alpha \\
&\quad + 8(2 - \nu)\tan^4\alpha]\, m^2 \\
&\quad + 64[(13 - 12\nu) - 2(4 - \nu)\tan^2\alpha]\, m^4 + 256m^6
\end{aligned} \tag{16}$$

$$g_0 = 9[(13 - 12\nu)(5 - 4\nu) - 8(8 - 7\nu)\tan^2\alpha + 16(4 - 3\nu^2)\tan^4\alpha]$$
$$+ 16[(215 - 412\nu + 192\nu^2) + 2(89 - 172\nu + 96\nu^2)\tan^2\alpha$$
$$+ 40(2 - \nu)\tan^4\alpha]\, m^2$$
$$- 32[(81 - 184\nu + 96\nu^2) + 4(16 - 13\nu)\tan^2\alpha - 8\tan^4\alpha]\, m^4$$
$$+ 256[(3 - 4\nu) - 2\tan^2\alpha]\, m^6 + 256m^8$$

In view of the approximation made in the derivation of equation (15), the following approximate method is suggested for this equation. Introducing

$$\lambda_n^2 = X_{no} + kX_{n1} \tag{17}$$

into equation (15) results in a sequence of equations associated with the various powers of k. The equations associated with the two lowest powers of k are

$$X_{no}^2 - 10X_{no} + 9 = 0$$

and

$$X_{no}^4 - g_6X_{no}^3 + g_4X_{no}^2 - g_2X_{no} + g_0 + 2G(X_{no} - 5)X_{n1} = 0$$

from which

$$X_{no} = 1 \text{ and } 9 \tag{18}$$

$$X_{n1} = -\frac{X_{no}^4 - g_6X_{no}^3 + g_4X_{no}^2 - g_2X_{no} + g_0}{2G(X_{no} - 5)} \tag{19}$$

Thus, one has two roots of λ_n^2 which are denoted by λ_{n1}^2 and λ_{n2}^2

$$\lambda_{n1}^2 = 1 + k\frac{1 - g_6 + g_4 - g_2 + g_0}{8G}$$
$$\lambda_{n2}^2 = 9 - k\frac{9^4 - 9^3g_6 + 9^2g_4 - 9g_2 + g_0}{8G} \tag{20}$$

Substituting them into equation (15) yields a quadratic equation of λ_n^2 which gives

$$\lambda^2_{n\,3,4} = \frac{1}{2}(g_6 - \lambda^2_{n2} - \lambda^2_{n1}) \pm i\sqrt{\frac{1}{\lambda^2_{n1}\lambda^2_{n2}}(g_0 + \frac{9G}{k}) - \frac{1}{4}(g_6 - \lambda^2_{n2} - \lambda^2_{n1})^2} \tag{21}$$

Hence the eight roots of λ_n group into two, four each. One group is of real numbers; the other is of complex numbers.

The next step, as a rountine, is to solve for A_n and B_n in terms of C_n for each root of λ_n from any two of the homogeneous equations of equations (12). The eight constants C_n shall be determined by eight conditions at $y = \sqrt{\frac{L_1}{L}}$ and 1. The boundary conditions along the generator edges are satisfied by the choice of sinusoidal functions of the angle θ. At the two circular edges one has the following four boundary conditions at each edge. For a build-in edge:

$$u = 0,\ v = 0,\ w = 0 \text{ and } w^{\cdot} = 0 \tag{22}$$

For a free edge:

$$N_s = 0,\ M_s = 0,\ S_s = 0 \text{ and } T_s = 0 \tag{23}$$

where

$$S_s = Q_s + \frac{1}{s} M'_{s\theta}\sec\alpha$$
$$T_s = N_{s\theta} - \frac{M_{s\theta}}{s}\tan\alpha \tag{24}$$

being transverse and tangential shearing forces at sections perpendicular to the s-direction respectively. The shearing force Q_s can be obtained from equations (4). For a simply supported edge:

$$w = 0, \quad M_s = 0 \quad N_s = 0 \quad \text{or} \quad v = 0$$

and

$$T_s = 0 \quad \text{or} \quad u = 0$$

Asymptotic Solutions

As the parameter k approaches zero, the two groups of roots λ_n reach at the following asymptotic values:

$$\lambda_{\substack{1\\2}} = \pm 1, \qquad \lambda_{\substack{3\\4}} = \pm 3 \tag{26}$$

$$\lambda_{\substack{5\\6}} = \rho(1 \pm i), \qquad \lambda_{\substack{7\\8}} = -\rho(1 \pm i) \tag{27}$$

where

$$\rho \equiv \left|\frac{\sqrt{2}}{2}\left(\frac{G}{k}\right)^{\frac{1}{4}}\right| \tag{28}$$

The subscript n has been and henceforth will be dropped for simplicity.

When the first group of λ, λ_i ($i = 1, 2, 3,$ and 4) is substituted into the first two equations (12) to eliminate A_i and B_i, and keeping the leading terms only solutions (8) assume the following forms:

$$u^I = \mp\, m\tan\alpha \left\{ \frac{C_1}{m^2 - 1} + \frac{C_2}{m^2 - 2(1 - \nu)} \frac{1}{y^2} + \frac{C_3}{m^2} y^2 + \frac{4 + 4\nu - m^2}{m^2(7 - 2\nu - m^2)} \frac{C_4}{y^4} \right\} \begin{matrix} \sin \\ \cos \end{matrix} \frac{n\pi\theta}{\theta_1} \tag{29}$$

$$v^I = \tan\alpha \left\{ \frac{C_1}{m^2 - 1} + \frac{2C_2}{m^2 - 2(1 - \nu)} \frac{1}{y^2} + \frac{3C_4}{m^2 - 7 + 2\nu} \frac{1}{y^4} \right\} \begin{matrix} \cos \\ \sin \end{matrix} \frac{n\pi\theta}{\theta_1}$$

$$w^I = \left\{ C_1 + C_2 y^{-2} + C_3 y^2 + C_4 y^{-4} \right\} \begin{matrix} \cos \\ \sin \end{matrix} \frac{n\pi\theta}{\theta_1}$$

When the second group of λ, λ_j, $(j = 5,6,7, \text{ and } 8)$ is used, following the similar procedure, and using some identities to convert the complex expressions into real, one obtains the following solutions:

$$u^{II} = \mp 2(2+\nu)m\tan\alpha\frac{1}{\rho^2}y^{-1}\Big\{y^{\rho}[C_6\cos(\rho\ell ny) - C_5\sin(\rho\ell ny)] - y^{-\rho}[C_8\cos(\rho\ell ny) - C_7\sin(\rho\ell ny)]\Big\}\begin{matrix}\sin\\ \cos\end{matrix}\frac{n\pi\theta}{\theta_1}$$

$$v^{II} = -\nu\tan\alpha\frac{1}{\rho}y^{-1}\Big\{y^{\rho}[(C_5 - C_6)\cos(\rho\ell ny) + (C_5 + C_6)\sin(\rho\ell ny)] - y^{-\rho}[(C_7 + C_8)\cos(\rho\ell ny) - (C_7 - C_8)\sin(\rho\ell ny)]\Big\}\begin{matrix}\cos\\ \sin\end{matrix}\frac{n\pi\theta}{\theta_1} \tag{30}$$

$$w^{II} = y^{-1}\Big\{y^{\rho}[C_5\cos(\rho\ell ny) + C_6\sin(\rho\ell ny)] + y^{-\rho}[C_7\cos(\rho\ell ny) + C_8\sin(\rho\ell ny)]\Big\}\begin{matrix}\cos\\ \sin\end{matrix}\frac{n\pi\theta}{\theta_1}$$

It is noted that the solutions of the first group are simply that of membrane theory.

Based on solutions (29) and (30) one may establish the orders of magnitude of the displacement components[4]

$$u^I,\ v^I,\ w^I,\ w^{II} = 0\left(\frac{1}{\rho^0}\right)$$
$$v^{II} = 0\left(\frac{1}{\rho}\right) \quad \text{and} \quad u^{II} = 0\left(\frac{1}{\rho^2}\right) \tag{31}$$

Due to u^I, v^I and w^I, the magnitudes of the corresponding stresses N_s^I, N_θ^I and $N_{\theta s}^I$ obtained by use of relations (2) are also of the order of $(\frac{1}{\rho^0})$ and the moments are of $(\frac{1}{\rho^3})$ and higher. The order properties of the stresses due to u^{II}, v^{II} and w^{II} are not quite obvious and will be examined as follows.

[4]It is assumed as usual that the parameter m defined by (14) is limited to small values such that the differentiation with respect to the θ does not affect the order of magnitude.

Changing the variable s to y according to (7) and then to η such that

$$y = \eta^{1/\rho} \tag{32}$$

and neglecting the terms which are of the order of $\frac{1}{\rho^3}$ and higher, the stress-displacement relations (2) assume the following forms:

$$
\begin{aligned}
N_s &= \mathcal{D}[\tfrac{1}{2}\rho\eta v,_{\eta} + \nu(u,_{\theta}\sec\alpha + v + w\tan\alpha)] \\
N_{\theta} &= \mathcal{D}[(u,_{\theta}\sec\alpha + v + w\tan\alpha) + \tfrac{1}{2}\nu\rho\eta v,_{\eta}] \\
N_{\theta s} = N_{s\theta} &= \mathcal{D}\frac{1-\nu}{2}[\tfrac{1}{2}\rho\eta u,_{\eta} - u + v,_{\theta}\sec\alpha] \\
M_s &= \mathcal{D}kL\left\{\tfrac{1}{4}\rho^2[\eta^2 w,_{\eta\eta} + (1 - \tfrac{1}{\rho})\eta w,_{\eta}] - \tfrac{1}{4}\rho\eta w,_{\eta} - \tfrac{1}{2}\rho\eta v,_{\eta}\tan\alpha \right. \\
&\quad \left. + \nu(w,_{\theta\theta}\sec^2\alpha + \tfrac{1}{2}\rho\eta w,_{\eta} - u,_{\theta}\sec\alpha\tan\alpha)\right\} \\
M_{\theta} &= \mathcal{D}kL[w,_{\theta\theta}\sec^2\alpha + \tfrac{1}{2}\rho\eta w,_{\eta} + w\tan^2\alpha + v\tan\alpha \\
&\quad + \tfrac{\nu}{4}\left\{\rho^2[\eta^2 w,_{\eta\eta} + (1 - \tfrac{1}{\rho})\eta w,_{\eta}] - \rho\eta w,_{\eta}\right\}] \\
M_{s\theta} &= \mathcal{D}k(1-\nu)L[\tfrac{1}{2}\rho\eta w,_{\eta\theta}\sec\alpha - w,_{\theta}\sec\alpha - \tfrac{1}{2}\rho\eta u,_{\eta}\tan\alpha \\
&\quad + u\tan\alpha] \\
M_{\theta s} &= \mathcal{D}kL(1-\nu)[\tfrac{1}{2}\rho\eta w,_{\eta\theta}\sec\alpha - w,_{\theta}\sec\alpha - \tfrac{1}{4}\rho\eta u,_{\eta}\tan\alpha \\
&\quad + \tfrac{1}{2}u\tan\alpha + \tfrac{1}{2}v,_{\theta}\tan\alpha\sec\alpha]
\end{aligned}
\tag{33}
$$

where a subscript preceded by a comma represents the appropriate derivative.

When the displacements

$$
\begin{aligned}
u &= u^{II} = \frac{1}{\rho^2}\,U \\
v &= v^{II} = \frac{1}{\rho}\,V \\
w &= w^{II} = W
\end{aligned}
\tag{34}
$$

are substituted into relationships (33) and only the terms with the lowest order of $(\frac{1}{\rho})$ are retained the following relationships are obtained

$$
\begin{aligned}
N_s^{II} &= \mathcal{D}[\tfrac{1}{2}\eta V,_{\eta} + \nu W \tan\alpha] \\
N_\theta^{II} &= \mathcal{D}[W \tan\alpha + \tfrac{1}{2}\nu\eta V,_{\eta}] \\
N_{s\theta}^{II} &= N_{\theta s}^{II} = \mathcal{D}\frac{1-\nu}{2}\,\frac{1}{\rho}\,[\tfrac{1}{2}\eta U,_{\eta} + V,_{\theta}\sec\alpha] \\
M_s^{II} &= \mathcal{D}L(1-\nu^2)\tan^2\alpha\frac{1}{\rho^2}\,[\eta^2 W,_{\eta\eta} + \eta W,_{\eta}] \\
M_\theta^{II} &= \nu M_s^{II} \\
M_{s\theta}^{II} &= \mathcal{D}2L(1-\nu)\tan^2\alpha\frac{1}{\rho^3}\,\eta\; W,_{\theta\eta}\,\sec\alpha
\end{aligned}
\tag{35}
$$

in which the relation

$$
k = \frac{4}{\rho^4}\,(1-\nu^2)\,\tan^2\alpha \tag{36}
$$

obtained from expression (28) has been used.

Note that the normal stresses N_s^{II} and N_θ^{II} are of the same order as that of N_s^{I} and N_θ^{I}. It can be shown, however, that N_θ^{I} and N_s^{II} vanish identically. When only the terms of the lowest order of $(\frac{1}{\rho})$ are retained, one has

$$
\begin{aligned}
&u = u^I, \qquad v = v^I, \qquad w = w^I + w^{II} \\
&N_s = N_s^{I}, \quad N_\theta = N_\theta^{II}, \; N_{s\theta} = N_{\theta s} = N_{s\theta}^{I} \\
&M_s = M_s^{II}, \quad M_\theta = M_\theta^{II}, \; M_{s\theta} = M_{\theta s} = M_{s\theta}^{II}
\end{aligned}
\tag{37}
$$

By the similar comparison of order property one can show that the transverse and tangential shearing forces defined by equations

(10) and (24) are

$$S_\theta = S_\theta^{\ II}, \qquad S_s = S_s^{\ II}, \qquad T_s = T_s^{\ I} = N_{s\theta}^{\ \ I} \tag{38}$$

Thus the two sets of solution, membrane and bending are coupled by the lateral deflection w; otherwise, they would be separable.

In view of equations (37), (38) and (34), and when solutions (29) and (30) are used, the stresses and moments may be given in the following final explicit forms:

$$N_s = -2E\delta \tan\alpha \left[\frac{C_2}{m^2 - 2(1-\nu^2)} y^{-2} + \frac{3C_4}{m^2 - 7 + 2\nu} y^{-4}\right] \begin{matrix}\cos\\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1}$$

$$N_\theta = E\delta y^{-1} \tan\alpha \Big\{ y^{\rho}[C_5\cos(\rho \ell n y) + C_6 \sin(\rho \ell n y)]$$
$$+ y^{-\rho}[C_7\cos(\rho \ell n y) + C_8\sin(\rho \ell n y)]\Big\} \begin{matrix}\cos\\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1}$$

$$N_{s\theta} = T_s = \mp E\delta \left\{ \frac{6\tan\alpha}{m(m^2 - 7 + 2\nu)} C_4 y^{-4} \right\} \begin{matrix}\sin\\ \cos\end{matrix} \frac{n\pi\theta}{\theta_1}$$

$$M_s = \frac{2E\delta}{\rho^2} \tan^2\alpha \; Ly \Big\{ y^{\rho}[C_6\cos(\rho \ell n y) - C_5\sin(\rho \ell n y)]$$
$$+ y^{-\rho}[-C_8\cos(\rho \ell n y) + C_7\sin(\rho\ ny)]\Big\} \quad \begin{matrix}\cos\\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1}$$

$$M_\theta = \nu M_s$$

$$S_\theta = \mp \frac{2E\delta}{\rho^2} m(2-\nu)\tan^2\alpha y^{-1} \Big\{ y^{\rho}[C_6\cos(\rho \ell n y) - C_5\sin(\rho \ell n y)]$$
$$+ y^{-\rho}[-C_8\cos(\rho \ell n y) + C_7\sin(\rho \ell n y)]\Big\} \quad \begin{matrix}\sin\\ \cos\end{matrix} \frac{n\pi\theta}{\theta_1} \tag{39}$$

$$S_s = \frac{E\delta}{\rho} \tan^2\alpha y^{-1} \Big\{ y^{\rho}[(-C_5 + C_6)\cos(\rho \ell n y) - (C_5 + C_6)\sin(\rho \ell n y)]$$
$$+ y^{-\rho}[(C_7 + C_8)\cos(\rho \ell n y) - (C_7 + C_8)\sin(\rho \ell n y)]\Big\} \begin{matrix}\cos\\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1}$$

and

$$\frac{\partial w}{\partial s} = \frac{\partial w^{II}}{\partial s}$$

$$= \frac{1}{2L}\rho y^{-3} \Big\{ y^{\rho}[(C_5 + C_6)\cos(\rho \ell ny) - (C_5 - C_6)\sin(\rho \ell ny)]$$

$$-y^{-\rho} [(C_7 - C_8)\cos(\rho \ell ny) + (C_7 + C_8)\sin(\rho \ell ny)]\Big\} \begin{matrix}\cos\\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1}$$

Particular Solutions due to Lateral Normal Loads

Let the lateral normal load given by(11) be confined in a form

$$P_{rn}(y) = a_n L^{\beta} y^{2\beta} \tag{40}$$

i.e.

$$P_{rn}(s) = a_n s^{\beta}$$

where a_n and β are prescribed.

One may assume a set of particular solution in the similar form as given by expressions (8) except λ_n, in this case λ_n shall be replaced by

$$\lambda^* = 2\beta + 3 \tag{41}$$

a known number. Then the particular solutions are readily obtained by solving the three algebraic equations (12) simultaneously provided that λ^* is none of the roots of the determinant. However, in one of the most common loadings the load is uniformly distributed along meridians, $\beta = 0$ hence $\lambda^* = 3$ which is one of the roots at the asymptotic case. In the case such as this the approach needs to be modified. In what follows the particular solution due to this kind of uniform load is given.

Since in this case λ^* is a finite constant when the parameter k approaches zero, the corresponding particular solution may be obtained from the equations of membrane theory of the system.

Setting $k = 0$ and having the independent variable s transformed to y, equations (6) reduce to the following equations of equilibrium of membrane theory with a lateral load P_r:

$$\frac{1-\nu}{8}[y^2u,_{yy} + 3yu,_y - 8u] + \frac{1+\nu}{4}yu,_{\theta y}\sec\alpha + u,_{\theta\theta}\sec^2\alpha$$
$$+ (2-\nu)v,_\theta\sec\alpha + w,_\theta\sec\alpha\tan\alpha = 0$$
$$\frac{1+\nu}{8}yu,_{\theta y}\sec\alpha - \frac{3}{2}(1-\nu)u,_\theta\sec\alpha + \frac{1}{4}y^2v,_{yy} + \frac{3}{4}yv,_y$$
$$+ \frac{1-\nu}{2}v,_{\theta\theta}\sec^2\alpha - (1-\nu)v + \frac{1}{2}\nu yw,_y\tan\alpha - (1-\nu)w\tan\alpha = 0 \tag{42}$$
$$u,_\theta\sec\alpha + \frac{1}{2}\nu yv,_y + v + w\tan\alpha = \frac{L}{\mathcal{D}}P_r y^2$$

where

$$P_r = a_n \begin{matrix}\cos\\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1} \tag{43}$$

Let the particular solutions of equations (42) be assumed as below:

$$u^P = (\mp d_1 + d_2\ell ny)y^2 \begin{matrix}\sin\\ \cos\end{matrix} \frac{n\pi\theta}{\theta_1}$$
$$v^P = (b_1 + b_2\ell ny)y^2 \begin{matrix}\cos\\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1} \tag{44}$$
$$w^P = e_1(1 + \ell ny)y^2 \begin{matrix}\cos\\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1}$$

in which d_1, d_2, b_1, b_2 and e_1 are constants to be determined. When these assumed solutions are substituted into equations (42) and after the sinusoidal functions and y^2 are cancelled, one will have three equations in the following fashion.

$$f_\phi \ell ny + h_\phi = \frac{a_n}{\tan\alpha}\frac{L}{\ell}\delta_{\phi 3} \tag{45}$$

where the subscript ϕ (= 1, 2, 3) indicates the three equations of (42) respectively, f_ϕ and h_ϕ are expressions of the physical and to-be-determined constants, and $\delta_{\phi 3}$ is the Kronecker delta.

By making the coefficients of both sides of equations (45) equal, there are two sets of algebraic equations; each contains three equations that

$$f_\phi = 0 \tag{46}$$

$$h_\phi = \frac{a_n}{\tan\alpha}\frac{L}{\ell}\delta_{\phi 3} \tag{47}$$

There are, however, only two of equations (46) that are independent because $\lambda^* = 3$ is one of the roots of the determinant. Thus the five constants may be determined by the five independent equations of (46) and (47). The results are

$$u^P = \mp \frac{a_n}{\tan\alpha}\frac{L}{E\delta}\frac{m}{3}\left\{\frac{1}{2m^2}\left[2m^4 - 3(5-\nu)m^2 - 3(1+\nu)\right] \mp (m^2 - 7 + 2\nu)\ell ny\right\} y^2 \begin{matrix}\sin\\ \cos\end{matrix}\frac{n\pi\theta}{\theta_1} \tag{48}$$

$$v^P = \frac{a_n}{\tan\alpha}\frac{L}{E\delta}\frac{1}{6}\left[3(1-2\nu) - m^2\right]y^2 \begin{matrix}\cos\\ \sin\end{matrix}\frac{n\pi\theta}{\theta_1}$$

$$w^P = \frac{a_n}{\tan^2\alpha}\frac{L}{E\delta}\frac{1}{3}m^2\left[m^2 - 7 + 2\nu\right](1 + \ell ny) \begin{matrix}\cos\\ \sin\end{matrix}\frac{n\pi\theta}{\theta_1}$$

When these displacements are substituted into the expressions (2) with $k = 0$ the corresponding stresses are

$$N_s^P = \frac{a_n L}{\tan\alpha}\left\{\frac{1}{6}(3 - m^2)y^2 + \frac{\nu}{1-\nu^2}\frac{m^2}{3}(m^2 - 7 + 2\nu)(1 \pm 1)\ell ny\right\} \begin{matrix}\cos\\ \sin\end{matrix}\frac{n\pi\theta}{\theta_1} \tag{49}$$

$$N_\theta^{\,P} = \frac{a_n L}{\tan\alpha}\left[y^2 + \frac{1}{1-\nu^2}\frac{m^2}{2}(m^2 - 7 + 2\nu)(1 \pm 1)\,\ell n y\right] \begin{matrix}\cos\\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1}$$

$$N_{s\theta}^{P} = \frac{a_n L}{\tan\alpha}\,\frac{m\,y^2}{12(1+\nu)}\left[(m^2 - 7 + 2\nu) \mp (3 - 6\nu - m^2)\right] \begin{matrix}\sin\\ \cos\end{matrix} \frac{n\pi\theta}{\theta_1}$$

These particular solutions combined with those given by solutions (29) (30) and (39) constitute the complete solutions.

An Example

For purpose of illustration, take a truncated semi-circular cone with the two generators simply supported. Thus the lower set of solutions (29) (30) (39), (48) and (49) are to be used. Let it be clamped at the smaller end at $s = L_1$ and free at the other end where $s = L$ so that

$$u = v = w = \frac{\partial w}{\partial s} = 0 \qquad \text{at} \quad y = \sqrt{\frac{L_1}{L}} \tag{50}$$

$$N_s = T_s = M_s = S_s = 0 \qquad \text{at } y = 1$$

By making use of the first two in each of the foregoing two sets of boundary conditions, constants C_1, C_2, C_3 and C_4 can be determined; then the other four constants can be determined by the remaining four boundary conditions.

The lateral normal loads are also known as wind loads. Usually there are two types of such loads: symmetrical and non-symmetrical. Since the asymptotic solutions hold for small values of n only, two cases of $n = 1$ and $n = 2$ are considered.

Let

$$a_n = p \qquad \text{for } n = 1$$
$$= 0 \qquad \text{for } n > 1 \tag{51}$$

that

$$P_r = p \sin\theta$$

represent a symmetrical load. For

$$a_n = \frac{4}{9}\sqrt{3}\, p \qquad \text{for } n = 1$$

$$= \frac{2}{9}\sqrt{3}\, p \qquad \text{for } n = 2$$

$$= 0 \qquad \text{for } n > 2$$

that

$$P_r = \frac{4}{9}\sqrt{3}\, p(\sin\theta + \frac{1}{2}\sin 2\theta) \tag{52}$$

represents a non-symmetrical load. These two types of loads are depicted in Fig. 1.

For numerical computations the following values are assumed

$$\alpha = 75^{\circ}, \qquad \nu = \frac{1}{3}, \qquad \sqrt{\frac{L_1}{L}} = 0.90 \tag{53}$$

Considering $\frac{t}{R}$ as a parameter where R is the principle radius at a section of thickness t, thus $\delta = \frac{t}{R}\cos\alpha$, the eight roots of λ computed from expressions (20) (21) and expressions (26) (27) for asymptotic values are listed in Table 1.

Table 1. The Values of λ

λ	$\frac{t}{R}$	n = 1	n = 2	Asymptotic Values
$\lambda_{\frac{1}{2}}$	0.004	± 0.999999	± 1.0523	± 1
	0.006	± 0.999997	± 1.1142	± 1
	0.008	± 0.999995	± 1.1955	± 1
$\lambda_{\frac{3}{4}}$	0.004	± 3.00003	± 2.9851	± 3
	0.006	± 3.00007	± 2.9663	± 3
	0.008	± 3.00013	± 2.9397	± 3
$\lambda_{\frac{57}{68}}$	0.004	$\pm 153.27(1.0027\pm i)$	$\pm 152.75(1.0099\pm i)$	$\pm 153.48(1\pm i)$
	0.006	$\pm 125.09(1.0035+i)$	$\pm 124.51(1.0149\pm i)$	$\pm 125.32(1\pm i)$
	0.008	$\pm 108.28(1.0045\pm i)$	$\pm 107.77(1.0198\pm i)$	$\pm 108.53(1\pm i)$

It is evident that as these roots are concerned for this case, the asymptotic results are quite satisfactory for practical use.

The asymptotic solutions of displacements, stresses and moments computed from expressions (29) (30), (39) combined with (48) and (49) may be given in the form:

$$F_n (y,\theta) = f_n(y) \begin{matrix} \sin \\ \cos \end{matrix} \frac{n\pi\theta}{\theta_1} \quad n = 1 \text{ and } 2 \tag{54}$$

in which the function $f_n(y)$ are presented in Figs. (2) to (11).

Closing Remarks

There are a number of approaches for solutions of shells of revolution available. A recent one was presented by Kalnins [2] by treating the system of equations as a series of initial-value problems. A comprehensive list of literature was also available there. Conical shells subjected to edge loads was studied by Clark and Garibotti [3] by using of edge effect approach.

The present solutions are in explicit forms and readily to be used for practical purpose; the asymptotic solutions are exact and applicable to conical shells

$$\left[\frac{1}{12}\left(\frac{t}{R}\cos\alpha\right)^2\right]^{\frac{1}{4}} \ll 1 .$$

When the above parameter is very small the solutions may be useful for conical shells whether of linearly varying thickness or of constant thickness such as those given in the example.

In the given example the bending effects diminish rapidly from the clamped edge as this is known as edge effect or boundary layer phenomenon. The moments and shearing forces due to the bending effect are of higher order as compared to the membrane stresses. However the membrane stress N_θ induced by the bending effect is of the same order as the other membrane stresses. Thus solutions of the membrane theory alone not only make the solutions incompatiable but also make some errors not negligible in the membrane stress N_θ.

The deflection particularly the lateral normal component at the free end in the given example is comparatively large to the thickness. For such large displacement, the theory is applicable provided the shell is not overstrained [4]. Thus the strain at the fixed end control the validity of the results.

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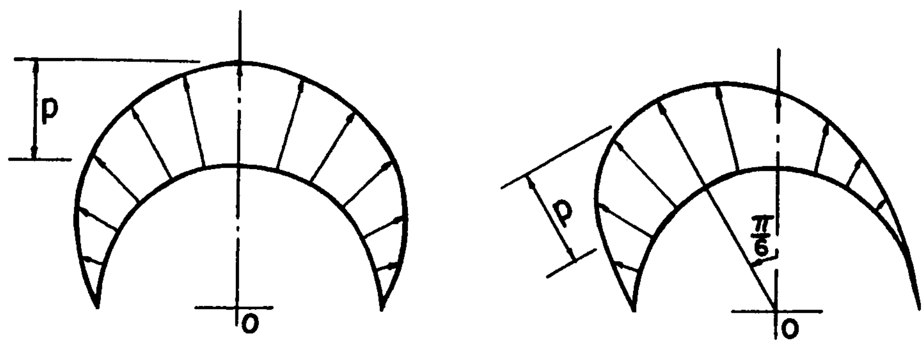

THE SYMMETRICAL LOAD THE ASYMMETRICAL LOAD

FIG. I

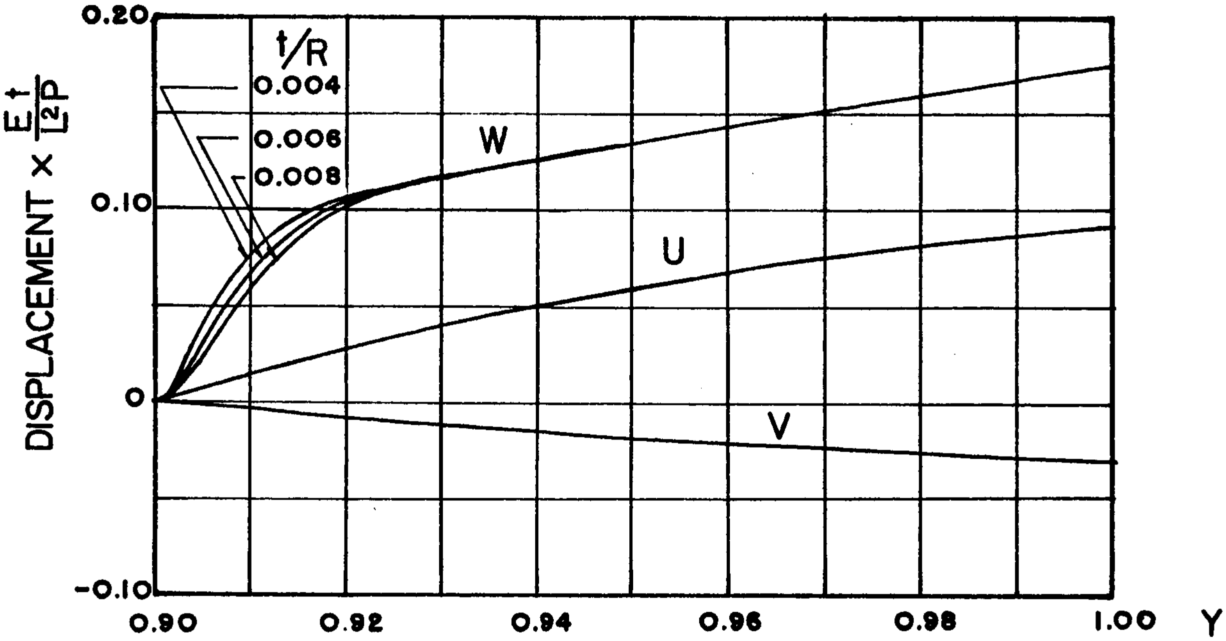

FIG. 2 DISPLACEMENTS U, V AND W (N=1)

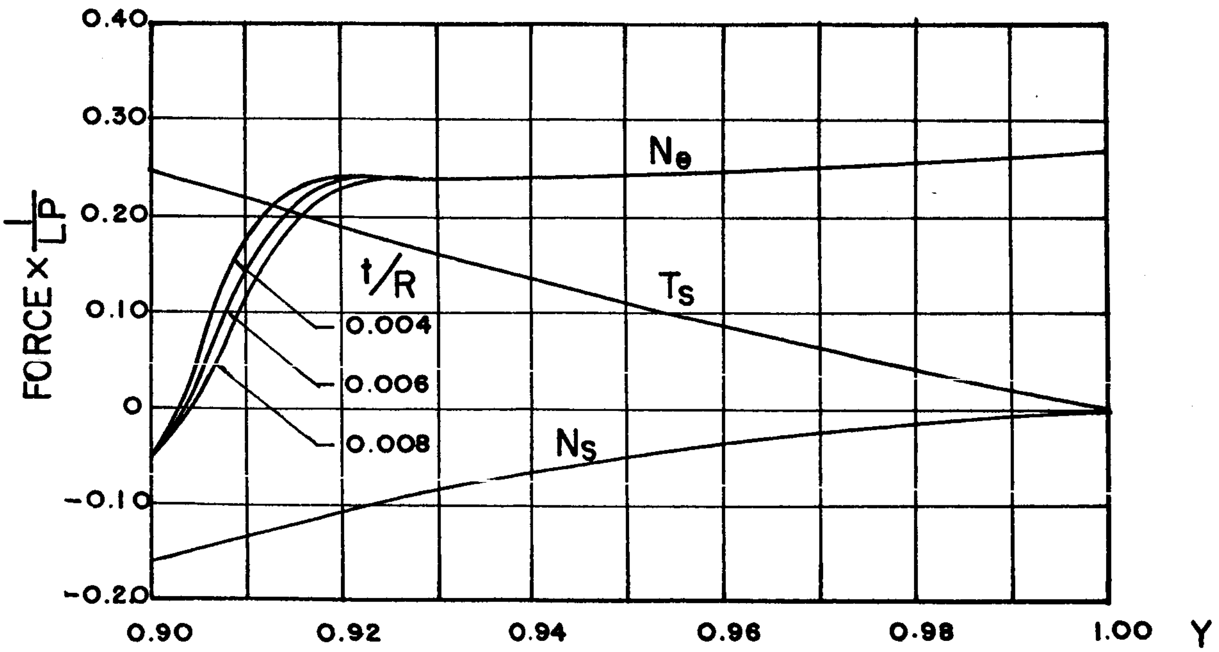

FIG. 3 MEMBRANE FORCES N_θ T_S AND N_S (N=1)

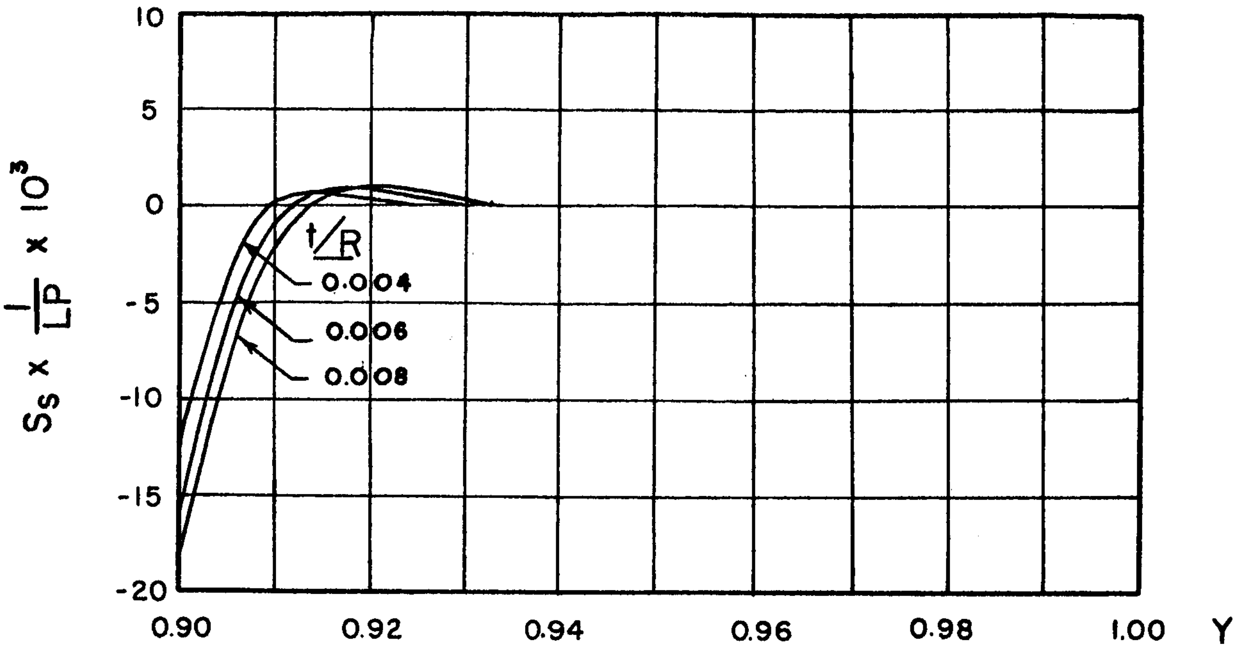

FIG. 4 TRANSVERSE SHEARING FORCE S_S (N=1)

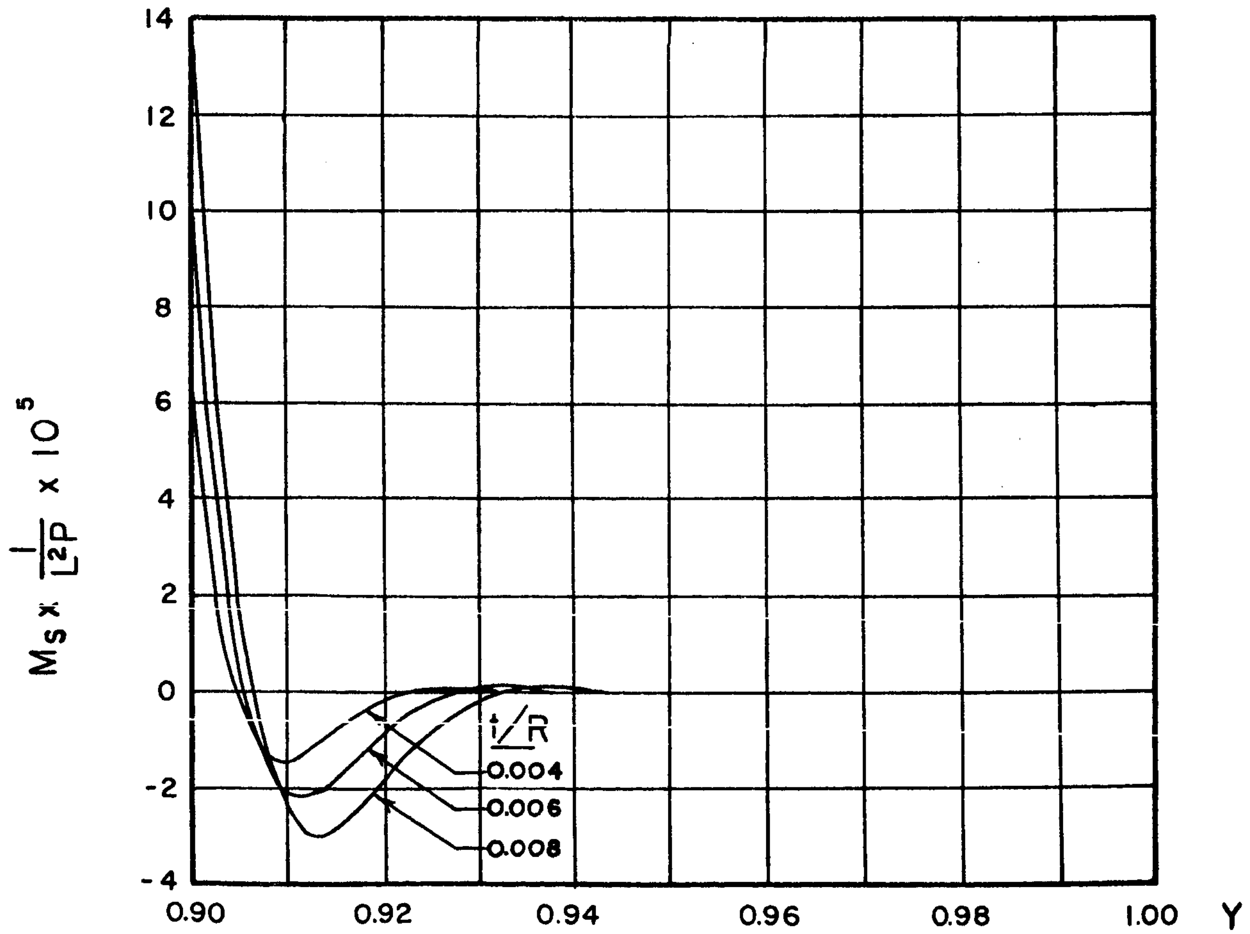

FIG. 5 NORMAL MOMENT M_S (N = 1)

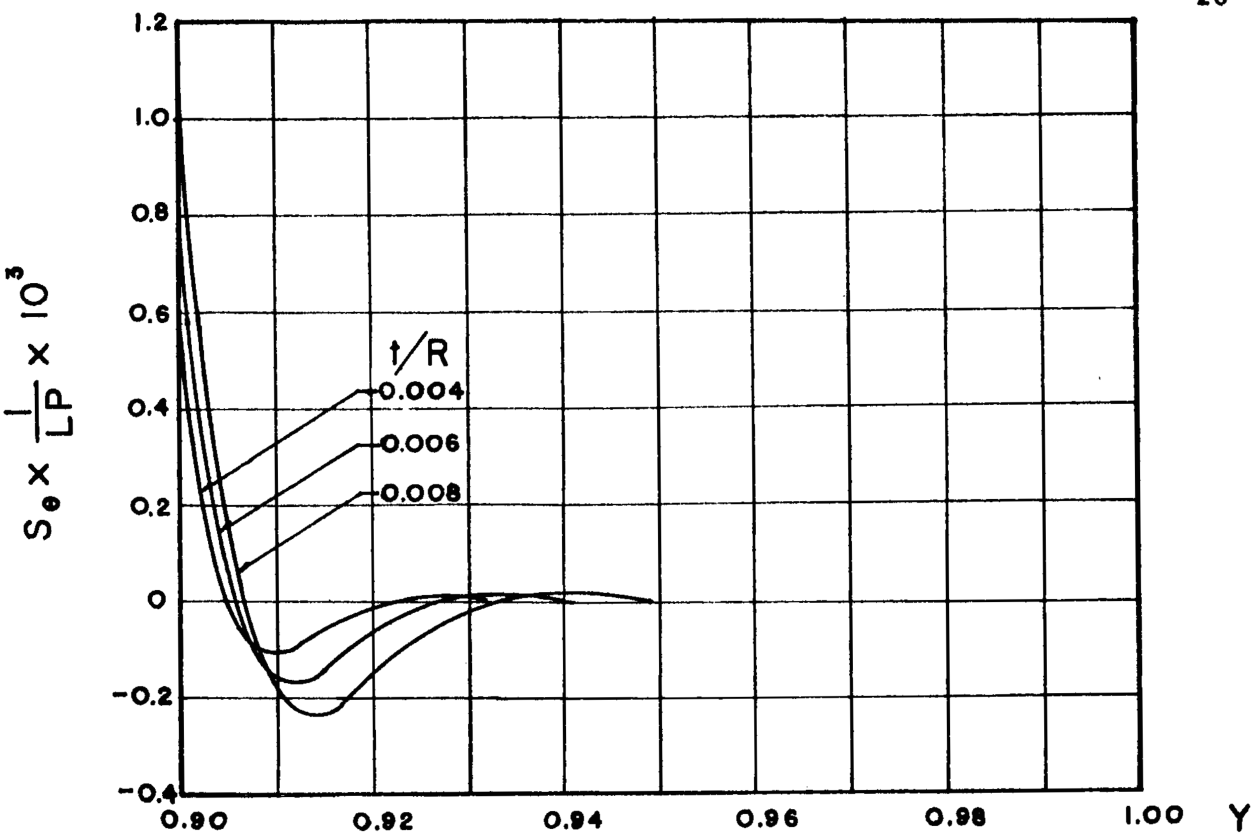

FIG. 6 TRANSVERSE SHEARING FORCE S_θ (N=1)

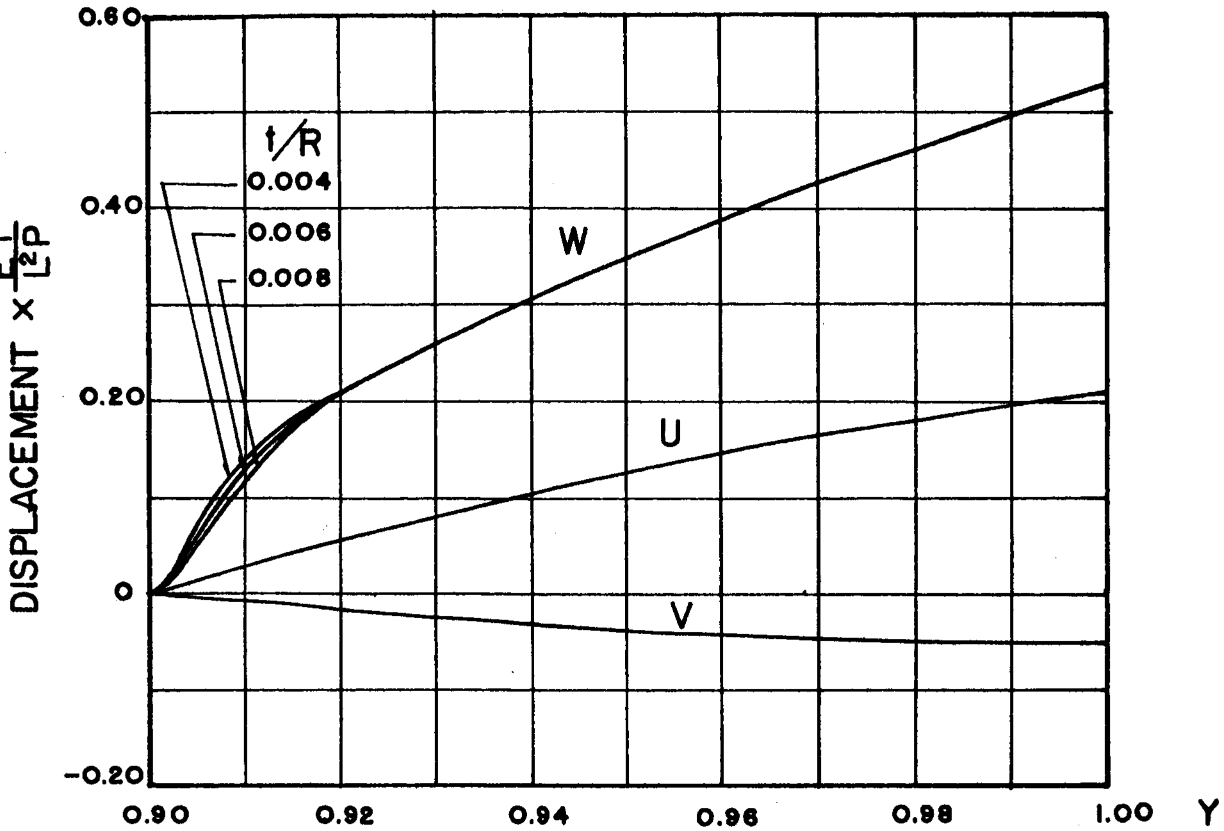

FIG. 7 DISPLACEMENTS U, V AND W (N=2)

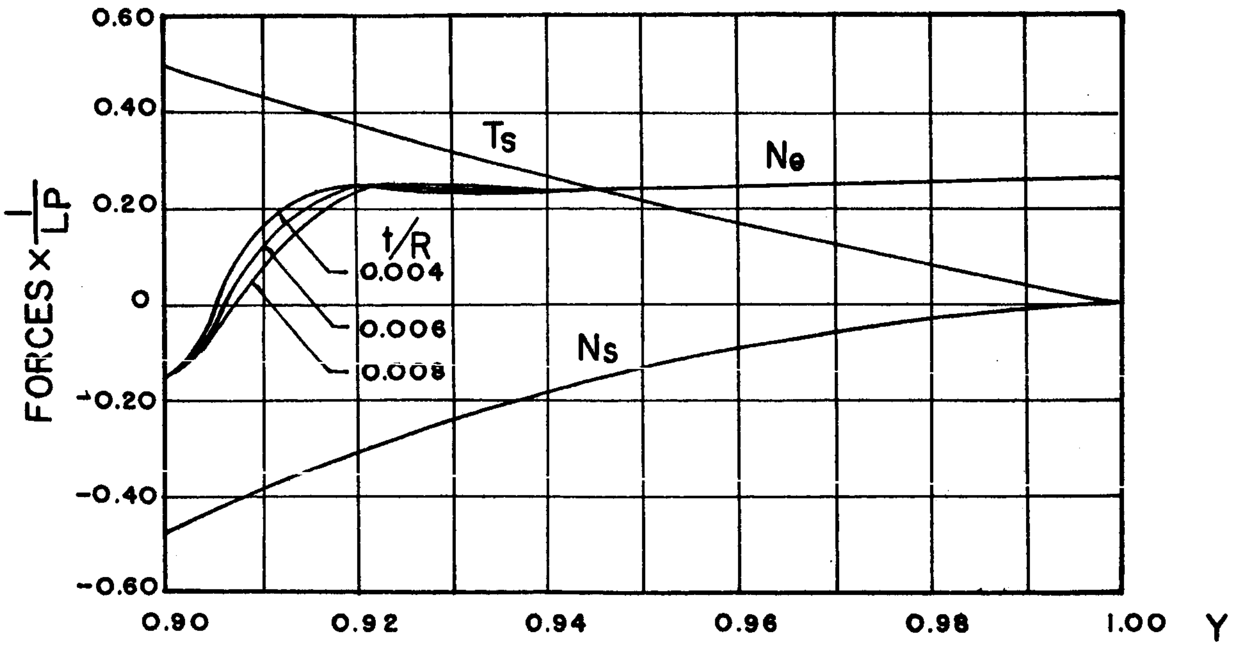

FIG. 8 MEMBRANE FORCES N_θ T_s AND N_s (N=2)

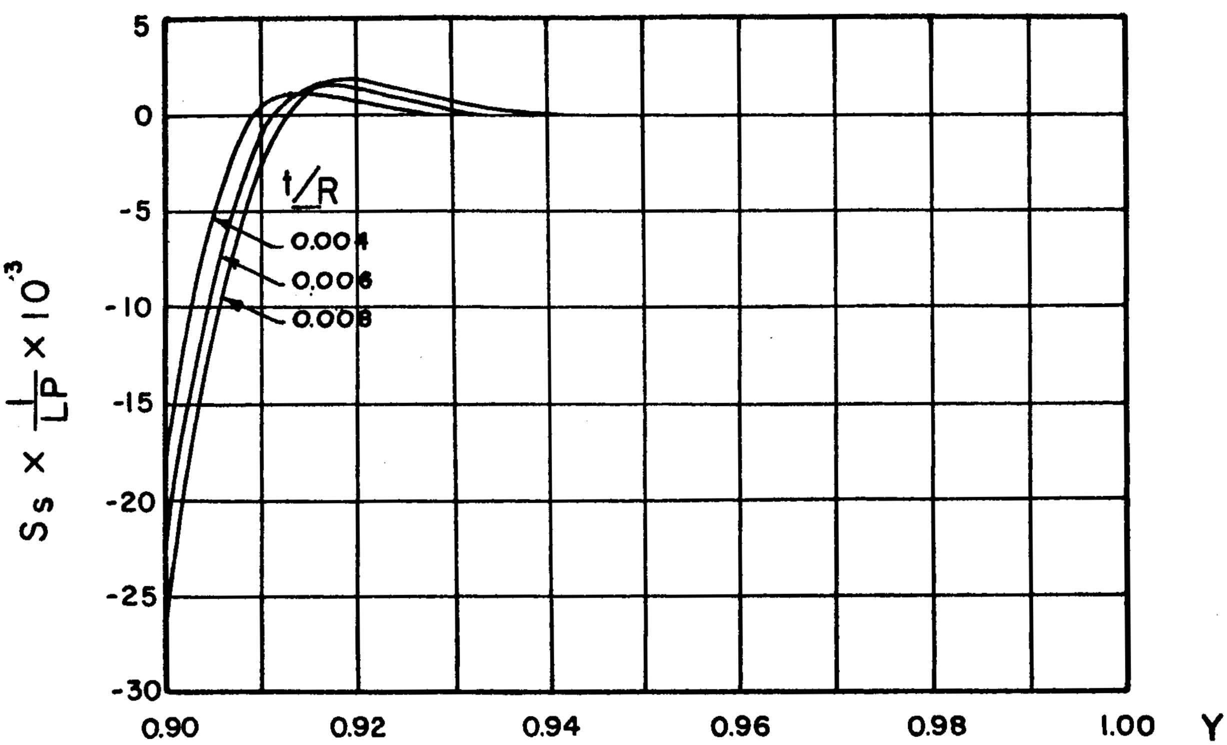

FIG. 9 TRANSVERSE SHEARING FORCE S_s (N=2)

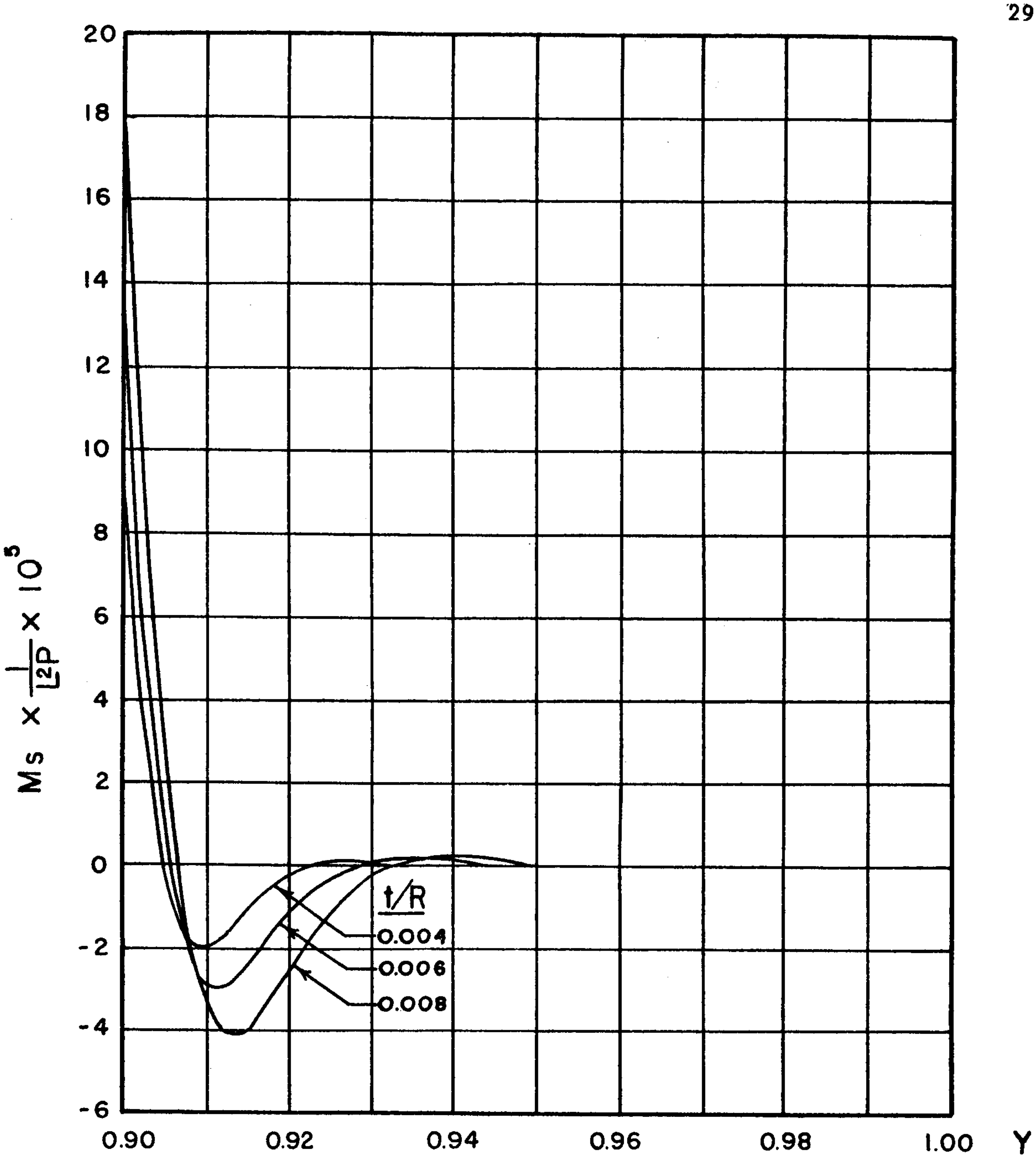

FIG. 10 NORMAL MOMENT M_S (N=2)

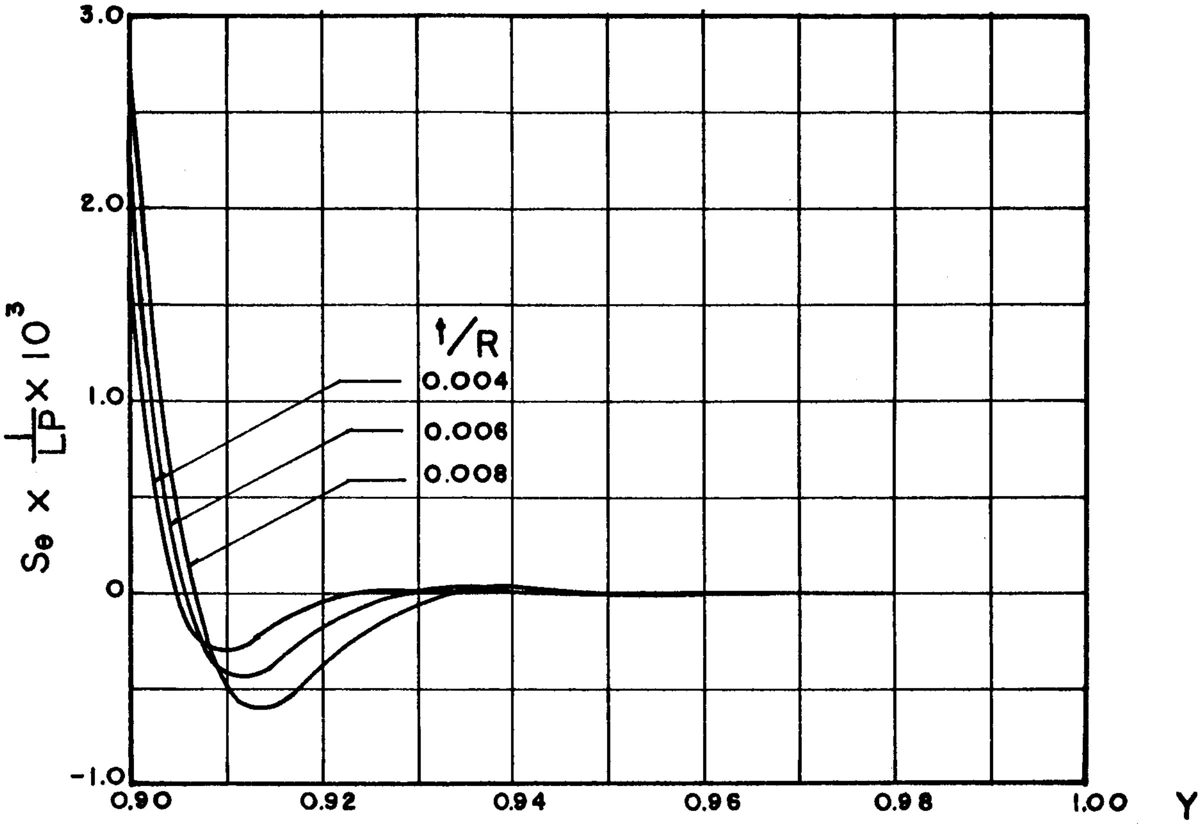

FIG. 11 TRANSVERSE SHEARING FORCE S_θ (N=2)