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SATELLITE MOTION FOR ALL INCLINATIONS AROUND AN OBLATE PLANET

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ERRATA FOR DOUGLAS PAPER NO. 3078

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Page	Equation	Should Read
23	(3.5e)	$p = r^2 \sin^2\theta \frac{d\phi}{dt}$
23	(3.5g)	$\frac{d\phi}{dt} = \frac{pu^2}{\cos i} + \frac{\cos^3 i \cos\theta}{p \sin^2 i \sin\theta} [\frac{\partial U}{\partial\theta} + \tan i \frac{\cos\phi}{\sin\theta} \frac{\partial U}{\partial\phi}]$
26	(3.8a)	$i(\phi;\varepsilon) = \sum_{n=0} i_{n/2}(\phi,\tilde{\phi};\varepsilon)\cdot\varepsilon^{n/2}$
26	(3.8b)	$u(\phi;\varepsilon) = \sum_{n=0} u_{n/2}(\phi,\tilde{\phi};\varepsilon)\cdot\varepsilon^{n/2}$
27	(3.12a)	$\frac{\partial i_1}{\partial\phi} = \frac{di_{oo}}{\partial\tilde{\phi}} - \frac{1}{p^4} \cos^5 i_{oo} \sin i_{oo} \sin 2\phi[1 + e_o \cos(\phi - \omega)]$
29	line above (3.18a)	i_o, ω_1, and e_1: (The asterisk refers to the note on next page)
30	line above (3.20a)	at $t = \tau$ are
30	(3.20b)	$e_1 = \eta_1 + \frac{B_2}{2S_o} (\cos 2v - \cos 2\omega)$
33	(3.29c)	$\frac{d\omega_{1/2}}{d\tilde{\phi}} = \frac{1}{2} S_2(i_{o1/2}^*)^2 + S_1 i_{o1}^* + A_o^* + A_2^* \cos 2\omega^*$
34	(3.33)	$i_{o1/2}^* = \frac{1}{S_1} [\pm(\bar{\kappa}_o - \kappa_1 \cos 2\omega^*)^{1/2} - \bar{S}_o]$
34	(3.34)	should have a $\pm$ on the right side next to the = sign

Page	Equation	Should Read
34	(3.35)	should have a $\pm$ on the right side next to the = sign
34	(3.36)	should have $\pm$ on the right side next to the = sign
34	(3.37)	$e_1^* = \pm \frac{1}{\kappa_1} (\bar{\kappa}_o - \kappa_1 \cos 2\omega^*)[- \frac{\bar{\bar{S}}_o}{S_1} B_{21}^* + E_{1/2} B_{22}$ $- \frac{e_o^*}{4} \frac{\bar{S}_o}{p^4} (1 - 3 \cos^2 i_{oo}^*) \cos^4 i_{oo}^*$ $- \frac{1}{2} [\frac{B_{21}^*}{S_1} + \frac{B_{21}^* B_2^*}{\kappa_1}] \cos 2\omega^* + E_1$
36	(3.43)	$i_{oc} = i_{oo}^* + \varepsilon^{1/2} i_{o1/2}^* + \varepsilon i_{o1}^*$
37	line above (3.48a)	. . . expansions to $O(\varepsilon)$ for all inclinations . . .
38	(3.49)	should have $\pm$ on the right side next to the = sign
38	below (3.49)	$2 \sin \omega^* \cos \omega^* = \frac{d\nu}{d\omega^*} = \pm 2[\nu(1 - \nu)]^{1/2}$
38	(3.50)	should have $\pm$ in front of the integral
39	(3.54a)	should have $\pm$ on the right side next to the = sign
39	third line below (3.55)	. . . motion around $\pi/2$ or $3\pi/2$ with a maximum amplitude $\omega^*_{max} = \pm \cos^{-1} \lambda$. . . .
39	(3.56)	$\lambda = \sin^2 w^* - (\bar{S}_o + S_1 J_{1/2}^*)^2 / 2\kappa_1$

Page	Equation	Should Read
40	(3.58)	$\bar{\phi} - \bar{\phi}_o = \pm \int \frac{d\omega^*}{(2\kappa_1)^{1/2} \sin \omega^*} = \pm \frac{1}{(2\kappa_1)^{1/2}} \log \lvert \tan \frac{\omega^*}{2} \rvert$
40	(3.59)	$\lvert \tan \frac{\omega^*}{2} \rvert = e^{\pm(2\kappa_1)^{1/2}(\bar{\phi} - \bar{\phi}_o)}$
40	(3.60)	$\lambda = \sin^2 w^* - (\bar{S}_o + S_1 J_{1/2}^*)^2 /2\kappa_1 = 0$
40	(3.61)	$\sin w^* = \pm(\bar{S}_o - s_1 J_{1/2})/(2\kappa_1)^{1/2}$
40	(3.62)	$w^* = \pm \sin^{-1}[(\bar{S}_o - S_1 J_{1/2})/(2\kappa_1)^{1/2}]$
40	lines below (3.62) down to the botton of the page	should be replaced by: . . . are assumed. This means that Case II happens only if initially the apse and the inclination are chosen properly as to satisfy the condition (3.62).
41	(3.64b)	$k_2 = [\frac{2\kappa_1}{\kappa_1 + \bar{\kappa}_o}]^{1/2}$
41	(3.64c)	should have $\pm$ on the right side next to the = sign
44	second line of (3.77)	$- \cos 2\phi - \frac{e_o}{2} \cos (\phi + \omega_c) - \frac{e_o}{2} \cos (3\phi - \omega_c)]$
46	(3.84)	$\Omega_{o1} = D_1 \omega_o^* + \int_o^{\omega_c} \ldots$
53	(A.28)	$g_1 = - \frac{S_2}{2S_1^2} \bar{S}_o + \frac{1}{\kappa_1} [- C_{21} \bar{S}_o + C_{22} S_1 E_{1/2}]$

SATELLITE MOTION FOR ALL INCLINATIONS

AROUND AN OBLATE PLANET*

M. Eckstein†, Y. Shi†† and J. Kevorkian†††

ABSTRACT

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A uniformly valid solution for the motion of a satellite around an oblate planet is presented. The Two Variable Expansion Procedure as earlier developed at Caltech was applied to obtain a solution valid for all inclinations including the critical. This solution is correct to order ϵ, where ϵ is a small parameter proportional to the oblateness parameter J_2. The reciprocal of the radius vector, eccentricity, perigee, inclination, and node of the satellite orbit are given as functions of the central angle ϕ between node and satellite. The results are based on a potential which includes the second and fourth zonal harmonics. The solution for the case of critical inclination is first obtained separately and then matched with the solution of the noncritical case to establish a solution uniformly valid for all inclinations.

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1. Introduction

The motion of a satellite around an oblate planet has received considerable attention in the literature after the advent of artificial satellites of the earth. The early theories, of which Brouwer's (1959) is the most comprehensive, were not valid for initial orbital inclinations close to the critical value $\cos^{-1}(5)^{-1/2} = 63.4°$ from the equatorial plane of symmetry. The non-validity of the solution at this angle exhibited itself by the occurrence of a divisor which tended to zero at the critical inclination.

Later, Hori (1960) and others (cf. Garfinkel (1960), Mersman (1962), and Izsak (1963) using diverse approaches, studied the behavior of the solution near the critical inclination. Though "a direct analytic comparison of the various treatments of the critical inclination problem is almost impossible because of the multiplicity of notations, approximations and starting points" (Mersman (1962), there is general agreement about the necessity of studying an expansion in powers of $J^{1/2}$ (where J is the small parameter measuring the oblateness perturbations). Furthermore, at least the qualitative behavior of the motion near the critical inclination, as first described by Hori, has been repeatedly substantiated. This statement by Mersman quite correctly reflects the inherent algebraic complexity of the main problem and the necessarily involved nature of its solution. However, the basic mathematical problem that gives rise to the singularity at the critical inclination is quite simple and was recognized by many authors. In particular, Struble (1961) has pointed out that for inclinations close to the critical the equations governing the slow variations of the apse and inclination angle are coupled by virtue of a regrouping of terms which otherwise have different orders of magnitude.

This phenomenon can be duplicated exactly in a particularly simple model equation corresponding to the forced oscillations of a system with an appropriate small non-linearity. The connection between non-linear oscillations and satellite motions with small pertubations is, of course, well known since it was first proposed by Laplace in his study of the motion of the moon. Therefore, in order to fix ideas the proposed model equation is first studied in detail, and the techniques are then directly applied to the main problem. The aim of the present paper is to develop the solution both near and away from the critical inclination in asymptotic series with respect to J. These series are uniformly valid for long times, but the primary goal is the achievement of uniform validity for all inclination angles as well.

The approach adopted here proceeds from the formulation proposed by Struble (1960) and (1961). It is first shown that two distinct asymptotic expansions (corresponding to two regimes of the initial inclination near and away from the critical) can be constructed and rendered uniformly valid for long times by the two-variable expansion procedure of Kevorkian (1962). It is then demonstrated that each of the above generalized asymptotic expansions, depending upon the initial inclination, individually describe the motion for all times. In addition, the two expansions match in an overlap domain of the inclination parameter lying between the critical and non-critical regimes. This matching is in the sense of the theory of Kaplun and Lagerstrom (1957), hence the uniformly valid asymptotic representation of the motion follows easily.

Furthermore, the analytic dependence of the solution on $J^{1/2}$, as first suggested by Hori (1960), is justified by the techniques of singular perturbation theory and the matching process.

The present solution includes the second and fourth zonal harmonics of the earth's potential. All secular and long-period terms are included up to $O(J^{5/2})$, while short-period terms are retained up to $O(J)$. The results are exhibited in the form of the reciprocal radius, eccentricity, perigee, inclination, and node as functions of the central angle between the ascending node and radius vector. The equation for the time is not given here but will be included in a future publication. A detailed comparison of the present results with at least the work of Struble (1961) and (1962) will also be provided there.

2. Model Equation

2.1 General Discussion

In order to demonstrate the essential mathematical features of the main problem and the expansion procedures, the following model equation is first studied in detail

$$(2.1) \qquad \frac{d^2y}{dt^2} + y + 2\varepsilon y[1 - 5\cos^2\{y^2 + (\frac{dy}{dt})^2\}^{1/2}] = \varepsilon^2\{y^2 + (\frac{dy}{dt})^2\}^{1/2}\cos t$$

where $\varepsilon << 1$.

In the absence of the forcing function, this equation can be integrated exactly and exhibits the following behavior in the phase-plane of y and dy/dt. Whenever the radius $r = [y^2 + \dot{y}^2]^{1/2}$ in the phase-plane takes on the critical values $r_c = \cos^{-1}(5)^{-1/2}$, the motion reduces to simple harmonic oscillations with amplitude r_c and unit frequency. For each annular region bounded by two consecutive values of r_c, the integral curves are ovals with their axes aligned alternately paralled either to y or to dy/dt. One would thus expect that the addition of the forcing term with unit frequency will cause local resonance in neighborhoods of the critical amplitudes r_c. As will be shown later on in this section, this will indeed be the case and will give rise to the problem of the "critical amplitude".

Using the two-variable method discussed by Cole and Kevorkian (1962), (1963), the following form of the asymptotic expansion is first assumed*

* Throughout this paper the omission of the upper index on a summation symbol will indicate an asymptotic expansion.

$$(2.2) \qquad y(t;\varepsilon) = \sum_{i=0} y_i(t,\tilde{t};\varepsilon)\varepsilon^i$$

where the slow variable $\tilde{t}$ is defined by

$$(2.3) \qquad \tilde{t} = \varepsilon t$$

as discussed by Kevorkian (1962). Then the governing equation for y_o is

$$(2.4) \qquad \frac{\partial^2 y_o}{\partial t^2} + y_o = 0$$

whose general solution is

$$(2.5) \qquad y_o(t,\tilde{t};\varepsilon) = \alpha(\tilde{t};\varepsilon)\cos[\tilde{t} - \beta(t;\varepsilon)]$$

The functions $\alpha(\tilde{t};\varepsilon)$ and $\beta(\tilde{t};\varepsilon)$ in (2.5) which will be called "integration constants" will be determined by requiring y_1 to be bounded. For the present case we always have the simple harmonic operator on the left-hand side of all higher order equations. For simplicity of calculations and for the explicit representation of the motion of the phase angle, we will expand the "integration constants" $\alpha(\tilde{t};\varepsilon)$ and $\beta(\tilde{t};\varepsilon)$ in the form:

$$(2.6) \qquad \alpha(\tilde{t};\varepsilon) = \sum_{i=0} \alpha_i(\tilde{t})\varepsilon^i \qquad\qquad \beta(\tilde{t};\varepsilon) = \sum_{i=0} \beta_i(\tilde{t})\varepsilon^i$$

From (2.1) the following equation for y_1 can be calculated:

$$(2.7) \qquad \frac{\partial^2 y_1}{\partial t^2} + y_1 = 2\frac{d\alpha_o}{d\tilde{t}}\sin(t-\beta) - 2\alpha_o\left[\frac{d\beta_o}{d\tilde{t}} + (1 - 5\cos^2\alpha_o)\right]\cos(t-\beta)$$

The boundedness of y_1 requires

$$(2.8) \qquad \frac{d\alpha_o}{d\tilde{t}} = 0 \qquad\qquad \frac{d\beta_o}{d\tilde{t}} = -(1 - 5\cos^2\alpha_o) \equiv s_o$$

These equations give

(2.9) $\quad a_0 = \text{const.} \qquad \beta_0 = s_0\bar{t} + b_0$

where b_0 is a constant depending on the initial condition. The solution for y_1 is then

(2.10) $\quad y_1(t,\bar{t};\varepsilon) = 0$

with no loss of generality because the homogeneous solution is already accounted for in the expansion of α and β in y_0;

Now the equation of $O(\varepsilon^2)$ for y_2 is

$$(2.11) \quad \frac{\partial^2 y_2}{\partial t^2} + y_2 = [2\frac{d\alpha_1}{d\bar{t}} - \alpha_0 \sin\beta]\sin(t-\beta) + \alpha_0 [s_0^2 - \frac{5}{2}\alpha_0 s_0 \sin 2\alpha_0$$
$$+ \cos\beta + 2s_0'\alpha_1 - 2\frac{d\beta_1}{d\bar{t}}]\cos(t-\beta) - \frac{5}{2}\alpha_0^2 s_0 \sin 2\alpha_0 \cos 3(t-\beta)$$

where

$$(2.12) \quad s_0' = \frac{ds_0}{d\alpha} = -5\sin 2\alpha_0$$

By the boundedness requirement on y_2 we must set

$$(2.13a) \quad \frac{d\alpha_1}{d\bar{t}} = \frac{a_0}{2}\sin\beta$$

$$(2.13b) \quad \frac{d\beta_1}{d\bar{t}} = \frac{s_0^2}{2} - \frac{5}{4}\alpha_0 s_0 \sin 2\alpha_0 + \frac{1}{2}\cos\beta + s_0'\alpha_1$$

Since for $s_0 \to 0$ (i.e. $\alpha_0 = \cos^{-1}(5)^{-1/2}$) $\beta = b_0 + O(\varepsilon)$, we see immediately from (2.13a) that α_1 becomes unbounded for large values of $\bar{t}$. Thus, the expansion procedure assumed in (2.2) is not uniformly valid near the critical amplitudes.

In this simple model the cause of the difficulty is easy to discern and remedy. As was pointed out earlier, whenever $\alpha = \alpha_c = \cos^{-1}(5)^{-1/2}$ the non-linear system degenerates to simple harmonic motion with a frequency equal to that of the forcing function. Therefore, in some neighborhood of α_c the amplitude must increase appreciably before the non-linear term comes into play and destroys the resonance of the forcing function. Due to this effect of local resonance the forcing function, which would otherwise be of order ε^2, now takes on a more important role. This fact is exhibited mathematically in equations (2.8). When s_o is small one cannot neglect the higher order forcing function in solving for β_o and α_c, since in this case the right-hand sides of (2.8) are exclusively composed of small terms. This fact was first pointed out by Struble (1961) in connection with the main problem.

In view of this, we anticipate the importance of the forcing function and introduce it immediately in the equations of order ε. This means equations (2.8) for α and β now become

$$(2.14) \qquad \frac{d\alpha}{d\tilde{t}} = \frac{\varepsilon\alpha}{2}\sin\beta \qquad\qquad \frac{d\beta}{d\tilde{t}} = -(1 - 5\cos^2\alpha) + \frac{\varepsilon}{2}\cos\beta$$

The terms of order ε in (2.14), which are exclusively the contributions of the right-hand side of (2.1), will radically alter the behavior of α and β near the critical amplitudes.

Equations (2.14) are Hamiltonian, hence along an integral curve

$$(2.15) \qquad 2H = 3\alpha + \frac{5}{2}\sin 2\alpha + \varepsilon\alpha\cos\beta = \text{const.}$$

With the aid of (2.15), the integral curves in the α, β plane can be easily calculated. The singular points are located at $\beta = \beta_s = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ and $\cos \alpha = \cos \alpha_s \equiv \pm(1/5 \mp \epsilon/10)^{1/2}$*. These points form an alternating pattern of centers and saddle-points with solution curves as shown qualitatively in Figure 1.

We observe three possible types of motion if we consider the integral curves in vertical strips with a width of order $\epsilon^{1/2}$ centered about any of the critical amplitudes.

The integral curves which pass through two adjacent saddle-points for a given value of α_s form the boundaries of oval regions with a width also of $O(\epsilon^{1/2})$ inside which both α and β undergo bounded oscillations. For example the motion in the neighborhood of the point $\beta = 0$ and $\alpha = \alpha_s = \cos^{-1}(1/5 - \epsilon/10)^{1/2}$ has the form

$$(2.16) \qquad \alpha = \alpha_s + C_1 \epsilon^{1/2} \cos\left[(2\epsilon\alpha_s)^{1/2}\, \tilde{t} + C_2\right]$$

$$(2.17) \qquad = -4C_1(2\alpha_s)^{-1/2} \sin\left[(2\epsilon\alpha_s)^{1/2}\, \tilde{t} + C_2\right]$$

where C_1 and C_2 are small constants depending on initial conditions which allow us to linearize equations (2.14).

The separatrix forming the above boundary corresponds to motion where α and β approach the value at the saddle-point asymptotically as $\tilde{t} \to \infty$. In fact, by use of (2.15) it is easy to show that the separatrix around the point $\beta = 0$

* The upper or lower signs in the radical are to be taken when β is an even or odd multiple of π respectively.

and $\alpha_s = \cos^{-1}(1/5 - \varepsilon/10)^{1/2}$ for $0 < \alpha_s < \pi/2$, intersects the α axis at a distance $(\varepsilon/2)^{1/2} \cos^{-1}(5)^{-1/2} + O(\varepsilon)$ from the singular point. Finally, the motion just outside the oscillatory regions is characterized by the fact that α undergoes bounded oscillations, while β has a secular motion superimposed on its oscillations. In all three of the above motions the characteristic frequency is $O(\varepsilon^{3/2})$ in the natural time variable whereas the amplitudes of oscillation are $O(\varepsilon^{1/2})$ (cf. equations (2.16) and (2.17)). This immediately suggests that the slow time scale appropriate for motion near the critical amplitudes is $\bar{t} = \varepsilon^{3/2}t$, and that one must seek an expansion for y in powers of $\varepsilon^{1/2}$.

As for the motion away from the critical amplitudes, we note from (2.8) and (2.13) that α oscillates with amplitude and frequency of order ε, and that the oscillatory as well as secular components of β behave similarly.

The above intuitive construction will next be analyzed systematically by the use of two different expansions and their roles established in terms of all possible initial conditions.

2.2 Outer expansion

In order to account for the most general form of initial conditions, we represent the motion away from the critical amplitude by an expansion in powers of $\varepsilon^{1/2}$, called the outer expansion:

(2.18) $$y(t;\varepsilon) = \sum_{i=0} y_{i/2}(t,\tilde{t};\varepsilon)\ {}^{1/2}$$

As before the leading term of (2.18) is

$$y_o(t,\tilde{t};\varepsilon) = \alpha(\tilde{t};\varepsilon)\cos[t - \beta(\tilde{t};\varepsilon)] \tag{2.19}$$

where we set

$$\alpha(\tilde{t};\varepsilon) = \sum_{i=0} \alpha_{i/2}(\tilde{t})\varepsilon^{i/2} \tag{2.20}$$

$$\beta(\tilde{t};\varepsilon) = \sum_{i=0} \beta_{i/2}(\tilde{t})\varepsilon^{i/2} \tag{2.21}$$

It is then easy to show that $y_{1/2} = y_1 = y_{3/2} = 0$ after having defined the $\alpha_{i/2}$, $\beta_{i/2}$ by the following boundedness requirements:

$$\frac{d\alpha_o}{d\tilde{t}} = 0 \tag{2.22}$$

$$\frac{d\beta_o}{d\tilde{t}} = -(1 - 5\cos^2\alpha_o) \equiv s_o \tag{2.23}$$

$$\frac{d\alpha_{1/2}}{d\tilde{t}} = 0 \tag{2.24}$$

$$\frac{d\beta_{1/2}}{d\tilde{t}} = s_o'\alpha_{1/2} = -5(\sin 2\alpha_o)\alpha_{1/2} \tag{2.25}$$

$$\frac{d\alpha_1}{d\tilde{t}} = \frac{\alpha_o}{2}\sin\beta \tag{2.26}$$

$$\frac{d\beta_1}{d\tilde{t}} = \frac{1}{2}s_o^2 - \frac{5}{4}\alpha_o s_o \sin 2\alpha_o + \frac{1}{2}\cos\beta + s_o'\alpha_1 + \frac{s_o''}{2}\alpha_{1/2}^2 \tag{2.27}$$

where

$$s_o'' = \frac{d^2 s_o}{d\alpha_o^2} = -10\cos 2\alpha_o$$

Note that trigonometric functions with β as argument are not expanded to avoid trivial non-uniformities as the expansion of β in (2.21) need not involve bounded functions. It is only the phase velocity $d\beta/d\tilde{t}$ that must be bounded.

The solutions of the above equations are:

(2.28) $\alpha_o = \text{const.} = a_o$ (2.29) $\beta_o = s_o\tilde{t} + b_o$

and

(2.30 $\alpha_{1/2} = \text{const.} = a_{1/2}$ (2.31) $\beta_{1/2} = s_o' a_{1/2}\tilde{t} + b_{1/2}$

and

(2.32) $$\alpha_1 = -\frac{\alpha_o}{2s_o}\left[1 - \epsilon^{1/2}\frac{s_o' a_{1/2}}{s_o} + \epsilon\left(\frac{s_o' a_{1/2}}{s_o}\right)^2 + \ldots\right](\cos\beta - \cos b) + a_1$$

and equation (2.27) reduces to

(2.33) $$\frac{d\beta_1}{d\tilde{t}} = \frac{1}{2}s_o^2 - \frac{5}{4}\alpha_o s_o \sin 2\alpha_o + \frac{1}{2}\cos\beta - \frac{s_o'\alpha_o}{2s_o}\left[1 - \epsilon^{1/2}\frac{s_o' a_{1/2}}{s_o}\right.$$

$$\left. + \epsilon\left(\frac{s_o' a_{1/2}}{s_o}\right)^2 + \ldots\right](\cos\beta - \cos b) + s_o' a_1 + \frac{s_o''}{2} a_{1/2}^2$$

if the initial conditions are given as

(2.34) $$\beta = b = b_o + \epsilon^{1/2} b_{1/2} + \epsilon b_1 + \ldots$$

$$\alpha = a = a_o + \epsilon^{1/2} a_{1/2} + \epsilon a_1 + \ldots$$

at $t = 0$.

Equation (2.32) for α_1 exhibits the non-uniformity of the expansion near $s_o = 0$. Note that $\alpha_{1/2}$ would be identically zero if the initial amplitude did not contain a term proportional to $\epsilon^{1/2}$.

2.3 Inner expansion

As mentioned previously, the outer expansion fails to be valid as $s_o \to 0$. We now seek a solution which is valid and does not become unbounded at the critical amplitudes. This expansion will be called the "inner expansion". We let (cf. discussion after Fig. 1)

$$s_o = \varepsilon^{1/2}\,\bar{s}_o \tag{2.35}$$

and assume the following expansion for y

$$y(t;\varepsilon) = \sum_{i=0} y_{i/2}^{*}\,(t,\bar{t};\varepsilon)\varepsilon^{i/2} \tag{2.36}$$

where a new slow variable

$$\bar{t} = \varepsilon^{3/2} t = \varepsilon^{i/2}\,\tilde{t} \tag{2.37}$$

has been chosen. The equation for y_o^{*} is again

$$\frac{\partial^2 y_o^{*}}{\partial t^2} + y_o^{*} = 0 \tag{2.38}$$

whose general solution can be written in the form:

$$y_o^{*}(t,\bar{t};\varepsilon) = \alpha^{*}(\bar{t};\varepsilon)\cos\,[t - \beta^{*}(\bar{t};\varepsilon)] \tag{2.39}$$

We also expand the slowly varying functions $\alpha^{*}(\bar{t};\varepsilon)$ and $\beta^{*}(\bar{t};\varepsilon)$ in the following form in order to account for the homogeneous solutions of all higher orders.

$$\alpha^{*}(\bar{t};\varepsilon) = \sum_{i=0} \alpha_{i/2}^{*}(\bar{t})\varepsilon^{i/2} \qquad \beta^{*}(\bar{t};\varepsilon) = \sum_{i=0} \beta_{i/2}^{*}(\bar{t})\varepsilon^{i/2} \tag{2.40}$$

Substitution of the above expansions into (2.1) and the requirement that the $y_{i/2}^*$ be bounded gives the following ordinary differential equations for the $\alpha_{i/2}^*$ and $\beta_{i/2}^*$:

$$(2.41) \qquad \frac{d\alpha_o^*}{d\bar{t}} = 0 \qquad\qquad \frac{d\beta_o^*}{d\bar{t}} = s_o'\alpha_{1/2}^*$$

$$(2.42a) \qquad \frac{d\alpha_{1/2}^*}{d\bar{t}} = \frac{\alpha_o^*}{2} \sin \beta^*$$

$$(2.42b) \qquad \frac{d\beta_{1/2}^*}{d\bar{t}} = s_o'\alpha_1^* + \frac{s_o''}{2}(\alpha_{1/2}^*)^2 + \frac{1}{2}\cos \beta^*$$

$$(2.43a) \qquad \frac{d\alpha_1^*}{d\bar{t}} = \frac{\alpha_{1/2}^*}{2} \sin \beta^*$$

$$(2.43b) \qquad \frac{d\beta_1^*}{d\bar{t}} = s_o'\alpha_{3/2}^* + s_o''\alpha_{1/2}^*\alpha_1^* + \frac{s_o'''}{6}(\alpha_{1/2}^*)^3 - \frac{5}{2}\alpha_o^*\bar{s}_o \sin 2\alpha_o^*$$

with the additional results that

$$(2.44) \qquad y_{1/2}^* = y_1^* = t_{3/2}^* = y_2^* = 0$$

and that only in $y_{5/2}^*$ do we have higher harmonics in the fast variable.

We note that equations (2.42) are precisely the equations one would obtain in the inner limit from (2.14). Equations (2.41-2.43) can be solved successively for the $\alpha_{i/2}^*$ and the results are summarized below.

$$(2.45a) \qquad \alpha_o^* = \text{const.} = a_o^*$$

$$(2.45b) \qquad \alpha_{1/2}^* = \bar{K}_o + \frac{1}{s_o'}[\bar{K}_1^2 + \bar{K}_2(\cos \beta^* - \cos b^*)]^{1/2}$$

$$(2.45c) \qquad \alpha_1^* = -\frac{\bar{K}_o}{\bar{K}_2}[\bar{K}_1^2 + \bar{K}_2(\cos \beta^* - \cos b^*)]^{1/2} - (\cos \beta^* - \cos b^*)/2s_o' + a_1^*$$

where

$$(2.46) \qquad \bar{K}_o = -\bar{s}_o/s_o' \qquad \bar{K}_1^{\,2} = (\bar{s}_o^{\,2} + s_o' a_{1/2}^{*})^2 \qquad \bar{K}_2 = -a_o^{*} s_o'$$

and the initial conditions at $\bar{t} = 0$

$$(2.47) \qquad \alpha_{i/2}^{*} = a_{i/2}^{*} \qquad \beta^{*} = b^{*}$$

have been imposed.

With the $\alpha_{i/2}^{*}$ so defined the solution for the $\beta_{i/2}^{*}$ reduces to quadratures. These details will not be carried out here as the qualitative behavior of both α and β have already been discussed.

2.4 Matching of solutions and composite expansion

In the standard singular perturbation problem in which two limit process expansions can be derived in their respective domains, either one or both of these expansions is defined incompletely prior to the matching (cf. Kaplun and Lagerstrom (1957)). For example, the initial conditions for the inner solution would depend upon the values taken on by the outer solution in the inner region if the motion spans both regimes (cf. Lagerstrom and Kevorkian (1963)). In this case, the matching will define the motion in the inner region and the behavior of the two limit-process expansions in their common overlap domain will provide the basis for deriving a composite expansion which is uniformly valid everywhere.

In the present example, as well as in the main problem, the motion depending upon the initial condition on α lies for all times in either the outer or inner regions. Furthermore, the parameter which establishes the appropriate expansion does not vary in order of magnitude with time. The purpose of matching is then two-fold. First, the direct matching of the two expansions will prove the existence of a common overlap domain and rule out the possibility of an even third limit-process expansion for some value of μ such that $s_o = O(\varepsilon^{\mu})$, $1 < \mu < 1/2$. Secondly, the matching will provide the necessary information for obtaining a representation of the motion for all values of s_o in the above order interval once the behaviors at the end-points of this order interval have been calculated. General principles of matching are discussed by Kaplun and Lagerstrom (1957). For the present examples, as well as for the main problem, it is sufficient to show that the inner solution for large values of $\bar{s}_o$ agrees with the inner limit of the outer expansion. In this event the derivation of a composite expansion which is uniformly valid for all s_o in the order interval ord $\epsilon^{1/2} \leq$ ord $s_o \leq$ ord 1 becomes particularly straightforward.

The matching between α and α^* is very simple. If we rewrite α^* in terms of outer variables and expand for $\bar{s}_o \to \infty$, we obtain

$$\alpha^* = \alpha_o^* + \epsilon^{1/2}\alpha_{1/2}^* + \epsilon\alpha_1^* + O(\epsilon^{3/2}) = a_o^* + \epsilon^{1/2}a_{1/2}^* + \epsilon a_1^* \tag{2.48}$$
$$- \frac{\epsilon}{2}\frac{\alpha_o^*}{s_o}(\cos\beta^* - \cos b^*) + \frac{1}{2}\epsilon^{3/2}\left(\frac{s_o'}{s_o}\alpha_o^* - 1\right) a_{1/2}^*(\cos\beta^* - \cos b^*)$$
$$- \frac{\epsilon^2}{2}\left(\frac{s_o'}{s_o}\alpha_o^* - 1\right)\left[\frac{s_o'}{s_o^2}a_{1/2}^*(\cos\beta^* - \cos b^*) + \frac{1}{4}\frac{\alpha_o^*}{s_o^2}(\cos^2 b^*\right.$$
$$\left. - 2\cos\beta^*\cos b^* + \frac{1}{2} + \frac{1}{2}\cos 2\beta^*)\right] + O(\epsilon^{5/2})$$

From the outer expansion we have

$$\alpha = \alpha_o + \epsilon^{1/2}\alpha_{1/2} + \epsilon\alpha_1 + O(\epsilon^{3/2}) = a_o + \epsilon^{1/2}a_{1/2} + \epsilon a_1 \tag{2.49}$$
$$- \frac{\epsilon\alpha_o}{2s_o}\left[1 - \epsilon^{1/2}\frac{s_o' a_{1/2}}{s_o} + \epsilon\frac{s_o'^2 a_{1/2}^2}{s_o^2} + \ldots\right](\cos\beta - \cos b) + O(\epsilon^{5/2})$$

By comparing equations (2.48) and (2.49) we see that the inner expansion contains the outer expansion explicitly to order ϵ^2. Note that in the overlap domain we have $a_{i/2} = a_{i/2}^*$. In fact, all terms in the outer expansion to order ϵ^2 are contained in $\alpha_o^* + \epsilon^{1/2}\alpha_{1/2}^*$. The outer expansion of α_1^* is entirely of higher order. Thus, the composite expansion which is uniformly valid to $O(\epsilon)$ everywhere is:

$$\alpha_c = \alpha_o^* + \epsilon^{1/2}\alpha_{1/2} + \epsilon\alpha_1^* \tag{2.50}$$

In this matching, we have assumed that both β and β^* are matched. This will be shown in the subsequent discussion. For simplicity, we will discuss the matching between $d\beta/d\tilde{t}$ and $d\beta^*/d\bar{t}$ instead.

To summarize, we have already obtained

$$\text{(2.51)} \qquad \frac{d\beta}{d\tilde{t}} = s_o + \varepsilon^{1/2} s_o' a_{1/2} + \varepsilon\Big(\frac{1}{2} s_o^2 - \frac{5}{4}\alpha_o^2 s_o \sin 2\alpha_o + \frac{1}{2}\cos\beta + s_o'\alpha_1 + \frac{s_o''}{2} a_{1/2}^2\Big) + O(\varepsilon^{3/2})$$

and

$$\text{(2.52)} \qquad \frac{d\beta^*}{d\bar{t}} = \bar{s}_o + s_o'\alpha_{1/2}^* + \varepsilon^{1/2}\Big[s_o'\alpha_1^* + \frac{1}{2}\cos\beta^* + \frac{s_o''}{2}(\alpha_{1/2}^*)^2\Big] + O(\varepsilon)$$

We note that the inner expansion of (2.52) for $d\beta^*/d\bar{t}$ contains all the terms that appear in the outer expansion (2.51) with the exception of the two terms $-\frac{5}{4}\varepsilon\alpha_o^2 s_o \sin 2\alpha_o$ and $\frac{1}{2}\varepsilon s_o^2$. This is consistent, because when the above terms are expressed in terms of the inner parameter $\bar{s}_o$, they become of order $\varepsilon^{3/2}$ and ε^2 respectively. Thus, they should appear in the expressions for $d\beta_1^*/d\bar{t}$ and $d\beta_{3/2}/d\bar{t}$ respectively. The first term does appear in the expression (2.43b) for $d\beta_1^*/d\bar{t}$ and one would recover the second term if $d\beta_{3/2}^*/d\bar{t}$ were evaluated.

Conversely, many terms in the inner expansion, e.g. $s_o'\alpha_1^*$ and $s_o''(\alpha_{1/2}^*)^2/2$, are of orders higher than we considered in the outer expansion and will appear in the corresponding higher order terms. Having carried out the calculations to the present order we can easily derive the following composite expansion for $d\beta_c/d\bar{t}$ which is uniformly valid to order ε^2 for all s_o.

$$(2.53) \qquad \frac{d\beta_c}{d\bar{t}} = \bar{s}_o + s_o{}'\alpha_{1/2}{}^* + \varepsilon^{1/2}[s_o{}'\alpha_1{}^* + \frac{1}{2}\cos\beta^* + \frac{s_o{}^*}{2}(\alpha_{1/2}{}^*)^2]$$

$$- \frac{5}{4}\varepsilon(\alpha_o{}^*)^2\bar{s}_o \sin 2\alpha_o{}^* + \frac{\varepsilon^2}{2}\bar{s}_o{}^2$$

In deriving (2.53) we have used the customary construction of adding the inner and outer representations for $d\beta/d\bar{t}$ and subtracting those terms which are common to both expansions in the intermediate region. These terms are the two higher order terms appearing at the end of (2.53). Thus, to order $\varepsilon^{1/2}$ the inner expansion $d\beta^*/d\bar{t}$ is itself uniformly valid for all s_o. It is only in deriving an expression valid to orders higher than $\varepsilon^{1/2}$ that one needs consideration of terms contributed by the outer expansion.

Finally, the solution of (2.1) for y which is uniformly valid to $O(\varepsilon)$ for all s_o is

$$(2.54) \qquad y(t,\varepsilon) = (\alpha_o{}^* + \varepsilon^{1/2}\alpha_{1/2}{}^* + \varepsilon\alpha_1{}^*)\cos[t - \beta_c(t;\varepsilon)] + O(\varepsilon^{3/2})$$

The behavior of the amplitude and phase to $O(\varepsilon^{1/2})$ was discussed earlier in connection with (2.14). The higher order terms will not alter the general qualitative nature of the solution. The detailed and systematic development of the expansions for α and β was carried out here to serve as a guideline for the study of the main problem for which there is no a priori knowledge of the particular higher order terms which cause local resonance. Hence one must rely on a more formal construction analagous to the process used in sections 2.2-2.4.

3. The Main Problem

Once a suitable choice of variables is made, the motion of a satellite around an oblate planet reduces in principle to the solution of a problem in non-linear oscillations analogous to the model discussed in Section 2. Of course, instead of the two slowly varying functions α and β, we now have six slowly varying orbital elements. However, it will be shown that the main problem hinges on solving the coupled equations for the inclination and apse which will be the analogues of α and β, and that the remainder of the elements will then be given by quadratures.

3.1 Formulation of the problem, coordinate system

Consider an inertial frame with origin at the center of an oblate planet having a radius R in the equatorial plane of symmetry. We normalize distances by the radius R and the time by $(R^3/GM)^{1/2}$, where G is the universal gravitational constant and M is the mass of the planet. The dimensionless equation of motion for a satellite is then

$$\frac{d^2\vec{x}}{dt^2} = \text{grad } U \tag{3.1}$$

where $\vec{x}$ is the dimensionless distance vector from the origin and the potential U has the following form in spherical polar coordinates with respect to the polar axis of symmetry:

$$U = \frac{1}{r} + \frac{\varepsilon}{3r^3}(1 - 3\cos^2\theta) + \frac{c\varepsilon^2}{5r^5}(35\cos^4\theta - 30\cos^2\theta + 3) + O(\varepsilon^3) \tag{3.2}$$

where θ is the polar angle.

It has been assumed that the planet is an ellipsoid of revolution and for the earth the constants ϵ and c are approximately (cf. Jeffries (1959) and Shi (1963))

$$\epsilon = J = 1.623 \times 10^{-3} \qquad c = 4/7$$

In the conventional spherical polar coordinates:

(3.3a) $\quad x = r \cos \psi \sin \theta$

(3.3b) $\quad y = r \sin \psi \sin \theta$

(3.3c) $\quad z = r \cos \theta$

where

(3.3d) $\quad \vec{x} = (x,y,z), \ |\vec{x}| = r$

Equation (3.1) for any potential U has the following component form:

(3.4a) $$\frac{d}{dt}\left(r^2 \sin^2\theta \frac{d\psi}{dt}\right) = \frac{\partial U}{\partial \psi}$$

(3.4b) $$\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) - r^2 \sin\theta \cos\theta \left(\frac{d\psi}{dt}\right)^2 = \frac{\partial U}{\partial \theta}$$

(3.4c) $$\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r \sin^2\theta \left(\frac{d\psi}{dt}\right)^2 = \frac{\partial U}{\partial r}$$

Since the satellite can be considered to move in an instantaneous plane defined by the distance and velocity vectors, one may also define the motion by the following variables proposed by Struble (1960) and (1961) (cf. Fig. 2 for the geometry).

i = angle between instantaneous orbital and equatorial planes

Ω = angle in the equatorial plane between some fixed direction, say x pointing towards the vernal equinox, and the ascending node

r = the radius

ϕ = angle between the ascending node and the distance vector.

Struble (1960) has shown that equations (3.4) transform to the following fifth-order system after elimination of the time.*

$$(3.5a)\qquad \frac{dp}{d\phi} = \frac{\dfrac{\partial U}{\partial \psi}}{\dfrac{pu^2}{\cos i} \bullet \dfrac{\cos^3 i \cos \theta}{p \sin^2 i \sin \theta}\left[\dfrac{\partial U}{\partial \theta} + \tan i \dfrac{\cos \phi}{\sin \theta}\dfrac{\partial U}{\partial \psi}\right]}$$

$$(3.5b)\qquad \frac{d\Omega}{d\phi} = \frac{-\cos^3 i \cos \theta\left[\dfrac{\partial U}{\partial \theta} + \tan i \dfrac{\cos \phi}{\sin \theta}\dfrac{\partial U}{\partial \psi}\right]}{p^2 u^2 \sin^2 i \sin \theta + \cos^4 i \cos \theta\left[\dfrac{\partial U}{\partial \theta} + \tan i \dfrac{\cos \phi}{\sin \theta}\dfrac{\partial U}{\partial \psi}\right]}$$

$$(3.5c)\qquad \frac{di}{d\phi} = -\frac{\sin^2 i \cos^3 i \cos \phi\left[\dfrac{\partial U}{\partial \theta} + \tan i \dfrac{\cos \phi}{\sin \theta}\dfrac{\partial U}{\partial \psi}\right]}{p^2 u^2 \sin^2 i \sin \theta + \cos^4 i \cos \theta\left[\dfrac{\partial U}{\partial \theta} + \tan i \dfrac{\cos \phi}{\sin \theta}\dfrac{\partial U}{\partial \psi}\right]}$$

$$(3.5d)\qquad \frac{d^2 u}{d\phi^2} - \frac{2}{u}\left(\frac{du}{d\phi}\right)^2 + \frac{\dfrac{du}{d\phi} \cdot \dfrac{d}{d\phi}\left(\dfrac{d\phi}{dt}\right)}{\dfrac{d\phi}{dt}} = -\frac{\dfrac{p^2 u^5}{\cos^2 i} + u^2 \dfrac{\partial U}{\partial r}}{\left(\dfrac{d\phi}{dt}\right)^2}$$

where p is the component of angular momentum along the polar axis and is defined by

$$(3.5e)\qquad p = r^2 \sin^2 \frac{d\phi}{dt}$$

$$(3.5f)\qquad u = \frac{1}{r}$$

In equation (3.5d) $\frac{d\phi}{dt}$ and θ are defined by

$$(3.5g)\qquad \frac{d\phi}{dt} = \frac{pu^2}{\cos i} + \frac{\cos^3 i \cos \theta}{p \sin^2 i \sin \theta}\left[\frac{\partial U}{\partial \theta} + \tan i \frac{\cos \phi}{\sin}\frac{\partial U}{\partial \psi}\right]$$

---

* Note that Struble (1960) defines the node in the opposite sense.

(3.5h) $\cos\theta = \sin i \sin\phi$

If we now use (3.2) for U and retain terms up to $O(\varepsilon^2)$ only, (3.5) simplify to

(3.6a) $$\frac{dp}{d\phi} = 0$$

(3.6b) $$\frac{d\Omega}{d\phi} = \frac{-2\varepsilon u\cos^3 i\cos^2\theta\,[1-2c\varepsilon u^2(7\cos^2\theta-3)]}{p^2\sin^2 i + 2\varepsilon u\cos^4 i\cos^2\theta\,[1-2c\varepsilon u^2(7\cos^2\theta-3)]}$$

(3.6c) $$\frac{di}{d\phi} = \frac{-2\varepsilon u\sin^2 i\cos^3 i\cos\phi\cos\theta\,[1-2c\varepsilon u^2(7\cos^2\theta-3)]}{p^2\sin^2 i + 2\varepsilon u\cos^4 i\cos^2\theta\,[1-2c\varepsilon u^2(7\cos^2\theta-3)]}$$

(3.6d) $$\frac{d^2u}{d\phi^2} - \frac{2}{u}\left(\frac{du}{d\phi}\right)^2 + \frac{\frac{du}{d\phi}\cdot\frac{d}{d\phi}\left(\frac{d\phi}{dt}\right)}{\frac{d\phi}{dt}} = -\frac{p^2u^5}{\left(\frac{d\phi}{dt}\right)^2\cos^2 i}$$
$$+\frac{1+\varepsilon u^2(1-3\cos^2\theta)+c\varepsilon^2u^4(35\cos^4\theta-30\cos^2\theta+3)}{\left(\frac{d\phi}{dt}\right)^2}\cdot u^4$$

where $\frac{d\phi}{dt}$ is given by (3.5g) with $\frac{\partial U}{\partial\psi} = 0$.

According to (3.6a) p is a constant, a consequence of the independence of U on ϕ. Furthermore, equation (3.6b) for the node is uncoupled from (3.6c) and (3.6d) and can hence be solved independently once u and i have been determined.

Making use of the identities (3.5g) and (3.5h) and retaining terms up to $O(\varepsilon^2)$ in (3.6c) and (3.6d) yields:

(3.7a) $$\frac{di}{d\phi} = -\varepsilon\frac{u}{p^2}\cos^3 i\sin i\sin 2\phi + 2\varepsilon^2\frac{u^2}{p^2}\cos^3 i\sin i\left[\frac{1}{p^2}\cos^4 i\sin^2\phi\right.$$
$$\left.-3cu+7cu\sin^2 i\sin^2\phi\right]\sin 2\phi + O(\varepsilon^3)$$

$$(3.7b)\qquad \frac{d^2u}{d\phi^2} + u = \frac{\cos^2 i}{p^2} + \varepsilon\Big[\frac{4u^2}{p^2}\cos^4 i\,\sin^2\phi + \frac{u}{p^2}\left(\frac{du}{d\phi}\right)\cos^2 i(1 - 3\cos^2 i)\sin 2\phi$$

$$- \frac{2}{p^2}\left(\frac{du}{d\phi}\right)^2 \cos^4 i\,\sin^2\phi + \frac{u^2}{p^2}\cos^2 i(1 - 3\sin^2 i\,\sin^2\phi) - 4\,\frac{u}{p^4}\cos^6 i\,\sin^2\phi\Big]$$

$$+ \varepsilon^2\Big[- 4\,\frac{\cos^4 i}{p^2}\,u^3\sin^2\phi\,\Big\{\frac{3\cos^4 i}{p^2}\sin^2\phi - 2uc(3 - 7\sin^2 i\,\sin^2\phi)\Big\}$$

$$- 8\,\frac{u^2}{p^4}\frac{du}{d\phi}\sin^2 i\,\cos^6 i\,\sin^2\phi\,\sin 2\phi - 2c\,\frac{u^3}{p^2}\frac{du}{d\phi}\{- 3\sin^2 i\,\cos^2 i$$

$$+ 7\sin^4 i\,\cos^2 i\,\sin^2\phi + 6\cos^4 i - 28\cos^4 i\,\sin^2 i\,\sin^2\phi\}\sin 2\phi$$

$$- 12c\,\frac{u^2}{p^2}\left(\frac{du}{d\phi}\right)^2\cos^4 i\,\sin^2\phi\,(3 - 7\sin^2 i\,\sin^2\phi)$$

$$+ 2\,\frac{u^2}{p^4}\frac{du}{d\phi}\cos^6 i\,(3\cos^2 i - 1)\sin^2\phi\,\sin 2\phi + 4\,\frac{u}{p^4}\left(\frac{du}{d\phi}\right)^2\cos^8 i\,\sin^4\phi$$

$$+ c\,\frac{u^4}{p^2}\cos^2 i\,\{35\sin^4 i\,\sin^4\phi - 30\sin^2 i\,\sin^2\phi + 3\}$$

$$- 4\,\frac{u^3}{p^4}\cos^6 i\,\sin^2\phi\,(1 - 3\sin^2 i\,\sin^2\phi) + 4\,\frac{u^2}{p^4}\cos^6 i\,\sin^2\phi\,\Big\{\frac{3}{p^2}\cos^4 i\,\sin^2\phi$$

$$- 2cu\,(3 - 7\sin^2 i\,\sin^2\phi)\Big\}\Big] + O(\varepsilon^3)$$

It is mentioned in passing that Struble (1961) chose a modified variable analogous to ϕ in order to eliminate certain non-uniformities in the solution. With the present approach this is unnecessary, since all the required scale changes are automatically accounted for by the two-variable procedure.

3.2 Outer expansion

The main problem to which we have previously referred is the solution of equations (3.7a) and (3.7b). Since $\cos^2 i/p^2$ is constant to order unity, we see from (3.7b) that this problem reduces to solving the motion of an oscillator with small non-linearities and a weak coupling because i is constant to order unity. The somewhat lengthy nature of the perturbation terms in (3.7b) does not alter the fact that the system in question is qualitatively analogous to the model equation studied in Section 2. We therefore proceed as in Section 2.2 by assuming the following expansions for i and u:

(3.8a) $$i(\phi;\varepsilon) = \sum_{n=0} i_n(\phi,\tilde{\phi};\varepsilon)\cdot\varepsilon^{n/2}$$

(3.8b) $$u(\phi;\varepsilon) = \sum_{n=0} u_n(\phi,\tilde{\phi};\varepsilon)\cdot\varepsilon^{n/2}$$

where $\tilde{\phi}$, analogous to the slow time variable, is defined by

(3.8c) $$\tilde{\phi} = \varepsilon\phi$$

Substitution of (3.8) into (3.7) gives to order unity

(3.9a) $$\frac{\partial i_o}{\partial \phi} = 0$$

(3.9b) $$\frac{\partial^2 u_o}{\partial \phi^2} + u_o = \frac{\cos^2 i_o}{p^2}$$

whose general solution is

(3.10a) $$i_o = i_o(\tilde{\phi};\varepsilon)$$

(3.10b) $$u_o = \frac{\cos^2 i_o}{p^2}\{1 + e(\tilde{\phi};\varepsilon)\cos[\phi - \omega(\tilde{\phi};\varepsilon)]\}$$

In (3.10b) the two "constants of integration" have been expressed in terms of the conventional Keplerian elements.

e = eccentricity

ω = apse angle measured in the counterclockwise sense from the ascending node to perigee in the instantaneous orbital plane.

As before we assume i_o, e and ω have the following expressions

(3.11a) $$i_o(\tilde{\phi};\varepsilon) = \sum_{n=0} i_{on/2}(\tilde{\phi})\varepsilon^{n/2}$$

(3.11b) $$e(\tilde{\phi};\varepsilon) = \sum_{n=0} e_{n/2}(\tilde{\phi})\varepsilon^{n/2}$$

(3.11c) $$\omega(\tilde{\phi};\varepsilon) = \sum_{n=0} \omega_{n/2}(\tilde{\phi})\varepsilon^{n/2}$$

in order to account for the homogeneous solutions of the higher order terms in i and u. It is easy to see that since terms of $O(\varepsilon^{1/2})$ are absent in (3.7), $i_{1/2} = u_{1/2} = 0$. The following equations for i_1 and u_1 can then be derived.

(3.12a) $$\frac{\partial i_1}{\partial \phi} = \frac{di_{oo}}{\partial \tilde{\phi}} - \frac{1}{p^4} \cos^5 i_{oo} \sin i_{oo} \sin 2\phi [1 - e_o \cos(\phi - \omega)]$$

(3.12b) $$\frac{\partial^2 u_1}{\partial \phi^2} + u_1 = [- \frac{2e_o}{p^2} \frac{d\omega_o}{d\tilde{\phi}} \cos^2 i_{oo} - \frac{e_o}{p^6} \cos^6 i_{oo} (1 - 5\cos^2 i_{oo})] \cos(\phi - \omega)$$

$$+ \frac{2}{p^2} \cos^2 i_{oo} \frac{de_o}{d\tilde{\phi}} \sin(\phi - \omega) - \frac{1}{p^6} \cos^6 i_{oo} \sin^2 i_{oo} [\cos 2\phi + \frac{e_o}{3} \cos(3\phi - \omega)]$$

$$+ \frac{\cos^6 i_{oo}}{p^6} (- \frac{1}{2} + \frac{7}{2} \cos^2 i_{oo})[1 + \frac{e_o^2}{2} + \frac{e_o^2}{2} \cos 2(\phi - \omega)]$$

$$+ \frac{\cos^6 i_{oo}}{2p^6} (3 - 7\cos^2 i_{oo})[(1 + \frac{e_o^2}{2})\cos 2\phi + e_o \cos(3\phi - \omega)$$

$$+ \frac{e_o^2}{4} \cos(4\phi - 2\omega) + \frac{e_o^2}{4} \cos 2\omega] - \frac{2}{p^6} \cos^8 i_{oo} \, [1 - \cos 2\phi - \frac{e_o}{2} \cos(3\phi - \omega)]$$

$$- \frac{e_o}{2} \frac{\cos^6 i_{oo}}{p^6} (1 - 3\cos^2 i_{oo})[- \cos(3\phi - \omega) - \frac{e_o}{2} \cos(4\phi - 2\omega) + \frac{e_o}{2} \cos 2\omega]$$

$$- \frac{e_o^2}{2} \frac{\cos^8 i_{oo}}{p^6} [1 - \cos 2(\phi - \omega) - \cos 2\phi + \frac{1}{2} \cos(4\phi - 2\omega) + \frac{1}{2} \cos 2\omega]$$

In order that i_1 and u_1 be bounded we must set

(3.13a) $$\frac{di_{oo}}{d\tilde{\phi}} = 0$$

(3.13b) $$\frac{de_o}{d\tilde{\phi}} = 0$$

(3.13c) $$\frac{d\omega_o}{d\tilde{\phi}} = - \frac{\cos^4 i_{oo}}{2p^4} (1 - 5\cos^2 i_{oo}) = S_o$$

Note the similarity of (3.13a) and (3.13c) to (2.22) and (2.23) establishing the analogy between α and β of the model equation with i and ω respectively for the main problem

Thus, the elements to first order become

(3.14) $$e_o = \text{const.} \qquad i_{oo} = j_o = \text{const.} \qquad \omega_o = S_o\tilde{\phi} + w_{oo}$$

where w_{oo} is a constant depending upon the initial conditions.

Equations (3.12) can now be solved to give

(3.15a) $$i_1 = \frac{1}{2p^4} \cos^5 i_{oo} \sin i_{oo} \, [\cos 2\phi + e_o \cos(\phi + \omega) + \frac{e_o}{3} \cos(3\phi - \omega)]$$

(3.15b) $$u_1 = \frac{\cos^6 i_{oo}}{2p^6} [- 1 + 3\cos^2 i_{oo} - \frac{e_o^2}{2}(1 - 5\cos^2 i_{oo})$$

$$+\frac{e_o^2}{4}\,(1 - 3\cos^2 i_{oo})\,\cos 2\omega - \left(\frac{1}{3}\sin^2 i_{oo} - \frac{e_o^2}{3} + \frac{5}{6}\,e_o^2\,\sin^2 i_{oo}\right)\cos 2\phi$$

$$+\frac{e_o^2}{6}\,(1 - 9\cos^2 i_{oo})\,\cos 2(\phi - \omega) - \frac{e_o}{12}\,(5 - 11\cos^2 i_{oo})\,\cos(3\phi - \omega)$$

$$-\frac{e_o^2}{12}\,(1 - 3\cos^2 i_{oo})\,\cos(4\phi - 2\omega)]$$

To $O(\varepsilon^{3/2})$ all the forcing terms on the right-hand sides of the equations for $u_{3/2}$ must be removed for boundedness, giving

(3.16) $$\frac{di_{o1/2}}{d\bar{\phi}} = 0 \qquad \frac{de_{1/2}}{d\bar{\phi}} = 0 \qquad \frac{d\omega_{1/2}}{d\bar{\phi}} = S_1 i_{o1/2}$$

$$S_n = \frac{d^n S_o}{di_{oo}^{\,n}},\ n = 1,\ 2,\ \ldots$$

which implies that

(3.17) $$i_{o1/2} = j_{1/2} = \text{constant} \qquad e_{1/2} = \text{constant} \qquad \omega_{1/2} = S_1 j_{1/2}\bar{\phi} + \omega_{1/2o}$$

where $\omega_{1/2o}$ is a constant depending on the initial condition.

The requirement that i_2 and u_2 be bounded provides the following equations for i_{o1}, ${}_1$, and e_1:*

(3.18a) $$\frac{di_{o1}}{d\bar{\phi}} = C_2 \sin 2\omega$$

(3.18b) $$\frac{d\omega_1}{d\bar{\phi}} = \frac{1}{2}\,S_2(i_{o1/2})^2 + S_1 i_{o1} + A_o + A_2 \cos 2\omega$$

(3.18c) $$\frac{de_1}{d\bar{\phi}} = B_2 \sin 2\omega$$

The solutions of (3.18) subject to the initial conditions

$$(3.19) \qquad \omega = v \qquad e_n = \eta_n \qquad i_{on} = j_n$$

at t = are

$$(3.20a) \qquad i_{o1} = j_1 + \frac{C_2}{2S_o}(\cos 2v - \cos 2\omega)$$

$$(3.20b) \qquad e_1 = \ _1 + \frac{B_2}{2S_o}(\cos 2v - \cos 2\omega)$$

$$(3.20c) \qquad \frac{d\omega_1}{d\tilde{\phi}} = \frac{1}{2} S_2 j_{1/2}{}^2 + S_1[j_1 + \frac{C_2}{2S_o}(\cos 2v - \cos 2\omega)] + A_o + A_2 \cos 2\omega$$

The non-uniformities of the outer solution near $S_o = 0$ are exhibited above and are a consequence of the non-validity of the expansions assumed in (3.8) near the critical inclination.

3.3 Inner expansion

As shown in Section 2, the expansion procedure for inclinations close to the critical value should be of the form

$$(3.21a) \qquad u(\phi;\varepsilon) = \sum_{n=0} u_{n/2}{}^*(\phi,\bar{\phi};\varepsilon)\varepsilon^{n/2}$$

$$(3.21b) \qquad i(\phi;\varepsilon) = \sum_{n=0} i_{n/2}{}^*(\phi,\bar{\phi};\varepsilon)\varepsilon^{n/2}$$

where

$$(3.21c) \qquad \bar{\phi} = \varepsilon^{3/2}\phi = \varepsilon^{1/2}\tilde{\phi}$$

* Henceforth all constants not defined in the text will be found in the Appendix with no additional reference.

and we are interested in the case where

$$(3.21d) \qquad S_o = \epsilon^{1/2}\bar{S}_o, \text{ with } \bar{S}_o = O(1)$$

Upon substitution of (3.21) into (3.7) we obtain the following equations for the leading terms:

$$(3.22a) \qquad \frac{\partial^2 u_o^*}{\partial \phi^2} + u_o^* = \frac{\cos^2 i_o^*}{p^2}$$

$$(3.22b) \qquad \frac{\partial i_o^*}{\partial \phi} = 0$$

whose general solution is of the form:

$$(3.23a) \qquad i_o^* = i_o^*(\bar{\phi};\epsilon)$$

$$(3.23b) \qquad u_o^* = \frac{\cos^2 i_o^*}{p^2}\{1 + e^*(\bar{\phi},\epsilon) \cos[\phi - \omega^*(\bar{\phi};\epsilon)]\}$$

We also expand the elements of the inner solution in the form:

$$(3.24a) \qquad i_o^*(\bar{\phi};\epsilon) = \sum_{n=0} i_{on/2}^*(\bar{\phi})\epsilon^{n/2}$$

$$(3.24b) \qquad e^*(\bar{\phi};\epsilon) = \sum_{n=0} e_{n/2}^*(\bar{\phi})\epsilon^{n/2}$$

$$(3.24c) \qquad \omega^*(\bar{\phi};\epsilon) = \sum_{n=0} \omega_{n/2}^*(\bar{\phi})\epsilon^{n/2}$$

Since the homogeneous solution to $O(\epsilon^{1/2})$ is already accounted for by the expansion of the elements, we find $u_{1/2}^* = i_{1/2}^* = 0$ and can derive the following equations for the terms of $O(\epsilon)$.

$$(3.25a)\qquad \frac{\partial^2 u_1^*}{\partial\phi^2} + u_1^* = -\frac{1}{p^6}\cos^6 i_{oo}^*\sin^2 i_{oo}[\cos 2\phi + \frac{e_o^*}{3}\cos(3\phi - \omega^*)]$$

$$+ \frac{\cos^6 i_{oo}^*}{p^6}(-\frac{1}{2} + \frac{7}{2}\cos^2 i_{oo}^*)[1 + \frac{e_o^{*2}}{2} + \frac{e_o^{*2}}{2}\cos 2(\phi - \omega^*)]$$

$$+ \frac{\cos^6 i_{oo}^*}{p^6}(3 - 7\cos^2 i_{oo}^*)[(1 + \frac{e_o^{*2}}{2})\cos 2\phi + e_o^*\cos(3\phi - \omega^*)$$

$$+ \frac{e_o^{*2}}{4}\cos(4\phi - 2\omega^*) + \frac{e_o^{*2}}{4}\cos 2\omega^*] - \frac{2}{p^6}\cos^8 i_{oo}^*[1 - \cos 2\phi$$

$$- \frac{e_o^*}{2}\cos(3\phi - \omega^*)] - \frac{e_o^*\cos^6 i_{oo}^*}{2p^6}(1 - 3\cos^2 i_{oo}^*)[-\cos(3\phi - \omega^*)$$

$$- \frac{e_o^*}{2}\cos(4\phi - 2\omega^*) + \frac{e_o^*}{2}\cos 2\omega^*] - \frac{e_o^{*2}}{2p^6}\cos^8 i_{oo}^*[1 - \cos 2(\phi - \omega^*)$$

$$- \cos 2\phi + \frac{1}{2}\cos(4\phi - 2\omega^*) + \frac{1}{2}\cos 2\omega^*]$$

$$(3.25b)\qquad \frac{\partial i^*_1}{\partial\phi} = -\frac{1}{p^4}\cos^5 i_{oo}^*\sin i_{oo}^*\sin 2\phi[1 + e_o^*\cos(\phi - \omega^*)]$$

There are no terms proportional to $\sin\phi$ or $\cos\phi$ in (3.25a) and no terms which depend on $\bar{\phi}$ in (3.25b), so (3.25) can be solved directly to yield

$$(3.26a)\qquad i^*_1 = \frac{1}{2p^4}\cos^5 i_{oo}^*\sin i_{oo}^*[\cos 2\phi + e_o^*\cos(\phi + \omega^*) + \frac{e_o^*}{3}\cos(3\phi - \omega^*)]$$

$$(3.26b)\qquad u^*_1 = \frac{\cos^6 i_{oo}^*}{2p^6}[-1 + 3\cos^2 i_{oo}^* - \frac{e_o^{*2}}{2}(1 - 5\cos^2 i_{oo}^*)$$

$$+ \frac{e_o^{*2}}{4}(1 - 3\cos^2 i_{oo}^*)\cos 2\omega^* - (\frac{1}{3}\sin^2 i_{oo}^* - \frac{e_o^{*2}}{3}$$

$$+\frac{5}{6} e_o^{*2} \sin^2 i_{oo}^*) \cos 2\phi + \frac{e_o^{*2}}{6}(1 - 9\cos^2 i_{oo}^*) \cos 2(\phi - \omega^*)$$

$$-\frac{e_o^*}{12}(5 - 11\cos^2 i_{oo}^*) \cos(3\phi - \omega^*) - \frac{e_o^{*2}}{12}(1 - 3\cos^2 i_{oo}^*) \cos(4\phi - 2\omega^*)]$$

Since we are only interested in obtaining a solution correct to $O(\varepsilon)$, we only give the boundedness conditions for the higher order terms.

Requiring $u^*_{3/2}$ and $i^*_{3/2}$ to be bounded gives

(3.27a) $\dfrac{di_{oo}^*}{d\bar{\phi}} = 0$ (3.27b) $\dfrac{de_o^*}{d\bar{\phi}} = 0$ (3.27c) $\dfrac{d\omega_o^*}{d\bar{\phi}} = \bar{S}_o + S_1 i_{o1/2}^*$

and this implies

(3.28) $i_{oo}^* = \text{constant} = j_o^*$ $e_o^* = \text{constant} = \eta_o^*$

The boundedness of u_2^* and i_2^* requires

(3.29a) $$\frac{di_{o1/2}^*}{d\bar{\phi}} = C_2^* \sin 2\omega^*$$

(3.29b) $$\frac{de_{1/2}^*}{d\bar{\phi}} = B_2^* \sin 2\omega^*$$

(3.29c) $$\frac{d\omega_{1/2}^*}{d\bar{\phi}} = \frac{1}{2} S_2(i_{o1/2}^*) + S_1 i_{o1}^* + A_o^* + A_2^* \cos 2\omega^*$$

and finally in order to make $u_{5/2}^*$ and $i_{5/2}^*$ bounded we must set

(3.30a) $$\frac{di_{o1}^*}{d\bar{\phi}} = (C_{21}^* i_{o1/2}^* + C_{22}^* e_{1/2}^*) \sin 2\omega^*$$

(3.30b) $$\frac{de_1^*}{d\bar{\phi}} = [B_{21}^* i_{o1/2}^* + B_{22}^* e_{1/2}^* - \frac{e_o^*}{4} \bar{S}_o \frac{\cos^4 i_{oo}^*}{p^4}(1 - 3\cos^2 i_{oo}^*)] \sin 2\omega^*$$

3.4 Solution of the inner equations

From equations (3.27c and (3.29a) we obtain

$$(3.31)\qquad \frac{di_{o1/2}^*}{d\omega_o^*} = \frac{C_2^* \sin 2\omega^*}{\bar{S}_o + S_1 i_{o1/2}^*}$$

If the initial conditions are given as

$$(3.32)\qquad \omega^* = \nu^* \qquad i_{on}^* = j_n^* \qquad e_n^* = \eta_n^*$$

at $t = \tau$, equation (3.31) has the solution

$$(3.33)\qquad i_{o1/2}^* = \frac{1}{S_1}\left[(\bar{\kappa}_o - \kappa_1 \cos 2\omega^*)^{1/2} - \bar{S}_o\right]$$

which upon substitution into (3.27c) gives

$$(3.34)\qquad \frac{d\omega_o^*}{d\phi} = (\bar{\kappa}_o - \kappa_1 \cos 2\omega^*)^{1/2}$$

By use of equation (3.34) and equation (3.29b) we now find

$$(3.35)\qquad e_{1/2}^* = \frac{B_2^*}{\kappa_1}(\bar{\kappa}_o - \kappa_1 \cos 2\omega^*)^{1/2} + E_{1/2}$$

Similarly, from equation (3.30a) we calculate

$$(3.36)\qquad i_{o1}^* = \frac{1}{\kappa_1}\left[- C_{21}^* \frac{\bar{S}_o}{S_1} + C_{22}^* E_{1/2}\right](\bar{\kappa}_o - \kappa_1 \cos 2\omega^*)^{1/2} - \frac{1}{2S_1}\left[C_{21}^* + \frac{C_{22}^*}{C_2^*} B_2^*\right] \cos 2\omega^* + I_1$$

Equation (3.30b) can next be integrated to

$$(3.37)\qquad e_1^* = (\bar{\kappa}_o - \kappa_1 \cos 2\omega^*)^{1/2}\left[- \frac{\bar{S}_o}{S_1} B_{21}^* + E_{1/2} B_{22}^* - \frac{e_o^*}{4}\frac{\bar{S}_o}{p^4}(1 - 3\cos^2 i_{oo}^*)\cos^4 i_{oo}^*\right] - \frac{1}{2}\left\{\frac{B_{21}^*}{S_1} + \frac{B_{22}^* B_2^*}{\kappa_1}\right\}\cos 2\omega^* + E_1$$

The solution of the apsidal motion will be considered in Section 3.6.

3.5 Matching and composite expansions

The problem of matching is essentially the same as the case discussed in Section 2 for the model equation. It must be remembered that in the overlap domain, the initial conditions are the same for both inner and outer expansions; thus

$$i_{oo}^* = i_{oo} = j_{oo} = j_o^*, \quad j_n^* = j_n, \quad e_o^* = e_o, \quad \eta_n^* = \eta_n, \quad w^* = w \tag{3.38}$$

One can then calculate the following relations between the constants appearing in the inner and outer expansions:

$$A_o^* = A_o - \frac{1}{2} S_o^2 \tag{3.39a}$$

$$A_2^* = A_2 - \frac{S_o}{4} \frac{\cos^4 i_{oo}}{p^4} (1 - 3\cos^2 i_{oo}) \tag{3.39b}$$

$$B_2^* = B_2 - \frac{S_o}{4} e_o \frac{\cos^4 i_{oo}}{p^4} (1 - 3\cos^2 i_{oo}) \tag{3.39c}$$

$$C_2^* = C_2 \tag{3.39d}$$

The matching between e and e^* can easily be realized by finding the outer expansion of $e_o^* + \epsilon^{1/2} e_{1/2}^* + \epsilon e_1^*$. This is simply

$$e_o^* + \epsilon^{1/2} e_{1/2}^* + \epsilon e_1^* = \eta_o^* + \epsilon^{1/2} \eta_{1/2}^* + \epsilon \eta_1^* - \epsilon \frac{B_2^*}{2S_o} (\cos 2\omega^* - \cos 2v^*) - \epsilon S_o \frac{\cos^4 i_{oo}^*}{4p^4} (1 - 3\cos^2 i_{oo}) + O(\epsilon^{3/2}) \tag{3.40}$$

The last term of $O(\varepsilon)$ in equation (3.40) arises from the outer expansion of εe_1^*. Thus, $e_o^* + \varepsilon^{1/2} e_{1/2}^*$ almost contain every term in the outer expansion and the outer expansion of εe_1^* is mostly of higher order.

By comparing equation (3.40) with the equation (3.20b), we note that in addition to matching directly, the inner expansion contains the outer. It then follows that the composite expansion for e which is uniformly valid to $O(\varepsilon)$ for all i is

$$(3.41) \qquad e_c = e_o^* + \varepsilon^{1/2} e_{1/2}^* + \varepsilon e_1^*$$

The matching between i^* and i proceeds in a similar way. The outer expansion of $i_{oo}^* + \varepsilon^{1/2} i_{o1/2}^*$ is

$$(3.42) \qquad i_{oo}^* + \varepsilon^{1/2} i_{o1/2}^* = j_o^* + \varepsilon^{1/2} j_{1/2}^* + \varepsilon \frac{C_2^*}{2S_o} (\cos 2w^* - \cos 2\omega) + O(\varepsilon^{3/2})$$

Comparison of equation (3.42) with equation (3.20a) shows that the inner expansion again contains the outer with the additional result that the outer expansion of i_{o1}^* is $O(\varepsilon^{3/2})$. Thus, the composite expansion for i_o is

$$(3.43) \qquad i_{oc}^* = i_{oo}^* + \varepsilon^{1/2} i_{o1/2}^* + \varepsilon i_{o1}^*$$

The above statements for e and i hold provided that ω and ω^* are matched, this will be considered next.

From equations (3.13c), (3.16), and (3.18b), we have

$$(3.44) \qquad \frac{d\omega}{d\bar{\phi}} = S_o + \varepsilon^{1/2} S_1 j_{1/2} + \varepsilon[\tfrac{1}{2} S_2 j_{1/2}^2 + S_1 i_{o1} + \Lambda_o + A_2 \cos 2\omega] + O(\varepsilon^{3/2})$$

$$(3.45) \qquad \frac{d\omega^*}{d\bar{\phi}} = \bar{S}_o + S_1 i^*_{o1/2} + \varepsilon^{1/2}[\tfrac{1}{2} S_2 (i^*_{o1/2})^2 + S_1 i^*_{o1} + A_o^*$$

$$+ A_2^* \cos 2\omega^*] + O(\varepsilon)$$

According to equation (3.42) the outer expansion of $i^*_{o1/2}$ contains i_{o1}. Use of this result leads to the following outer expansion for $\frac{d\omega^*}{d\bar{\phi}}$:

$$(3.46) \qquad \frac{d\omega^*}{d\bar{\phi}} = S_o + \varepsilon^{1/2} S_1 j^*_{1/2} + \varepsilon[\tfrac{1}{2} S_2 (j^*_{1/2})^2 + S_1 i_{o1} + A_o^*$$

$$+ A_2^* \cos 2\omega] + O(\varepsilon^{3/2})$$

Comparing equations (3.46) with (3.44) we note that they are matched in any overlap domain $S_o = O(\varepsilon^{\mu})$ with $0 < \mu < \frac{1}{2}$ because those terms not contained in the outer expansion of $d\omega^*/d\bar{\phi}$ have S_o as a factor (cf. Eq. 3.39) and are obviously small in the overlap domain. The composite expansion for the motion of the apse is therefore

$$(3.47) \qquad \frac{d\omega_c}{d\bar{\phi}} = \bar{S}_o + S_1 i^*_{o1/2} + \varepsilon^{1/2} [\tfrac{1}{2} S_2 (i^*_{o1/2}) + S_1 i^*_{o1} + A_o$$

$$+ A_2 \cos 2\omega_c] + O(\varepsilon)$$

uniformly to order $\varepsilon^{1/2}$ for all inclinations.

From the assumed forms for u and i it is easily seen that the uniformly valid expansions to $O(\varepsilon)$ for all incalinations for these variables are

$$(3.48a) \qquad u_c = \frac{\cos^2 i_{oc}}{p^2} [1 + e_c \cos(\phi - \omega_c)] + \varepsilon^{1/2} u_{1/2} + \varepsilon u_1$$

$$(3.48b) \qquad i_c = i_{oc} + \varepsilon^{1/2} i_{1/2} + \varepsilon i_1$$

where i_{oc}, e_c and ω_c are used instead of i_o, e and ω in $u_{1/2}$, u_1, $i_{1/2}$ and i_1 in equations (3.48a) and (3.48b) and ω_c can be obtained by integrating equation (3.47).

3.6 Apsidal motion

The dominant behavior of the apsidal motion is described by the leading term. We have from (3.34)

$$(3.49) \qquad d\bar{\phi} = (\bar{\kappa}_o - \kappa_1 + 2\kappa_1 \sin^2\omega^*)^{-1/2} d\omega^* + O(\epsilon^{1/2})$$

If we let

$$\sin^2\omega^* = \nu \qquad\qquad 2\sin\omega^*\cos\omega^* = \frac{d\nu}{d\omega^*} = 2[\nu(1-\nu)]^{1/2}$$

and consider only the leading term we obtain

$$(3.50) \qquad \bar{\phi} - \bar{\phi}_o = \int_o^\nu \frac{d\xi}{[(-8\kappa_1)(\xi - \lambda)\xi(\xi - 1)]^{1/2}}$$

$$(3.51) \qquad \lambda = (\kappa_1 - \bar{\kappa}_o)/2\kappa_1$$

For the earth's potential the quantity κ_1 is positive* near the critical inclination. Thus the square root appearing in the above expression is real only if

$$(3.52) \qquad \nu - \lambda > 0, \qquad \sin^2\omega^* > \lambda$$

* Because $c = 4/7$ for the earth's potential. It is interesting to note that for Vinti's (1959) potential $c = 5/18$ (cf. Shi (1963)) which implies $\kappa_1 = 0$ for the motion at the critical inclination.

Now we have to distinguish the following three cases:

<u>Case 1</u>

(3.53) $\quad - \kappa_1 < \bar{\kappa}_o < \kappa_1 \qquad$ or $\qquad 0 < \lambda < 1$

In this case (3.50) becomes an elliptic integral of the first kind.

(3.54a) $\quad \bar{\phi} - \bar{\phi}_o = (2\kappa_1)^{-1/2} F(\chi_1, \kappa_1)$

where the amplitude χ_1 is

(3.54b) $\quad \chi_1 = \pm \tan^{-1} \left[\frac{\kappa_1 + \bar{\kappa}_o}{\kappa_1 - \bar{\kappa}_o} \tan^2 \omega^* - 1\right]^{1/2}$

and the modulus is

(3.54c) $k_1 = [(\bar{\kappa}_o + \kappa_1)/2\kappa_1]^{1/2}$

Using elliptic functions we may express ω^* explicitly as

(3.55) $\quad \omega^* = \pm \tan^{-1} \left[\frac{\kappa_1 - \bar{\kappa}_o}{\kappa_1 + \bar{\kappa}_o} \{1 + \mathrm{tn}^2[(2\kappa_1)^{1/2}(\bar{\phi} - \bar{\phi}_o)]\}\right]^{1/2}$

where the modulus of tn is k_1.

The interpretation of this result is that the perigee performs a pendulum motion around $\pi/2$ or $3/2\pi$ with a maximum amplitude $\omega^*_{max} = \pm \sin^{-1}\lambda$. λ depends on the initial conditions because after substituting the expression (A.27) for $\bar{\kappa}_o$ we obtain

(3.56) $\quad \lambda = \sin^2 \omega^* + (\bar{S}_o + S_1 J_{1/2}^*)^2/2\kappa_1$

Case 2

$$(3.57) \qquad \bar{\kappa}_o = \kappa_1 \quad \text{or} \quad \lambda = 0$$

In this case we have

$$(3.58) \qquad \bar{\phi} - \bar{\phi}_o = \pm \int_o^{\omega^*} \frac{d\xi}{(2\kappa_1)^{1/2}\cos\xi} = \frac{\pm 1}{(8\kappa_1)^{1/2}} \log \frac{1 + \sin\omega^*}{1 - \sin\omega^*}$$

or after some manipulations

$$(3.59) \qquad \sin\omega^* = -\frac{1 - e^{\pm(8\kappa_1)^{1/2}(\bar{\phi} - \bar{\phi}_o)}}{1 + e^{\pm(8\kappa_1)^{1/2}(\bar{\phi} - \bar{\phi}_o)}}$$

This means that ω^* approaches 0 or π asymptotically as ϕ goes to infinity. This case represents the boundary between oscillatory and secular motion of the perigee. The boundary depends on the initial conditions. We have

$$(3.60) \qquad \lambda = \sin^2 w^* + (\bar{S}_o + S_1 j_{1/2}{}^*)/2\kappa_1 = 0$$

which is possible only when the initial values

$$(3.61) \qquad w^* = 0 \quad \text{or} \quad \pi$$

and

$$(3.62) \qquad \bar{S}_o + S_1 j_{1/2}{}^* = 0$$

are assumed. This means that initially the apse has to coincide with the line of the nodes, and the inclination is exactly critical at least to the order kept in our calculations, because (3.62) is evidently the expansion of the initial value of the small divisor.

Case 3

$$(3.63) \qquad \bar{\kappa}_o > \kappa_1 \qquad \text{or } \lambda < 0$$

In this case we obtain

$$(3.64a) \qquad \bar{\phi} - \bar{\phi}_o = (\bar{\kappa}_o + \kappa_1)^{-1/2} F(\chi_2, k_2)$$

where the modulus is

$$(3.64b) \qquad k_2 = [\kappa_1/\bar{\kappa}_o + \kappa_1]^{1/2}$$

and the amplitude is

$$(3.64c) \qquad \chi_2 = \tan^{-1} \{[(\bar{\kappa}_o + \kappa_1)/(\bar{\kappa}_o - \kappa_1)]^{1/2} \tan\omega^*\}$$

The use of elliptic functions gives

$$(3.65) \qquad \omega^* = \tan^{-1} \{[(\bar{\kappa}_o - \kappa_1)/(\bar{\kappa}_o + \kappa_1)]^{1/2} \mathrm{tn}[(\bar{\kappa}_o + \kappa_1)^{1/2}(\bar{\phi} - \bar{\phi}_o)]\}$$

where the modulus of tn is k_2.

The apse angle may assume any value in this case and the motion of the perigee is secular. For large $\bar{\kappa}_o$, k_2^2 becomes small and we may expand $F(\chi_2, k_2)$. This gives

$$(3.66) \qquad (\bar{\kappa}_o + \kappa_1)^{1/2}(\bar{\phi} - \bar{\phi}_o) = \left(1 + \frac{1^2}{2^2} k_2^2 + \frac{1^2\, 3^2}{2^2\, 4^2} k_2^4 + \ldots\right) \chi_2 - \frac{1}{8} k_2^2 \sin 2\chi_2 - O(k_2^4)$$

Since $\chi_2 \to \omega^*$ for large $\bar{\kappa}_o$ (cf. (3.64c)) this shows that the motion of the perigee is secular with small additional oscillations. In the previous discussion of the behavior of the apsidal motion we have considered only the solution of ω_o*. A solution using all available information to $O(\varepsilon^2)$ must make use of the composite expansion as obtained in (3.47).

By substitution of $i_{o1/2}$* and i_{o1}* in (3.47), we obtain the following result which is uniformly valid to $O(\varepsilon^{1/2})$:

$$(3.67) \qquad \frac{d\omega_c}{d\bar{\phi}} = (1 + \varepsilon^{1/2} g_1)(\bar{\kappa}_o - \kappa_1 \cos 2\omega_c)^{1/2} + \varepsilon^{1/2}[g_o + g_2 \cos 2\omega_c]$$

After integration we have

$$(3.68) \qquad \bar{\phi} - \bar{\phi}_o = \int_o^{\omega_c} \frac{d\xi}{(1 + \varepsilon^{1/2} g_1)(\bar{\kappa}_o - \kappa_1 \cos 2\xi)^{1/2} + \varepsilon^{1/2}[g_o + g_2 \cos 2\xi]}$$

The evaluation of this integral leads to elliptic functions and a highly transcendental relation between ω_c and $\bar{\phi}$.

3.7 Motion of the node

Equation (3.6b) for the node can be brought to the following form:

$$(3.69) \qquad \frac{d\Omega}{d\phi} = - \quad [\frac{2}{p^2} u \cos^3 i \sin^2\phi] - \varepsilon^2[\frac{4}{p^2} u^2(3cu - 7cu \sin^2 i \sin^2\phi$$

$$- \frac{1}{p^2} \cos^4 i \sin^2\phi)] \cos^3 i \sin^2\phi + O(\varepsilon^3)$$

Applying the composite expansions for u and i and substituting the known results we obtain

* Note that $u_{1/2}$ and $i_{1/2}$ are zero.

$$(3.70)\qquad \frac{d\Omega}{d\phi} = -\varepsilon\left[\frac{2}{p^2}u_o\cos^3 i_{oc}\sin^2\phi\right] - \varepsilon^2\Big\{\frac{2}{p^2}u_1\cos^3 i_{oc}\sin^2\phi - \frac{6}{p^2}u_o i_1\cos^2 i_{oc}\sin i_{oc}\sin^2\phi + \frac{4}{p^2}u_o^2\Big[3cu_o - 7cu_o\sin^2 i_{oc}\sin^2\phi - \frac{1}{p^2}\cos^4 i_{oc}\sin^2\phi\Big]\cos^3 i_{oc}\sin^2\phi\Big\} + O(\varepsilon^{5/2})$$

Since all quantities on the right-hand side of (3.70) are already known as functions of ϕ, the node could be found by straightforward integration. However, for the sake of simplicity and a more systematic approach that avoids the shifting of orders of magnitude due to integration of long-period terms, we will also solve (3.70) by the two-variable expansion procedure.

We use the slow variable $\bar{\phi} = \varepsilon^{3/2}\phi$ and assume the following expansion for the node:

$$(3.71)\qquad \Omega = \frac{1}{\varepsilon^{1/2}}\sum_{n=0}\Omega_{n/2}(\phi,\bar{\phi};\varepsilon)\varepsilon^{n/2}$$

The factor $\varepsilon^{-1/2}$ in front of the summation in (3.71) is suggested because the leading term of the nodal velocity is of order ε at all inclinations, which forces us to make the leading term of the node itself of order $\varepsilon^{-1/2}$ to insure that the derivative with respect to $\bar{\phi}$ be of order unity.

Using the same procedure as for the other variables we obtain the following equations:

$$(3.72a)\qquad \frac{\partial\Omega_o}{\partial\phi} = 0$$

which implies that

$$(3.72b) \qquad \Omega_o = \Omega_o(\bar{\phi};\varepsilon).$$

Again, we expand Ω_o in the form:

$$(3.73) \qquad \Omega_o(\bar{\phi};\varepsilon) = \sum_{n=0} \Omega_{on/2}(\bar{\phi})\varepsilon^{n/2}$$

Since the right-hand side of equation (3.70) is $O(\varepsilon)$, we obtain

$$(3.74) \qquad \frac{\partial\Omega_{1/2}}{\partial\phi} = 0 \qquad \text{and (3.75)} \qquad \frac{\partial\Omega_1}{\partial\phi} = 0$$

implying that

$$(3.76) \qquad \Omega_{1/2} = \Omega_1 = 0$$

because the integration constants are already included in the expansion (3.73).

Collecting the terms of order ε we obtain

$$(3.77) \qquad \frac{\partial\Omega_{3/2}}{\partial\phi} = -\frac{\partial\Omega_o}{\partial\bar{\phi}} - \frac{2}{p^2}u_o\cos^3 i_{oo}\sin^2\phi = -\frac{\partial\Omega_o}{\partial\bar{\phi}} - \frac{\cos^5 i_{oo}}{p^4}[1 + e_o\cos(\phi - \omega_c)$$

$$- \cos 2\phi - \frac{e_o}{2}\cos(\phi - \omega_c) - \frac{e_o}{2}\cos(3\phi - \omega_c)]$$

The terms depending on $\bar{\phi}$ in (3.77) are $\partial\Omega_o/\partial\bar{\phi} + \cos^5 i_{oo}/p^4$. After substituting the expansion for Ω_o (the expansion for i_{oc} has already been substituted in (3.77)), we require for boundedness

$$(3.78) \qquad \frac{\partial\Omega_{oo}}{\partial\bar{\phi}} = -\frac{\cos^5 i_{oo}}{p^4}$$

The higher order terms in the expansion of Ω_o and i_{oc} are hence shifted to the next order. Integration of (3.78) gives

$$\Omega_{oo} = -\frac{\cos^5 i_{oo}}{p^4}\bar{\phi} + L_o \tag{3.79}$$

where L_o is a constant depending on the initial conditions. The solution for $\Omega_{3/2}$ is

$$\Omega_{3/2} = -\frac{1}{p^4}\cos^5 i_{oo}\Big[-\frac{1}{2}\sin 2\phi + e_o\sin(\phi - \omega_c) - \frac{e_o}{2}\sin(\phi + \omega_c) - \frac{e_o}{6}\sin(3\phi - \omega_c)\Big] \tag{3.80}$$

In order to make Ω_2 bounded, we must set

$$\frac{\partial\Omega_{o1/2}}{\partial\bar{\phi}} = 5\,\frac{i^*_{o1/2}}{p^4}\cos^4 i_{oo}\sin i_{oo} = -\frac{5}{p^4 S_1}\cos^4 i_{oo}\sin i_{oo}\Big[\bar{S}_o - (\bar{\kappa}_o - \kappa_1\cos 2\omega_c)^{1/2}\Big] \tag{3.81}$$

or

$$\Omega_{o1/2} = -\frac{5}{p^4 S_1}\cos^4 i_{oo}\sin i_{oo}[\bar{\phi} - \omega^*] + L_{1/2} \tag{3.82}$$

where $L_{1/2}$ is an integration constant and ω^* is given by (3.55), (3.59) or (3.65) depending on the values of $\bar{\kappa}_o$ and κ_1.

The terms of $O(\epsilon^2)$ depending on $\bar{\phi}$ only are: $\partial\Omega_{o1}/\partial\bar{\phi} + d_o + d_2\cos 2\omega_c + (5/2p^4)\cos^3 i_{oo}(4 - 5\cos^2 i_{oo})(i_{o1/2}^*)^2 - (5/p^4)i_{o1}^*\cos^4 i_{oo}\sin i_{oo}$.

The boundedness requirement on $\Omega_{5/2}$ implies, after substituting for $i_{o1/2}^*$ and i_{o1}^*, that

$$\frac{\partial\Omega_{o1}}{\partial\bar{\phi}} = D_o + D_1(\bar{\kappa}_o - \kappa_1\cos 2\omega_c)^{1/2} + D_2\cos 2\omega_c \tag{3.83}$$

The integration of (3.83) yields

$$\Omega_{o1} = D_1\bar{\phi} + \int_o^{\omega_c} \frac{D_o + D_2\cos 2\xi}{(\bar{\kappa}_o - \kappa_1\cos 2\xi)^{1/2}}\, d\xi + L_1 \tag{3.84}$$

where L_1 is an integration constant and the integral depends on the values of $\bar{\kappa}_o$ and κ_1. The evaluation of this integral leads to elliptic functions of the first and second kind and will not be exhibited here.

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Appendix

(Definition of Constants)

$$\text{(A.1)} \qquad A_o = + \frac{\cos^8 i_{oo}}{2p^8} \Big[-\frac{17}{24} + \frac{9}{2} c - \frac{25}{48} e_o^{\,2} + \frac{27}{8} e_o^{\,2} c$$

$$+ \left(\frac{17}{4} - 54c + \frac{21}{8} e_o^{\,2} - \frac{189}{4} e_o^{\,2} c\right) \cos^2 i_{oo}$$

$$+ \left(-\frac{85}{24} + \frac{147}{2} c - \frac{15}{16} e_o^{\,2} + \frac{567}{8} e_o^{\,2} c\right) \cos^4 i_{oo} \Big]$$

$$\text{(A.2)} \qquad A_2 = + \frac{\cos^8 i_{oo}}{2p^8} \Big[-\frac{1}{6} - \frac{3}{2} c + \frac{5}{24} e_o^{\,2} - \frac{15}{4} e_o^{\,2} c$$

$$+ \left(-\frac{1}{3} + 12c - \frac{14}{3} e_o^{\,2} + 42\, e_o^{\,2} c\right) \cos^2 i_{oo}$$

$$+ \left(\frac{5}{2} - \frac{21}{2} c + \frac{45}{8} e_o^{\,2} - \frac{189}{4} e_o^{\,2} c\right) \cos^4 i_{oo} \Big]$$

$$\text{(A.3)} \qquad B_2 = + \frac{\cos^8 i_{oo}}{2p^8} e_o \Big[-\frac{1}{2} - \frac{3}{2} c - \frac{1}{12} e_o^{\,2} + \frac{3}{2} e_o^{\,2}$$

$$+ \left(-\frac{1}{3} + 12\, c + \frac{4}{3} e_o^{\,2} - 12\, e_o^{\,2} c\right) \cos^2 i_{oo}$$

$$+ \left(\frac{5}{2} - \frac{21}{2} c - \frac{5}{4} e_o^{\,2} + \frac{21}{2} e_o^{\,2}\right) \cos^4 i_{oo} \Big]$$

$$\text{(A.4)} \qquad B_{21} = - \frac{e_o}{2p^8} \cos^7 i_{oo} \sin i_{oo} \Big[-\frac{4}{3} - 12c - \frac{2}{3} e_o^{\,2} + 12\, e_o^{\,2} c$$

$$+ \left(-\frac{10}{3} + 120\, c + \frac{40}{3} e_o^{\,2} - 120\, e_o^{\,2} c\right) \cos^2 i_{oo}$$

$$+ \left(30 - 126\, c - 15\, e_o^{\,2} + 126\, e_o^{\,2} c\right) \cos^4 i_{oo} \Big]$$

(A.5) $$B_{22} = \frac{\cos^8 i_{oo}}{2p^8} \left[-\frac{1}{6} - \frac{3}{2}c - \frac{1}{4}e_o^2 + \frac{9}{2}e_o^2 c \right.$$
$$+ \left(-\frac{1}{3} + 12c + 4e_o^2 - 36e_o^2 c\right)\cos^2 i_{oo}$$
$$\left. + \left(\frac{5}{2} - \frac{21}{2}c - \frac{15}{4}e_o^2 + \frac{63}{2}ce_o^2\right)\cos^4 i_{oo} \right]$$

(A.6) $$C_2 = \frac{1}{4}\frac{e_o^2}{p^8}\cos^9 i_{oo}\sin i_{oo}\left[-\frac{1}{6} + 3c + \left(\frac{5}{2} - 21c\right)\cos^2 i_{oo}\right]$$

(A.7) $$C_{21} = \frac{1}{4}\frac{e_o^2}{p^8}\cos^8 i_{oo}\left[\frac{3}{2} - 27c - \left(\frac{175}{6} - 261c\right)\cos^2 i_{oo}\right.$$
$$\left. + (30 - 252c)\cos^4 i_{oo}\right]$$

(A.8) $$C_{22} = \frac{1}{2}\frac{e_o}{p^8}\cos^9 i_{oo}\sin i_{oo}\left[-\frac{1}{6} + 3c + \left(\frac{5}{2} - 21c\right)\cos^2 i_{oo}\right]$$

(A.9) $$A_o^* = +\frac{\cos^8 i_{oo}^*}{2p^8}\left[-\frac{23}{24} + \frac{9}{2}c - \frac{25}{48}e_o^{*2} + \frac{27}{8}e_o^{*2}c\right.$$
$$+ \left(\frac{27}{4} - 54c + \frac{21}{8}e_o^{*2} - \frac{189}{4}e_o^{*2}c\right)\cos^2 i_{oo}^*$$
$$\left. + \left(-\frac{235}{24} + \frac{147}{2}c - \frac{15}{16}e_o^{*2} + \frac{567}{8}e_o^{*2}c\right)\cos^4 i_{oo}^*\right]$$

(A.10) $$A_2^* = +\frac{\cos^8 i_{oo}^*}{2p^8}\left[+\frac{1}{12} - \frac{3}{2}c + \frac{5}{24}e_o^{*2} - \frac{15}{4}e_o^{*2}c\right.$$
$$+ \left(-\frac{7}{3} + 12c - \frac{14}{3}e_o^{*2} + 42e_o^{*2}c\right)\cos^2 i_{oo}^*$$
$$\left. + \left(\frac{25}{4} - \frac{21}{2}c + \frac{45}{8}e_o^{*2} - \frac{189}{4}e_o^{*2}c\right)\cos^4 i_{oo}^*\right]$$

$$(A.11)\qquad B_2^* = + \frac{\cos^6 i_{oo}^*}{2p^8} e_o^* \Big[+ \frac{1}{12} - \frac{3}{2} c - \frac{1}{12} e_o^{*2} + \frac{3}{2} e_o^{*2} c$$

$$+ \left(- \frac{7}{3} + 12c + \frac{4}{3} e_o^{*2} - 12 e_o^{*2} c\right) \cos^2 i_{oo}^*$$

$$+ \left(\frac{25}{4} - \frac{21}{2} c - \frac{5}{4} e_o^{*2} + \frac{21}{2} e_o^{*2} c\right) \cos^4 i_{oo}^* \Big]$$

$$(A.12)\qquad B_{21}^* = - \frac{e_o^*}{2p^8} \cos^7 i_{oo}^* \sin i_{oo}^* \Big[+ \frac{2}{3} - 12c - \frac{2}{3} e_o^{*2} + 12 e_o^{*2} c$$

$$+ \left(- \frac{70}{3} + 120c + \frac{40}{3} e_o^{*2} - 120 e_o^{*2} c\right) \cos^2 i_{oo}^*$$

$$+ \left(75 - 126c - 15 e_o^{*2} + 126 e_o^{*2} c\right) \cos^4 i_{oo}^* \Big]$$

$$(A.13)\qquad B_{22}^* = + \frac{\cos^8 i_{oo}^*}{2p^8} \Big[\frac{1}{12} - \frac{3}{2} c - \frac{1}{4} e_o^{*2} + \frac{9}{2} e_o^{*2} c$$

$$+ \left(- \frac{7}{3} + 12 c + 4 e_o^{*2} - 36 e_o^{*2} c\right) \cos^2 i_{oo}^*$$

$$+ \left(\frac{25}{4} - \frac{21}{2} c - \frac{15}{4} e_o^{*2} + \frac{63}{2} e_o^{*2} c\right) \cos^4 i_{oo}^* \Big]$$

$$(A.14)\qquad C_2^* = + \frac{1}{4} \frac{e_o^{*2}}{p^8} \cos^9 i_{oo}^* \sin i_{oo}^* \left[- \frac{1}{6} + 3c + \left(\frac{5}{2} - 21c\right) \cos^2 i_{oo}^*\right]$$

$$(A.15)\qquad C_{21}^* = + \frac{1}{4} \frac{e_o^{*2}}{p^8} \cos^8 i_{oo}^* \Big[\frac{3}{2} - 27c - \left(\frac{175}{6} - 261c\right) \cos^2 i_{oo}^*$$

$$+ (30 - 252c) \cos^4 i_{oo}^* \Big]$$

$$(A.16)\qquad C_{22}^* = + \frac{1}{2} \frac{e_o^*}{p^8} \cos^9 i_{oo}^* \sin i_{oo}^* \left[- \frac{1}{6} + 3c + \left(\frac{5}{2} - 21c\right) \cos^2 i_{oo}^*\right]$$

$$(A.17)\qquad S_o = - \frac{\cos^4 i_{oo}}{2p^4} \left(1 - 5 \cos^2 i_{oo}\right)$$

$$(A.18)\qquad S_1 = + \frac{\cos^3 i_{oo}}{2p^4} \sin i_{oo} \, (2 - 15 \cos^2 i_{oo})$$

$$(A.19)\qquad S_2 = - \frac{\cos^2 i_{oo}}{p^4} (6 - 83 \cos^2 i_{oo} + 90 \cos^4 i_{oo})$$

$$(A.20)\qquad \bar{S}_o = \frac{1}{\varepsilon^{1/2}} \left(- \frac{\cos^4 i_{oo}^*}{2p^4} (1 - 5 \cos^2 i_{oo}^*) \right)$$

$$(A.21)\qquad \bar{\kappa}_o = (\bar{S}_o + S_1 j_{1/2}^*)^2 + \kappa_1 \cos 2w^*$$

$$(A.22)\qquad \kappa_1 = S_1 C_2^* = + \frac{1}{8} \frac{e_o^{*2}}{p^{12}} \cos^{12} i_{oo}^* \sin^2 i_{oo}^* (2 - 15 \cos^2 i_{oo}^*)$$

$$\times \left[- \frac{1}{6} + 3c + \left(\frac{5}{2} - 21c\right) \cos^2 i_{oo}^* \right]$$

$$(A.23)\qquad E_{1/2} = \eta_{1/2}^* - \frac{B_2^*}{\kappa_1} (\bar{S}_o + S_1 j_{1/2}^*)$$

$$(A.24)\qquad E_1 = \eta_1^* - \frac{\bar{S}_o + j_{1/2}^* S_1}{\kappa_1} \left[- \frac{\bar{S}_o}{S_1} B_{21}^* + E_{1/2} B_{22}^* \right.$$

$$\left. - \frac{e_o^*}{4} \frac{\bar{S}_o}{p^4} (1 - 3 \cos^2 i_{oo}^*) \cos^4 i_{oo}^* \right]$$

$$+ \frac{1}{2} \left[\frac{B_{21}^*}{S_1} + \frac{B_{22}^*}{\kappa_1} B_2^* \right] \cos 2w^*$$

$$(A.25)\qquad I_1 = j_1^* - \frac{1}{\kappa_1} \left[- C_{21}^* \frac{\bar{S}_o}{S_1} + C_{22}^* E_{1/2} \right] (\bar{S}_o + j_{1/2}^* S_1)$$

$$+ \frac{1}{2S_1} \left[C_{21}^* + \frac{C_{22}^*}{C_2^*} B_2^* \right] \cos 2w^*$$

$$(A.26)\qquad \lambda = \sin^2 w^* + \frac{(\bar{S}_o + S_1 j_{1/2}^*)^2}{2\kappa_1}$$

$$(\text{A.27}) \qquad g_o = \frac{S_2}{2S_1^{\,2}}\left(\bar{\kappa}_o + \bar{S}_o^{\,2}\right) + S_1\bar{I}_1 + A_o$$

$$(\text{A.28}) \qquad g_1 = -\frac{S_2}{S_1^{\,2}}\bar{S}_o + \frac{1}{I_1}\left[-C_{21}\bar{S}_o + C_{22}S_1E_{1/2}\right]$$

$$(\text{A.29}) \qquad g_2 = A_2 - \frac{1}{2}\left[\frac{S_2}{S_1^{\,2}}\kappa_1 + C_{21} + \frac{C_{22}}{C_2}B_2^{\,*}\right]$$

$$(\text{A.30}) \qquad d_o = \frac{\cos^9 i_{oo}}{p^6}\left[\frac{1}{3} - \frac{9}{2}c + \frac{3}{8}e_o^{\,2} - \frac{27}{4}e_o^{\,2}c\right.$$

$$\left. + \left(-\frac{5}{6} + \frac{21}{2}c - \frac{5}{24}e_o^{\,2} + \frac{63}{4}e_o^{\,2}c\right)\cos^2 i_{oo}\right]$$

$$(\text{A.31}) \qquad d_2 = \frac{\cos^9 i_{oo}}{p^6}e_o^{\,2}\left[-\frac{2}{3} + 6c + \left(\frac{5}{4} - \frac{21}{2}c\right)\cos^2 i_{oo}\right]$$

$$(\text{A.32}) \qquad D_o = -d_o - \frac{5}{2}\frac{\cos^3 i_{oo}}{S_1^{\,2}p^4}\left(4 - 5\cos^2 i_{oo}\right)\left(\bar{\kappa}_o + \bar{S}_o^{\,2}\right)$$

$$+ 5\frac{I_1}{p^4}\cos^4 i_{oo}\sin i_{oo}$$

$$(\text{A.33}) \qquad D_1 = +5\frac{\cos^3 i_{oo}}{p^4S_1}\left[\left(4 - 5\cos^2 i_{oo}\right)\frac{\bar{S}_o}{S_1} + \frac{\cos i_{oo}\sin i_{oo}}{C_2}\left(-C_{21}\frac{\bar{S}_o}{S_1}\right.\right.$$

$$\left.\left. + C_{22}E_{1/2}\right)\right]$$

$$(\text{A.34}) \qquad D_2 = -d_2 - \frac{5}{2}\frac{\cos^3 i_{oo}}{S_1p^4}\left[\cos i_{oo}\sin i_{oo}\left(C_{21} + \frac{C_{22}}{C_2}B_2^{\,*}\right)\right.$$

$$\left. - \left(4 - 5\cos^2 i_{oo}\right)C_2\right]$$

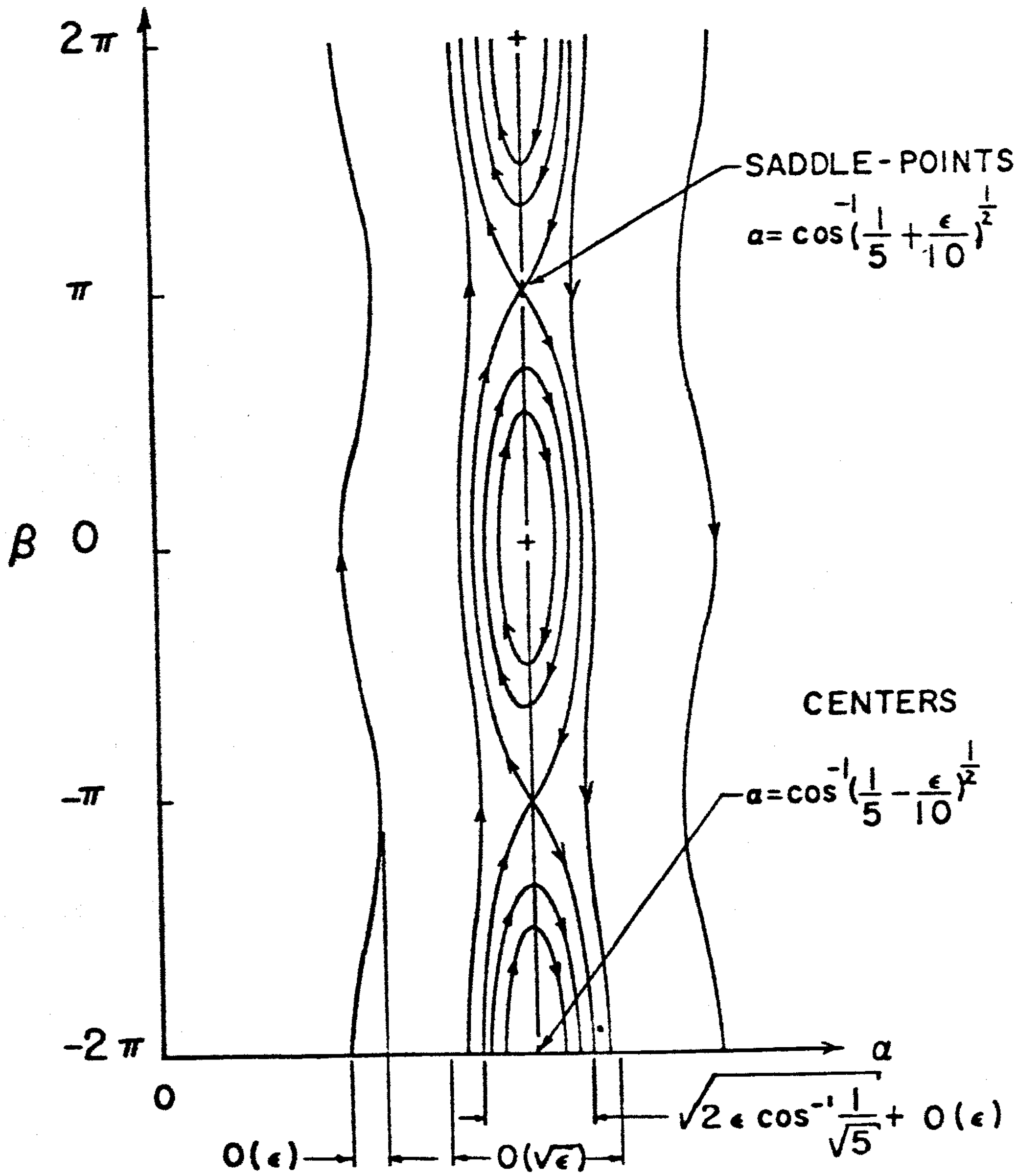
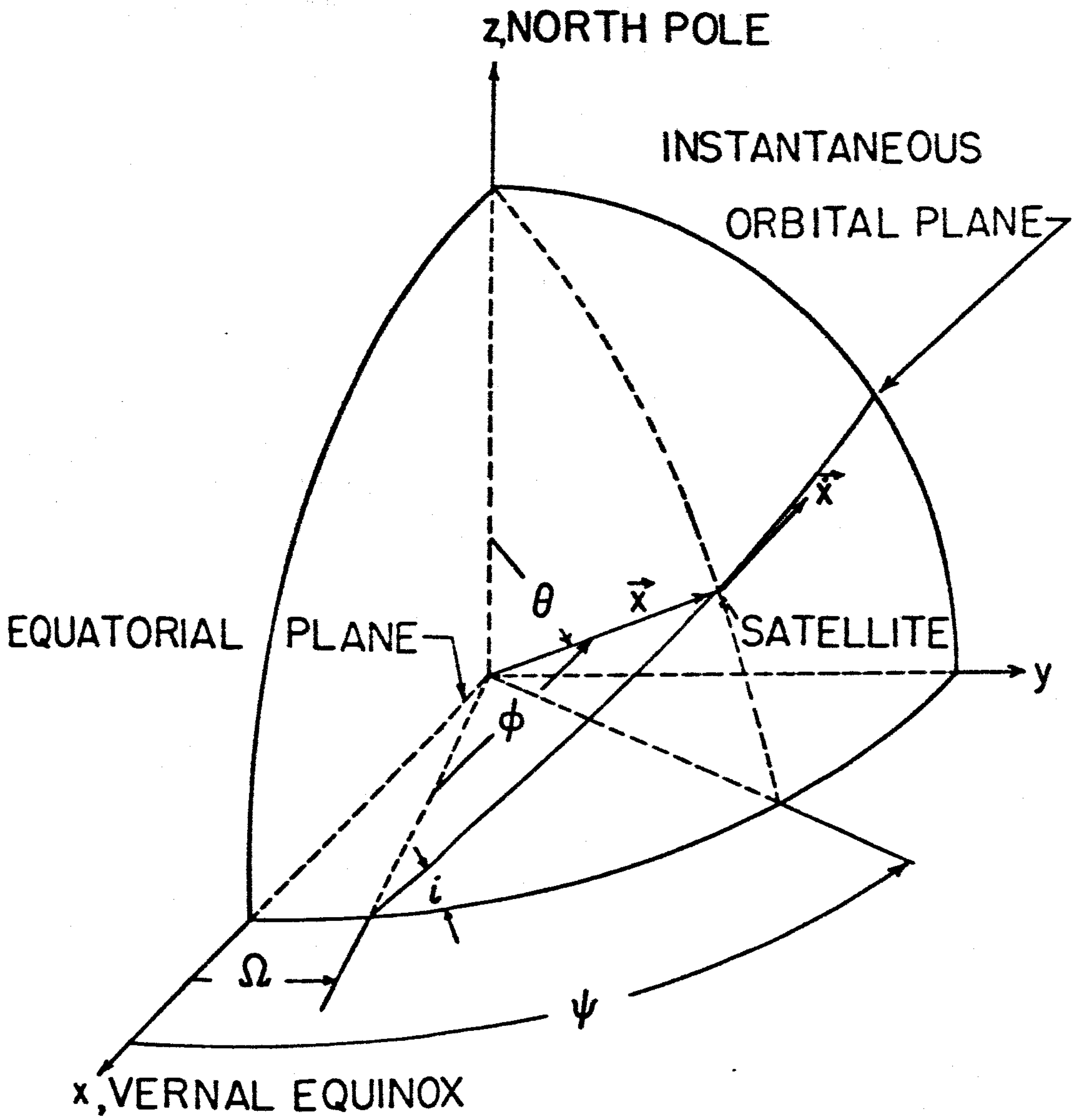

2π
π
β 0
−π
−2π
0
α
SADDLE-POINTS
$\alpha = \cos^{-1}\left(\frac{1}{5}+\frac{\epsilon}{10}\right)^{\frac{1}{2}}$
CENTERS
$\alpha = \cos^{-1}\left(\frac{1}{5}-\frac{\epsilon}{10}\right)^{\frac{1}{2}}$
$\sqrt{2\epsilon\cos^{-1}\frac{1}{\sqrt{5}}} + O(\epsilon)$
$O(\epsilon)$
$O(\sqrt{\epsilon})$

FIGURE 1



z,NORTH POLE
INSTANTANEOUS
ORBITAL PLANE
$\vec{\dot{x}}$
θ
$\vec{x}$
EQUATORIAL PLANE
SATELLITE
y
ϕ
i
Ω
ψ
x, VERNAL EQUINOX

FIGURE 2