NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report No. 32-839

Injection Accuracy Characteristics for Lunar Missions

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ABSTRACT

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This Report investigates a technique for expressing launch vehicle injection accuracy in terms of spacecraft midcourse correction requirements for Earth–Moon missions. A figure-of-merit is defined which can be used as a measure of injection accuracy. The variables important to guidance accuracy such as injection energy, launch azimuth, lunar declination, and Earth–Moon distance are discussed. A "best" target coordinate system is developed and numerical results which enable the engineer to relate launch vehicle injection accuracy to spacecraft midcourse correction requirements are presented. Both "miss-only" type midcourse corrections and "miss-plus-time" corrections are considered.

Author

I. INTRODUCTION

In the design of an Earth–Moon mission, a nominal trajectory (or trajectories) is selected which satisfies the mission constraints. This nominal trajectory consists of (1) a boost phase which extends from launch to injection, defined as the final termination of booster thrust, (2) a coast phase where the spacecraft moves from injection to the Moon, and (3) a terminal phase where the spacecraft is in the close vicinity of the Moon. With present-day launch vehicles, the inaccuracy of the injection guidance system is such that injection coordinate errors are generally large enough to require a spacecraft midcourse correction during the coast phase of the mission in order to null the resulting target errors. The statistical description of the magnitude of this correction offers a convenient means for expressing the launch vehicle injection accuracy. Indeed, it is the most reasonable way of expressing injection accuracy, since the goal of the injection guidance system is really to minimize the magnitude of the midcourse correction. Furthermore, the statistical magnitude of the midcourse correction offers a single number for expressing the injection accuracy, while a complete description of the accuracy at the injection point would require a 6×6 covariance matrix.

This Report is concerned with investigating the technique of expressing launch vehicle injection accuracy in terms of the spacecraft midcourse correction requirements for Earth–Moon missions. The analysis deals mainly with the coast phase of the mission and discusses the differential corrections relating injection errors to the target errors.

In the development of the Report, three coordinate systems are defined. In Section III it is shown that of these systems one is a "best" target coordinate system in the sense that it gives the reader the capability to discern the important variables of the subject. In addition, when numerical results are presented in this coordinate system, the reader is able to select a "worst-case" trajectory from the point of view of requiring the maximum midcourse correction for a given set of injection errors.

II. COORDINATE SYSTEMS

It is convenient to define the following three coordinate systems: (1) the injection coordinate system, (2) the **T–R–S**$_I$ coordinate system, which has been widely used as a target coordinate system, and (3) a "best" target coordinate system.

A. *Injection Coordinate System*

The injection coordinate system is chosen such that four of the coordinates lie in the nominal plane of motion of the spacecraft at injection and the remaining two coordinates are normal to this plane of motion. This system is illustrated in Fig. 1, where x is the distance measured along the (spherical) Earth's surface from the launch pad to the spacecraft; R is the distance between the spacecraft and the center of the Earth; V is the component of the inertial velocity vector in the nominal plane of motion, Γ is the angle between V and the local horizontal, W is the lateral position of the spacecraft from the nominal plane of motion, and $\dot{W}$ is the lateral component of inertial velocity (note that for a nominal trajectory, W and $\dot{W}$ are both zero). R_E is the radius of the Earth.

Note that with this convention a perturbation in x causes a change in the direction of the inertial velocity vector, since Γ is measured from the local horizontal.

Furthermore, to first-order, perturbations in x, R, V, and Γ cause no perturbations in W and $\dot{W}$; likewise, perturbations in W and $\dot{W}$ cause no changes in the four in-plane coordinates.

B. T–R–S$_I$ *Coordinate System*

The two-dimensional miss parameter, defined as **B**, is generally used to measure miss distances for lunar trajectories and is described by W. Kizner in Ref. 1. **B** has the desirable feature of being very nearly a linear function of changes in injection conditions over the range of reasonable injection errors. The osculating conic at closest approach to the Moon is used in defining **B**. **B** is the vector from the Moon's center of mass perpendicular to the incoming asymptote. Let **S**$_I$ be a unit vector in the direction of the incoming asymptote. The orientation of **B** in the plane normal to **S**$_I$ is described in terms of two unit vectors **R** and **T**, both normal to **S**$_I$. **T** is conventionally taken parallel to a fixed reference plane and **R** completes a right-handed orthogonal system. Figure 2 illustrates the coordinate system.

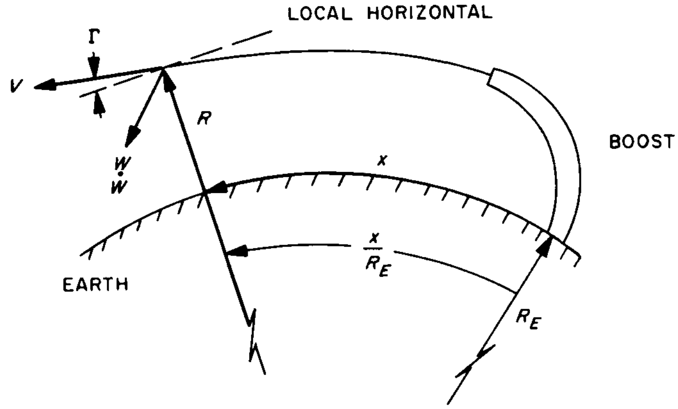

Fig. 1. Injection coordinate system

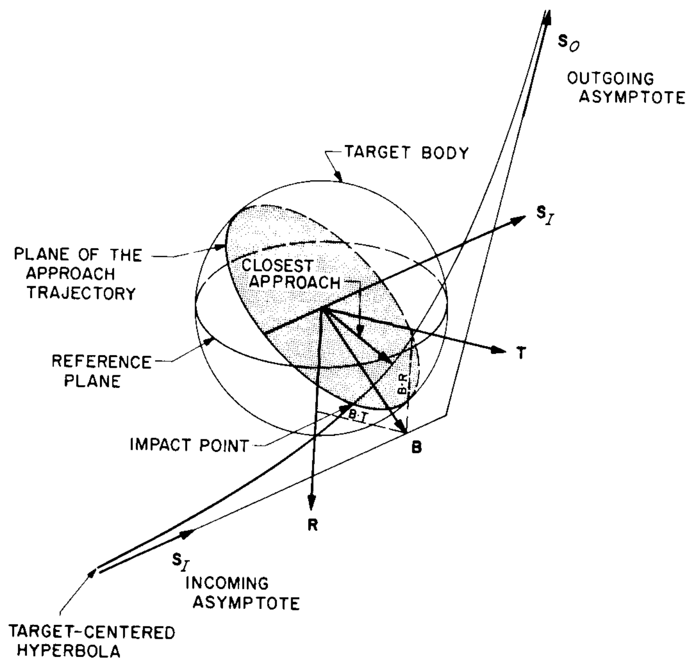

Fig. 2. T–R–S$_I$ coordinate system

The third component of the system is the target error in the S-direction computed under the assumption that the Moon's mass is zero and is directly related to the error in the actual flight time to closest approach (Ref. 2).

C. *"Best" Target Coordinate System*

In the **T–R–S**$_I$ system we observe that the differential corrections relating the miss at the target to injection errors vary with declination of the Moon and with injection launch azimuth. Seeking to study the statistical effect of injection guidance errors for a set of nominal trajectories, we find that it is difficult to choose a worst-case trajectory from data computed in this system.

For these reasons we wish to investigate a new set of coordinates and consider the inertial $\mathbf{X}_T$–$\mathbf{Y}_T$–$\mathbf{W}_T$ coordinate system shown in Fig. 3. We assume that the Moon is a massless point constrained to move in the orbit of the Moon and we select a selenocentric coordinate system at the standard massless impact time. The $\mathbf{Y}_T$-axis is in the instantaneous direction of the Earth and the $\mathbf{W}_T$-axis is normal to the plane of the Earth-centered spacecraft orbit. The $\mathbf{X}_T$-axis completes the right-handed system ($\mathbf{X}_T = \mathbf{Y}_T \times \mathbf{W}_T$). Since the orientation of the $\mathbf{X}_T$–$\mathbf{Y}_T$–$\mathbf{W}_T$ system is always the same relative to the spacecraft plane of motion (relative to the Earth), it seems reasonable to expect that the differential corrections defined in this system will be independent of the Moon's declination and of the injection launch azimuth. This fact will enable us to draw important conclusions later in the Report.

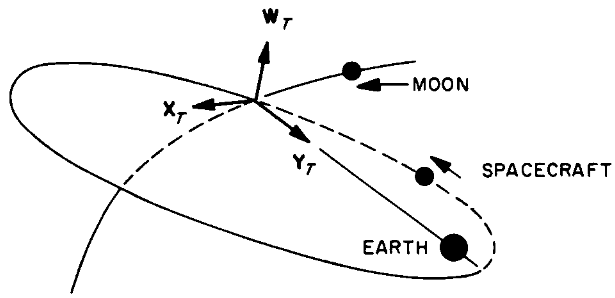

Fig. 3. "Best" target coordinate system

III. BASIC EQUATIONS AND IMPORTANT VARIABLES

In considering present-day lunar missions and analyzing expected injection errors, we find that the perturbed transfer trajectories have three important characteristics (Ref. 3): (1) the trajectories which are in error differ only slightly from the standard, hence linear perturbation theory may be used; (2) the sources of the injection errors — for example, gyro drift and accelerometer scale factor — are independent random variables with zero means; and (3) the magnitudes of the individual errors are distributed approximately to the Gaussian law. These three characteristics are exploited in the computation of injection accuracy and the related spacecraft midcourse correction requirements. The injection error vector is defined as δq, where

$$\delta q = \begin{bmatrix} \delta x \\ \delta R \\ \delta V \\ \delta \Gamma \\ \delta W \\ \delta \dot{W} \end{bmatrix}$$

and is obtained by linear perturbation analysis:

$$\delta q = A \delta e$$

where δe is an $n \times 1$ vector of launch vehicle component errors and A is a $6 \times n$ matrix of differential corrections relating injection errors to launch vehicle guidance component errors.

The miss at the Moon resulting from the injection errors can be approximated with linear perturbation theory as

$$m = U \delta q$$

where m is a 3×1 target error vector and U is a 3×6 matrix of differential corrections which map injection errors to target errors. Defining K to be the 3×3 mapping matrix which maps midcourse velocity perturbations to perturbations in position at the target, the midcourse velocity correction vector δV required to null the target miss due to injection errors can be written as

$$\delta V = -K^{-1} U A \delta e$$

where K^{-1} is the inverse of K. Denoting the expectation operation by $E\{\cdot\}$, the covariance matrix of the midcourse correction is given by

$$\Lambda_V = K^{-1} U E \left\{ A \delta e \, \delta e^T A^T \right\} U^T (K^{-1})^T$$

where U^T is the transpose of U. Let Λ_I be the injection covariance matrix; then

$$\Lambda_V = K^{-1}\, U\, \Lambda_I U^T\, (K^{-1})^T \qquad (1)$$

Equation (1) is used to describe the injection accuracy in terms of the 3×3 midcourse correction covariance matrix. This matrix gives the statistics of the midcourse correction required to place the spacecraft on a trajectory having the desired, or nominal, terminal conditions. Under the assumption that the error sources described in the δe vector are independent Gaussian random variables with zero means, the Λ_V matrix can be related to the probability of having a sufficient midcourse correction capability. Figure 4 is an example of probability of sufficient capability as a function of the capability for three Λ_V's: (1) a pencil-shaped distribution (one-dimensional Gaussian), (2) a pancake-shaped distribution (Rayleigh), and (3) a spherical distribution (chi-squared with three degrees of freedom).

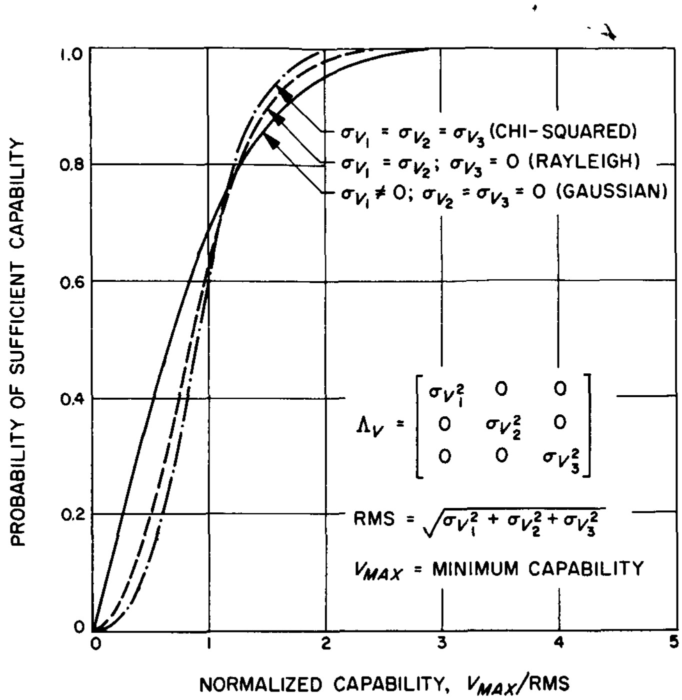

Fig. 4. Probability of sufficient capability

A. *Figure-of-Merit*

From Fig. 4, it is apparent that the trace of the midcourse covariance matrix relates injection accuracy to spacecraft midcourse correction requirements and can be used as a measure of the accuracy of the injection guidance system.* With this in mind, we define a spacecraft midcourse figure-of-merit (FOM) as the square root of the trace of the midcourse correction covariance matrix. With the pencil-shaped distribution in mind, we conclude that it is desirable to have a midcourse correction capability of at least three times the FOM, where the FOM is computed for a set of worst-case trajectory parameters. There are two figure-of-merits commonly in use: (1) miss-only figure-of-merit (FOM_M) which assumes that the midcourse correction is designed to null the **B** plane errors disregarding the errors in flight time, and (2) miss-plus-time figure-of-merit (FOM_{M+T}) which assumes that the midcourse will correct for errors both in miss and time of flight.

B. *Important Variables: "Miss-plus-Time" Correction*

We define a miss-plus-time midcourse correction figure-of-merit (FOM_{M+T}) to be the square root of the trace of the midcourse covariance matrix. From Eq. (1)

$$FOM_{M+T} = \sqrt{\text{TRACE}}\ \{K^{-1} U_{XYW}\, \Lambda_I\, U^T_{XYW}\, (K^{-1})^T\} \qquad (2)$$

*First pointed out in an unpublished paper by C. G. Pfeiffer of the Jet Propulsion Laboratory.

where U_{XYW} maps perturbations at a fixed injection time in the x–R–V–Γ–W–$\dot{W}$ system to perturbations at a fixed impact time in the $\mathbf{X}_T$–$\mathbf{Y}_T$–$\mathbf{W}_T$ target system.

$$U_{XYW} = \begin{bmatrix} \frac{\partial X_T}{\partial x} & \frac{\partial X_T}{\partial R} & \frac{\partial X_T}{\partial V} & \frac{\partial X_T}{\partial \Gamma} & 0 & 0 \\ 0 & \frac{\partial Y_T}{\partial R} & \frac{\partial Y_T}{\partial V} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial W_T}{\partial W} & \frac{\partial W_T}{\partial \dot{W}} \end{bmatrix}$$

$\partial Y_T/\partial\Gamma$ is zero only when the injection true anomaly is zero. However, for true anomalies between -10 and $+20$ deg,

$$\frac{\partial X_T}{\partial \Gamma} \gg \frac{\partial Y_T}{\partial \Gamma}$$

hence $\partial Y_T/\partial\Gamma$ can be assumed zero for the purpose of this Report. Since the $\mathbf{X}_T$–$\mathbf{Y}_T$–$\mathbf{W}_T$ system considers the Moon to be a massless point moving in space, U_{XYW} is a function of only the in-plane standard trajectory parameters: (1) injection energy C_3, (2) transfer trajectory perigee altitude R_P, (3) injection true anomaly η_I, and (4) Earth–Moon distance R_{EM} at the standard impact time.

From Ref. 4, the K matrix may be approximated by time-to-go, t_{go}, times the identity matrix, where t_{go} is the standard impact time less the midcourse maneuver time.

$$K \approx t_{go} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equation (2) then may be written as

$$FOM_{M+T} = \frac{1}{t_{go}} \sqrt{\text{TRACE}} \left\{ U_{XYW} \Lambda_I U_{XYW}^T \right\} \quad (3)$$

For present-day lunar missions (either direct ascent or parking orbit) the perigee altitude of the transfer trajectory is relatively fixed. In addition, the midcourse maneuver is planned for a fixed time; hence t_{go} may be considered a known constant for any given trajectory. It therefore follows that for a given mission (that is, fixed perigee altitude and fixed maneuver time) FOM_{M+T} is a function of (1) injection energy, (2) injection true anomaly, (3) Earth–Moon distance at the impact time, and (4) the injection covariance matrix.

$$FOM_{M+T} = \text{function}\,(C_3, \eta_I, R_{EM}, \Lambda_I) \quad (4)$$

C. *Important Variables: "Miss-Only" Correction*

To consider the important variables for a miss-only midcourse correction (flight time not to be corrected), Eq. (3) is rewritten as

$$FOM_{M+T} = \frac{1}{t_{go}} \sqrt{\text{TRACE}} \left\{ U_{TRS} \Lambda_I U_{TRS}^T \right\} \quad (5)$$

or

$$FOM_{M+T} = \frac{1}{t_{go}} \sqrt{\sigma_T^2 + \sigma_R^2 + \sigma_S^2} \quad (6)$$

where U_{TRS} is the matrix of differential corrections relating perturbations at a fixed injection time in the x–R–V–Γ–W–$\dot{W}$ system to perturbations at a fixed impact time in the **T–R–S**$_I$ massless target coordinate system, and where

$$U_{TRS} \Lambda_I U_{TRS}^T = \begin{bmatrix} \sigma_T^2 & \rho_{TR}\,\sigma_T\,\sigma_R & \rho_{TS}\,\sigma_T\,\sigma_S \\ & \sigma_R^2 & \rho_{RS}\,\sigma_R\,\sigma_S \\ & (\text{SYM}) & \sigma_S^2 \end{bmatrix}$$

A miss-only maneuver corrects errors only in the **R–T** plane and does not correct errors in the S-direction. Hence, we define the miss-only figure-of-merit (FOM_M) as

$$FOM_M = \frac{1}{t_{go}} \sqrt{\sigma_T^2 + \sigma_R^2} \quad (7)$$

Noting that

$$U_{TRS} = \begin{bmatrix} T_X & T_Y & T_W \\ R_X & R_Y & R_W \\ S_X & S_Y & S_W \end{bmatrix} U_{XYW}$$

and that

$$\sigma_S^2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} U_{TRS} \Lambda_I U_{TRS}^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

it can be shown that

$$FOM_M^2 = FOM_{M+T}^2 - \frac{\mathbf{S}^T U_{XYW} \Lambda_I U_{XYW}^T \mathbf{S}}{t_{go}^2} \quad (8)$$

where

$$\mathbf{S} = \begin{bmatrix} S_X \\ S_Y \\ S_W \end{bmatrix}$$

It follows from Eq. (8) that for a given mission the miss-only *FOM* is a function of (1) injection energy, (2) injection true anomaly, (3) Earth–Moon distance at the impact time, (4) injection covariance matrix, and (5) *direction of the* **S** *vector.*

$$FOM_M = \text{function}\,(C_3, \eta_I, R_{EM}, \Lambda_I, \mathbf{S}) \quad (9)$$

With a few minor assumptions, the direction of **S** can be obtained analytically.

D. *Direction of* S

The **S**-vector is in the direction of the incoming asymptote of the selenocentric conic. Considering the Moon to be a massless point moving in space, **S** is a unit vector in the direction of the velocity of the spacecraft relative to the Moon, $\mathbf{V}_{rel}$. For simplicity, the Moon is assumed to move in a circular orbit.

From Fig. 5

$$\mathbf{S} = \frac{\mathbf{V}_{rel}}{V_{rel}}$$

and

$$\mathbf{V} = (V \cos \Gamma, -V \sin \Gamma, 0)$$
$$\mathbf{V}_M = (V_M \cos \beta, 0, -V_M \sin \beta)$$

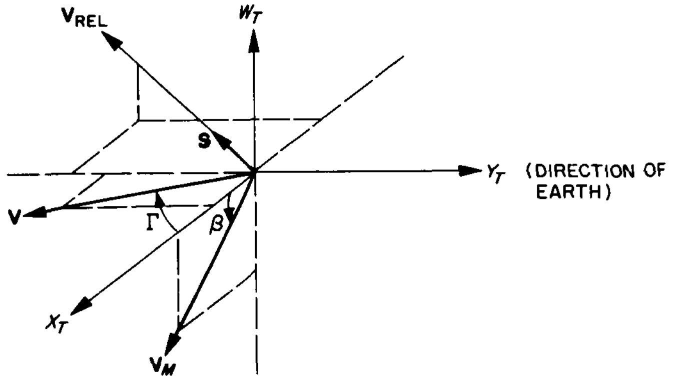

Fig. 5. Direction of S-vector

where Γ is the flight path angle of the spacecraft with respect to the Earth, **V** is the velocity of the spacecraft, $\mathbf{V}_M$ is the velocity of the Moon, and β is the angle between the Moon's plane of motion and the spacecraft's plane of motion (Moon assumed massless). After some manipulation we have

$$(S_X, S_Y, S_W) = \frac{\left(\frac{V}{V_M}\cos\Gamma - \cos\beta, -\frac{V}{V_M}\sin\Gamma, \sin\beta\right)}{\sqrt{\left(\frac{V}{V_M}\right)^2 + 1 - 2\left(\frac{V}{V_M}\right)\cos\Gamma\cos\beta}} \quad (10)$$

where

$$\Gamma = \cos^{-1}\left[\frac{R_P}{R_{EM}}\left(\frac{R_P C_3 + 2K_E}{R_{EM} C_3 + 2K_E}\right)\right]^{1/2}$$

$$\frac{V}{V_M} = \left[\frac{C_3 R_{EM}}{K_E} + 2\right]^{1/2}$$

where K_E is the gravitation constant of the Earth. Figure 6 presents **S** as a function of C_3 and β, for an R_P of 90 nautical miles and for maximum and minimum values of R_{EM}.

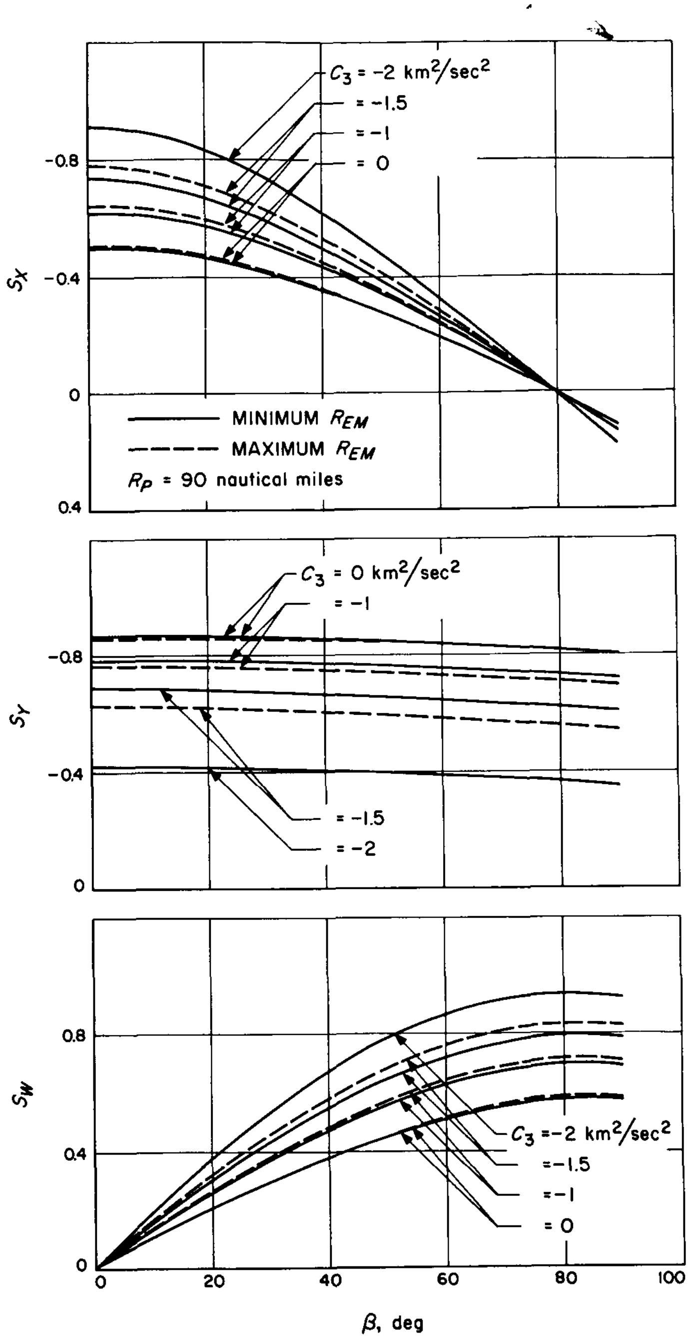

Fig. 6. Components of S-vector

By noting that the Earth–Moon line of centers at time of standard impact is common to both the Moon plane of motion and the spacecraft plane of motion, one can approximate β by

$$\cos\beta = \frac{\cos i_M \cos\theta_L \sin\Sigma_L \pm \sqrt{(\cos^2\theta_M - \cos^2\theta_L \sin^2\Sigma_L)(\sin^2 i_M - \sin^2\theta_M)}}{\cos^2\theta_M} \quad (11)$$

where i_M is the inclination of the Moon's orbit, θ_L is the launch latitude, Σ_L is the launch azimuth, and θ_M is the declination of the Moon. Figures 7 and 8 present β as a function of lunar declination and launch azimuth

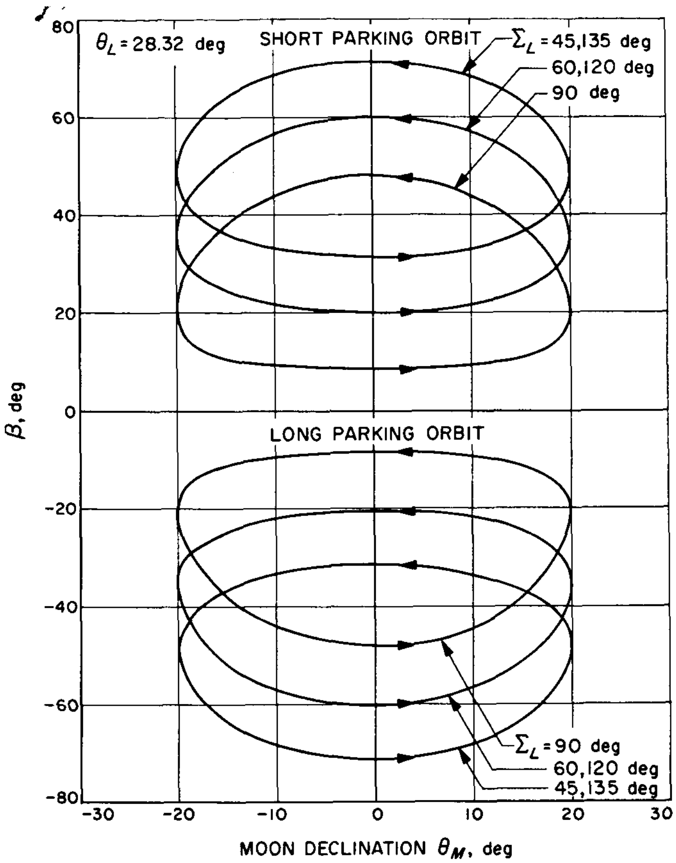

Fig. 7. Angle between planes of motion. Inclination of Moon 20 deg

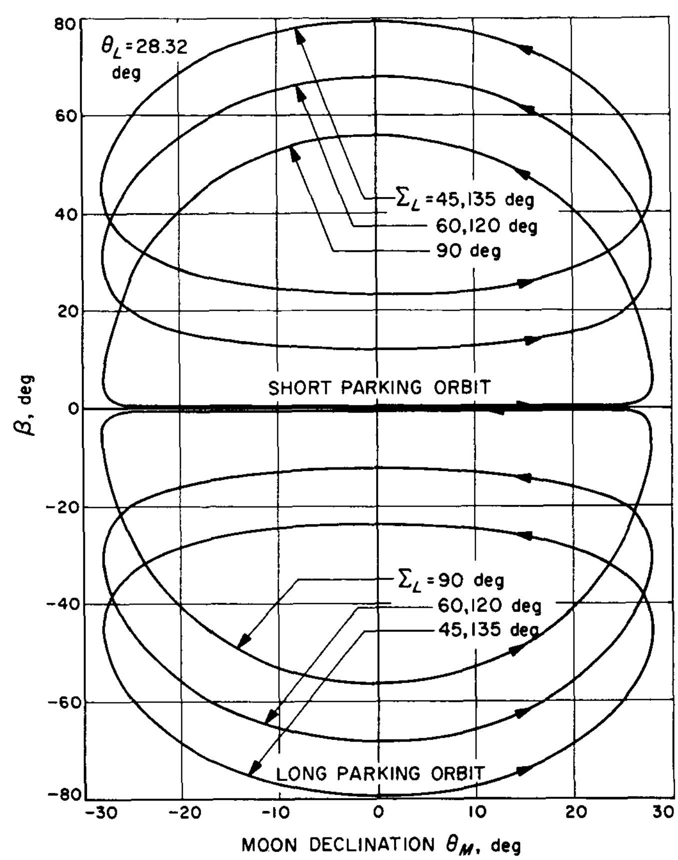

Fig. 8. Angle between planes of motion. Inclination of Moon 28 deg

for declinations of the Moon of 20 and 28 deg respectively ($\theta_L = 28.32$ deg).

For parking orbit missions when one has the flexibility to launch at any lunar declination, β generally takes on four values for any given value of declination. For example, when $\theta_M = 10$ deg, $\Sigma_L = 90$ deg, and $i_M = 20$ deg, $\beta = \pm 9$ and ± 44 deg. β has two positive values (and two negative values) because a given value of lunar declination occurs twice in a given lunar month. There are two possible parking orbit coast times for any given transfer trajectory: a "long" coast and a "short" coast. Both positive values of β are associated with the short parking orbit coast time and both negative values of β are associated with the long parking orbit coast time. It follows, then, that for a given launch azimuth, extreme values of β occur at zero declination; for the short parking orbit coast time, β has its maximum positive value at zero declination descending node and its minimum positive value at zero declination ascending node; for the long parking orbit coast time, β has its maximum negative value at zero declination ascending node and its minimum negative value at zero declination descending node.

For direct ascent missions, launch is possible only when the Moon is near minimum (maximum negative) declination. For this case, which is really a subset of the parking orbit case, β is always positive (the short parking orbit) and takes on two positive values for a given value of negative declination (it is assumed that $\theta_L \geq i_M$). For a given launch azimuth and a direct ascent launch mode, β has its maximum positive value near zero declination descending node and its minimum positive value near zero declination ascending node.

E. Effect of Declination of the Moon on FOM_M

The effect of lunar declination on FOM_M can be seen from the preceding analysis. Consider an injection error in velocity (or altitude) only. From Eq. (8)

$$FOM_M^2 = FOM_{M+T}^2 - \frac{\sigma_V^2}{t_{go}^2}\left[S_X \frac{\partial X_T}{\partial V} + S_Y \frac{\partial Y_T}{\partial V}\right]^2$$

Now FOM_{M+T} is not a function of declination; hence the trajectory which maximizes FOM_M for a given injection error in velocity is one which minimizes $[S_X \partial X_T/\partial V + S_Y \partial Y_T/\partial V]$. Referring to Figs. 6, 9, and 10, it can be seen that S_X, $\partial X_T/\partial V$, S_Y, and $\partial Y_T/\partial V$ all have the same sign and FOM_M is maximized when β is maximized. For a parking orbit this occurs at zero declination descending node for the short parking orbit coast arc and zero declination ascending node for the long parking orbit coast arc. Consider an injection error x (or Γ).

$$FOM_M^2 = FOM_{M+T}^2 - \frac{\sigma_T^2}{t_{go}^2}\left(S_X \frac{\partial X_T}{\partial x}\right)^2$$

Again FOM_M is maximized when β is maximized, which for parking orbit ascent implies zero declination descending node for the short coast solution and zero declination ascending node for the long coast solution.

For an injection error in either W or $\dot{W}$, FOM_M is maximized at zero declination ascending node for the short parking orbit coast case. However, from a practical point of view, out-of-plane injection errors are insignificant, leading to the conclusion that the trajectory which maximizes FOM_M for a given FOM_{M+T} is one which maximizes β, which, as pointed out above, impacts the Moon at zero declination descending node for the short parking orbit coast solution and zero declination ascending node for the long parking orbit coast solution. For direct ascent, FOM_M is maximized when the Moon is as near zero declination descending node as possible.

IV. NUMERICAL RESULTS

Figures 9–16 present the differential corrections in the U_{XYW} matrix as a function of injection energy, C_3; Earth–Moon distance at the standard impact time, R_{EM}; and injection time anomaly, η_I. These results were obtained from a JPL conic trajectory computer program developed by W. Kirhofer and D. J. Roek. The range of C_3 chosen represents trajectories with flight times between 50 and 80 hr.

With one exception, $\partial W_T/\partial \dot{W}$, the variations in the differential corrections are all consistent in that minimum injection energy and the maximum Earth–Moon distance yields the largest magnitude of each derivative. If one is

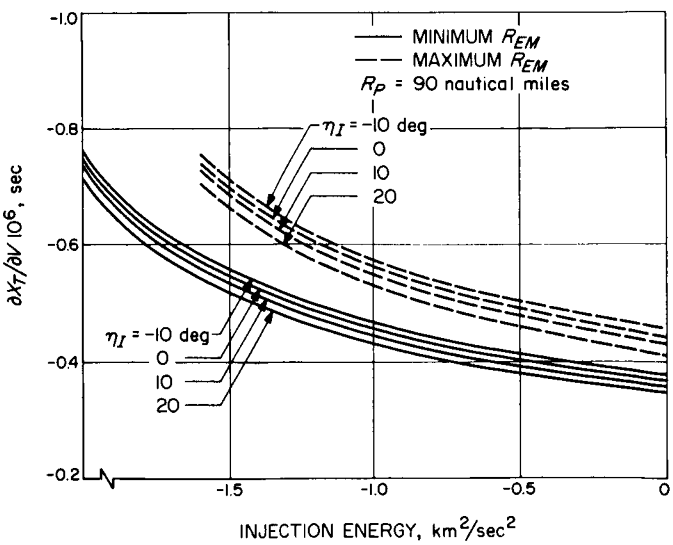

Fig. 9. Variation of $\partial X_T/\partial V$ with injection energy

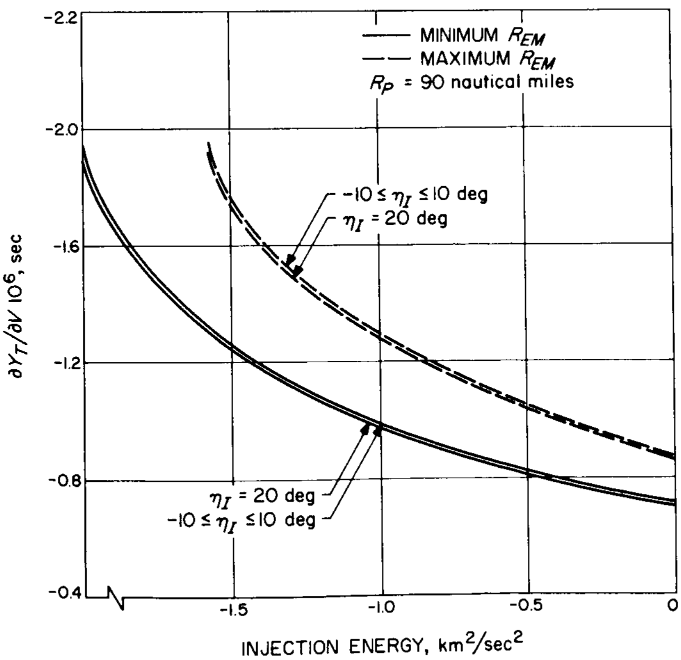

Fig. 10. Variation of $\partial Y_T/\partial V$ with injection energy

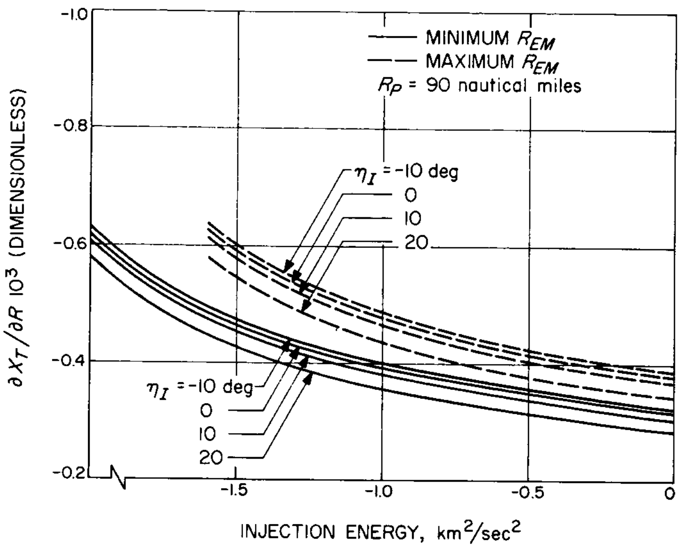

Fig. 11. Variation of $\partial X_T/\partial R$ with injection energy

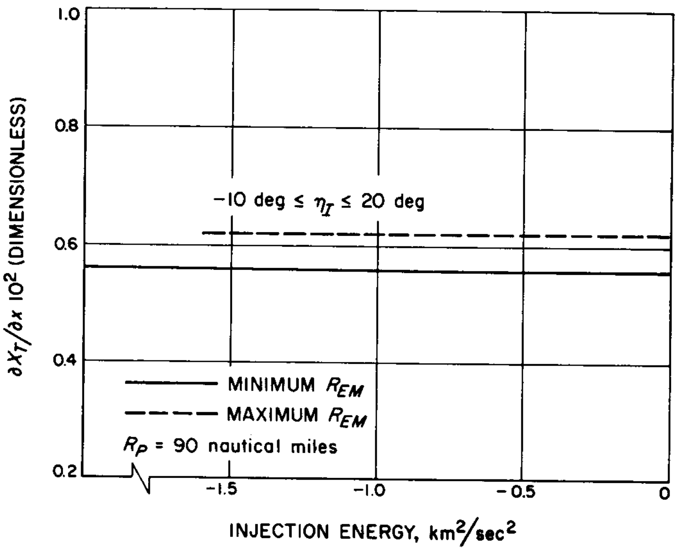

Fig. 13. Variation of $\partial X_T/\partial x$ with injection energy

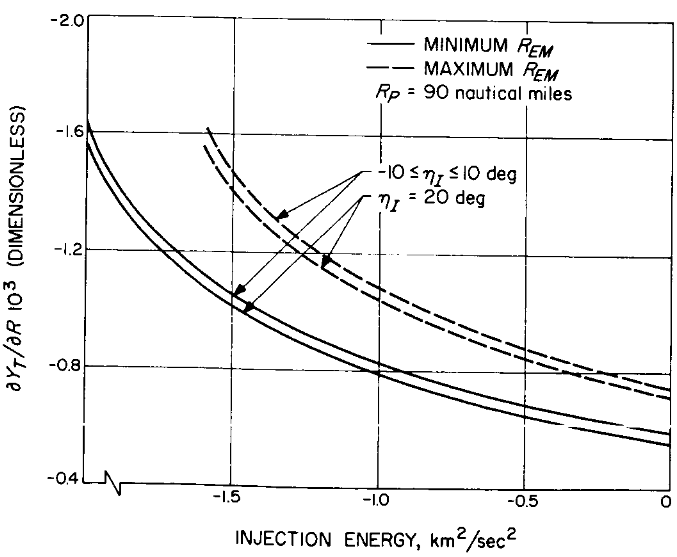

Fig. 12. Variation of $\partial Y_T/\partial R$ with injection energy

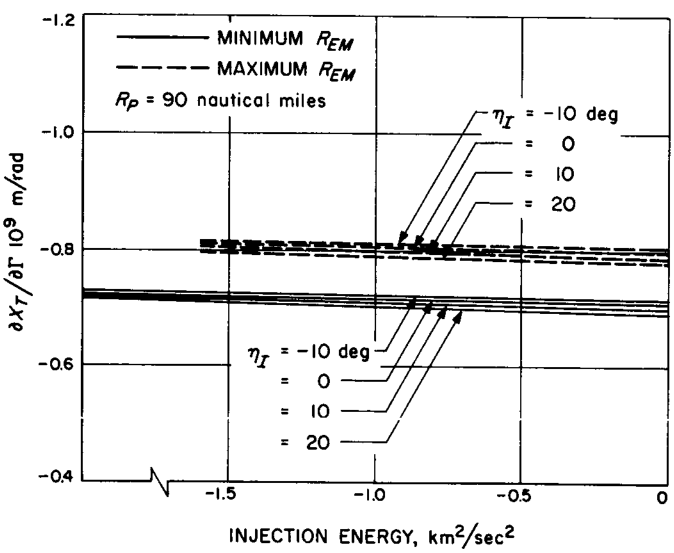

Fig. 14. Variation of $\partial X_T/\partial \Gamma$ with injection energy

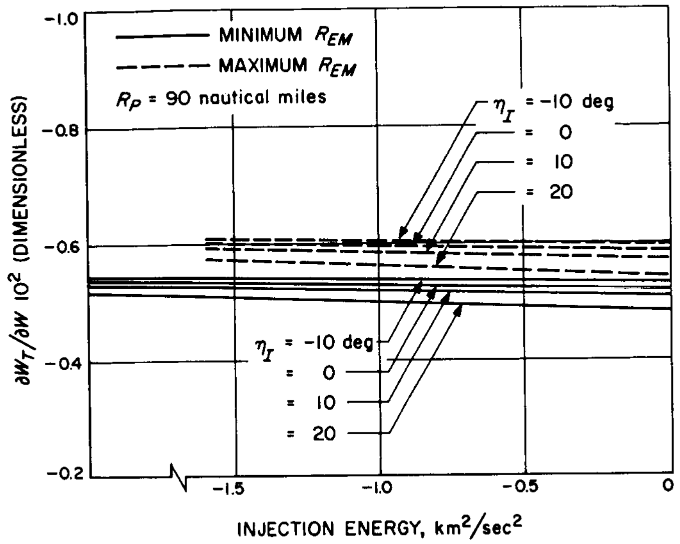

Fig. 15. Variation of $\partial W_T/\partial W$ with injection energy

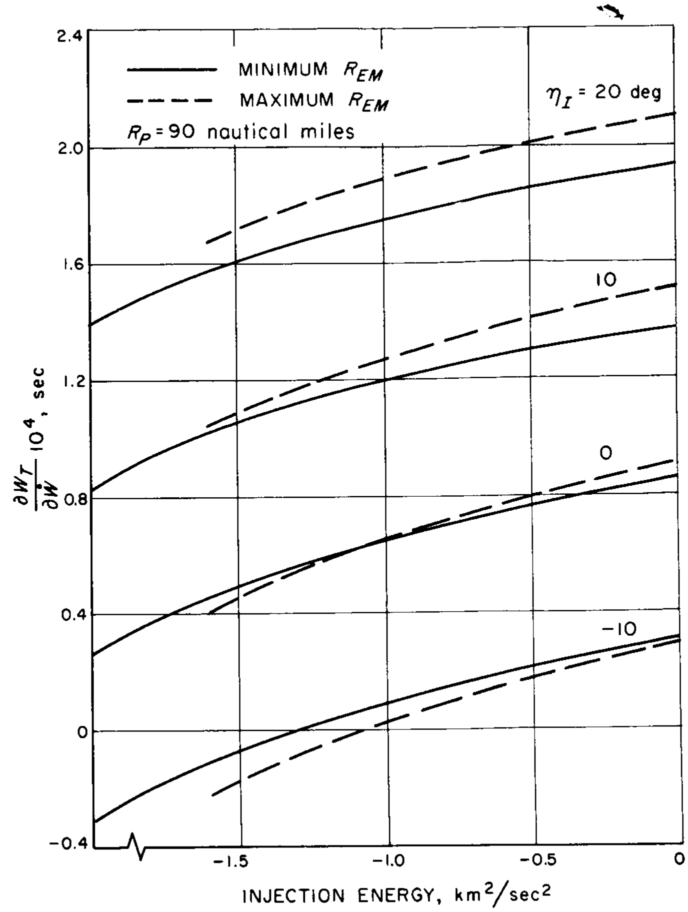

Fig. 16. Variation of $\partial W_T/\partial \dot{W}$ with injection energy

not concerned with the insignificant contribution of errors in $\dot{W}$, then the conclusion is that for a given set of injection errors, the trajectory with minimum injection energy and maximum Earth–Moon distance gives the largest FOM_{M+T}. The variation with injection true anomaly is minor.

In order to check the conic differential corrections, these data are compared with data obtained from an

Table 1. Definition of comparison trajectories

	Trajectory				
	1	2	3	4	5
Launch Date	9 March '71	23 Jan '62	31 Jan '62	6 Feb '62	12 Feb '62
C_3, km²/sec²	−1.50	−0.96	−1.57	−1.58	−1.14
Flight time, hr	79	65	65	65	65
Lunar declination	Zero ↓	Zero ↓	Minimum	Zero ↑	Maximum
i_M, deg	28	20	20	20	20
Coast arc	Long	Short	Short	Short	Short
R_{EM}, km	$0.406 \cdot 10^6$	$0.404 \cdot 10^6$	$0.367 \cdot 10^6$	$0.366 \cdot 10^6$	$0.396 \cdot 10^6$
R_P, nautical miles	90	100	100	100	100
Σ_L, deg	115	90	90	90	90
η_I, deg	0	3.38	3.39	3.27	3.29
β, deg (Eq. 11)	−8.2	48.3	20.5	8.3	20.5
S_X (Eq. 10)	−0.77	−0.36	−0.66	−0.72	−0.62
S_Y (Eq. 10)	−0.63	−0.74	−0.68	−0.68	−0.74
S_W (Eq. 10)	−0.15	0.56	0.31	0.15	0.29

Table 2. Comparison of conic and *n*-body data[a]

q	Source	Trajectory 1 $\frac{\partial\lvert\mathbf{B}\rvert}{\partial q}$	Trajectory 1 $\frac{\partial S}{\partial q}$	Trajectory 2 $\frac{\partial\lvert\mathbf{B}\rvert}{\partial q}$	Trajectory 2 $\frac{\partial S}{\partial q}$	Trajectory 3 $\frac{\partial\lvert\mathbf{B}\rvert}{\partial q}$	Trajectory 3 $\frac{\partial S}{\partial q}$	Trajectory 4 $\frac{\partial\lvert\mathbf{B}\rvert}{\partial q}$	Trajectory 4 $\frac{\partial S}{\partial q}$	Trajectory 5 $\frac{\partial\lvert\mathbf{B}\rvert}{\partial q}$	Trajectory 5 $\frac{\partial S}{\partial q}$
x	Conic	0.40×10^2	-0.48×10^2	0.58×10^2	-0.22×10^2	0.42×10^2	-0.37×10^2	0.39×10^2	-0.40×10^2	0.49×10^2	-0.38×10^2
	n-body	0.41×10^2	-0.44×10^2	0.60×10^2	-0.20×10^2	0.43×10^2	-0.36×10^2	0.36×10^2	-0.41×10^2	0.50×10^2	-0.34×10^2
R	Conic	0.81×10^3	0.14×10^4	0.66×10^3	0.95×10^3	0.57×10^3	0.10×10^4	0.50×10^3	0.11×10^4	0.38×10^3	0.12×10^4
	n-body	0.97×10^3	0.13×10^4	0.72×10^3	0.92×10^3	0.68×10^3	0.10×10^4	0.73×10^3	0.10×10^4	0.55×10^3	0.10×10^4
V	Conic	0.10×10^7	0.16×10^7	0.79×10^6	0.11×10^7	0.71×10^6	0.12×10^7	0.59×10^6	0.13×10^7	0.77×10^6	0.13×10^7
	n-body	0.11×10^7	0.16×10^7	0.81×10^6	0.11×10^7	0.74×10^6	0.12×10^7	0.78×10^6	0.12×10^7	0.59×10^6	0.12×10^7
Γ	Conic	0.52×10^9	0.62×10^9	0.75×10^9	0.29×10^9	0.55×10^9	0.47×10^9	0.50×10^9	0.52×10^9	0.63×10^9	0.49×10^9
	n-body	0.53×10^9	0.58×10^9	0.76×10^9	0.26×10^9	0.56×10^9	0.45×10^9	0.47×10^9	0.53×10^9	0.64×10^9	0.45×10^9
W	Conic	0.60×10^2	0.90×10^1	0.49×10^2	-0.33×10^2	0.51×10^2	-0.16×10^2	0.53×10^2	-0.80×10^1	0.57×10^2	-0.17×10^2
	n-body	0.63×10^2	0.95×10^1	0.50×10^2	-0.32×10^2	0.53×10^2	-0.17×10^2	0.56×10^2	-0.93×10^1	0.57×10^2	-0.18×10^2
$\dot{W}$	Conic	0.45×10^4	-0.67×10^1	0.72×10^4	0.50×10^4	0.62×10^4	0.20×10^4	0.64×10^4	0.10×10^4	0.76×10^4	0.23×10^4
	n-body	0.46×10^4	0.86×10^1	0.64×10^4	0.57×10^4	0.56×10^4	0.32×10^4	0.61×10^4	0.21×10^4	0.73×10^4	0.35×10^4

[a] Units are in seconds, meters, and radians.

n-body trajectory program (Ref. 5). This comparison is made for five trajectories, the characteristics of each being given in Table 1. Table 2 presents the numerical comparison. For the conic case (Ref. 6)

$$\frac{\partial S}{\partial q} = \frac{\partial X_T}{\partial q} S_X + \frac{\partial Y_T}{\partial q} S_Y + \frac{\partial W_T}{\partial q} S_W$$

and

$$\frac{\partial\,|\,\mathbf{B}\,|}{\partial q} = \sqrt{\left(\frac{\partial X_T}{\partial q}\right)^2 + \left(\frac{\partial Y_T}{\partial q}\right)^2 + \left(\frac{\partial W_T}{\partial q}\right)^2 - \left(\frac{\partial S}{\partial q}\right)^2}$$

where (S_X, S_Y, S_W) is obtained from Eq. (10) and $\partial X_T/\partial q$, $\partial Y_T/\partial q$, $\partial W_T/\partial q$ from Figs. 9–16. It can be seen that the results agree quite well.

V. CONCLUSIONS

From this analysis, the following conclusions can be stated:

1. For a given mission (fixed perigee altitude and fixed midcourse maneuver time), the miss-plus-time midcourse figure-of-merit is a function of (1) injection energy, (2) injection true anomaly, (3) Earth–Moon distance at the impact time, and (4) the injection covariance matrix.

2. For a given mission, the miss-only figure-of-merit is a function of (1) injection energy, (2) injection true anomaly, (3) Earth–Moon distance at the impact time, (4) injection covariance matrix, (5) launch azimuth, and (6) lunar declination at the impact time.

3. For a given injection covariance matrix, the trajectory which maximizes both FOM_{M+T} and FOM_M is one which is launched with minimum injection energy and arrives at the Moon when the Earth–Moon distance is maximum and the Moon is at zero declination descending node for the short parking orbit coast, zero declination ascending node for the long parking orbit coast, and near zero declination descending node for direct ascent. Injection true

anomaly is of minor significance. For an injection energy of $C_3 = -1.5$ km²/sec², one such parking orbit worst-case trajectory occurs with a launch date 9 March 1971 (short coast).

In addition, it appears that differential corrections obtained from conic data are sufficiently accurate for expressing launch vehicle accuracy in terms of a spacecraft midcourse correction figure-of-merit.

NOMENCLATURE

- A — $6 \times n$ matrix of differential corrections relating injection errors to launch vehicle guidance component errors
- $\mathbf{B}$ — Two-dimensional miss vector
- C_3 — Injection energy $\left(C_3 = V^2 - \frac{2K_E}{R} \right)$
- FOM_M — Miss-only figure-of-merit
- FOM_{M+T} — Miss-plus-time figure-of-merit
- i_M — Inclination of Moon
- K — 3×3 matrix of differential corrections relating midcourse velocity perturbations to miss perturbations at target
- K_E — Gravitational constant of Earth
- m — 3×1 target error vector
- R — Distance between center of Earth and spacecraft
- R_E — Radius of Earth
- R_{EM} — Earth–Moon distance
- R_P — Perigee altitude of transfer orbit
- RMS — Root-mean-square
- $\mathbf{R}$ — Unit vector in $\mathbf{T}$–$\mathbf{R}$–$\mathbf{S}_I$ target coordinate system
- $\mathbf{S}$ — Unit vector in the direction of selenocentric incoming asymptote expressed in $\mathbf{X}_T$–$\mathbf{Y}_T$–$\mathbf{W}_T$ coordinate system
- $\mathbf{S}_I$ — Unit vector in direction of selenocentric incoming asymptote
- $\mathbf{S}_O$ — Unit vector in direction of selenocentric outgoing asymptote
- t_{go} — Time-to-go (time of impact less time of midcourse correction)
- $\mathbf{T}$ — Unit vector in $\mathbf{T}$–$\mathbf{R}$–$\mathbf{S}_I$ target coordinate system
- U — 3×6 matrix of differential corrections relating injection errors to target errors
- V — Component of injection velocity in standard plane of motion
- V_{max} — Maximum midcourse correction capability
- $\mathbf{V}_M$ — Velocity of Moon
- $\mathbf{V}_{rel}$ — Velocity of spacecraft relative to Moon
- W — Lateral position at injection
- $\dot{W}$ — Lateral speed at injection
- $\mathbf{W}_T$ — Unit vector in $\mathbf{X}_T$–$\mathbf{Y}_T$–$\mathbf{W}_T$ target coordinate system
- x — Down-range distance at injection (spherical Earth)
- $\mathbf{X}_T$ — Unit vector in $\mathbf{X}_T$–$\mathbf{Y}_T$–$\mathbf{W}_T$ target coordinate system
- $\mathbf{Y}_T$ — Unit vector in $\mathbf{X}_T$–$\mathbf{Y}_T$–$\mathbf{W}_T$ target coordinate system
- β — Angle between Moon plane of motion and spacecraft plane of motion (massless Moon)
- Γ — Flight path angle
- δe — $n \times 1$ vector of launch vehicle component errors
- δq — 6×1 injection coordinate error vector
- δV — Midcourse velocity correction vector
- η_I — Injection time anomaly
- θ_L — Latitude of launch pad

NOMENCLATURE (Cont'd)

θ_M Declination of Moon

Λ_I Injection covariance matrix

Λ_V Midcourse velocity correction covariance matrix

ρ Linear correlation coefficient

Σ_L Launch azimuth

σ^2 Variance

$(\cdot)^T$ Transpose of matrix $(\cdot)$

$(\cdot)^{-1}$ Inverse of matrix $(\cdot)$

$E\{\cdot\}$ Expectation operator of matrix $(\cdot)$

$\sqrt{\mathrm{TRACE}}\{\cdot\}$ Square root of trace of matrix $(\cdot)$

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