NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report No. 32-786

A Photometric Method for Deriving Lunar Topographic Information

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September 15, 1965

Prepared Under Contract No. NAS 7-100
National Aeronautics & Space Administration

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A general and rigorous treatment is given of the photometric method for deriving lunar surface elevation information from pictures of the surface. In the course of the derivation certain shortcomings inherent in the method are pointed out. The resulting equations are then applied to the *Ranger* pictures as part of a digital processing procedure and examples are given.

author

I. INTRODUCTION

Pictures of the lunar surface taken from spacecraft such as *Ranger* and *Orbiter* have two major purposes. The first is simply the presentation of a visual record of the surface for qualitative interpretation by an observer.

The second purpose, which is of greater interest here, is the extraction of quantitative surface data. In particular, elevation information is sought.

Quantitative elevation data can be found both by stereoscopic and by photometric methods. The photometric method, first suggested by van Diggelen (Ref. 1), uses the surface reflectance properties to interpret the surface shape in terms of the picture data. A rigorous derivation of the necessary equations and a discussion of the shortcomings of the photometric method are presented. The equations are then used to analyze the *Ranger* television pictures, and typical results are shown.

II. CALCULATION OF ELEVATIONS IN TERMS OF A LENS-CENTERED COORDINATE SYSTEM

The problem to be considered is the derivation of geometric information about a lunar scene from a picture using photometric considerations. Utilizing the imaging geometry, the facsimile system transfer characteristics, and the scene photometric properties, it is desired to reconstruct quantitatively the shape of the surface being viewed. Here it will be assumed that the surface possesses homogeneous photometric properties, at least over the local area being viewed.

A summary of the observed lunar photometry for mare regions is given in Ref. 2 using the following notation, which will be adopted here (see Fig. 1). The incidence angle i (angle between the direction of incident light and

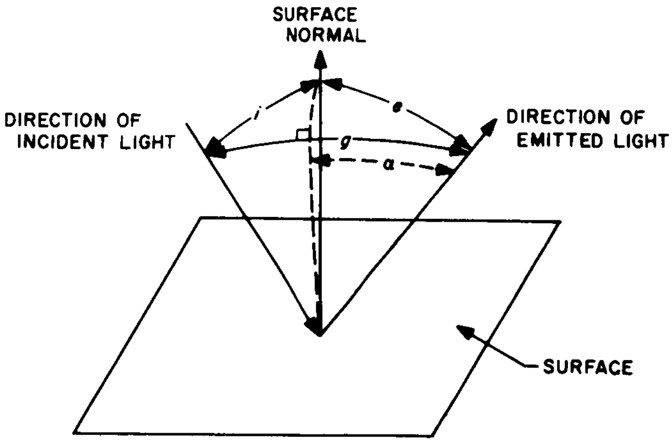

Fig. 1. Surface photometric geometry

the surface normal), the emission angle e (angle between the direction of emitted light and the surface normal), and the phase angle g (angle between the directions of incident and emitted light) completely define the photometric geometry, so that the scene luminance b can be written

$$b\,(i, e, g) = \rho_0\, E_0\, \phi\,(i, e, g)$$

where ρ_0 is the normal albedo of the surface, E_0 is the solar constant at the surface, and $\phi\,(i, e, g)$ is the surface photometric function. The normalization of ϕ is such that

$$\phi\,(0, 0, 0) = 1$$

It is observed that a degeneracy exists in the lunar photometric function, so that it is completely defined by two angles only. These are taken to be g and a new angle α, which is the projection of the emission angle e in the plane containing g (see Fig. 1). Thus, in what follows, it will be assumed that

$$b\,(g, \alpha) = \rho_0\, E_0\, \phi\,(g, \alpha) \tag{1}$$

Consider now an enumeration of the information known about the output scene reproduction and the geometry of its formation. The output reproduction may be a photographic print or transparency, or simply a magnetic tape recording of video information from which, of course, it is assumed that the proper two-dimensional geometric relationship of the signals can be derived. It is assumed that the static or uniform field transfer function of the facsimile system is known, so that an output signal, be it the print reflectance, the transparency transmittance, or the tape voltage output, is directly relatable to the object scene luminance. This relationship is not quite so simple since the reproduction process is not perfect, i.e., the input scene image is degraded by the facsimile system, so that an output signal is not truly related to the input luminance through the static transfer function. This is particularly true for steep luminance gradients, e.g., in small surface features. For the present, however, it is assumed that these degradations have been corrected by digital processing methods or by spatial filtering, so that the reproduction system may be considered perfect and the static transfer characteristic indeed applies. In this case, the output can just as well be considered as a two-dimensional array of luminances corresponding to the input scene.

To be more specific, let a right-handed Cartesian coordinate system be located with its origin at the center of the image-forming lens and with the lens in the xy-plane (see Fig. 2). With the convention that primed coordinates refer to points in image space and unprimed coordinates refer to points in object space, the output facsimile can be considered as an array of luminances $b(x', y')$. From the image-forming process and the known coordinates of points in the image plane, one knows the direction $\hat{\mathbf{r}}'$ from every object scene point to its image.* This follows since for a simple lens, the position vector $\mathbf{r}'$ of the image of an object point with position vector $\mathbf{r}$ is given by (assuming distant objects)

$$\mathbf{r}' = -\frac{F}{\mathbf{r}\cdot\hat{\mathbf{z}}}\,\mathbf{r} \tag{2}$$

Here F is the focal length of the lens, $\hat{\mathbf{z}}$ is a unit vector along the optical axis normal to the image plane, and the distant object assumption is formally written

$$\mathbf{r}\cdot\hat{\mathbf{z}} >> F$$

One furthermore knows a unit vector $\hat{\mathbf{R}}_{MS}$ along the line from the center of the Moon to the center of the Sun, which is opposite to the direction of the incident illumination on the scene. Thus, the phase angle g is known for every point in the image since

$$g = \cos^{-1}(\hat{\mathbf{r}}' \cdot \hat{\mathbf{R}}_{MS}), \quad 0 \leqq g \leqq \pi \tag{3}$$

Then from Eq. (1), given the luminance b and the phase angle g for some image point, one can calculate the proper auxiliary angle α.

*A bold-face letter indicates a vector of, in general, nonunit length. A caret over a bold-face letter indicates a unit vector.

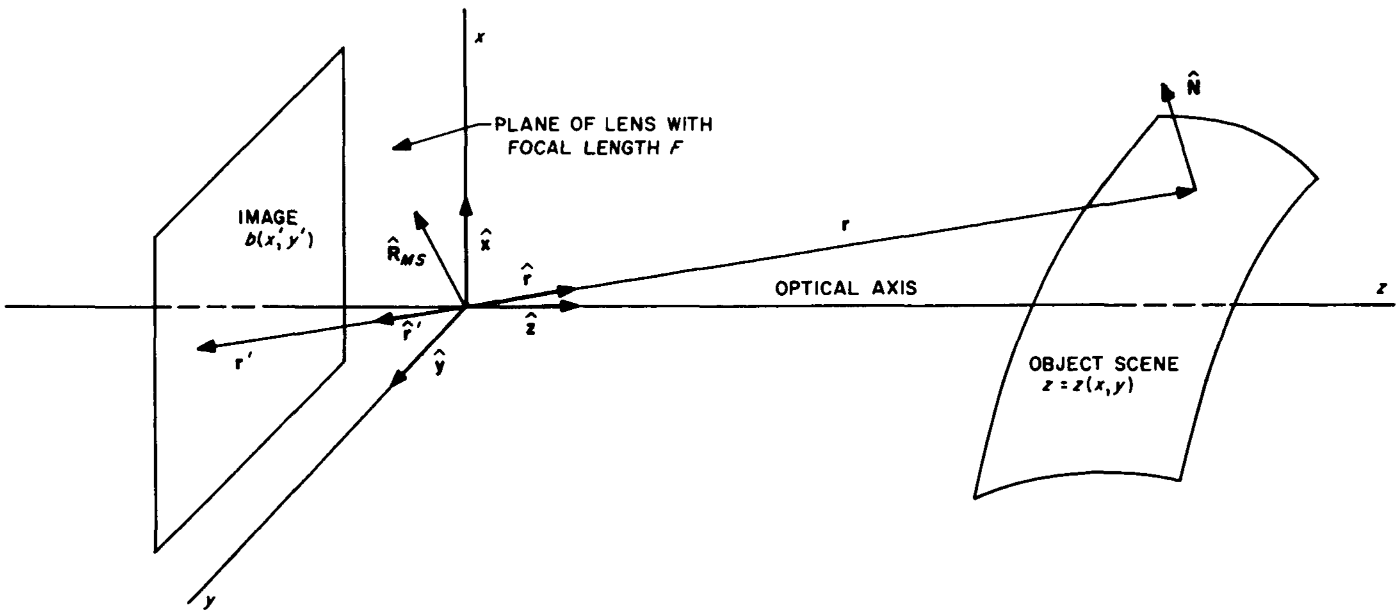

Fig. 2. Spacecraft viewing geometry in a lens-centered coordinate system

Therefore, the totality of geometric information available in the image may be summarized as follows:

F = the lens focal length

$\hat{\mathbf{z}}$ = the direction of the optical axis

$\hat{\mathbf{R}}_{MS}$ = the incident illumination direction

$\hat{\mathbf{r}}'(x', y')$ = the direction from an object point to its image

$g(x', y')$ = the phase angle as a function of position in the image plane

$\alpha(x', y')$ = the auxiliary photometric angle as a function of position in the image facsimile

Consider now a representation of the object scene from which one can write down the necessary analysis. Since the lens-centered coordinate system is already available and is of particular pertinence to this part of the problem, a representation in terms of it will be used. A Moon-centered coordinate system of perhaps more interest for surface interpretation will be introduced later. In the lens-centered coordinate system, an equation for the object scene can be written (see Fig. 2)

$$z = z(x, y) \tag{4}$$

where, of course, the functional form of the expression is to be determined. From the picture, lateral (x, y) geometric information can be derived about the scene. It remains to determine the distance z and ultimately the length r of the position vector to the object surface. Thus, the problem can be stated as trying to find some path S' with a path length variable s' along it in the output picture such that the derivative dz/ds' can be written as a function h only of the known picture information,

$$\frac{dz}{ds'} = h(F, \hat{\mathbf{z}}, \hat{\mathbf{R}}_{MS}, \hat{\mathbf{r}}', g, \alpha) \tag{5}$$

Then, since the position vector $\mathbf{r}$ to an object point can be written in terms of z and $\mathbf{r}'$, an expression for dr/ds' can be determined, where r is the length of $\mathbf{r}$. So if one knows the length r_0 of an object position vector for some point on the path, this differential equation can be integrated to find r, and finally $\mathbf{r}$, everywhere along the path thereby reconstructing a portion of the object surface. What follows is aimed at this goal.

Let the path S' in the image have a parametric representation

$$\begin{aligned} x' &= x'(s') \\ y' &= y'(s') \end{aligned} \tag{6}$$

From Eq. (2), one can write

$$\begin{aligned} x' &= -\frac{F}{z(x, y)}x \\ y' &= -\frac{F}{z(x, y)}y \end{aligned} \tag{7}$$

and for completeness

$$z' = -F$$

Thus

$$\frac{dz}{ds'} = \left(\frac{\partial z}{\partial x'}\right)_{y'} \frac{dx'}{ds'} + \left(\frac{\partial z}{\partial y'}\right)_{x'} \frac{dy'}{ds'} = \nabla' z(x, y) \cdot \hat{\mathbf{S}}'_T \tag{8}$$

where ∇' is the nabla or gradient operator with respect to the primed coordinates,

$$\nabla' = \left(\frac{\partial}{\partial x'}\right)_{y' z'} \hat{\mathbf{x}} + \left(\frac{\partial}{\partial y'}\right)_{x' z'} \hat{\mathbf{y}} + \left(\frac{\partial}{\partial z'}\right)_{x' y'} \hat{\mathbf{z}} \tag{9}$$

$\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit vectors along the principal Cartesian axes; and $\hat{\mathbf{S}}'_T$ is a unit vector tangent to the path S',

$$\hat{\mathbf{S}}'_T = \frac{dx'}{ds'} \hat{\mathbf{x}} + \frac{dy'}{ds'} \hat{\mathbf{y}}$$

Now the quantities $(\partial z/\partial x')_{y'}$ and $(\partial z/\partial y')_{x'}$ must be calculated. Since z is a function of x and y, using the chain rule, one has

$$\begin{aligned} \left(\frac{\partial z}{\partial x'}\right)_{y'} &= \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial x'}\right)_{y'} + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial x'}\right)_{y'} \\ \left(\frac{\partial z}{\partial y'}\right)_{x'} &= \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y'}\right)_{x'} + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial y'}\right)_{x'} \end{aligned} \tag{10}$$

The derivatives $(\partial x/\partial x')_{y'}$, $(\partial y/\partial x')_{y'}$, $(\partial x/\partial y')_{x'}$ and $(\partial y/\partial y')_{x'}$ are calculated using Eq. (7). Let

$$\begin{aligned} x' &= \xi(x, y) = -\frac{Fx}{z(x, y)} \\ y' &= \eta(x, y) = -\frac{Fy}{z(x, y)} \end{aligned} \tag{11}$$

Then differentiating these with respect to x', one has

$$1 = \left(\frac{\partial \xi}{\partial x}\right)_y \left(\frac{\partial x}{\partial x'}\right)_{y'} + \left(\frac{\partial \xi}{\partial y}\right)_x \left(\frac{\partial y}{\partial x'}\right)_{y'}$$

$$0 = \left(\frac{\partial \eta}{\partial x}\right)_y \left(\frac{\partial x}{\partial x'}\right)_{y'} + \left(\frac{\partial \eta}{\partial y}\right)_x \left(\frac{\partial y}{\partial x'}\right)_{y'}$$

since x' and y' are independent variables. These are then simultaneous equations for $(\partial x/\partial x')_{y'}$ and $(\partial y/\partial x')_{y'}$. They may be solved using Kramer's rule of determinants provided the Jacobian of the transformation is different from zero. Thus, one has

$$\left(\frac{\partial x}{\partial x'}\right)_{y'} = \frac{\left(\frac{\partial \eta}{\partial y}\right)_x}{J}; \qquad \left(\frac{\partial y}{\partial x'}\right)_{y'} = -\frac{\left(\frac{\partial \eta}{\partial x}\right)_y}{J}$$

where J is the Jacobian

$$J = \begin{vmatrix} \left(\frac{\partial \xi}{\partial x}\right)_y & \left(\frac{\partial \xi}{\partial y}\right)_x \\ \left(\frac{\partial \eta}{\partial x}\right)_y & \left(\frac{\partial \eta}{\partial y}\right)_x \end{vmatrix} = \left(\frac{\partial \xi}{\partial x}\right)_y \left(\frac{\partial \eta}{\partial y}\right)_x - \left(\frac{\partial \xi}{\partial y}\right)_x \left(\frac{\partial \eta}{\partial x}\right)_y$$

By a similar process, one can show

$$\left(\frac{\partial x}{\partial y'}\right)_{x'} = -\frac{\left(\frac{\partial \xi}{\partial y}\right)_x}{J}; \qquad \left(\frac{\partial y}{\partial y'}\right)_{x'} = \frac{\left(\frac{\partial \xi}{\partial x}\right)_y}{J}$$

Using the definitions of $\xi(x, y)$ and $\eta(x, y)$ in Eq. (11), one has

$$\begin{aligned} \left(\frac{\partial \xi}{\partial x}\right)_y &= -\frac{F}{z^2}\left[z - x\left(\frac{\partial z}{\partial x}\right)_y\right] \\ \left(\frac{\partial \xi}{\partial y}\right)_x &= \frac{F}{z^2} x \left(\frac{\partial z}{\partial y}\right)_x \\ \left(\frac{\partial \eta}{\partial x}\right)_y &= \frac{F}{z^2} y \left(\frac{\partial z}{\partial x}\right)_y \\ \left(\frac{\partial \eta}{\partial y}\right)_x &= -\frac{F}{z^2}\left[z - y\left(\frac{\partial z}{\partial y}\right)_x\right] \end{aligned}$$

and one can show that

$$J = \frac{F^2}{z^3}\left[z - x\left(\frac{\partial z}{\partial x}\right)_y - y\left(\frac{\partial z}{\partial y}\right)_x\right]$$

Now, from the calculus of three dimensions, if a surface has a representation

$$G(x, y, z) = 0$$

a unit normal $\hat{\mathbf{N}}$ to the surface at the point (x, y, z) is given by

$$\hat{\mathbf{N}} = \frac{\nabla G(x, y, z)}{|\nabla G(x, y, z)|}$$

In the present case, then, with the surface represented by

$$z - z(x, y) = 0$$

the unit normal becomes

$$\hat{\mathbf{N}} = \frac{\hat{\mathbf{z}} - \nabla z(x, y)}{\sqrt{1 + (\nabla z)^2}}$$

or, since

$$\hat{\mathbf{N}} \cdot \hat{\mathbf{z}} = \frac{1}{\sqrt{1 + (\nabla z)^2}}$$

one has

$$\hat{\mathbf{N}} = (\hat{\mathbf{N}} \cdot \hat{\mathbf{z}}) \, [\hat{\mathbf{z}} - \nabla z(x, y)] \tag{12}$$

And noting that, by definition,

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z(x, y)\,\hat{\mathbf{z}}$$

it follows from Eq. (12) that

$$\frac{\hat{\mathbf{N}} \cdot \mathbf{r}}{\hat{\mathbf{N}} \cdot \hat{\mathbf{z}}} = z - x\left(\frac{\partial z}{\partial x}\right)_y - y\left(\frac{\partial z}{\partial y}\right)_x$$

so that the expression for the Jacobian becomes

$$J = \frac{F^2}{z^3} \frac{\hat{\mathbf{N}} \cdot \mathbf{r}}{\hat{\mathbf{N}} \cdot \hat{\mathbf{z}}}$$

Thus, one has finally

$$\begin{aligned}
\left(\frac{\partial x}{\partial x'}\right)_{y'} &= -\frac{z(\hat{\mathbf{N}} \cdot \hat{\mathbf{z}})}{F(\hat{\mathbf{N}} \cdot \mathbf{r})}\left[z - y\left(\frac{\partial z}{\partial y}\right)_x\right] \\
\left(\frac{\partial y}{\partial x'}\right)_{y'} &= -\frac{z(\hat{\mathbf{N}} \cdot \hat{\mathbf{z}})}{F(\hat{\mathbf{N}} \cdot \mathbf{r})}\, y\left(\frac{\partial z}{\partial x}\right)_y \\
\left(\frac{\partial x}{\partial y'}\right)_{x'} &= -\frac{z(\hat{\mathbf{N}} \cdot \hat{\mathbf{z}})}{F(\hat{\mathbf{N}} \cdot \mathbf{r})}\, x\left(\frac{\partial z}{\partial y}\right)_x \\
\left(\frac{\partial y}{\partial y'}\right)_{x'} &= -\frac{z(\hat{\mathbf{N}} \cdot \hat{\mathbf{z}})}{F(\hat{\mathbf{N}} \cdot \mathbf{r})}\left[z - x\left(\frac{\partial z}{\partial x}\right)_y\right]
\end{aligned} \tag{13}$$

Then, from Eqs. (9) and (10),

$$\begin{aligned}
\nabla' z(x, y) &= \left(\frac{\partial z}{\partial x'}\right)_{y'} \mathbf{x} + \left(\frac{\partial z}{\partial y'}\right)_{x'} \hat{\mathbf{y}} \\
&= \left[\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial x'}\right)_{y'} + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial x'}\right)_{y'}\right]\hat{\mathbf{x}} \\
&\quad + \left[\left(\frac{\partial z}{\partial x}\right)_{y} \left(\frac{\partial x}{\partial y'}\right)_{x'} + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial y'}\right)_{x'}\right]\hat{\mathbf{y}}
\end{aligned}$$

and using Eq. (13), one has

$$\nabla' z(x, y) = -\frac{z^2(\hat{\mathbf{N}} \cdot \hat{\mathbf{z}})}{F(\hat{\mathbf{N}} \cdot \mathbf{r})}\left[\left(\frac{\partial z}{\partial x}\right)_y \hat{\mathbf{x}} + \left(\frac{\partial z}{\partial y}\right)_x \hat{\mathbf{y}}\right]$$

or

$$\nabla' z(x, y) = -\frac{z(x, y)}{F} \frac{(\hat{\mathbf{N}} \cdot \hat{\mathbf{z}})(\hat{\mathbf{r}} \cdot \hat{\mathbf{z}})}{(\hat{\mathbf{N}} \cdot \hat{\mathbf{r}})} \nabla z(x, y) \tag{14}$$

From Eq. (12), $\nabla z(x, y)$ can be written in terms of $\hat{\mathbf{N}}$ as

$$\nabla z(x, y) = \hat{\mathbf{z}} - \frac{\hat{\mathbf{N}}}{(\hat{\mathbf{N}} \cdot \hat{\mathbf{z}})}$$

so that from Eq. (14),

$$\nabla' z(x, y) = -\frac{z(x, y)}{F} \frac{(\hat{\mathbf{N}} \cdot \hat{\mathbf{z}})(\hat{\mathbf{r}} \cdot \hat{\mathbf{z}})}{(\hat{\mathbf{N}} \cdot \hat{\mathbf{r}})}\left[\hat{\mathbf{z}} - \frac{\hat{\mathbf{N}}}{\hat{\mathbf{N}} \cdot \hat{\mathbf{z}}}\right]$$

Then the desired expression for dz/ds' is, using Eq. (8),

$$\frac{dz}{ds'} = \frac{z(x, y)}{F} \frac{(\hat{\mathbf{N}} \cdot \hat{\mathbf{S}}'_T)(\hat{\mathbf{r}}' \cdot \hat{\mathbf{z}})}{\hat{\mathbf{N}} \cdot \hat{\mathbf{r}}'} \tag{15}$$

where the substitution $\hat{\mathbf{r}} = -\hat{\mathbf{r}}'$ has been made.

What remains now is to calculate the surface normal $\hat{\mathbf{N}}$ in terms of the photometric geometry. For this purpose, the standard angles i, e, and g, will not be used but rather a more useful trio. These are taken to be α, β, and g, where α and g are as before and β is the angle between $\hat{\mathbf{N}}$ and the normal to the plane containing g (see Fig. 3).

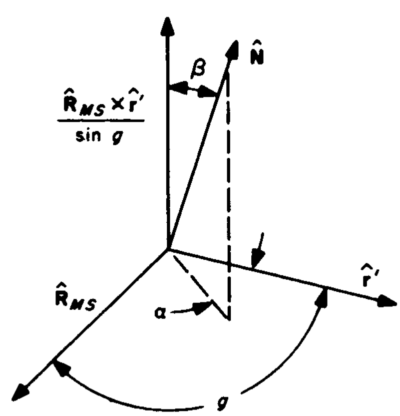

Fig. 3. Decomposition of the surface normal in terms of photometric angles

It is easy to decompose $\hat{\mathbf{N}}$ into components along the three axes shown, noting that they are not all mutually orthogonal. In terms of the photometric angles,

$$\hat{\mathbf{N}} = \frac{\sin\beta\sin(\alpha+g)}{\sin g}\hat{\mathbf{r}}' - \frac{\sin\beta\sin\alpha}{\sin g}\hat{\mathbf{R}}_{MS} + \frac{\cos\beta}{\sin g}(\hat{\mathbf{R}}_{MS}\times\hat{\mathbf{r}}') \tag{16}$$

where the sign convention for α used in Ref. 1 has been adopted, i.e., α is positive if it does not overlap g and is negative if it does, as is the case in Fig. 3. Then, substituting for $\hat{\mathbf{N}}$ in Eq. (15), one has

$$\frac{dz}{ds'} = \frac{z(x,y)}{F}\frac{(\hat{\mathbf{r}}'\cdot\hat{\mathbf{z}})}{\cos\alpha\sin g}[\sin(\alpha+g)(\hat{\mathbf{r}}'\cdot\hat{\mathbf{S}}'_T) - \sin\alpha(\hat{\mathbf{R}}_{MS}\cdot\hat{\mathbf{S}}'_T) + \cot\beta(\hat{\mathbf{R}}_{MS}\times\hat{\mathbf{r}}')\cdot\hat{\mathbf{S}}'_T] \tag{17}$$

where it is assumed that the angle β is not zero.

Now to reconstruct the object scene, one needs to know the length r of the position vector to each of its points. From the picture geometry, the direction from an object point to its image point is $\hat{\mathbf{r}}'$ so that the position vector $\mathbf{r}$ of the object point can be written (assuming z is known)

$$\mathbf{r} = \frac{z(x,y)}{\hat{\mathbf{r}}'\cdot\hat{\mathbf{z}}}\hat{\mathbf{r}}'$$

and its length can be written simply

$$r = -\frac{z}{\hat{\mathbf{r}}'\cdot\hat{\mathbf{z}}} \tag{18}$$

remembering that $\hat{\mathbf{r}}'\cdot\hat{\mathbf{z}} < 0$. Thus, a differential equation for r can be written

$$\frac{dr}{ds'} = \frac{-1}{\hat{\mathbf{r}}'\cdot\hat{\mathbf{z}}}\frac{dz}{ds'} + \frac{z}{(\hat{\mathbf{r}}'\cdot\hat{\mathbf{z}})^2}\frac{d(\hat{\mathbf{r}}'\cdot\hat{\mathbf{z}})}{ds'}$$

and it is easy to show that

$$\frac{d(\hat{\mathbf{r}}'\cdot\hat{\mathbf{z}})}{ds'} = -\frac{1}{r'}(\hat{\mathbf{r}}'\cdot\hat{\mathbf{z}})(\hat{\mathbf{r}}'\cdot\hat{\mathbf{S}}'_T)$$

Thus, one has

$$\frac{dr}{ds'} = -\frac{1}{\hat{\mathbf{r}}'\cdot\hat{\mathbf{z}}}\left[\frac{dz}{ds'} + \frac{z}{r'}(\hat{\mathbf{r}}'\cdot\hat{\mathbf{S}}'_T)\right]$$

or

$$\frac{dr}{ds'} = -\frac{1}{\hat{\mathbf{r}}'\cdot\hat{\mathbf{z}}}\left[\frac{dz}{ds'} - \frac{z}{F}(\hat{\mathbf{r}}'\cdot\hat{\mathbf{z}})(\hat{\mathbf{r}}'\cdot\hat{\mathbf{S}}'_T)\right] \tag{19}$$

where Eq. (2) has been used. Then, substituting Eq. (15) in Eq. (19), one has

$$\frac{dr}{ds'} = -\frac{z}{F(\hat{\mathbf{N}}\cdot\hat{\mathbf{r}}')}[\hat{\mathbf{N}} - (\hat{\mathbf{N}}\cdot\hat{\mathbf{r}}')\hat{\mathbf{r}}']\cdot\hat{\mathbf{S}}'_T$$

or

$$\frac{dr}{ds'} = \frac{r}{F}\frac{(\hat{\mathbf{r}}'\cdot\hat{\mathbf{z}})}{(\hat{\mathbf{N}}\cdot\hat{\mathbf{r}}')}[\hat{\mathbf{r}}'\times(\hat{\mathbf{N}}\times\hat{\mathbf{r}}')]\cdot\hat{\mathbf{S}}'_T \tag{20}$$

and, using Eq. (16), this can be written in terms of the photometric angles,

$$\frac{dr}{ds'} = \frac{r}{F}\frac{(\hat{\mathbf{r}}'\cdot\hat{\mathbf{z}})}{\cos\alpha\sin g}[\sin\alpha\cos g(\hat{\mathbf{r}}'\cdot\hat{\mathbf{S}}'_T) - \sin\alpha(\hat{\mathbf{R}}_{MS}\cdot\hat{\mathbf{S}}'_T) + \cot\beta(\hat{\mathbf{R}}_{MS}\times\hat{\mathbf{r}}')\cdot\hat{\mathbf{S}}'_T] \tag{21}$$

where again it is assumed that $\beta \neq 0$.

Along any path S', nothing is known about the angle β since the photometric function is degenerate in that direction. Thus, to derive exact elevation information, the path S' must be chosen such that (assuming $\beta \neq 0$)

$$(\hat{\mathbf{R}}_{MS}\times\hat{\mathbf{r}}')\cdot\hat{\mathbf{S}}'_T = 0 \tag{22}$$

This represents a differential equation for the path S', and it is easy to see that the solutions are a family of straight lines all passing through the image point for which $\hat{\mathbf{r}}' = \hat{\mathbf{R}}_{MS}$ or the zero phase point. These paths correspond to the intersections of the family of planes containing the zero phase point and the center of the lens with the image plane $z = -F$.

Consider a member of this family of planes with unit normal $\hat{\mathbf{N}}_0$ and with corresponding path $S'(\hat{\mathbf{N}}_0)$ (note that $\hat{\mathbf{N}}_0$ is defined except for its sign). Choosing this sign, one can write

$$\hat{\mathbf{N}}_0 = \frac{\hat{\mathbf{R}}_{MS}\times\hat{\mathbf{r}}'}{\sin g}$$

where $\hat{\mathbf{r}}'$ corresponds to any point along $S'(\hat{\mathbf{N}}_0)$ except the zero phase point. Then, since $\hat{\mathbf{S}}'_T(\hat{\mathbf{N}}_0)$ is normal to $\hat{\mathbf{N}}_0$ by condition (22) and also since $\hat{\mathbf{S}}'_T(\hat{\mathbf{N}}_0)$ is normal to $\hat{\mathbf{z}}$,

$$\hat{\mathbf{S}}'_T(\hat{\mathbf{N}}_0) = \frac{\hat{\mathbf{z}} \times \hat{\mathbf{N}}_0}{|\hat{\mathbf{z}} \times \hat{\mathbf{N}}_0|} = \frac{\hat{\mathbf{z}} \times (\hat{\mathbf{R}}_{MS} \times \hat{\mathbf{r}}')}{\sin g\, |\hat{\mathbf{z}} \times \hat{\mathbf{N}}_0|} \tag{23}$$

where the direction of $\hat{\mathbf{S}}'_T(\hat{\mathbf{N}}_0)$ is along the path $S'(\hat{\mathbf{N}}_0)$ and away from the zero phase point (see Fig. 4). Then, substituting Eq. (23) in Eq. (21), one has for the path $S'(\hat{\mathbf{N}}_0)$

$$\frac{dr}{ds'} = -\frac{r}{F}\frac{(\hat{\mathbf{r}}' \cdot \hat{\mathbf{z}})^2}{|\hat{\mathbf{z}} \times \hat{\mathbf{N}}_0|}\tan\alpha \tag{24}$$

This is a particularly simple differential equation. Since F, $\hat{\mathbf{z}}$, and $\hat{\mathbf{N}}_0$ are constants along the path and $\hat{\mathbf{r}}'$ and α are known as functions of s', the solution is simply

$$\frac{r(P)}{r(P_0)} = \exp\left\{\frac{-1}{F\,|\hat{\mathbf{z}} \times \hat{\mathbf{N}}_0|}\int_{P_0}^{P} (\hat{\mathbf{r}}' \cdot \hat{\mathbf{z}})^2 \tan\alpha\, ds'\right\} \tag{25}$$

where s' is measured positive away from the zero phase point, and P and P_0 are points along the path of integration.

It may be useful at this point to list the symbols found in Eq. (25) and their definitions. For some integration path, a straight line in the image plane passing through the zero phase point (the image point for which the photometric geometry gives a phase angle of zero), the symbols mean:

P, P_0 = points along the integration path; P_0 is the reference point for the integral

$r(P), r(P_0)$ = the lengths of the position vectors to the object points corresponding to the image points P and P_0

F = the imaging lens focal length

$\hat{\mathbf{z}}$ = a unit vector along the optical axis

$\hat{\mathbf{N}}_0$ = a unit normal to the plane containing the integration path and the center of the lens

$\hat{\mathbf{r}}'$ = a unit vector from the center of the lens to a point on the integration path

α = the auxiliary photometric angle determined from the object scene luminance and the phase angle g through the surface photometric function

s' = a path length variable along the integration path and measured positive away from the zero phase point

To summarize the results obtained so far, it has been shown that exact elevation data can be derived only along straight line paths in the image plane which pass through the zero phase point. Along each of these paths, then, an integral can be evaluated which depends on the photometric angle α derived from the scene luminances, the geometry of the imaging process, and the particular path geometry. This integral, Eq. (25), relates the lengths

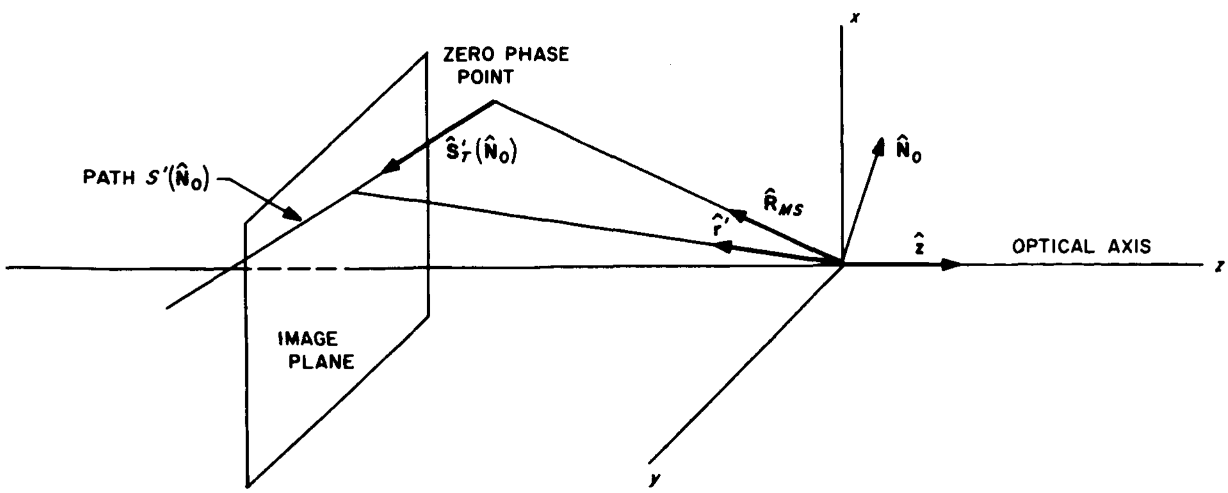

Fig. 4. Integration path vector conventions

of position vectors of points on the object surface, corresponding to image points along the path, to the presumably known length of one of these position vectors. It is apparent from the above that if the zero phase point lies in the picture, then all the position vector lengths can be related to the one for that point. Unfortunately, the zero phase point is undesirable to photograph from satellites since only details due to nonuniformities in the surface albedo are visible (assumed nonexistent in the present treatment) because of the lunar photometric properties, and the zero phase point cannot be photographed from Earth. Thus, in practical applications of this method of extracting surface elevation data, one can relate elevations only along disconnected lines through the image and one has no photometric way of relating elevations from one line to the other.

III. CALCULATION OF ELEVATIONS IN TERMS OF A MOON-CENTERED COORDINATE SYSTEM

Now as mentioned earlier, surface interpretations are far simpler in terms of a Moon-centered coordinate system rather than one centered at the camera lens. That is, one would like to have the surface represented in terms of elevation changes above some reference spherical datum about the Moon's center. Let $\mathbf{r}_M$ be the position vector of the center of the Moon in the lens-centered coordinate system (see Fig. 5). Then, if an object point has position vector $\mathbf{r}(P)$ in the lens-centered coordinate system (P is the corresponding point in the image plane), and $\mathbf{R}(P)$ in a Moon-centered coordinate system,

$$\mathbf{R}(P) = \mathbf{r}(P) - \mathbf{r}_M \tag{26}$$

Now it will be assumed that the field of view of the camera system covers a small enough area that locally

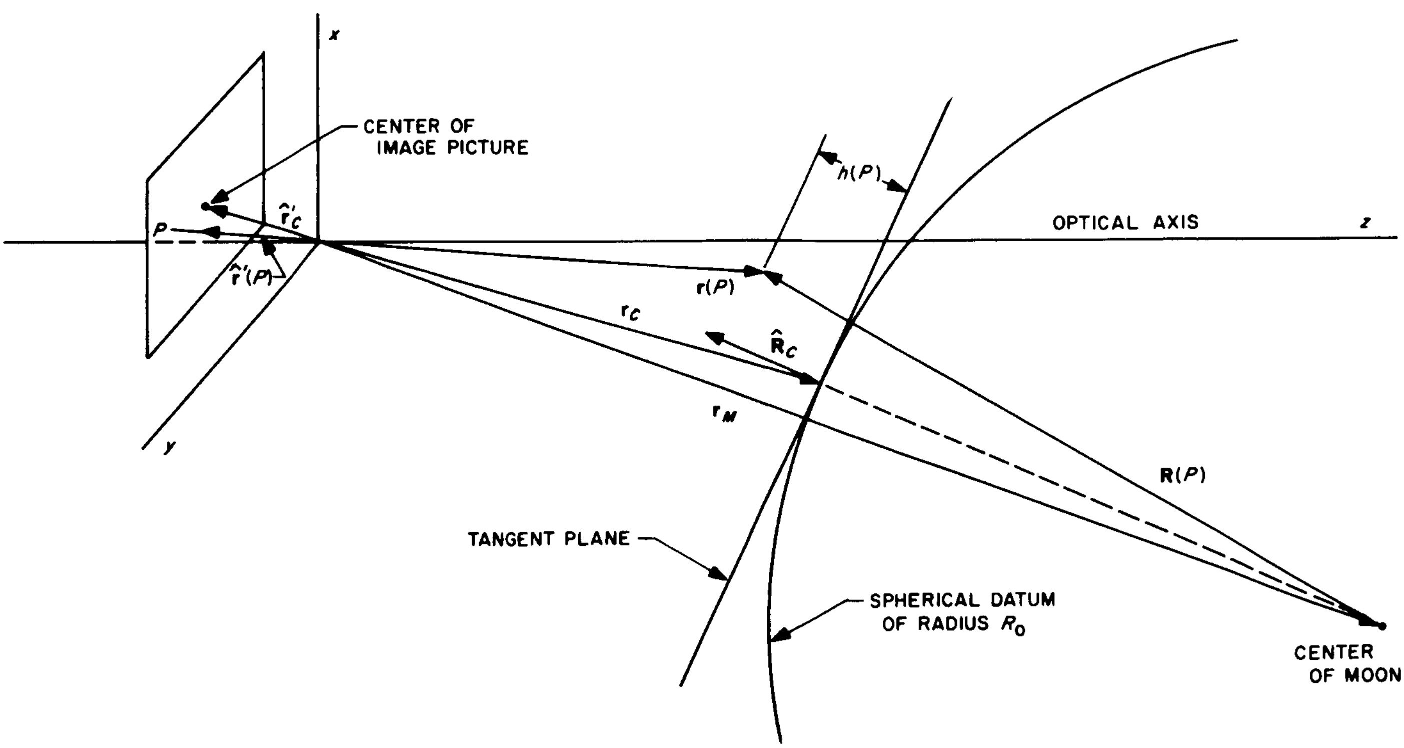

Fig. 5. Geometry for calculating surface heights above the spherical datum

the spherical datum approximates a plane. This means that the vector **R** has essentially the same direction (normal to the tangent plane) over the area covered. The direction is taken to be the normal to the spherical datum at the point corresponding to the center of the output picture. Note that the center of the picture need not be where the optical axis intersects the image plane, e.g., in television systems where the raster can be displaced. So let the center of the picture be located at a direction $\hat{\mathbf{r}}'_c$ from the lens center. Then the length r_c of the position vector to the corresponding point on the spherical datum of radius R_0 is

$$r_c = -(\hat{\mathbf{r}}_c' \cdot \mathbf{r}_M) - [R_0^2 + (\hat{\mathbf{r}}_c' \cdot \mathbf{r}_M)^2 - r_M^2]^{1/2}$$

and the normal $\hat{\mathbf{R}}_c$ to the datum at this point is (see Fig. 5)

$$\hat{\mathbf{R}}_c = -\frac{1}{R_0}(r_c\hat{\mathbf{r}}_c' + \mathbf{r}_M) \tag{27}$$

Then, over the area covered by the field of view

$$\mathbf{R} \approx R\hat{\mathbf{R}}_c$$

where R is the length of **R** and from Eq. (26)

$$R(P) \approx \mathbf{r}(P) \cdot \hat{\mathbf{R}}_c - \mathbf{r}_M \cdot \hat{\mathbf{R}}_c$$

Similarly for the tangent point one can write

$$R_0 = \mathbf{r}_c \cdot \hat{\mathbf{R}}_c - \mathbf{r}_M \cdot \hat{\mathbf{R}}_c$$

Thus, the height $h(P)$ above the datum of the object point at $\mathbf{r}(P)$ is given by

$$h(P) \approx R(P) - R_0 \approx \mathbf{r}(P) \cdot \hat{\mathbf{R}}_c - \mathbf{r}_c \cdot \hat{\mathbf{R}}_c$$

Now P falls on some integration path S' described earlier so that only the quantity $r(P)/r(P_0)$ is known from Eq. (25) where P_0 is a reference point along the path. The reference position vector length $r(P_0)$ is not accurately known except that it corresponds to an object point assumed close to the reference datum, so that

$$\mathbf{r}(P_0) \cdot \hat{\mathbf{R}}_c \approx \mathbf{r}_c \cdot \hat{\mathbf{R}}_c$$

Then, if $\hat{\mathbf{r}}'(P)$ and $\hat{\mathbf{r}}'(P_0)$ are the directions from the lens center to the image points P and P_0, one has

$$h(P) \approx [\mathbf{r}(P_0) \cdot \hat{\mathbf{R}}_c] \frac{r(P)}{r(P_0)} \frac{[\hat{\mathbf{r}}'(P) \cdot \hat{\mathbf{R}}_c]}{[\hat{\mathbf{r}}'(P_0) \cdot \hat{\mathbf{R}}_c]} - \mathbf{r}_c \cdot \hat{\mathbf{R}}_c$$

or

$$h(P) \approx -(\mathbf{r}_c \cdot \hat{\mathbf{R}}_c)\left\{1 - \frac{r(P)}{r(P_0)} \frac{[\hat{\mathbf{r}}'(P) \cdot \hat{\mathbf{R}}_c]}{[\hat{\mathbf{r}}'(P_0) \cdot \hat{\mathbf{R}}_c]}\right\} + \{[\mathbf{r}(P_0) - \mathbf{r}_c] \cdot \hat{\mathbf{R}}_c\} \frac{[\mathbf{r}(P) \cdot \hat{\mathbf{R}}_c]}{[\mathbf{r}(P_0) \cdot \hat{\mathbf{R}}_c]} \tag{28}$$

For visual spacecraft approaching or orbiting the Moon, pictures are taken high above the surface datum compared to the surface fluctuations about the datum. Thus, the factor $[\mathbf{r}(P) \cdot \hat{\mathbf{R}}_c]/[\mathbf{r}(P_0) \cdot \hat{\mathbf{R}}_c]$ in Eq. (28) can be written as

$$\frac{[\mathbf{r}(P) \cdot \hat{\mathbf{R}}_c]}{[\mathbf{r}(P_0) \cdot \hat{\mathbf{R}}_c]} = 1 + \delta h(P)$$

where

$$\delta h(P) = \frac{[\mathbf{r}(P) - \mathbf{r}(P_0)] \cdot \hat{\mathbf{R}}_c}{\mathbf{r}(P_0) \cdot \hat{\mathbf{R}}_c}$$

is the relative surface height fluctuation between the object points corresponding to the image points P and P_0 and is small compared to unity, from the above. Similarly,

$$\Delta h(P_0) = [\mathbf{r}(P_0) - \mathbf{r}_c] \cdot \hat{\mathbf{R}}_c$$

is the absolute height of the object point corresponding to the reference point P_0 above the lunar datum and is assumed to be small compared to the height of the spacecraft above the datum. Thus, one can write Eq. (28) as

$$h(P) \approx -(\mathbf{r}_c \cdot \hat{\mathbf{R}}_c)\left\{1 - \frac{r(P)}{r(P_0)} \frac{[\hat{\mathbf{r}}'(P) \cdot \hat{\mathbf{R}}_c]}{[\hat{\mathbf{r}}'(P_0) \cdot \hat{\mathbf{R}}_c]}\right\} + \Delta h(P_0)[1 + \delta h(P)]$$

or to lowest order in small quantities

$$h(P) \approx -(\mathbf{r}_c \cdot \hat{\mathbf{R}}_c)\left\{1 - \frac{r(P)}{r(P_0)} \frac{[\hat{\mathbf{r}}'(P) \cdot \hat{\mathbf{R}}_c]}{[\hat{\mathbf{r}}'(P_0) \cdot \hat{\mathbf{R}}_c]}\right\} + \Delta h(P_0) \tag{29}$$

So, consistent with the previous result, since $\Delta h(P_0)$ involves the unknown $r(P_0)$ for an integration path, only a relative height $h_{rel}(P)$ above the lunar datum can be found for each object point corresponding to an image point P along the integration path where

$$h_{rel}(P) \approx -(\mathbf{r}_c \cdot \hat{\mathbf{R}}_c)\left\{1 - \frac{r(P)}{r(P_0)} \frac{[\hat{\mathbf{r}}'(P) \cdot \hat{\mathbf{R}}_c]}{[\hat{\mathbf{r}}'(P_0) \cdot \hat{\mathbf{R}}_c]}\right\} \tag{30}$$

Due to the linearization procedure inherent in the above approximations, however, the present relative elevations along a given integration path differ from the actual values by an additive constant $\Delta h(P_0)$ which depends only on the path. This is in contrast to the ratios of values obtained in the analysis resulting in Eq. (25).

It is perhaps helpful to summarize the symbols appearing in Eq. (30):

P, P_0 = points along a given integration path in the image plane as defined for Eq. (25). P_0 is the reference point for the integration

$h_{rel}(P)$ = the relative height of the object point corresponding to the image point P above a spherical lunar datum (approximated by a plane over the field of view)

$\mathbf{r}_c$ = the position vector of the point on the spherical datum imaged at the center of the picture

$\hat{\mathbf{R}}_c$ = the unit normal to the spherical datum and hence to the tangent plane at the point at position $\mathbf{r}_c$

$r(P)/r(P_0)$ = the ratio of position vector lengths to the object points corresponding to P and P_0. This ratio is determined from Eq. (25)

$\hat{\mathbf{r}}'(P), \hat{\mathbf{r}}'(P_0)$ = the unit vectors giving the directions from the center of the lens to the image points P and P_0

IV. APPLICATION TO THE *RANGER* PICTURES

The underlying purpose of the above analysis is to extract quantitative topographic information about the lunar surface from pictures such as those *Ranger* took.

The necessary calculations lend themselves quite nicely to digital computation and were incorporated in the digital picture processing program currently being developed at JPL. Since the zero phase point never appears in the *Ranger* pictures for reasons discussed above, and since no stereoscopic data exists, an assumption to relate relative elevations across the integration paths is needed. For a complex picture with much detail, short of manipulating the relative elevations by hand, this assumption must be statistical in nature. The simplest assumption, and one difficult to improve upon without extreme complication, is that the average elevations along the respective integration paths are all equal. This assumption obviously relies for accuracy on the existence of many random elevation fluctuations along each path and neglects any general slope of the surface in a direction normal to the paths. Such a slope is not detectable by these methods due to the lunar photometric properties and must be measured by stereoscopy. Using this assumption to relate relative elevations across the integration paths, a topographic map of the area covered by a picture can be calculated. Examples of the results of this procedure are shown in Figs. 6 through 8. Figure 6 shows the last P-3 frame from *Ranger VIII* in an unrectified form with certain noises removed and with the sine-wave response falloff of the camera corrected. Below the picture is an elevation profile along the line shown in the picture, depicted at 10 times vertical exaggeration and at true scale. It can be seen that the profile follows the picture shadings very well. Analysis of the profile, taking into account the lighting direction when the picture was taken, gives good agreement with the shadow areas actually found in the picture. Figure 7 shows a rectified picture of the second from the last *Ranger VIII* frame with the same noises removed but with no sine-wave response correction applied. Figure 8 then shows the same frame as Fig. 7 in the form of a complete contoured elevation map. The horizontal scale is the same as in Fig. 7 and the contour interval is 65 cm. The shading between the contours is to be interpreted as elevations with the convention that the darker the area the higher the surface. It will be noted that the contours are quite consistent with a subjective interpretation of the brightness picture in Fig. 7 in terms of elevations.

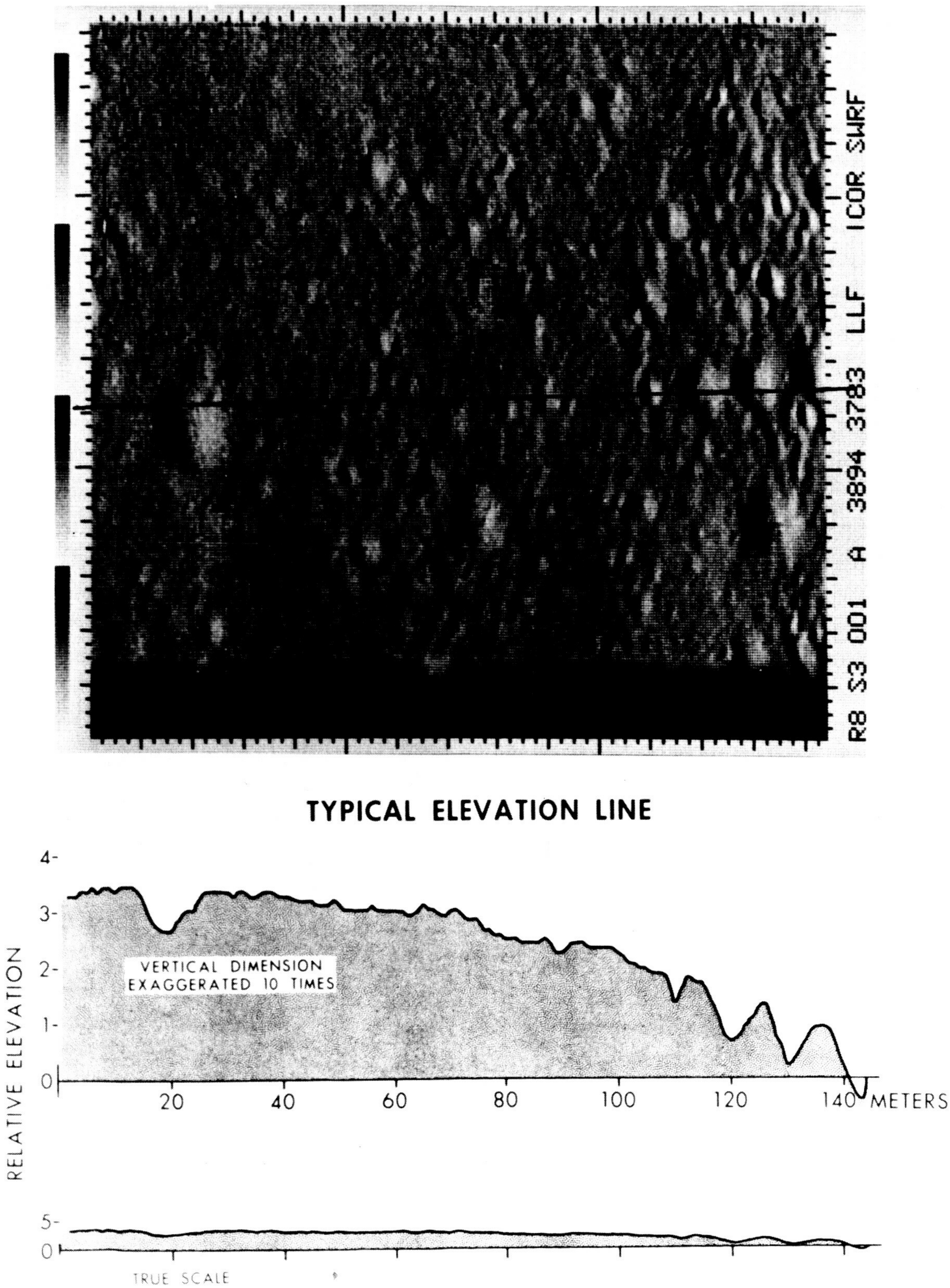

Fig. 6. Elevation profile through the last P-3 picture from *Ranger VIII*

Fig. 7. Corrected and rectified P-3 picture from *Ranger VIII*

Fig. 8. Rectified contour-map of the same P-3 picture from *Ranger VIII* (Shading across the frame represents elevation changes, with the convention that, the darker the area, the higher the surface.)

V. CONCLUSIONS

In conclusion, a solution has been found to the original problem set forth; namely, the derivation of quantitative geometric information about an object scene using a picture of it and a knowledge of the surface photometry. It has been shown that for the lunar photometric properties, there exist straight line paths through the image picture along which an integral (Eq. 25) can be evaluated to relate the lengths of the position vectors of object points corresponding to image points on the path. These position vectors originate from the imaging lens center. The relationship between their lengths is in the form of a ratio, i.e., the integral gives the ratio of the length of the object position vector at the end point of the integration interval to that at the beginning of the interval. If the integration interval always starts at the same point, all of the ratios are with respect to a single reference position vector length. Furthermore, since all of the integration paths pass through the image point for which the photometric phase angle is zero, if this point is contained in the picture, it may be used as the reference for all of the integration paths thereby completely and consistently reconstructing the object surface. Unfortunately, extremely low contrast pictures result from photographing the zero phase point, making it undesirable from the standpoint of visual interpretation. Thus, with the zero phase point not in the picture, the integration paths have no point in common so that each has a separate reference length. There is no photometric method for connecting the various integration path reference lengths, so that some other information or assumption about the surface must be imposed.

It has further been shown that these relative position vector lengths can be converted to relative heights above a spherical lunar datum. This calculation assumed that the field of view of the camera covers a small enough portion of the surface so that the datum is well approximated by a tangent plane. A further assumption was that the camera is very distant above the datum compared to surface elevation fluctuations about the datum. For a given integration path, then, and the corresponding object points, the calculation yields relative heights above the datum differing from the actual values only by a single additive constant for the path. Again, if the zero phase point lies in the picture, this constant can be chosen common to all of the integration paths. If it does not, though, each path will have a separate constant; all of these constants must be related by nonphotometric information.

The above results were applied to the *Ranger* partial-scan photographs of the Moon as part of the digital processing analysis currently under way at JPL. Using the simple assumption that the average elevation along any integration path is a constant for the picture to relate the elevation data across the paths, elevation maps of the pictures were produced. These resulting maps are extremely useful in studies of the lunar surface and in statistical analyses of safe-landing probabilities for spacecraft such as *Surveyor* and *Apollo*. They provide truly quantitative topographic information in regions where shadows are nonexistent.

NOMENCLATURE

- b surface luminance
- e emission angle for surface photometric geometry
- E_0 solar constant at Moon
- F focal length of lens
- g phase angle for surface photometric geometry
- h height of point in object space above spherical lunar datum which is assumed to locally approximate a plane
- $h_{\rm rel}$ relative height of point in object space above spherical lunar datum which is assumed to locally approximate a plane
- i incidence angle for surface photometric geometry
- J Jacobian of imaging transformation
- $\hat{\mathbf{N}}$ unit vector normal to object surface
- $\hat{\mathbf{N}}_0$ unit vector normal to plane containing zero phase point and center of lens
- P, P_0 points on path S' in image plane
- r length of position vector $\mathbf{r}$
- r' length of position vector $\mathbf{r}'$
- r_c length of $\mathbf{r}_c$
- $\mathbf{r}$ position vector from center of lens to point in object space
- $\hat{\mathbf{r}}$ unit vector from center of lens toward point in object space
- $\mathbf{r}'$ position vector from center of lens to point in image space
- $\hat{\mathbf{r}}'$ unit vector from center of lens toward point in image space
- $\mathbf{r}_c$ position vector from center of lens to object point on spherical lunar datum which is imaged at center of output format
- $\hat{\mathbf{r}}_c$ unit vector along $\mathbf{r}_c$
- $\mathbf{r}_M$ position vector from center of lens to center of Moon
- R length of $\mathbf{R}$
- R_0 radius of spherical lunar datum
- $\mathbf{R}$ position vector from center of Moon to point in object space
- $\hat{\mathbf{R}}_c$ unit vector normal to spherical lunar datum at object point which is imaged at center of output format
- $\hat{\mathbf{R}}_{MS}$ unit vector from center of Moon toward center of Sun
- s' path length variable along path S'
- S' path in image plane
- $\hat{\mathbf{S}}'_T$ unit vector tangent to path S'
- x, y, z coordinates of point in object space with respect to lens-centered coordinate system
- x', y', z' coordinates of point in image space with respect to lens-centered coordinate system
- $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ unit vectors along principal axes of right-handed Cartesian lens-centered coordinate system; $\mathbf{z}$ is along optical axis
- α projection of emission angle e in plane containing g for surface photometric geometry
- β colatitude of surface normal with respect to plane containing g
- ρ_0 surface normal albedo
- $\phi(i, e, g)$ lunar surface photometric function
- ∇ nabla or gradient operator with respect to unprimed coordinates
- ∇' nabla or gradient operator with respect to primed coordinates

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