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MISSILE DYNAMICS FOR STABILITY ANALYSIS
~~VOLUME I~~ - DERIVATION OF EQUATIONS

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ABSTRACT

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This document covers the derivation of a set of linearized dynamic equations for use in missile stability analyses. The equations are presented in matrix form and represent the transfer function from engine command signal to gyro output signals. The derivation includes the effects of propellant sloshing, elastic deformation of the vehicle structure, and the dynamics of gimballed engines. The effects of fixed thrusting engines and the effects of inertia corrections to the bending data have also been considered. The vehicle structure is taken as a multi-branched beam under the influence of bending deflections, shear deflections, rotary inertia, and axial accelerations.

The results are presented in terms of a set of generalized coordinates representing rigid body motion, bending deflections, engine deflections, and slosh mass deflections. The equations are transformed into normal coordinates and the results are also presented in terms of a set of orthogonal combined modes.

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I - 1 INTRODUCTION

This document covers the derivation and programming of a set of linearized dynamic equations for use in missile stability analysis. The work presented here is essentially an expansion of the equations and digital computer program described in Reference 1. Since the complete derivation is presented in detail, this document supersedes Reference 1. The revision was undertaken in order to increase the flexibility of the computer program with regard to the types of missiles to be studied and to incorporate changes in the nomenclature and data output found desirable as a result of the use of the previous program. The major changes incorporated are as follows:

(a) Bending data derived for multi-branched beams may be used.

(b) The output, including punched cards for the system matrix, may be obtained in either generalized coordinates or normal coordinates (combined mode representation.)

(c) A plotting option is provided so that either the input or the output modes may be machine plotted.

(d) Inertia corrections can be made at any of twenty locations on the missile independent of slosh tank locations.

(e) Gyro slope data is punched in the system matrix and provisions are made to mix the signals from two rate gyro locations.

(f) Input bending data need not be normalized to total missile mass.

(g) The equations and the program have been arranged to accept a massless engine for use with secondary injection type thrust vector control.

(h) In order to utilize the ability to reduce truncation errors with combined mode representation, the program has been arranged so that higher frequency combined modes may be dropped from the punched output.

(i) Changes may be made in bending data which has been introduced into the program from column binary bending decks.

(j) When studying the effects of variations in damping or performing a limit cycle study using non-linear damping it is necessary to obtain output data for various values of damping. Since damping has no effect on the input or output modes, and is only introduced during the calculations for the punched output, an option is provided to allow a number of output decks to be punched from a single run. This option provides a series of output decks for various values of damping with a minimum of machine time.

(k) An option is also provided to punch out the eigenvector or modal matrix on cards for use in the root analysis program used in limit cycle studies.

This document has been issued in two volumes. Volume I covers the formulation of the dynamic equations and the system matrix. Volume II describes the programming of the equations for solution by 7090 digital computers and contains detailed instructions for the use of the program. Section I of Volume I contains an introduction and a statement of the problem to be solved. In Section II the dynamic equations and system matrix are formulated in generalized coordinates. These equations and system matrix are then transformed into normal coordinates in Section III.

The development of the equations in this document includes all linear terms consistent with the following assumptions:

(a) All physical parameters of the missile such as mass, inertia, and thrust are considered constant. Although these parameters actually vary slowly with time, such variations will have negligible effect on the short time missile response and stability.

(b) Input bending modes are determined for free-free end conditions with engine and sloshing mass rigidity attached and include the effects of bending, shear deflections, rotary inertia and axial acceleration.

(c) Aerodynamic forces normal to the missile axis are assumed to be independent of the local structural bending slope and are considered to vary linearly with the angle of attack developed by the underformed elastic axis of the missile.

(d) Aerodynamic forces along the missile axis are assumed to be independent of local bending and angle of attack.

(e) The dynamic equations are developed for motions in the missile yaw plane, defined as a plane containing the missile velocity vector and a perpendicular to the local vertical. The trim conditions on α, θ, and δ will be zero in this plane. The equations are also applicable to the pitch plane if the trimmed values of these variables may be neglected. Such trimmed values for a gravity turn are negligible. These conditions are usually approached with the approximate gravity turns employed in practice.

I - 2 MISSILE STABILITY

At any instant of flight time, a missile in its nominal or trimmed condition may be considered as a system in dynamic equilibrium. All external forces are steady forces in equilibrium with inertia forces in accordance with D'Alembert's principle, and the internal kinetic and potential energies are constant. Under these conditions the missile is an M^{th} degree of freedom closed dynamic system and when displaced or perturbed from its nominal or trimmed condition, the resulting motion with respect to the trimmed condition will be divergent for an unstable missile or will damp out for a stable missile. A stable missile must be stable for the slightest displacements from the equilibrium position and a stability analysis is concerned with the stable and unstable tendencies of the system at the equilibrium position. Therefore, the analysis may be based upon the assumption of small displacements for which the effects of second and higher order terms may be neglected.

In the absence of external driving forces the motion of the system can be represented by M linearized homogeneous second order differential equations in M independent variables. When expressed in LaPlace notation these equations become M homogeneous linear algebraic equations in the M independent variables where the coefficients are linear combinations of the LaPlace operator, s, and s^2. In matrix form these equations may be written as

$$\left[R\right]\left\{Q_M\right\} = 0 \qquad \text{I-2.1}$$

where the M x M coefficient matrix $[R]$ is a function of s and is referred to as the system matrix. The column matrix $\{Q_M\}$ is a column of the independent generalized coordinates of the system. The system matrix $[R]$ can be written as

$$[R] = \left[s^2 [A] + s [B] + [C]\right] \qquad \text{I-2.2}$$

where $[A]$, $[B]$, $[C]$ are matrices of constants.

The system of dynamic equations expressed in LaPlace notation I-2.1 represents an eigenvalue problem which has 2M eigenvalues or roots found by equating the determinant of $[R]$ to zero. The determinant of $[R]$ is a 2Mth degree polynomial in s and when equated to zero may be written as the product of M quadratics

$$\prod_{1}^{M} (s^2 + A_i s + B_i) = 0 \qquad \text{I-2.3}$$

The roots may be real or complex. However, all complex roots appear in conjugate pairs and a plot of the roots in the s plane is symmetrical about the real axis. Corresponding to each eigenvalue or root, there is an associated eigenvector having M constant components. These components are the elements of the column matrix $\{q_M\}$ which satisfies equation I-2.1 when the corresponding value of s is substituted in the $[R]$ matrix. The ratios of the components of the associated eigenvector then represents the ratios of the generalized variables present in the motion for that particular root.

The system has 2M roots each with an associated eigenvector to represent the relative motion associated with that root. However, as stated previously, the system has M degrees of freedom and, therefore, M natural modes of motion which are orthogonal to each other or completely uncoupled. Each of these orthogonal or uncoupled modes of motion can be represented dynamically by a single degree of freedom system as shown in Figure I-1.

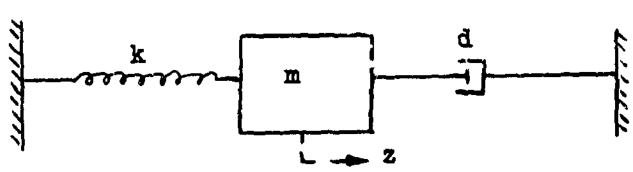


Figure No. I-1

Single Degree of Freedom System

where

z = normal coordinate for mode

m = normalized mass for mode

k = modal spring constant = $\omega^2 m$

ω = modal frequency

d = modal damping

$$m\ddot{z} + d\dot{z} + kz = 0$$

$$(ms^2 + ds + k)\, z = 0$$

$$s^2 = -\frac{d}{2m} \pm \sqrt{\frac{d^2}{4m^2} - \frac{k}{m}} \qquad \text{I-2.3}$$

$$s = \sigma \pm \lambda \qquad \text{I-2.4}$$

then

$$z = e^{\sigma t}\left[Ae^{\lambda t} + Be^{-\lambda t}\right] \qquad \text{I-2.5}$$

In the case of under-damped modes where the damping ratio is less than unity

$$\lambda = j\omega.$$

Substituting for λ reduces equation I-2.5 to

$$z = e^{\sigma t}\left[a \cos \omega t - b \sin \omega t\right] \qquad \text{I-2.6}$$

Therefore, each complex pair of roots produces a single mode of oscillatory motion defined in terms of the generalized variables by the components of the associated eigenvector. It should be noted that the negative frequency represented by $-j\omega$ results from the mathematical possibility of negative rotation of the amplitude vector generating sinusoidal functions. Since the components of the eigenvectors are complex numbers showing both amplitude and phase relationships between the generalized variables, the eigenvectors corresponding to a pair of conjugate roots will be identical except that the signs of the imaginary parts will be opposite. When the eigenvector is used to study the motion resulting from a pair of complex roots the eigenvector corresponding to positive frequency should be used.

In the case of over-damped modes where the damping ratio is greater than unity λ is real, and equation I-2.5 can be written

$$z = e^{\sigma t} \left[(A+B) \cosh \lambda t + (A-B) \sinh \lambda t \right] \qquad \text{I-2.7}$$

Therefore, each over-damped mode represents the motion of two real roots. The relative magnitudes of the generalized variables in the resultant motion can be described by a linear combination of the eigenvectors corresponding to the two roots. The only other type of mode which may occur is a damped mode for which the modal spring k is zero.

Then

$$\lambda = \sigma$$

and

$$s = 0$$

$$s = 2\sigma$$

equation I-2.5 reduces to

$$z = Ae^{2\sigma t} + B \qquad \text{I-2.8}$$

Therefore, each mode of this type represents the motion corresponding to two roots one of which is at the origin. The motion corresponding to the root at the origin is simply B equal to a constant, and since the motion has been taken as the motion with respect to the nominal or trimmed condition, this constant must be zero. It may be seen that in order for a root to appear at the origin, one of the equations of motion must contain s in all terms such that s may be divided out of the equation. Since the motion resulting from these roots is zero, this operation is justified, and the M equations of motion need not all be second degree in s in which case the number of roots may be less than 2M. However, the number of orthogonal natural modes of motion will always be M.

Any free or driven motion of the system must be a linear combination of the M natural modes of motion. In any such combination representing the free motion of the system, the stable natural modes of motion will damp out in accordance with their damping ratios, reducing the motion to a combination of any unstable modes present. Therefore, a stability analysis of the system for any form of excitation reduces to the study of the stability of each of the M natural modes of motion. A simple stable or unstable type of analysis for an existing missile could be performed by formulating the dynamic equations, obtaining the roots of the system matrix and noting the root locations in the s plane. However, such an analysis does not lend itself to the determination of stability margins, the study of system modifications, the design of new systems, or limit cycle studies. A considerable reduction in computer expense together with a greater insight into the physical aspects of the problem will result if the analysis is performed in a number of steps. In general these steps consist of the following:

(a) The free-free uncoupled bending modes and the rigid tank fluid sloshing parameters are obtained from separate computer programs.

(b) The bending and slosh data together with trajectory and engine data are introduced into a set of homogeneous dynamic equations representing the missile dynamics. These equations in LaPlace notation are expressed as a coefficient matrix where the coefficients are functions of the LaPlace operator, s.

(c) The coefficient matrix expressing missile dynamics is then expanded to the system matrix by the addition of the autopilot and guidance parameters.

(d) The system matrix is used to perform a number of different types of stability studies. It may be used to obtain the system roots and associated eigenvectors which in turn are used to indicate the stability of each mode of motion or the amplitudes of limit cycles. The system matrix is also used to obtain the poles and zeros of the open loop transfer function of the system. These data are in turn used to obtain open loop frequency response generally used in bending and rigid body studies or they may be used to obtain closed loop root loci, generally used in fluid slosh studies.

The equations formulated in Volume I of this document and the digital computer program described in Volume II are used in performing the work outlined in step (b). The input data consists of the bending, slosh, trajectory, and engine data, while the output is the missile dynamics coefficient matrix in the form of punched cards.

I - 3 SYSTEM MATRIX

A general block diagram for a missile control system is shown in Figure I-2. As noted in Section I-2, the system has M degrees of freedom and will have M natural modes of motion when θ_c is zero. Since the missile mass is considered as a distributed mass in the calculation of bending modes there are an infinite number of elastic modes. Likewise, integration over the total fluid to obtain the equivalent slosh parameters produces an infinite number of fluid modes for each fluid tank. It is therefore necessary to limit the number of bending and slosh modes which may be incorporated into the analysis in order that M remain a finite number. The effects of limiting the number of bending and slosh modes considered is discussed in Section II and it is assumed here that M has been reduced to a finite number.

In Section I-2 it was assumed the dynamics of the system was expressed by M second order linear differential equations in M generalized independent variables. Using LaPlace notation, these equations become M second degree linear algebraic equations in the same variables where the coefficients were linear functions of s and s^2.

The M x M square coefficient matrix for these equations was defined as the system matrix. However, the dynamic equations and the system matrix may be expanded by the addition of a number of dependent variables and an equal number of auxiliary equations. Therefore, the system matrix for the M degree of freedom system may be a square matrix of any order greater than M. The total number of roots of the system will remain 2M less any roots at the origin, and the system will have M orthogonal natural modes of motion independent of the order of the system matrix.

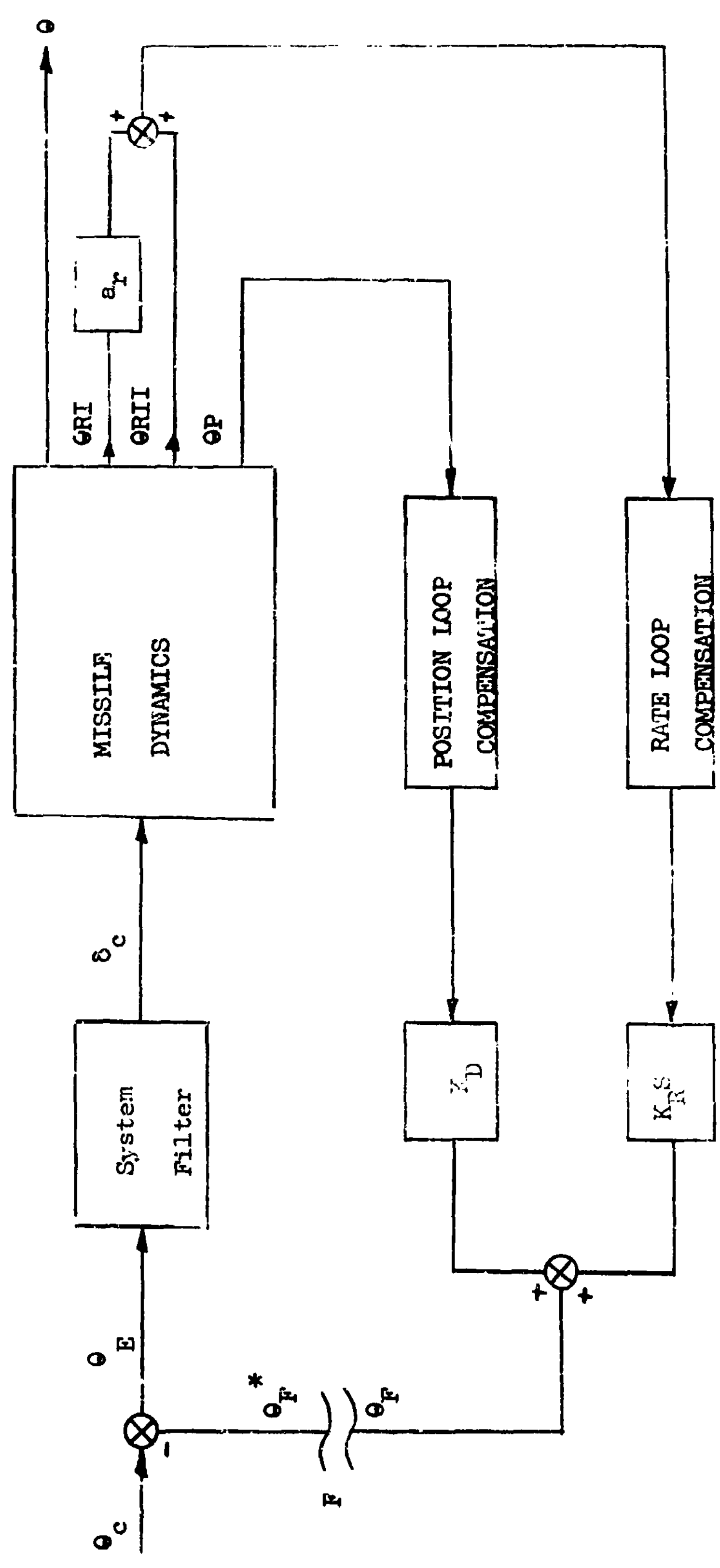


FIGURE I - 2

General Control System Block Diagram

That portion of the system matrix representing the equations formulated in this document and referred to as the missile dynamics matrix in Section I-2, corresponds to the transfer functions between the gyro angles θ_p and θ_R and the engine command signal, δ_c (Figure I-2). This matrix, which appears in the program output both in printed form and as a punched deck of cards, is not a square matrix. The number of columns will be one greater than the number of rows, therefore, the matrix represents the coefficients of a series of homogeneous equations for which the number of variables is one greater than the number of equations. The addition of an equation relating the engine command signal to the gyro outputs will produce the square system matrix from which the closed loop roots may be determined.

In practice the command signal, δ_c, is related to the gyro outputs by the addition of a number of auxiliary equations and dependent variables. The final equation relates the output variable to the input variable thereby closing the loop producing the square system matrix.

In order to determine the open loop characteristics of the system with the loop opened at point such as F in Figure I-2, the dependent variable θ_F^* is introduced together with an auxiliary equation relating θ_F to θ_F^*. This auxiliary equation should always be represented by the last row in the system matrix. The non-square matrix obtained by omitting the last row of the system matrix then represents the transfer function $\frac{\theta_F}{\theta_F^*}$ which is defined as the system transfer function.

The auxiliary equation relating θ_F^* to θ_F may be written

$$a\,\theta_F + b\,\theta_F^* = 0 \qquad \text{I-3.1}$$

therefore, the last row of the system matrix will have a in the column representing θ_F and b in the column representing θ_F^*. Substituting -1 for either a or b and +1 for the other will make θ_F equal to θ_F^* and the solution of the system matrix will give the closed loop roots and the corresponding modes of motion for the system.

Letting b equal unity and a equal zero reduces equation I-3.1 to

$$\theta_F^* = 0$$

and θ_F is arbitrary. The resulting motion when the input is zero must be the natural modes of the system, therefore, solutions of the system matrix will give the open loop poles and the corresponding modes of motion.

Since the system also has M degrees of freedom for the open loop condition there will be 2M roots and M natural modes of motion.

Letting a equal unity and b equal zero reduces equation I-3.1 to

$$\theta_F = 0$$

and θ_F^* is arbitrary. The solution of the system matrix under these conditions will give the roots and modes of motion required to produce zero output for an arbitrary input. These roots are defined as the open loop zeros of the system.

The system transfer function may be written

$$\frac{\theta_F}{\theta_F^*} = (K_R)_N \frac{\prod (s - z_i)}{\prod (s - p_j)} \qquad \text{I-3.2}$$

where $(K_R)_N$ = nominal gain of system

$\prod (s - z_i)$ = product of $(s - z_i)$

$\prod (s - p_j)$ = product of $(s - p_j)$

z_i = open loop zero

p_j = open loop pole

For the closed loop condition the transfer function is equal to unity and equation I-3.2 becomes

$$\prod (s - p_j) = (K_R)_N \prod (s - z_i) \qquad \text{I-3.3}$$

Letting the system gain become a variable, then for the open loop condition K_R is zero and I-3.3 becomes

$$\prod (s - p_j) = 0$$

and the open loop poles are the system roots. However, if K_R becomes infinite equation I-3.3 becomes

$$\prod (s - z_i) = 0$$

and it may be seen that the open loop zeros are also the roots of the closed loop system when the system gain is infinite. As the system gain is varied from zero to infinity the closed loop roots will move from the open loop poles to the open loop zeros passing through the nominal closed loop roots when the gain is equal to the nominal gain. As the gain approaches infinity in an M^{th} degrees of freedom system, the system may have less than M degrees of freedom and less than M natural modes of motion. In this case the roots must approach zeros which are at infinity. Solution of the system matrix for the zeros of the system will produce only finite zeros and the difference between the number of zeros found and 2M represents the number of zeros at infinity.

II-1 COORDINATE SYSTEM

It is assumed that at some time, t, the missile in its nominal or trimmed condition flies along an inertial axis, H, such that the missile pitch plane contains the local vertical and the missile yaw plane makes an angle γ with the local vertical, see figure II-1.1. The center of gravity of the missile is displaced a distance, Q_H, along the H axis measured from a second inertial axis, L, in the yaw plane perpendicular to the H axis.

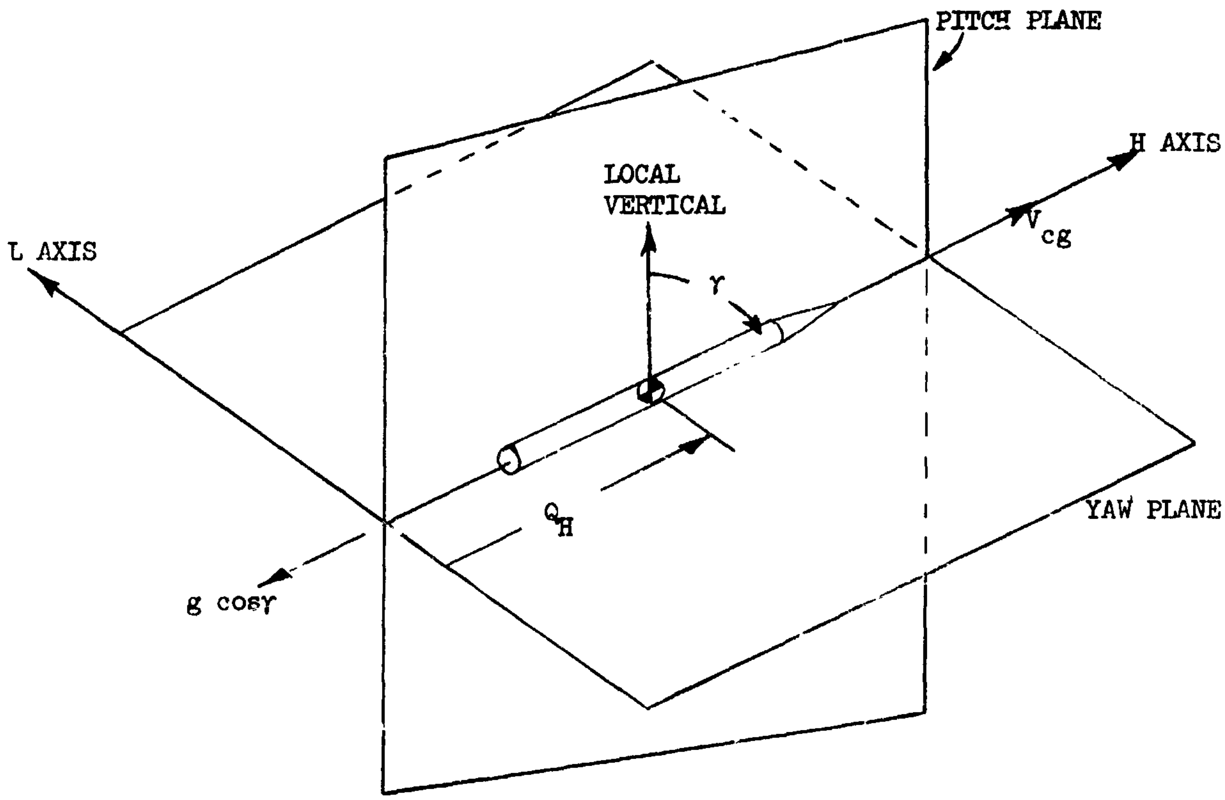
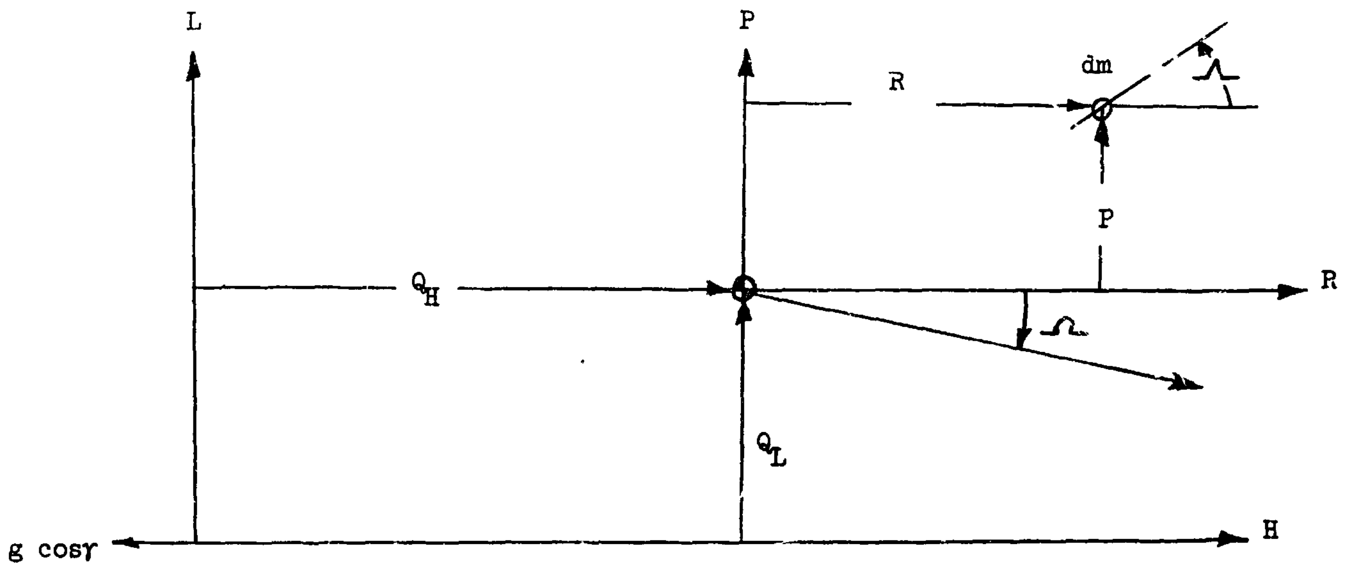


FIGURE II - 1.1

It is further assumed that the missile is displaced or perturbed in the yaw plane and acts as a system of mass elements, dm, each of which has relative motion with respect to the total system center of gravity. The displacement in the yaw plane of the total system center of gravity at time, t, is defined by the coordinates Q_H and Q_L measured along the H and L axes, see figures II-1.2.



YAW PLANE
FIGURE II - 1.2

The R and P axes are non-rotating in inertial space and are translating in the yaw plane with the total system c.g. at a velocity V_{cg} making an angle Ω with the R and H axes. The R and P axes remain parallel to the H and L axes. The yaw plane coordinates of the mass element dm in inertial space are H, L, and Λ

where Λ = rotational displacement of dm with respect to R

$H = Q_H + R$

$L = Q_L + P$

The component of the acceleration of gravity in the yaw plane is g cos γ directed along the negative H axis.

When the missile is in its nominal or trimmed condition, Q_L, Ω, and Λ are zero, R and P are constant, and Q_H is increasing along the H axis. The dynamics of the system is such that the magnitudes of these coordinates have oscillatory components when the missile has been displaced or perturbed. The oscillatory components are damped in a stable system and divergent in an unstable system.

The motion of the mass elements, dm, with respect to the system C.G. is defined with respect to a third set of coordinate axes defined in the yaw plane such that the origin is at the center of gravity and the axes are angularly displaced by an angle Θ with respect to the R and P axes, see figure II-1.3.

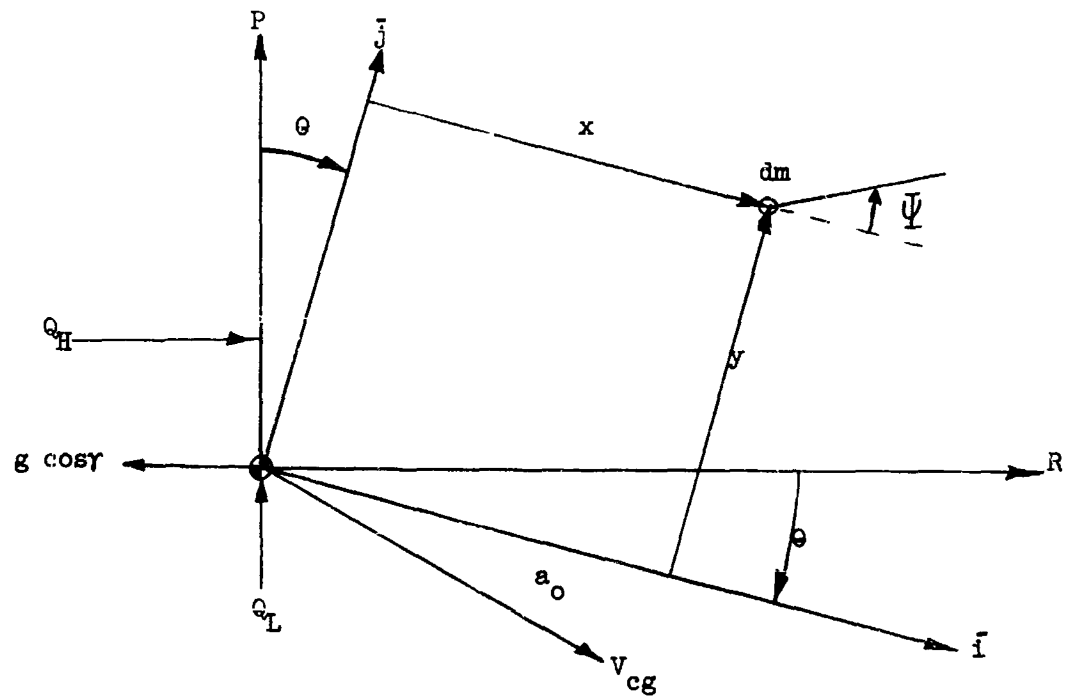


FIGURE II - 1.3

The displacements of the element of mass dm with respect to the $\bar{i}$ - $\bar{j}$ axes are x, y and Ψ.

Ω, Θ, and α_o are taken as positive for rotation about the - $\bar{k}$ axes while Λ and Ψ are taken as positive for rotation about the + $\bar{k}$ axis. It should be noted that no restrictions have been placed upon the selection of Θ, which is arbitrary. The only restriction placed upon the $\bar{i}$ - $\bar{j}$ coordinate system is that the origin be at the total c.g. of the system. This restriction requires that

$$\int_{TM} x\,dm = 0 \qquad \int_{TM} y\,dm = 0 \qquad \text{II-1.1}$$

where $\int_{TM}$ represents integration over the total mass or total missile.

From equation II-1.1

$$\int_{TM} \dot{x}\, dm = 0 = \text{total linear momentum along the } \bar{i} \text{ axis resulting from motion with respect to c.g.}$$

$$\int_{TM} \dot{y}\, dm = 0 = \text{total linear momentum along the } \bar{j} \text{ axis resulting from motion with respect to c.g.}$$ II-1.2

$$\int_{TM} \ddot{x}\, dm = 0 = \text{net inertia force in the } \bar{i} \text{ direction resulting from motion with respect to c.g.}$$

$$\int_{TM} \ddot{y}\, dm = 0 = \text{net inertia force in the } \bar{j} \text{ direction resulting from motion with respect to c.g.}$$

The angular momentum of the mass element dm about the $\bar{k}$ axis due to motion with respect to the c.g. is

$$d\,\mathcal{M}_{xy} = (\dot{y}x - \dot{x}y + r^2\,\dot{\Psi})\, dm \qquad \text{II-1.3}$$

where r = radius of gyration of the element of mass, dm.

From the geometry of the system

$$R = x \cos \theta + y \sin \theta$$

$$P = y \cos \theta - x \sin \theta$$

$$\Lambda = \Psi - \theta$$

$$\Omega = \theta + \alpha_o \qquad \text{II-1.4}$$

$$H = Q_H + x \cos \theta + y \sin \theta$$

$$L = Q_L + y \cos \theta - x \sin \theta$$

Placing the origin of the $\bar{i}$ - $\bar{j}$ coordinate system at the c.g. makes the net linear momentum in any direction, resulting from motion with respect to the c.g., equal to zero. Therefore, the sum of the external forces in any direction must equal the rate of change of linear momentum of the total mass located at the total c.g. This in effect uncouples the motion of the center of gravity from the motion of the mass elements with respect to the c.g.

Likewise the sum of the external moments about the c.g. equals the time rate of change of angular momentum about the c.g. and it would be desirable to define the angle θ such that the rotation of the $\bar{i}$ - $\bar{j}$ axes is a function of the external torques only and independent of the motion of the mass elements with respect to the c.g.

Letting $\mathcal{M}_T$ = angular momentum of total mass about c.g.

$$d\mathcal{M}_T = (\dot{P}R - \dot{R}P)dm + r^2 \dot{\Lambda} dm$$

substituting from equations II-1.4

$$d\mathcal{M}_T = (\dot{y}x - \dot{x}y + r^2\dot{\Psi})dm - \dot{\theta}(x^2 + y^2 + r^2)dm$$

$$\mathcal{M}_T = \int_{TM} (\dot{y}x - \dot{x}y + r^2\dot{\Psi})dm - \dot{\theta} \int_{TM} (x^2+y^2+r^2)dm$$

The first integral is the angular momentum of the total mass resulting from motion with respect to the c.g. and equals $\mathcal{M}_{xy}$ from equation II-1.3. The second integral is the total mass moment of inertia about the c.g. which is defined as I.

$$\mathcal{M}_T = \mathcal{M}_{xy} - I\dot{\theta} \qquad \text{II-1.5}$$

$$\text{Torque} = \dot{\mathcal{M}}_T = \dot{\mathcal{M}}_{xy} - \frac{d}{dt}(I\dot{\theta}) \qquad \text{II-1.6}$$

Equation II-1.6 shows that it is desirable to define θ such that

$$\dot{\mathcal{M}}_{xy} = 0 \qquad \text{II-1.7}$$

θ will then always take a value such that

$$\mathcal{M}_{xy} = \text{constant} \qquad \text{II-1.8}$$

and

$$\sum(\text{External torques}) = -\frac{d}{dt}(I\dot{\theta}) \qquad \text{II-1.9}$$

It will be noted that the rotation of the $\bar{i} - \bar{j}$ axes, θ, is still coupled to the motion with respect to the c.g. by the term $I\ddot{\theta}$ which results from the non-linear centrifugal and coriolis forces. However, for first order linearization, $\frac{d}{dt}(I\dot{\theta}) = I\ddot{\theta}$ and the motions are uncoupled. Furthermore, for a linear oscillatory system $\mathcal{M}_{xy}$ must be a sine function which can only have the constant value of zero.

The displacement of the total system, can be defined by the displacement of the c.g., Q_H and Q_L, the rotation of the $\bar{i} - \bar{j}$ coordinate system, θ, and the displacement of the system with respect to the $\bar{i} - \bar{j}$ axes. The motion of the system with respect to the $\bar{i} - \bar{j}$ axes will have N degrees of freedom and can be described by N generalized coordinates Q_n. The coordinates x, y, and Ψ for each element of mass can then be defined in terms of the Q_n's and the constraints of the system such that

$$\begin{aligned} x &= x(Q_n\text{'s}) \\ y &= y(Q_n\text{'s}) \\ \Psi &= \Psi(Q_n\text{'s}) \end{aligned} \qquad \text{II-1.10}$$

where equations II-1.10 need not be linear.

The total system then has N + 3 degrees of freedom and can be described by the generalized coordinates Q_H, Q_L, θ, and N coordinates Q_n. It has already been indirectly shown that the dynamic equations associated with the Q_H, Q_L and θ coordinates are

$$\sum P_H = M\ddot{Q}_H = \text{sum of external force along H axes}$$

$$\sum P_y = M\ddot{Q}_L = \text{sum of external forces along L axes}$$

$$-\frac{d}{dt}(I\dot{\theta}) = \text{sum of external moments about c.g.}$$

It will also be shown that the motion of the system with respect to the $\bar{i}$ - $\bar{j}$ axes can be defined in terms of N normal coordinates q_a. The dynamic equations derived in terms of the generalized coordinates Q_n will then be transformed in terms of the normal coordinates q_a.

The displacements of the mass element dm in inertial space are given by equation II-1.4 as

$$H = Q_H + x \cos \Theta + y \sin \Theta$$

$$L = Q_L + y \cos \Theta - x \sin \Theta$$

$$\Lambda = \Psi - \Theta$$

The coordinates of the $\bar{i}$ - $\bar{j}$ axes are Q_H, Q_L, and Θ which are considered as generalized coordinate and are only a function of time. The coordinates of the mass element dm with respect to the $\bar{i}$ - $\bar{j}$ axes are x, y, and Ψ. The motion with respect to the $\bar{i}$ - $\bar{j}$ axes has not been defined in terms of the constraints of the system. Therefore, x, y, and Ψ cannot be considered generalized coordinates and the general dynamic relationships discussed in this section will apply to any system of mass elements.

II-2.1 VELOCITIES

Differentiating equations II-1.4 with respect to time

$$\dot{H} = \dot{Q}_H + \dot{x} \cos \Theta - x\dot{\Theta} \sin \Theta + y\dot{\Theta} \cos \Theta$$

$$\dot{L} = \dot{Q}_L + \dot{y} \cos \Theta - y\dot{\Theta} \sin \Theta - \dot{x} \sin \Theta - x\dot{\Theta} \cos \Theta \qquad \text{II-2.1}$$

$$\dot{\Omega} = \dot{\Theta} + \dot{a}_o$$

$$\dot{\Lambda} = \dot{\Psi} - \dot{\Theta}$$

The total velocity of the mass element, dm with respect to the fixed H and L axis, expressed as components along the rotating $\bar{i}$ - $\bar{j}$ axes is

$$V = \bar{i} \left[\dot{H} \cos \Theta - \dot{L} \sin \Theta \right] + \bar{j} \left[\dot{H} \sin \Theta + \dot{L} \cos \Theta \right]$$

substituting for $\dot{H}$ and $\dot{L}$

$$V = \bar{i} \left[(\dot{Q}_H \cos \Theta - \dot{Q}_L \sin \Theta) + \dot{x} + y\dot{\Theta} \right] + \bar{j} \left[(\dot{Q}_H \sin \Theta + \dot{Q}_L \cos \Theta) + \dot{y} - x\dot{\Theta} \right] \qquad \text{II-2.2}$$

since V_{cg} is the velocity of c.g. at time t

$$\dot{Q}_H = V_{cg} \cos \Omega = V_{cg} \cos (\Theta + \alpha_o)$$
$$\dot{Q}_L = -V_{cg} \sin \Omega = V_{cg} \sin (\Theta + \alpha_o) \qquad \text{II-2.3}$$

Therefore

$$V = \bar{i} \left[V_{cg} \cos \alpha_o + \dot{x} + \dot{\Theta} y \right] + \bar{j} \left[-V_{cg} \sin \alpha_o + \dot{y} - \dot{\Theta} x \right] \qquad \text{II-2.4}$$

II-2.2 KINETIC ENERGY

The kinetic energy of the mass element dm is

$$d\tau = \frac{1}{2} V^2 \, dm + \frac{1}{2} r^2 (\dot{\Psi} - \dot{\Theta})^2 \, dm$$

substituting from equations II-2.2 and using II-1.2

$$\tau = \frac{1}{2}(\dot{Q}_H^2 + \dot{Q}_L^2) \int_{TM} dm + \frac{1}{2} \dot{\Theta}^2 \int_{TM} (x^2+y^2+ r^2)dm + \frac{1}{2} \int_{TM} (\dot{x}^2+\dot{y}^2+r^2\dot{\Psi}^2)dm$$
$$-\dot{\Theta} \int_{TM} (\dot{y}x - \dot{x}y + \dot{\Psi} r^2) \, dm$$

$$\tau = \frac{1}{2} M(\dot{Q}_H^2 + \dot{Q}_L^2) + \frac{1}{2} I\dot{\Theta}^2 + \tau_{xy} - \dot{\Theta} \mathfrak{m}_{xy}$$

$$\tau = \frac{1}{2} M \, V_{cg}^2 + \frac{1}{2} I\dot{\Theta}^2 + \tau_{xy} - \dot{\Theta} \mathfrak{m}_{xy} \qquad \text{II-2.5}$$

where τ_{xy} is the kinetic energy of the system due to motion with respect to the $\bar{i}$ - $\bar{j}$ axes and is only a function of the Q_n's and their derivatives. By equation II-1.8, $\mathfrak{m}_{xy}$ is a constant. The generalized momentum associated with each generalized coordinate is then

$$\frac{\partial \tau}{\partial \dot{Q}_H} = M\dot{Q}_H$$
$$\frac{\partial \tau}{\partial \dot{Q}_L} = M \, \dot{Q}_L \qquad \text{II-2.6}$$
$$\frac{\partial \tau}{\partial \dot{\Theta}} = I\dot{\Theta} - \mathfrak{m}_{xy}$$

$$\frac{\partial\tau}{\partial Q_\eta} = \frac{\partial \tau_{xy}}{\partial Q_n} \qquad \text{II-2.6}$$

The generalized inertia force associated with each generalized coordinate is then

$$\frac{d}{dt}\left(\frac{\partial\tau}{\partial \dot{Q}_H}\right) = M\ddot{Q}_H$$

$$\frac{d}{dt}\left(\frac{\partial\tau}{\partial \dot{Q}_L}\right) = M\ddot{Q}_L$$

$$\frac{d}{dt}\left(\frac{\partial\tau}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(I\dot{\theta}\right) \qquad \text{II-2.7}$$

$$\frac{d}{dt}\left(\frac{\partial\tau}{\partial \dot{Q}_\eta}\right) = \frac{d}{dt}\left(\frac{\partial \tau_{xy}}{\partial \dot{Q}_n}\right)$$

The generalized force associated with each generalized coordinate as a result of non-linearities are

$$\frac{\partial\tau}{\partial Q_H} = 0$$

$$\frac{\partial\tau}{\partial Q_L} = 0$$

$$\frac{\partial\tau}{\partial \theta} = 0 \qquad \text{II-2.8}$$

$$\frac{\partial\tau}{\partial Q_n} = \frac{\partial \tau_{xy}}{\partial Q_n}$$

For a linear system the last equation of II-2.8 is also zero.

II-2.3 POTENTIAL ENERGY AND DISSIPATION FUNCTION

The potential energy of the system will be divided into two types, the potential energy resulting from the positions of the mass elements in the gravity field, and the internal potential energy stored in elastic deformation of the structure.

$$dV_G = Hg \cos\gamma \, dm$$

substituting from II-1.4 and using II-1.2

$$V_G = M\,Q_H\,g\cos\gamma \qquad \text{II-2.9}$$

The potential energy due to gravity is taken as zero at the origin of the inertial axes L-H and the acceleration of gravity is assumed constant for all mass elements.

It is assumed that the internal potential energy V_I is a function of the x's and y's corresponding to the positions of the mass elements with respect to the $\bar{i}$ - $\bar{j}$ axes and is not a function of the generalized coordinates Q_H, Q_L, and Θ.

$$V_I = V_I(Q_n\text{'s}) \qquad \text{II-2.10}$$

The rate at which energy is dissipated in the system is defined as $2F_D$. If it is assumed that all of the energy dissipated results from relative motion of the mass elements with respect to the $\bar{i}$ - $\bar{j}$ axes, the dissipation function, F_D, is then independent of the generalized coordinates Q_H, Q_L and Θ.

$$F_D = F_D(Q_n\text{'s}) \qquad \text{II-2.11}$$

II-2.4 THE LAGRANGE EQUATIONS

The general form of Lagrange's equations of motions are

$$\frac{d}{dt}\left(\frac{\partial\tau}{\partial\dot{Q}}\right) - \frac{\partial\tau}{\partial Q} + \frac{\partial V_G}{\partial Q} + \frac{\partial V_I}{\partial Q} + \frac{\partial F_D}{\partial\dot{Q}} = P_Q \qquad \text{II-2.12}$$

where

$$P_Q = \sum_P \left[P_H \frac{\partial H_P}{\partial Q} + P_L \frac{\partial L_P}{\partial Q}\right]$$

P_H = Component of external force P along H
P_L = Component of external force P along L

The external force P can be defined by

$$P = \bar{i}\,P_x + \bar{j}\,P_y \qquad \text{II-2.13}$$

then

$$P_H = P_x \cos \Theta + P_y \sin \Theta$$

$$P_L = P_y \cos \Theta - P_x \sin \Theta$$

and

$$P_Q = \frac{\Gamma}{P} \left[(P_x \cos \Theta + P_y \sin \Theta) \frac{\partial H_P}{\partial Q} + (P_y \cos (P_y \cos \Theta - P_x \sin\Theta) \frac{\partial L_P}{\partial Q} \right]$$

The point of application of P is H_P and L_P defined by equation II-1.4 as

$$H_P = Q_H + x_P \cos \Theta - y_P \sin \Theta$$

$$L_P = Q_L + y_P \cos \Theta - x_P \sin \Theta$$

then

$$\frac{\partial H_P}{\partial Q_H} = 1$$

$$\frac{\partial L_P}{\partial Q_H} = 0$$

$$\frac{\partial H_P}{\partial Q_L} = 0$$

$$\frac{\partial L_P}{\partial Q_L} = 1 \qquad \text{II-2.14}$$

$$\frac{\partial H_P}{\partial \Theta} = -x_P \sin \Theta + y_P \cos \Theta$$

$$\frac{\partial L_P}{\partial \Theta} = -y_P \sin \Theta - x_P \cos \Theta$$

$$\frac{\partial H_P}{\partial Q_n} = \cos \Theta \frac{\partial x_P}{\partial Q_n} + \sin \Theta \frac{\partial y_P}{\partial Q_n}$$

$$\frac{\partial L_P}{\partial Q_n} = \cos \Theta \frac{\partial y_P}{\partial Q_n} - \sin \Theta \frac{x_P}{\partial Q_n}$$

$$P_{QH} = \cos\Theta \sum_P P_x + \sin\Theta \sum_P P_y$$

$$P_{QL} = \cos\Theta \sum_P P_y - \sin\Theta \sum_P P_x \qquad \text{II-2.15}$$

$$P_\Theta = -\sum_P \left[P_y x_P - P_x y_P\right] = - \text{moment of external forces about c.g.}$$

$$P_{Qn} = \sum_P \left[P_x \frac{\partial x_P}{\partial Q_n} + P_y \frac{\partial y_P}{\partial Q_n}\right]$$

From equations II-2.7, II-2.8, II-2.9, II-2.10, II-2.11 and II-2.15 the Lagrange equations of motion for coordinates Q_H, Q_L, Θ, and Q_n are

$$M\ddot{Q}_H + Mg\cos\gamma = \cos\Theta \sum_P P_x + \sin\Theta \sum_P P_y$$

$$M\ddot{Q}_L = \cos\Theta \sum_P P_y - \sin\Theta \sum_P P_x$$

$$\frac{d}{dt}(I\dot{\Theta}) = -\sum_P \left[P_y x_P - P_x y_P\right]$$

$$\frac{d}{dt}\left(\frac{\partial \tau_{xy}}{\partial \dot{Q}_n}\right) - \frac{\partial \tau_{xy}}{\partial Q_n} + \frac{\partial V_I}{\partial Q_n} + \frac{\partial F_D}{\partial \dot{Q}_n} = \sum_P \left[P_x \frac{\partial x_P}{\partial Q_n} + P_y \frac{\partial y_P}{\partial Q_n}\right]$$

$\ddot{Q}_H$ and $\ddot{Q}_L$ are the accelerations of the total center of gravity in the direction of the nominal flight path and normal to the nominal flight path. From equations II -2.16

$$\ddot{Q}_H = \cos\Theta \sum_P \frac{P_x}{M} + \sin\Theta \sum_P \frac{P_y}{M} - g\cos\gamma \qquad \text{II-2.17}$$

also by differentiating equations II - 2.3

$$\ddot{Q}_H = -V_{cg}(\dot{\Theta} + \dot{\alpha}_o)\sin(\Theta + \alpha_o) + \dot{V}_{cg}\cos(\Theta + \alpha_o)$$

$$\qquad \text{II-2.18}$$

$$\ddot{Q}_L = -V_{cg}(\dot{\Theta} + \dot{\alpha}_o)\cos(\Theta + \alpha_o) - \dot{V}_{cg}\sin(\Theta + \alpha_o)$$

The second equation of II-2.18 is the so called normal force equation and the first will be referred to as the axial force equation. The equations of motion can now be written

$$\ddot{Q}_H = - V_{cg}(\dot{\theta}+\dot{\alpha}_o) \sin(\theta+\alpha_o) + \dot{V}_{cg}\cos(\theta+\alpha_o) = \cos\theta \sum_P \frac{P_x}{M} + \sin\theta \sum_P \frac{P_y}{M} - g\cos\gamma \qquad \text{II-2.19}$$

$$\ddot{Q}_L = - V_{cg}(\dot{\theta}+\dot{\alpha}_o) \cos(\theta+\alpha_o) - \dot{V}_{cg}\sin(\theta+\alpha_o) = \cos\theta \sum_P \frac{P_y}{M} - \sin\theta \sum_P \frac{P_x}{M} \qquad \text{II-2.20}$$

$$\frac{d}{dt}(I\dot{\theta}) = - \sum_P \left[P_y x_p - P_x y_P\right] \qquad \text{II-2.21}$$

$$\frac{d}{dt}\left(\frac{\partial \tau_{xy}}{\partial \dot{Q}_n}\right) - \frac{\partial \tau_{xy}}{\partial Q_n} + \frac{\partial V_I}{\partial Q_n} + \frac{\partial F_D}{\partial \dot{Q}_n} = \sum_P \left[P_x \frac{\partial x_P}{\partial Q_n} + P_y \frac{\partial y_P}{\partial Q_n}\right] \qquad \text{II-2.22}$$

From the fact that the motion of the center of gravity is independent of the motion with respect to the c.g., and the definition of θ, equations II-2.19, II-2.20, and II-2.21 could have been simply derived from Newton's equations. Also equation II-2.22 is Lagrange's equations for the motion of the total mass with respect to the $\bar{i}$ - $\bar{j}$ axes and is independent of Q_H, Q_L and θ.

The acceleration of the center of mass can also be expressed as components along the $\bar{i}$ and $\bar{j}$ axes, a_x and a_y where

$$\begin{aligned} a_x &= \ddot{Q}_H \cos\theta - \ddot{Q}_L \sin\theta \\ a_y &= \ddot{Q}_L \cos\theta + \ddot{Q}_H \sin\theta \end{aligned} \qquad \text{II-2.23}$$

substituting for $\ddot{Q}_H$ and $\ddot{Q}_L$ equations II-2.19 and II-2.20 become

$$a_x = -V_{cg}(\dot{\theta}+\dot{\alpha}_o)\sin\alpha_o + \dot{V}_{cg}\cos\alpha_o = \sum_P \frac{P_x}{M} - \cos\theta\, g\cos\gamma \qquad \text{II-2.24}$$

$$a_y = -V_{cg}(\dot{\theta}+\dot{\alpha}_o)\cos\alpha_o - \dot{V}_{cg}\sin\alpha_o = \sum_P \frac{P_y}{M} - \sin\theta\, g\cos\gamma \qquad \text{II-2.25}$$

It should be noted that a linear system and small angles have not been assumed. The only restrictions placed on the system are that the origin of the $\bar{i}$ - $\bar{j}$ axes be at the total c.g. and η_{xy} = constant.

II-3 MISSILE CONFIGURATION

In Section II-2 general dynamic equations were derived for a system of mass elements having relative motion with respect to a set of rotating body axes, $\bar{i}$ - $\bar{j}$. The system was assumed to have N degrees of freedom described by N generalized coordinates Q_n. In this section the system of mass elements will be defined in terms of a missile configuration and the N generalized coordinates will be established.

The missile body is assumed to consist of a relative slender flexible body incorporating R fluid tanks which may be only partially filled. A number of thrusting engines are assumed to be attached to the missile body such that they may or may not be rotated with respect to the body center line. In addition provisions are provided to make inertia corrections at various points on the missile body. These inertia corrections in effect consist of the addition of a quantity of matter which has no mass but has a mass moment of inertia ΔI.

II-3.1 SLOSH REPRESENTATION

Translational or rotational motion of each partially filled fluid tank will result in relative motion between the fluid and tank, thus producing dynamic forces on the tank. The moving fluid represents an infinite degree of freedom system having an infinite number of modes of motion. It has been shown (References 2, 3, and 4) that equivalent tank loads for each fluid mode will result from a spring mass analogy. The analogy consists of placing a cap or bulkhead at the free fluid surface to confine the fluid and prevent sloshing and attaching a portion of the fluid mass to the structure by a simple spring. Equations for the calculation of the equivalent sloshing mass, natural frequency of the spring mass combination, and the required attach station on the fluid tank are derived in Reference 4 for tanks of arbitrary shape. The equations for tanks with axial symmetry have been programmed for computer computation. A description of this program and instructions for its use are contained in Reference 5. Results obtained from the slosh program show that the slosh masses associated with fluid modes above the first mode are small and in general these modes may be neglected.

For the calculation of the mass moment of inertia of the missile about its center of gravity and the computation of bending modes, it is necessary to consider the moment of inertia of the fluid in the capped tank. The equation for finding the mass moment of inertia about the fluid c.g. for a fluid in a cylindrical tank is derived in R ference 3. These data are presented as a correction factor, κ_I to be applied to the moment of inertia that would result if the fluid were frozen as a solid. This correction factor κ_I has been plotted against tank aspect ratio (length divided by diameter) in figure II-3.1. The frozen or solid inertia of a fluid mass m_f in a cylindrical tank of radius R_T and length L_T is

$$I_{solid} = m_f \left[\frac{R_T^2}{4} + \frac{L_T^2}{12} \right] \qquad \text{II-3.1}$$

The free fluid inertia I_f is then

$$I_f = \kappa_I \, I_{solid} \qquad \text{II-3.2}$$

The mass moment of inertia that results when the fluid is considered as a filament concentrated along the missile center line for a distance ℓ is given by

$$I_{FiL} = m_f \frac{\ell^2}{12} \qquad \text{II-3.3}$$

When the missile inertia or bending modes are based upon a filament of fluid, the correction factor that must be applied at the fluid cg is

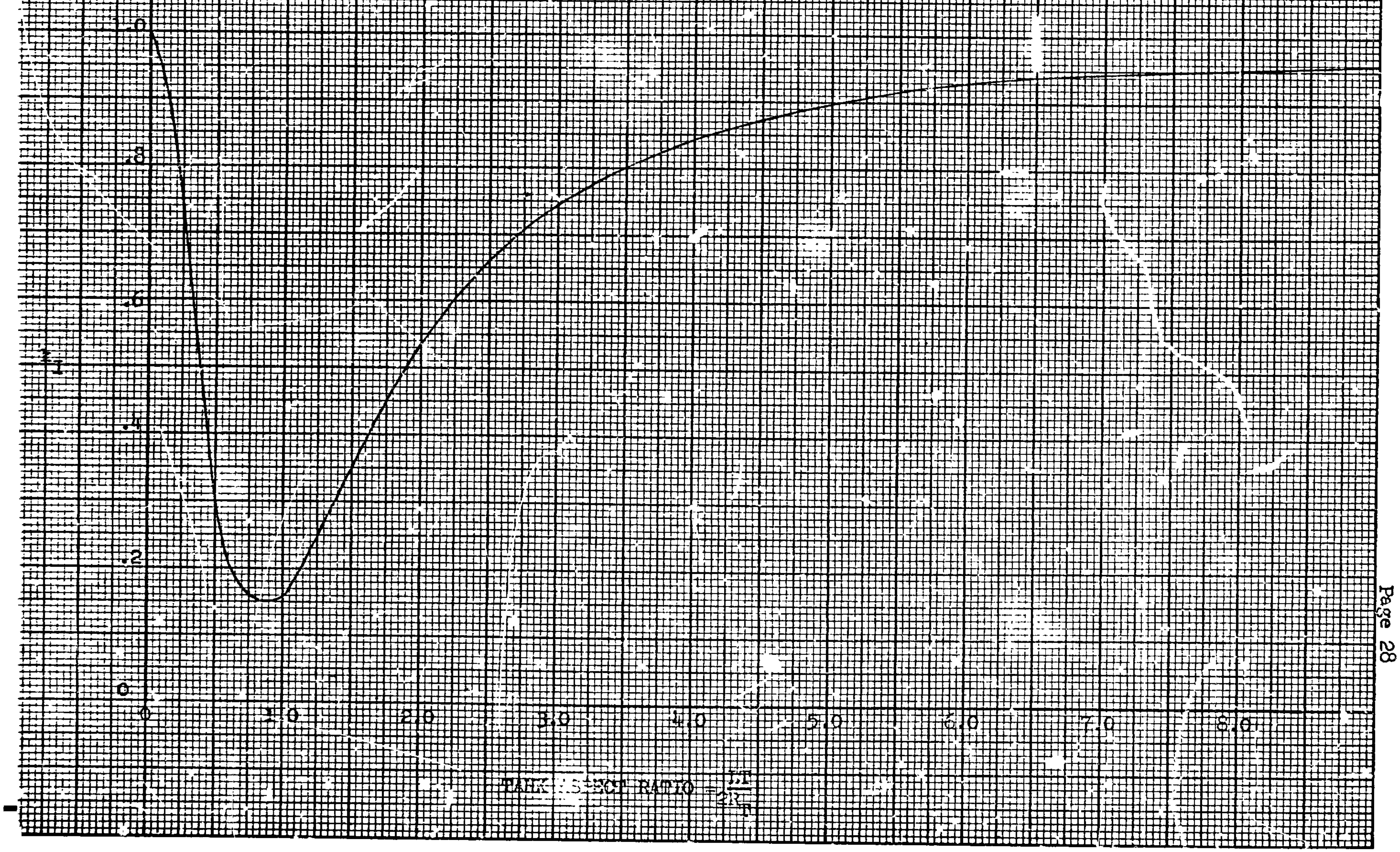
$$\Delta I = I_F - I_{FiL} \qquad \text{II-3.4}$$

where ΔI may be positive or negative.

It should be noted that the slosh mass amplitude ρ_j used in this analysis is the amplitude of motion of the equivalent slosh mass of the spring mass analogy. ρ_j does not represent the slosh amplitude of the fluid in the tank. The fluid amplitude is obtained by applying a correction factor to ρ_j. This correction is obtained from the output data of the sloshing program.

FIGURE II - 3.1

RATIO OF EFFECTIVE FLUID MOMENT OF INERTIA
SOLID MOMENT OF INERTIA AS A FUNCTION OF ASPECT RATIO
CYLINDRICAL TANKS



II - 3.2 BENDING MODES

The free-free flexible missile body moving in the yaw plane represents an infinite degree of freedom system consisting of three zero frequency or rigid body modes and an infinite number of elastic or bending modes. However, in an analysis of this type only a finite number, T, of the modes can be considered and it is assumed that the elastic deformation of the body can be described by a linear combination of the first T bending modes in order of increasing modal frequency. Experience has shown that the system being studied is relatively loosely coupled, that is, the resulting system natural frequencies are near the input bending, slosh, and engine frequencies and the resulting modes consist primarily of the corresponding bending, slosh, and engine motions. Therefore, if the T^{th} bending frequency is well above the highest slosh and engine input frequency, the driving functions for the higher modes above the T^{th} mode will be small and the truncation errors for all but the highest mode, T^{th}, should be negligible and the above assumption is considered justified. The output data for the T^{th} mode should never be used in a stability analysis. The input data should always include at least one mode above the highest mode to be studied.

All elastic deflections are taken with respect to the undeformed elastic axis (UEA). The UEA is defined as the body center line in the absence of elastic deformation and is assumed to be a straight line. The center of mass for every beam element for all beam branches must lie on the UEA in the absence of elastic deformation. Bending mode calculations are made assuming that all mass elements are distributed along the center line with radii of gyration, r. The location of a point on the body is defined as x_h measured forward along the UEA from the total system center of gravity, see figure II-3.2. For the purpose of this analysis it is assumed that the bending modes have been calculated for a multi-branched beam taking into account, shear deflection,

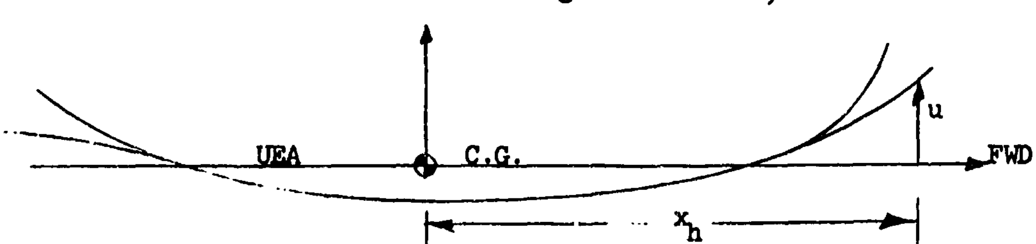


FIGURE II - 3.2
Total Deflection Due to Bending and Shear

rotary inertia, and axial acceleration, and that all engine and slosh masses are rigidly attached to the missile. The translational deflection of a beam element at x_h perpendicular to the UEA is taken as u and the total slope of the beam center line at x_h with respect to the UEA is u'. The total slope u' is the sum of the slope due to bending deformation and the slope due to shear deformation. The bending slope is defined as ψ and is the rotation of the beam element at x_h since shear deformation produces no rotation. The translational deflection in the direction of the UEA is neglected. The deflection u and slopes u' and ψ at any point x_h are defined by the following equations:

$$u = \sum_i \phi_i \, b_i$$

$$u' = \sum_i \phi_i' \, b_i \qquad \text{II-3.6}$$

$$\psi = \sum_i \lambda_i \, b_i$$

where

ϕ_i = deflection of the i^{th} bending mode at x_h

ϕ_i' = total slope of the i^{th} bending mode at x_h

λ_i = bending slope of the i^{th} bending mode at x_h

b_i = i^{th} bending coordinate

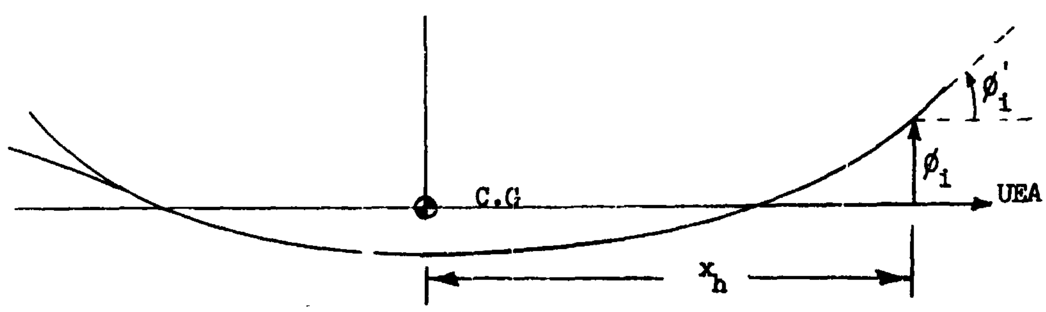


FIGURE II - 3.3

Bending Deflection Due i^{th} Mode

The bending deflection due to the i^{th} mode is shown in figure II-3.3. For multi-branched beams ϕ_i, ϕ_i' and λ_i will have values for each branch present at x_h. The normalized mass for the i^{th} mode is given by

$$\int_{TM} (\phi_i^2 + r^2 \lambda_i^2) dm = M_i \qquad \text{II-3.7}$$

where the integration is the sum of the integrals carried out over each branch of the beam. It is desirable to normalize each mode such that M_i is the total missile mass. Since the mode shapes normalized to mass M_i can be normalized to the mass M by multiplying ϕ_i, λ_i and ϕ_i by $\sqrt{k_i}$ were

$$k_i = \frac{M}{M_i} \qquad \text{II-3.8}$$

It will be assumed that all bending modes are normalized to total missile mass.

For the i^{th} mode shape shown in Figure II-3.3 the values of ϕ_i, ϕ_i' and λ_i are functions of x_h only. ϕ_i is the deflection when the bending coordinate is unity, and is non dimensional. Since ϕ_i' and λ_i are the slopes when the bending coordinate is unity they have the dimension of 1/ft. The bending coordinate b_i is then defined at the value of x_h where ϕ_i is unity for a mode shape normalized to total mass. The i^{th} bending mode represents the maximum deflections at each point along the beam for sinusoidal motion since b_i is a sine function of time t and the modal frequency. Therefore, for the input mode shown in figure II-3.3 the deflection and slopes could be multiplied by -1 and a new input mode slope would be defined which would shift the phase of the final b_i by 180 degrees. Either input mode can be used, however, in order to be consistent a positive input mode is defined as that phase of the mode which results in a positive deflection at the aft end of the missile.

Since the input modes are orthogonal

$$\int_{TM} (\phi_i \phi_t + r^2 \lambda_i \lambda_t) dm = 0 \quad t \neq i \qquad \text{II-3.9}$$

also since the bending modes produce no translation of the c.g. and no rotation of the UEA the linear momentum of the total system and the angular momentum about the c.g. must be zero. Therefore

$$\int_{TM} \phi_i \, dm = 0$$

$$\int_{TM} (\phi_i x_h + r^2 \lambda_i) dm = 0 \qquad \text{II-3.10}$$

and

$$\int_{TM} u dm = 0$$

$$\int_{TM} (u x_h + r^2 \psi) dm = 0$$

As previously noted, this analysis assumes that the bending modes have been calculated for a multi-branched beam taking into account, shear deflection, rotary inertia, and axial acceleration, and that all engines and slosh masses are rigidly attached to the beam. If the effects of the beam column loads due to axial acceleration are to be neglected, no changes in the input data are required. If the effects of shear deformation are neglected in the bending modes, the total slope is made equal to the bending slope. If the inertial energy stored in the beam elements due to rotation are to be neglected, the radius of gyration of each beam element becomes zero in all equations. However, the engine input data and the dynamic equations are based upon a mass moment of inertia about the gimbal point, I_{Ge}, for each gimballed engine and the rotary inertia of these engines cannot be neglected. The moment of inertia of the engine about its gimbal point is

$$I_{Ge} = M_{Ge} \ell_{GE}^2 + \int_e r^2 \, dm.$$

where M_{Ge} = mass of e^{th} engine

ℓ_{Ge} = distance from gimbal to c.g. of e^{th} engine

Therefore, when using bending data which neglects the effects of rotary inertia, it is necessary to introduce an inertia correction ΔI_e at each gimballed engine c.g. such that

$$\Delta I_e = \int_e r^2 \, dm = I_{Ge} - M_{Ge} \ell_{GE}^2$$

It is again noted that the engines and slosh masses are assumed to be rigidly attached to the beam when the bending mode are determined. It is important that the engine mass distribution be such that the static moment of the mass about the engine gimbal point be equal to $M_{Ge} \ell_{Ge}$ used as gimballed engine input data; and in addition each slosh mass must appear in the mass distribution as a concentrated mass at the spring mass attach point. Any other distribution of these masses will introduce errors in the analysis the magnitudes of which are a function of the effect of the distribution on the modal deflections and slopes at the attach points.

II-3.3 ENGINES

Two types of thrusting engines are considered, fixed engines, and gimballed engines. Fixed engines are those engines which are fixed or locked to the missile such that the thrust vector is always tangent to the body center line at the point of attachment. These engines do not produce modes of oscillation but do provide thrust forces which drive other modes and must therefore be treated separately from the gimballed engines each of which does produce an additional mode of oscillation. The e^{th} gimballed engine is considered free to rotate about its gimbal point such that the engine center line makes an angle δ_e with the body center line. It should be noted that a gimballed engine which is locked during some portion of flight time by locking the engine actuator or by making the command signal zero, must be considered gimballed at all times of flight since engine angles will result from the flexibility of the actuating mechanism. Although only fixed and gimballed engines are considered in the derivation of the dynamic equations, a third type of engine is of importance. For secondary injection, jet vane or jetavator thrust vector control systems the thrust vector is deflected through an angle but the engine mass remains fixed to the missile center line. This type of engine is essentially a fixed engine with controlled thrust deflections or a massless gimballed engine engine. The engine does not produce a mode of oscillation but is the controlled driving force for all other modes.

No difficulties are encountered when a gimballed engine mass and inertia of zero are introduced into the equations derived in this section. However, when using the combined mode representation covered in section III, a massless engine produces a mass matrix which is not positive and definite and which will be rejected by the eigenvalue subroutine used. A program subroutine provided to handle this case is discussed in Volume II.

II-3.4 MASS MOMENT OF INERTIA

In the introduction to this section it was noted that provisions are included to make inertia corrections at various points along the missile. These corrections are introduced primarily to compensate for the fluid inertia in the propellant tanks as discussed in section II-3.1. Equation II-3.5 gives the inertia correction that must be applied at the fluid center of gravity when the bending modes are based upon the fluid mass distributed along the missile center line. For the general type of bending data normally used, rotating inertia is taken in account and the fluid inertia correction can best be introduced in the inertia distribution, in which case no inertia corrections are required in the dynamic equations. In the case of bending data neglecting rotating inertia it is desirable to introduce the fluid inertia correction from Equation II-3.5. In addition if the inertia distribution for the non-fluid portion of the structure is known the effect of rotating inertia can be introduced. In any case where rotating inertia is neglected it is necessary to introduce the rotary inertia of all gimballed engines since the derivation of the dynamic equation assumes this inertia to be present. The inertia corrections are not system parameters which affect the dynamics of the system but should be considered correction factors introduced to correct the bending data. The mass moment of inertia of the system is

$$I = \int_{TM} (x^2 + y^2 + r^2)dm \qquad \text{II-3.11}$$

expressed in terms of the coordinates along the $\bar{i} - \bar{j}$ axes. In the previous derivation (reference 1) the inertia corrections were added to the value of I introduced into the equations. However, for the present derivation it is assumed that the value of I inputed has already been corrected for fluid inertia and the self inertia of the beam elements.

II-3.5 NOTATION

Input data pertaining to the engines, slosh masses, bending modes, and inertia corrections are designated by the following notation:

	GIMBALLED ENGINES	SLOSHING MASSES	BENDING MODES	FIXED ENGINES	INERTIA CORRECTIONS
Numbering	1 thru P	1 thru R	1 thru T	1 thru F	1 thru K
General Subscript	e	j	i	f	k
Specific Subscript	p	r	t	-	-
Attach point	x_{he}	x_{hj}	$(x_{hi}=0)$	x_{hf}	x_{hk}
Thrust	T_{Ge}	-	--	T_{Ff}	-
Deflection	δ_e	ρ_j	b_i	-	-

The subscript h used for the attach points indicates that the coordinate is measured along the UEA and not the i axis. The specific subscripts p, r, t have been introduced since general expressions contain summations of e, j, and i and must be differentiated with respect to specific engines, slosh masses and bending modes. For example, the general expression for kinetic energy contains the term $\sum_{e=1}^{P} \frac{1}{2} m_e \dot{\delta}_e^2$. Therefore the general expression for the p^{th} engine equations will contain the term

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\delta}_p} \sum_{e=1}^{P} \frac{1}{2} m_e \dot{\delta}_e^2\right) = m_p \ddot{\delta}_p \neq m_e \ddot{\delta}_e$$

The subscripts p, r, and t will appear in the p^{th} engine equation, the r^{th} sloshing equation, and the t^{th} bending equation.

The motion of the mass element dm with respect to the $\bar{i} - \bar{j}$ axes is the sum of the rigid body motion due to motion of the UEA, and the elastic deflection, due to motion with respect to the UEA. The rigid body motion of the UEA is a function of the engine angles δ_e and the sloshing deflections ρ_j and also the bending coordinate b_i when inertia corrections are applied to the bending modes. The bending deflections are only a function of b_i. Therefore, the motion of each element of mass can be defined in terms of the P values of δ_e, the R values of ρ_j and the T values of b_i. The total number of generalized coordinates N is then

$$N = P + R + T \qquad \text{II-3.12}$$

These N generalized coordinates, Q_n, are then defined as follows: the first P coordinates are the engine deflections, δ_e in radians, for the P gimballed engines in the order of input, the next R coordinates are the sloshing mass deflections ρ_j in feet, for the R slosh masses in the order of input: the last T coordinates are the input bending mode coordinates, b_i, in feet for the T input bending modes in the order of input.

$$\left\{Q_n\right\} = \underset{\text{Column matrix}}{N \times 1} = \left\{\begin{matrix}\left\{\delta_e\right\} \\ \\ \left\{\rho_j\right\} \\ \\ \left\{b_i\right\}\end{matrix}\right\} = \left\{\begin{matrix}\left\{\begin{matrix}Q_1 \\ \vdots \\ \vdots \\ Q_P\end{matrix}\right\} = \left\{\begin{matrix}\delta_1 \\ \vdots \\ \delta_e \\ \vdots \\ \delta_P\end{matrix}\right\} \\ \left\{\begin{matrix}Q_{P+1} \\ \vdots \\ \vdots \\ Q_{P+R}\end{matrix}\right\} = \left\{\begin{matrix}\rho_1 \\ \vdots \\ \rho_j \\ \vdots \\ \rho_R\end{matrix}\right\} \\ \left\{\begin{matrix}Q_{P+R+1} \\ \vdots \\ \vdots \\ Q_N\end{matrix}\right\} = \left\{\begin{matrix}b_1 \\ \vdots \\ b_i \\ \vdots \\ b_T\end{matrix}\right\}\end{matrix}\right\} \qquad \text{II - 3.13}$$

In this analysis the braces $\{\ \}$ will be used as a symbol denoting a column matrix and the subscript T will be used to denote a transposed matrix, therefore $\{\ \}_T$ is a row matrix composed of the elements of the $\{\ \}$ column matrix.

The resulting motion with respect to the $\bar{i} - \bar{j}$ axes when b_i is unity and all other generalized coordinates are zero has become known as the i^{th} input bending mode. Likewise the resulting motion when δ_e is unity and all other coordinates are zero is defined as the e^{th} input engine mode, and the resultant motion when ρ_j is unity and all other coordinates are zero is defined as the j^{th} input slosh mode. These so called input modes or non-orthogonal modes are not true modes of the system. However, for a linear system superposition is valid and the output modes are a linear combination of these modes. These input modes are numbered from 1 thru N in the same order as their corresponding generalized coordinate. The motion of the missile with respect to the $\bar{I}$ - $\bar{J}$ axes can then be defined by N dynamic equations in N generalized coordinates. This motion is a linear combination of N so called input modes. The subscripts m and n will be used to designate the m^{th} and n^{th} of these coordinates, equations and modes. Where possible n will be used for the coordinates and m for these equations and input modes.

It will be noted that with the definition of $x_{hi} = 0$, there is an attach point associated with each input mode and each generalized coordinate. Therefore, an attach point can be defined for each input mode or coordinate

$$\left\{x_m\right\} = \begin{array}{c} N \times 1 \\ \text{Column matrix} \end{array} = \left\{\begin{array}{ccc} \left\{\begin{array}{c} x_1 \\ \vdots \\ x_P \end{array}\right\} & = & \left\{x_{ne}\right\} \\ \left\{\begin{array}{c} x_{P+1} \\ \vdots \\ x_{P+R} \end{array}\right\} & = & \left\{x_{hj}\right\} \\ \left\{\begin{array}{c} x_{P+R+1} \\ \vdots \\ x_N \end{array}\right\} & = & \left\{x_{hi} = 0\right\} \end{array}\right\} \qquad \text{II-3.14}$$

Likewise for each fixed engine and each inertia correction

$$\left\{x_{Ff}\right\} = \begin{array}{c} F \times 1 \\ \text{Column matrix} \end{array} = \left\{x_{nf}\right\} \qquad \text{II-3.15}$$

$$\left\{x_{Ik}\right\} = \begin{array}{c} K \times 1 \\ \text{Column matrix} \end{array} = \left\{x_k\right\} \qquad \text{II-3.16}$$

The only input bending data required in the dynamic equations of the system are the deflection and bending slope at each of the attach points. Therefore, the required bending data can be defined by a system of subscripts where the first subscript denotes the attach station and the second subscript the mode or coordinate causing the deflection. For example

λ_{ei} = bending slope at e^{th} engine attach point due to i^{th} bending mode.

λ_{ji} = bending slope at j^{th} slosh attach point due to i^{th} bending mode.

λ_{Ffi} = bending slope at f^{th} fixed engine due to i^{th} bending mode.

λ_{Iki} = bending slope at k^{th} inertia correction due to i^{th} bending mode.

The same subscripts are used in connection with the deflection ϕ. However, in each case the slopes and deflections must be taken for the correct beam branch at each attach point. By further defining $\lambda_{ee}, \lambda_{ej}, \lambda_{je}, \lambda_{jj}, \lambda_{ii}, \lambda_{ie}, \lambda_{ij}, \lambda_{Ffe}, \lambda_{Ike}, \lambda_{Ikj} \lambda_{Ffj}$, and the corresponding deflections all equal to zero, general symbols for the bending slopes and deflections can be defined as

λ_{mn} = bending slope at m^{th} attach point due to n^{th} mode

λ_{nm} = bending slope at n^{th} attach point due to m^{th} mode

ϕ_{mn} = deflection at m^{th} attach point due to n^{th} mode

ϕ_{nm} = deflection at n^{th} attach point due to m^{th} mode

λ_{Ffn} = bending slope at f^{th} fixed engine attach point due to n^{th} mode

ϕ_{Ffn} = deflection at f^{th} fixed engine attach point due to n^{th} mode

λ_{Ikn} = bending slope at k^{th} inertia correction due to n^{th} mode

ϕ_{Ikn} = deflections at k^{th} inertia correction due to n^{th} mode

II-4 MOTION WITH RESPECT TO $\bar{i}$ - $\bar{j}$ AXES

In Section II-2 the dynamic equations were derived in terms of the x, y and Ψ coordinates of each element of mass with respect to the $\bar{i}$ - $\bar{j}$ axes. In section II-3 a missile configuration was developed for which the motion with respect to the $\bar{i}$-$\bar{j}$ axes can be described by N generalized coordinates Q_n. The motion of the missile with respect to the $\bar{i}$ - $\bar{j}$ axes will be developed in this section. The development is based upon first order linearization.

II-4.1 DISPLACEMENT

The displacement of the missile configuration with respect to the $\bar{i}$ - $\bar{j}$ axes is shown in Figure II-4.1.

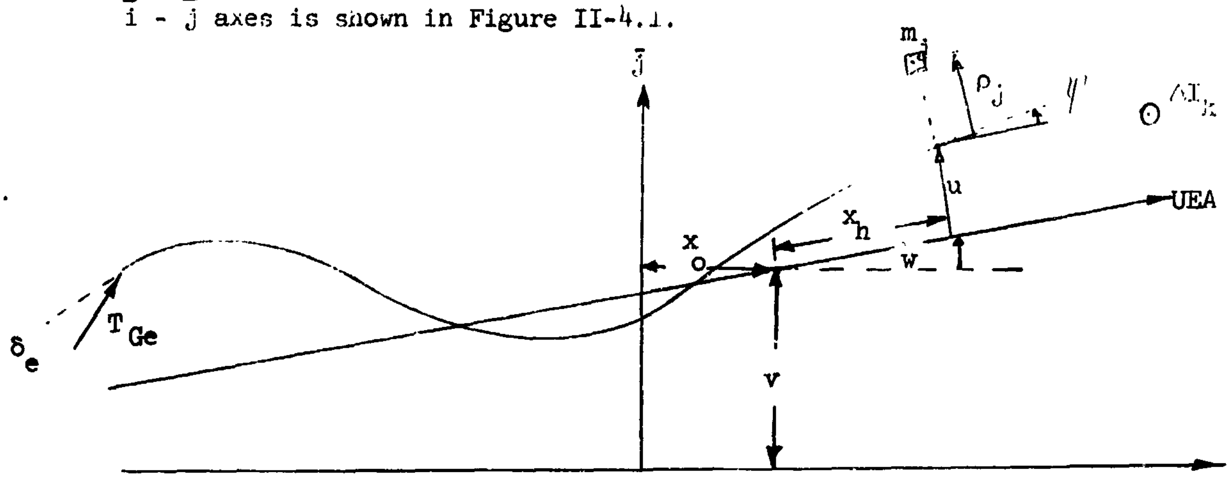


FIGURE II-4.1

The x and y displacements of the origin on the UEA are defined as x_o and v and the rotation of the UEA as w. The elastic displacements with respect to the UEA are u and ψ defined by equations II-3.6. The gimballed engine displacements δ_j are taken for positive rotation with respect to the + x axis. Sloshing displacements are defined positive upward, which is opposite to the definition used in reference 1. This change was made in order that a slosh motion in phase with rigid body motion would be in the same direction as the rigid body motion.

Coordinates of a Point on Beam

$$x_b = x_o + x_h \cos w - u \sin w$$

$$y_o = v + x_h \sin w + u \cos w \qquad \text{II-4.1}$$

$$\bar{\Psi}_b = w + \psi$$

Coordinates of the j^{th} Slosh Mass Attach Point

$$x_j = x_o + x_{hj} \cos w - u_j \sin w - \rho_j \sin(\psi_j + w) = x_{bj} - \rho_j \sin(\psi_j + w) \qquad \text{II-4.2}$$

$$y_j = v + x_{hj} \sin w + u_j \cos w + \rho_j \cos(\psi_j + w) = y_{bj} + \rho_j \cos(\psi_j + w)$$

$$\bar{\Psi}_j = w + \psi_j$$

Coordinates of the kth Inertia Correction Attach Point

$$x_k = x_o + x_{hk} \cos w - u_k \sin w = x_{bk}$$

$$y_k = v + x_{hk} \sin w + u_k \cos w = y_{bk} \qquad \text{II-4.3}$$

$$\Psi_k = w + \psi_{Ik}$$

Coordinates of a Point on eth Engine

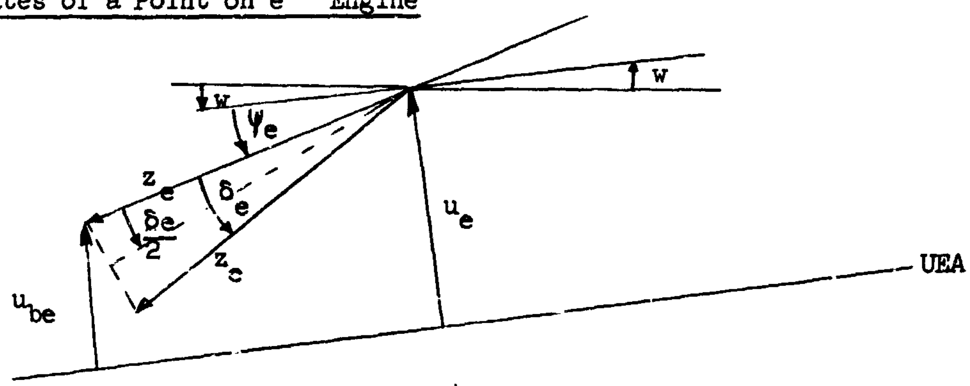


FIGURE II-4.2

$$x_E = x_{bE} + 2z_e \sin \frac{\delta_e}{2} \sin \left(w + \psi_e + \frac{\delta_e}{2}\right)$$

$$y_E = y_{bE} + 2z_e \sin \frac{\delta_e}{2} \cos \left(w + \psi_e + \frac{\delta_e}{2}\right) \qquad \text{II-4.4}$$

$$\Psi_E = w + \psi_e + \delta_e$$

ψ_e is the bending rotation of the beam at the gimbal of the eth engine. The angle required is the rotation of the rigid engine as a result of elastic deformation of the beam. The use of ψ_e for this angle assumes that the engine rotation is due to bending deflections only. If the engine actuator supporting structure is such that engine rotation results from shear deflection in addition to bending deflection, a small error will be introduced which may be corrected by modification of the input bending data.

The value of x_o may be determined from the fact that the origin of $\bar{i}$ - $\bar{j}$ axes remains at the center of gravity.

Therefore

$$\int_{TM} x\, dm = 0$$

$$\int_b x_b\, dm + \sum_j \int_j x_j\, dm + \sum_e \int_e x_E\, dm = 0$$

substituting from II-4.1, II-4.2 and II-4.3

and $$\int_e z_e\, dm = m_{Ge} \ell_{Ge}$$

m_{Ge} = mass of e^{th} engine

ℓ_{Ge} = distance from gimbal to c.g. of e^{th} engine

$$x_o = \frac{1}{M}\sum_j \left[m_j \rho_j \sin(\psi_j + w)\right] - \frac{1}{M} \sum_e 2m_{Ge} \ell_{Ge} \sin \frac{\delta_e}{2} \sin(w + \psi_e \frac{\delta_e}{2})$$

The displacement of the origin of the UEA in the direction of the $\bar{i}$ axes is the sum of second order terms and is therefore neglected in linearization of the system and

$$x_o = 0 \qquad \text{II-4.5}$$

The value of v may also be determined from the fact that the origin of $\bar{i} - \bar{j}$ axes remains at the center of gravity.

$$\int_{TM} y\, dm = 0$$

$$\int_b y_b\, dm + \sum_j \int_j y_j\, dm + \sum_e \int_e y_E\, dm = 0$$

again substituting for y_b, y_j and y_E

$$v = -\sum_j \frac{m_j}{M} \rho_j \cos(\psi_j + w) + \sum_e 2 \frac{m_{Ge} \ell_{Ge}}{M} \sin \frac{\delta_e}{2} \cos(w + \psi_e + \frac{\delta_e}{2})$$

assuming small angles

$$v = - \sum_j \frac{m_j}{M} \rho_j + \sum_e \frac{m_{Ge} \ell_{Ge}}{M} \delta_e \qquad \text{II-4.6}$$

Let $v_j = - \frac{m_j}{M}$

$$v_e = \frac{m_{Ge} \ell_{Ge}}{M}$$

$$v_i = 0$$

then $v = \sum_e v_e \delta_e + \sum_j V_j \rho_j + \sum_i v_i b_i$ II-4.7

$$v = \{v_e\}_T \{\delta_e\} + \{v_j\}_T \{\rho_j\} + \{v_i\}_T \{b_i\}$$

$$v = \begin{Bmatrix} \{v_e\} \\ \{v_j\} \\ \{v_i\} \end{Bmatrix}_T \begin{Bmatrix} \{\delta_e\} \\ \{\rho_j\} \\ \{b_i\} \end{Bmatrix} = \{v_m\}_T \{Q_n\} \qquad \text{II-4.8}$$

where

$$\{v_m\} = \begin{matrix} N \times 1 \\ \text{Column matrix} \end{matrix} = \begin{Bmatrix} \{v_e\} \\ \{v_j\} \\ \{v_i\} \end{Bmatrix}$$

where v_e, v_j, and v_i are constants for each input mode or coordinate. For the linear system the coordinates given by equations II-4.1 thru II-4.4 become

Coordinates of a Point on Beam

$$x_b = x_h$$

$$y_b = v + x_h w + u \qquad \text{II-4.9}$$

$$\Psi_b = w + \psi$$

Coordinates of the j^{th} Slosh Mass Attach Point

$$x_j = x_{hj}$$

$$y_j = v + x_{hj}w + u_j + \rho_j = x_{bj} + \rho_j \qquad \text{II-4.10}$$

$$\Psi_j = w + \psi_j$$

Coordinates of the k^{th} Inertia Corrections Attach Points

$$x_k = x_{hk} = x_{bk}$$

$$y_k = v + x_{hk}w + u_{Ik} = y_{bk} \qquad \text{II-4.11}$$

$$\Psi_k = w + \psi_{Ik} = \Psi_{bk}$$

Coordinates of a point on the e^{th} Engine

$$x_E = x_{he} - z_e$$

$$y_E = v + x_E w + u_E - z_e\delta_e = y_{be} - z_e\delta_e \qquad \text{II-4.12}$$

$$\Psi_e = w + \psi_e + \delta_e$$

Coordinate of Attach Point of f^{th} Fixed Engine

$$x_{Ff} = x_{hf}$$

$$y_{Ff} = v + x_{Ff}w + u_{Ff} = y_{bf} \qquad \text{II-4.13}$$

$$\Psi_{Ff} = w + \psi_{Ff}$$

Equations II-4.9 thru II-4.13 show that x coordinate along the $\bar{i}$ axis is equal to the x_h coordinate along the UEA for all input parameters. Therefore the column matrix defined by equation II-3.14 also represents the x coordinate along $\bar{i}$ axis for each attach point. Also since x_h at any point is a constant and not a function of time, all of the mass elements of the missile have no motion in the direction of the $\bar{i}$ axis and all motion is in the direction of the $\bar{j}$ axis. The value of w can be found from the fact that the angular momentum about the c.g. due to motion with respect to $\bar{i}$ - $\bar{j}$ axes is zero.

$\mathcal{M}xy = 0$ for linear system

$$\mathcal{M}xy = \int_{TM} (\dot{y}x - \dot{x}y)\, dm + \int_{TM} r^2(\dot{w} + \dot{\psi})\, dm + \sum_e \int_e r^2\, \dot{\delta}\, dm + \sum_k \Delta I_k(\dot{\psi}_k + \dot{w}) \quad \text{(II-4.14)}$$

Substituting for x, y, $\dot{x}$ and $\dot{y}$

$$\mathcal{M}xy = \dot{w}\left[\int_{TM} (x^2 + r^2)dm + \sum_k \Delta I_k\right] + \int_{TM} (\dot{u}x + r^2\dot{\psi})\, dm + \sum_j m_j\, \dot{\rho}_j x_j$$

$$+ \sum_e \dot{\delta}_e \left[\int_e (z_e^2 + r^2)dm - x_e \ell_{Ge} m_{Ge}\right] + \sum_k \Delta I_k \dot{\psi}_k$$

since $\int_{TM} (x^2 + r^2)\, dm + \sum_k \Delta I_k = I$

$\int_{TM} (\dot{u}x + r^2\dot{\psi})dm = 0$

$\int_e (z_e^2 + r^2)dm = I_{Ge}$ = mass moment of inertia of e^{th} engine about gimbal

$$\mathcal{M}\, xy = \dot{w}I + \sum_j x_j\, m_j \dot{\rho}_j + \sum_e \left[I_{Ge} - x_e \ell_{Ge} m_{Ge}\right] \dot{\delta}_e + \sum_k \Delta I_k \dot{\psi}_k$$

since $\mathcal{M}_{xy} = 0$ for a linear system

$$\dot{w} = -\sum_e \frac{I_{Ge} - x_e \ell_{Ge} m_{Ge}}{I}\dot{\delta}_e - \sum_j \frac{m_j x_j}{I}\dot{\rho}_j - \sum_k \frac{\Delta I_k}{I}\dot{\psi}_k$$

$$\dot{\psi}_k = \sum_i \lambda_{Iki}\, \dot{b}_i$$

$$\dot{w} = \sum_e \frac{I_{Ge} - x_e \ell_{Ge} m_{Ge}}{I}\dot{\delta}_e - \sum_j \frac{m_j x_j}{I}\dot{\rho}_j - \sum_i\left[\sum_k \frac{\Delta I_k}{I}\lambda_{Iki}\right]\dot{b}_i \qquad \text{II-4.15}$$

Define $w_e = -\left[\frac{I_{Ge} - x_e \ell_{Ge} m_{Ge}}{I}\right]$

$$w_j = -\frac{m_j x_j}{I} \qquad \text{II-4.16}$$

$$w_i = -\sum_k \frac{\Delta I_k}{I}\lambda_{Iki}$$

then

$$\dot{w} = \sum_e w_e \, \dot{\delta}_e + \sum_j w_j \, \dot{\rho}_j + \sum_i w_i \, \dot{b}_i$$

defining $w = o$ for $\delta_e = \rho_j = b_i = 0$

$$w = \sum_e w_e \, \delta_e + \sum_j w_j \, \rho_j + \sum_i w_i b_i \qquad \text{II-4.17}$$

$$w = \{w_e\}_T \{\delta_e\} + \{w_j\}_T \{\rho_j\} + \{w_i\}_T \{b_i\}$$

$$w = \begin{Bmatrix} \{w_e\} \\ \{w_j\} \\ \{w_i\} \end{Bmatrix}_T \begin{Bmatrix} \{\delta_e\} \\ \{\rho_j\} \\ \{b_i\} \end{Bmatrix} = \{w_m\}_T \{Q_n\} \qquad \text{II-4.18}$$

where

$$\{w_m\} = \underset{\text{column matrix}}{N \times 1} = \begin{Bmatrix} \{w_e\} \\ \{w_j\} \\ \{w_i\} \end{Bmatrix}$$

where w_m is a constant

II-4.2 <u>KINETIC ENERGY WITH RESPECT TO $\bar{i} - \bar{j}$ AXES</u>

$$\tau_{xy} = \tau_{xy}\,(\text{translation}) + \tau_{xy}\,(\text{rotation})$$

$$\tau_{xy}\,(\text{trans}) = \frac{1}{2}\int_b (\dot{x}_b^2 + \dot{y}_b^2)\,dm + \frac{1}{2}\sum_j \int_j (\dot{x}_j^2 + \dot{y}_j^2)\,dm + \frac{1}{2}\sum_e \int_e (\dot{x}_E^2 + \dot{y}_E^2)\,dm$$

$$\tau_{xy}\,(\text{rot}) = \frac{1}{2}\int_b r^2(\dot{\psi} + \dot{w})^2 dm + \frac{1}{2}\sum_j \int_j r^2(\dot{\psi}_j + \dot{w})^2 dm + \frac{1}{2}\sum_e \int_e r^2(\dot{\psi}_e + \dot{w} + \dot{\delta})^2 dm + \frac{1}{2}\sum_k \Delta I_k(\dot{\psi}_k + \dot{w})^2$$

$$\dot{x}_b = 0$$

$$\dot{y}_b = \dot{v} + x\dot{w} + \dot{u}$$

$$\dot{x}_j = 0$$

$$\dot{y}_j = \dot{y}_{bj} + \dot{\rho}_j$$

$$\dot{x}_E = 0$$

$$\dot{y}_E = \dot{y}_{bE} - z_e\dot{\delta}_e$$

$$x_E = x_e - z_e$$

$$u_E = u_e - \psi_e z_e$$

Substituting

$$2\tau_{xy}\ (\text{trans.}) = \dot{v}^2 M + \dot{w}^2 \int_{TM} x^2\, dm + \int_{TM} \dot{u}^2 dm + 2\dot{w}\int_{TM} x\dot{u}\ dm$$

$$+ \sum_j m_j \left(\dot{\rho}_j^2 + 2\dot{\rho}_j \left[\dot{v} + x_j\dot{w} + \dot{u}_j\right]\right)$$

$$+ \sum_e \Big[\dot{\delta}_e^2 \int_e z_e^2\, dm - 2\dot{v}\,\dot{\delta}_e\, m_{Ge}\ell_{Ge} - 2\dot{\delta}_e\dot{w}x_e\ell_{Ge}m_{Ge} + 2\dot{\delta}\dot{w}\int_e z_e^2\ dm$$

$$- 2\dot{\delta}_e\,\dot{u}_e\, m_{Ge}\ell_{Ge} + 2\dot{\delta}_e\dot{\psi}_e \int_e z_e^2\, dm\Big]$$

$$2\tau_{xy}\ (\text{rot}) = \int_{TM} r^2\dot{\psi}^2 dm + 2\dot{w}\int_{TM} r^2\dot{\psi} dm + \dot{w}^2\int_{TM} r^2 dm + \sum_e\int_e r^2\left[2\dot{\delta}\dot{\psi}_e + 2\dot{\delta}\dot{w} + \dot{\delta}^2\right] dm$$

$$+ \sum_k \Delta I_k\left[\dot{\psi}_k^2 + 2\dot{\psi}_k\,\dot{w} + \dot{w}^2\right]$$

adding

$$\tau_{xy} = \frac{1}{2} M\dot{v}^2 + \frac{1}{2} I\dot{w}^2 + \frac{1}{2}\sum_i M\dot{b}_i^2 + \frac{1}{2}\,{}_j m_j \dot{\rho}_j^2 + \sum_j m_j\dot{\rho}_j\left[\dot{v} + x_j\dot{w} + \dot{u}_j\right]$$

$$+ \frac{1}{2}\sum_e I_{Ge}\,\dot{\delta}_e^2 + \sum_e \dot{\delta}_e\left[\dot{w}(I_{Ge} - x_e\ell_{Ge}m_{Ge}) + \dot{\psi}_e\, I_{Ge} - m_{Ge}\ell_{Ge}\,\dot{u}_e - m_{Ge}\ell_{Ge}\dot{v}\right]$$

$$+ \sum_k \Delta I_k\left[\frac{1}{2}\dot{\psi}_k^2 + \dot{\psi}_k\dot{w}\right]$$

since $\frac{m_j}{M} = v_j$

$$x_j = \frac{I}{M}\ \frac{w}{v_j}$$

$$I_{Ge} - x_e\ell_{Ge}m_{Ge} = -Iw_e$$

$$\ell_{Ge}m_{Ge} = Mv_e$$

$$\frac{\tau_{xy}}{M} = \frac{1}{2}\dot{v}^2 + \frac{1}{2}\frac{I}{M}\dot{w}^2 + \frac{1}{2}\sum_i \dot{b}_i^2 - \frac{1}{2}\sum_j v_j \dot{\rho}_j^2 + \frac{1}{2}\sum_e\left(\frac{I_{Ge}}{M}\right)\dot{\delta}_e^2 + \frac{1}{2}\sum_k \frac{\Delta I_k}{M}\dot{\psi}_{Ik}^2$$

$$- \sum_e \dot{\delta}_e\left[\frac{I}{M} w_e\dot{w} + v_e\dot{v}\right] - \sum_j \dot{\rho}_j\left[\frac{I}{M} w_j\dot{w} + v_j\dot{v}\right] - \sum_i \dot{b}_i\left[\frac{I}{M} w_i\dot{w} + v_i\dot{v}\right]$$

$$+ \sum_e \dot{\delta}_e\left[\frac{I_{Ge}}{M}\dot{\psi}_e - v_e\dot{u}_e\right] - \sum_j \dot{\rho}_j\left[v_j\dot{u}_j\right] \qquad \text{II-4.19}$$

From equation II-2.22, dynamic equation associated with each of the generalized coordinates Q_n is the Lagrange Equation for motion with respect to the $\bar{1} - \bar{1}$ axes

$$\frac{d}{dt}\left(\frac{\partial \tau_{xy}}{\partial \dot{Q}_m}\right) - \frac{\partial \tau_{xy}}{\partial Q_m} + \frac{\partial V_I}{\partial Q_m} + \frac{\partial F_D}{\partial \dot{Q}_m} = P_m$$

The second term is zero for a linear system therefore the Lagrange equation divided by the total system mass becomes

$$\frac{1}{M}\frac{d}{dt}\left(\frac{\partial \tau_{xy}}{\partial \dot{Q}_m}\right) + \frac{1}{M}\frac{\partial V_I}{\partial Q_m} + \frac{1}{M}\frac{\partial F_D}{\partial \dot{Q}_m} = \frac{P_m}{M} \qquad \text{II-4.20}$$

Equation II-4.20 represents a series of N dynamic equations one for each Q_m. In matrix form II-4.20 becomes

$$\left\{\frac{1}{M}\frac{d}{dt}\frac{\partial \tau_{xy}}{\partial \dot{Q}_m}\right\} + \left\{\frac{1}{M}\frac{\partial V_I}{\partial Q_m}\right\} + \left\{\frac{1}{M}\frac{\partial F_D}{\partial \dot{Q}_m}\right\} = \left\{\frac{P_m}{M}\right\} \qquad \text{II-4.21}$$

The first column matrix represents the inertia forces associated with the Q_m coordinates. Where Q_m equals δ_p, the inertia force associated with the p^{th} engine is

$$\frac{1}{M}\frac{d}{dt}\left(\frac{\partial \tau_{xy}}{\partial \dot{\delta}_p}\right) = \frac{I_{Gp}}{M}\ddot{\delta}_p - \sum_e \ddot{\delta}_e\left[\frac{I}{M} w_e w_p + v_e v_p\right] - \sum_j \ddot{\rho}_j\left[\frac{I}{M} w_j w_p + v_j v_p\right]$$

$$- \sum_i \ddot{b}_i\left[\frac{I}{M} w_i w_p + v_i v_p\right] + \sum_i \ddot{b}_i\left[\frac{I_{Gp}}{M}\lambda_{pi} - v_p \phi_{pi}\right] \qquad \text{II-4.22}$$

The inertia force associated with the r^{th} sloshing coordinate ρ_r is

$$\frac{1}{M}\frac{d}{dt}\left(\frac{\partial^{\tau}xy}{\partial\dot{\zeta}_r}\right) = -v_r\ddot{\rho}_r - \sum_e \ddot{\delta}_e\left[\frac{I}{M}w_e w_r + v_e v_r\right] - \sum_j \ddot{\rho}_j\left[\frac{I}{M}w_j w_r + v_j v_r\right]$$

$$- \sum_i \ddot{b}_i\left[\frac{I}{M}w_i w_r + v_i v_r\right] - \sum_i \ddot{b}_i\left[v_r\,\phi_{ri}\right] \qquad \text{II-4.23}$$

The inertia force associated with the t^{th} bending coordinate b_t is

$$\frac{1}{M}\frac{d}{dt}\left(\frac{\partial^{\tau}xy}{\partial\dot{b}_t}\right) = \ddot{b}_t - \sum_e \ddot{\delta}_e\left[\frac{I}{M}w_e w_t + v_e v_t\right] - \sum_j \ddot{\rho}_j\left[\frac{I}{M}w_j w_t + v_j v_t\right]$$

$$- \ddot{b}_i\left[\frac{I}{M}w_i w_t + v_i v_t\right] + \sum_e \ddot{\delta}_e\left[\frac{I_{Ge}}{M}\lambda_{et} - v_e\,\phi_{et}\right]$$

$$- \sum_j \ddot{\rho}_j\,v_j\,\phi_{jt} + \sum_i \ddot{b}_i\left(\sum_k \frac{\Delta I_k}{M}\lambda_{Iki}\,\lambda_{Ikt}\right) \qquad \text{II-4.24}$$

Equations II-4.22, II-4.23, and II-4.24 can be written as

$$\frac{1}{M}\frac{d}{dt}\left(\frac{\partial^{\tau}xy}{\partial\dot{\delta}_p}\right) = \left\{B_{pn}\right\}_T\left\{\ddot{Q}_n\right\} = \frac{I_{Gp}}{M}\ddot{\delta}_p - \left\{\frac{I}{M}w_p w_n + v_p v_n\right\}_T\left\{\ddot{Q}_n\right\} + \left\{\frac{I_{Gp}}{M}\lambda_{pn} - v_p\phi_{pm}\right\}_T\left\{\ddot{Q}_n\right\}$$

$$\frac{1}{M}\frac{d}{dt}\left(\frac{\partial^{\tau}xy}{\partial\dot{\zeta}_r}\right) = \left\{B_{rn}\right\}_T\left\{\ddot{Q}_n\right\} = -v_r\ddot{\rho}_r - \left\{\frac{I}{M}w_r + w_n + v_r v_n\right\}_T\left\{\ddot{Q}_n\right\} - v_r\left\{\phi_{rn}\right\}_T\left\{\ddot{Q}_n\right\}$$

$$\frac{1}{M}\frac{d}{dt}\left(\frac{\partial^{\tau}xy}{\partial\dot{b}_t}\right) = \left\{B_{tn}\right\}_T\left\{\ddot{Q}_n\right\} = \ddot{b}_t - \left\{\frac{I}{M}w_t w_n + v_t v_n\right\}_T\left\{\ddot{Q}_n\right\} + \sum_e \ddot{\delta}_e\left[\frac{I_{Ge}}{M}\lambda_{et} - v_e\phi_{et}\right]$$

$$- \sum_j \ddot{\rho}_j\,v_j\phi_{jt} + \left\{\sum_i \frac{\Delta I_k}{M}\lambda_{Ikt}\lambda_{Ikn}\right\}_T\left\{\ddot{Q}_n\right\}$$

Defining B_m where $B_e = \frac{I_{Ge}}{M}$, $B_j = -v_j$, $B_t = 1.0$

C_m where $C_e = \frac{I_{Ge}}{M}$, $C_j = 0$, $C_i = 0$

$$\text{then} \left\{ \frac{1}{M} \frac{d}{dt} \frac{\partial \tau_{xy}}{\partial \dot{Q}_m} \right\} = \left[B \right] \left\{ \ddot{Q}_n \right\} \qquad \text{II-4.25}$$

and in LaPlace notation

$$\left\{ \frac{1}{M} \frac{d}{dt} \frac{\partial \tau_{xy}}{\partial \dot{Q}_m} \right\} = s^2 \left[B \right] \left\{ Q_n \right\}$$

where $\left[B \right]$ is an N x N square matrix $\left[B_{mn} \right]$

m = row

n = column

$$B_{mn} = B_m \hat{\delta}(m,n) - \left[\frac{I}{M} w_m w_n + v_m v_n \right] + C_m \lambda_{mn} + C_n \lambda_{nm} - v_m \phi_{mn}$$
$$-v_n \phi_{nm} + \sum_k \frac{\Delta I_k}{M} \lambda_{Ikm} \lambda_{Ikn} \qquad \text{II-4.26}$$

The B matrix is the dynamic coupling matrix where MB_{mn} gives the generalized force along Q_m resulting from unit acceleration of Q_n which is equal to the generalized force along Q_n resulting from unit acceleration of Q_m. Therefore,

$$B_{mn} = B_{nm}$$

and $\left[B \right]$ must be a symmetric matrix.

II-4.3 INTERNAL POTENTIAL ENERGY

In considering the power balance of the system, rate of energy input equals rate of energy absorbed, the energy of the engine actuators must be taken into account. When the actuator moment, $M_{\delta e}$ is in the same direction as the actuator velocity, $\dot{\delta}_{ae}$ the work rate or power $M_{\delta e} \dot{\delta}$ is positive and the actuator acts as an external force increasing the system energy. However, when the directions are opposite in sign the work rate or power is negative and the actuator acts as a damper decreasing the energy of the system. Since the actuator can introduce energy and absorb energy it also appears as a spring storing potential energy. Although each approach will produce the same dynamic equations, the energy of the actuator will be introduced as potential energy in this analysis.

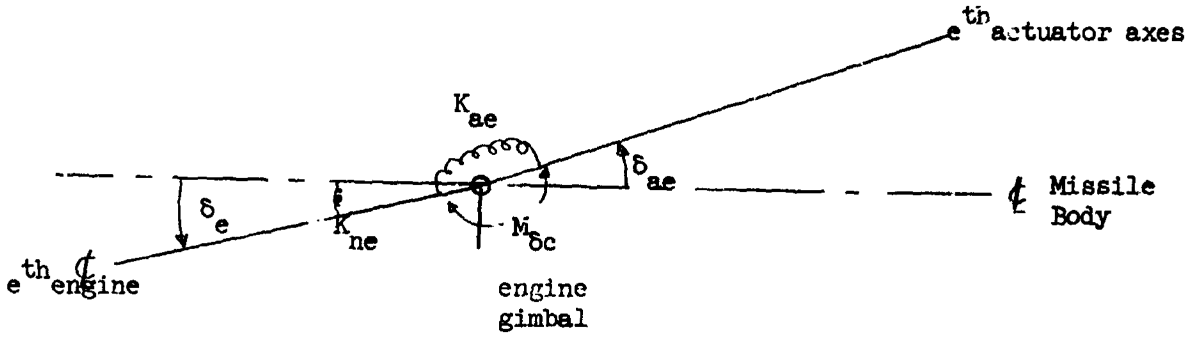


FIGURE II-4.3

K_{ae} = Torsional spring constant between e^{th} engine and its actuator in pound-feet per radian.

$M_{\delta e}$ = Actuator moment of the e^{th} actuator equal to reaction torque from the e^{th} engine.

$$M_{\delta}e = K_{ae}\,(\delta_{ae} - \delta_e)$$

$$V_{Ie_1} = \frac{1}{2} M_{\delta e}(\delta_{ae} - \delta_e) = \frac{1}{2} K_{ae}\,(\delta_{ae} - \delta_e)^2$$

For a given engine angle, δ_e, the magnitude of δ_{ae} determines the potential energy stored in the engine spring K_{ae}.

It is also assumed that due to fluid lines etc. there is a torsional spring, K_{ne}, between the e^{th} engine and the missile body. The energy stored in this spring is

$$V_{Ie_2} = \frac{1}{2} K_{ne}\,\delta_e^2$$

The total internal potential energy stored by the engine springs, due to both engine deflection and actuator deflection is

$$V_{Ie} = \sum_e \left[\frac{1}{2}\; K_{ae}(\delta_{ae} - \delta_e)^2 + \frac{1}{2} K_{ne}\,\delta_e^2\right] \qquad \text{II-4.27}$$

The total internal potential energy stored by the engines, slosh springs, and elastic deformation is

$$V_I = V_{Ie} + \frac{1}{2}\sum_j k_j\,\rho_j^2 + \frac{1}{2}\sum_i k_i\,b_i^2$$

but

$$k_j = m_j \omega_j^2$$

$$k_i = M\, \omega_i^2$$

Therefore,

$$V_I = \frac{1}{2} \sum_e K_{ae}(\delta_{ae} - \delta_e)^2 + \frac{1}{2} \sum_e K_{ne}\, \delta_e^2 + \frac{1}{2} \sum_j m_j \omega_j^2\, \rho_j^2 + \frac{1}{2} M \sum_i \omega_i^2\, b_i^2 \qquad \text{II-4.28}$$

The second term of equation II-4.21 then becomes:

For the p^{th} engine

$$\frac{1}{M} \frac{\partial V_I}{\partial \delta_p} = \frac{K_{ap} + K_{np}}{M}\, \delta_p - \frac{K_{ap}}{M}\, \delta_{ap} \qquad \text{II-4.29}$$

For the r^{th} slosh mass

$$\frac{1}{M} \frac{\partial V_I}{\partial \rho_r} = - V_r\, \omega_r^2\, \rho_r \qquad \text{II-4.30}$$

For the t^{th} bending mode

$$\frac{1}{M} \frac{V_I}{b_t} = \omega_t^2\, b_t \qquad \text{II-4.31}$$

Defining N x 1 column matrices $\{A_m\}$ $\{k_m\}$ and $\{\delta_{an}\}$ such that

$$\{A_m\} = \begin{Bmatrix} \left\{A_e = \dfrac{K_{ae} + K_{ne}}{M}\right\} \\ \left\{A_j = - V_r \omega_r^2\right\} \\ \left\{A_i = \omega_i^2\right\} \end{Bmatrix} \qquad \text{II-4.32}$$

$$\{k_m\} = \begin{Bmatrix} \left\{k_e = \dfrac{K_{ae}}{M}\right\} \\ \left\{k_j = 0\right\} \\ \left\{k_i = 0\right\} \end{Bmatrix} \qquad \text{II-4.33}$$

$$\left\{\delta_{an}\right\} = \begin{Bmatrix} \delta a1 \\ \delta a2 \\ \vdots \\ \delta aP \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \text{II-4.34}$$

then

$$\frac{1}{M} \frac{\partial V_I}{\partial Q_m} = \left[A\right]\left\{Q_n\right\} - \left[k\right]\left\{\delta_{an}\right\} \qquad \text{II-4.35}$$

where

$$A_{mn} = A_m \hat{\delta}(m,n) \qquad \text{II-4.36}$$

$$k_{mn} = k_m \hat{\delta}(m,n) \qquad \text{II-4.37}$$

The A matrix is the static coupling matrix where MA_{mn} gives the generalized force along Q_m resulting from unit deflection of Q_n, which is equal to the generalized force along Q_n resulting from unit deflection of Q_m. Therefore $\left[A\right]$ must be symmetric, however the system is statically uncoupled and

$$A_{mn} = A_{nm} = 0 \text{ when } n \neq m$$

hence $\left[A\right]$ is a diagonal matrix.

II-4.4 DISSIPATION FUNCTION

The dissipation function F_D appearing in the third term of equation II-4.21 is defined as one half of the rate at which energy is dissipated in the system. It is assumed that all damping forces result from bending, slosh, and engine velocities such that

$$F_D = \frac{1}{2} \sum_e D_{Ge} \dot{\delta}_e^2 + \frac{1}{2} \sum_j D_{sj} \dot{\zeta}_j^2 + \frac{1}{2} \sum_i D_{bi} \dot{b}_i^2$$

where D_{Ge} = engine damping in pound feet per radian

D_{sj} = slosh damping in pounds per feet per second

D_{bi} = bending damping in pound per feet per second.

but

$$D_{sj} = 2\, m_j\, \omega_j \zeta_j$$

$$D_{Bi} = 2M\, \omega_i \zeta_i$$

where ζ_j is the damping ratio associated with the j slosh mass and ζ_i is the damping ratio associated with the i bending mode

then

$$F_D = \frac{1}{2} \sum_e D_{Ge}\, \dot{\delta}_e^2 + \sum_j m_j \omega_j \zeta_j\, \dot{\rho}_j^2 + M_i\, \omega_i \zeta_i\, \dot{b}_i^2 \qquad \text{II-4.38}$$

Since the engine damping is usually given as torque per deflection rate as found from tests, the engine damping will be left as D_{Ge}. It should be noted that ω_j is not the frequency of the j^{th} slosh spring and mass when attached to the missile body but is the frequency when the spring mass is attached to a fixed point. ζ_j equal to unity does not represent critical damping for the missile system when the j^{th} sloshing mode only is present but represents critical damping for the j^{th} tank when rigidly held fixed. Likewise ω_i is not the frequency of the input bending mode when inertia corrections have been made.

The third term of equation II-4.21 then becomes for p^{th} engine

$$\frac{1}{M} \frac{\partial F_D}{\partial \dot{\delta}_p} = \frac{D_{Gp}}{M} \dot{\delta}_p \qquad \text{II-4.39}$$

For the r^{th} slosh mass

$$\frac{1}{M} \frac{\partial F_D}{\partial \dot{\rho}_r} = -\, 2 v_r \omega_r \zeta_r\, \dot{\rho}_r \qquad \text{II-4.40}$$

For the t^{th} bending mode

$$\frac{1}{M} \frac{\partial F_D}{\partial \dot{b}_i} = 2\omega_t \zeta_t\, \dot{b}_t \qquad \text{II-4.41}$$

Defining a N x 1 column matrix $\left\{D_m\right\}$ such that

$$\left\{D_m\right\} \quad \left\{\begin{array}{l} \left\{D_e = \dfrac{D_{Ge}}{M}\right\} \\ \left\{D_j = -\, 2 v_j \omega_j \zeta_j\right\} \\ \left\{D_i = 2\omega_i \zeta_i\right\} \end{array}\right\}$$

then

$$\left\{\frac{1}{M} \; \frac{F_D}{\dot{Q}_m}\right\} = s \left[D\right]\left\{Q_n\right\} \qquad \text{II-4.42}$$

where

$$D_{mn} = D_m \hat{\delta}\,(m,n)$$

The $\left[D\right]$ matrix is the damping matrix where MD_{mn} gives the generalized force along Q_m resulting from unit velocity of Q_n which is equal to the generalized force along Q_n resulting from unit velocity Q_m. Therefore, $\left[B\right]$ must be symmetric. However, for the coordinate system used $\left[D\right]$ is a diagonal matrix.

From equations II-4.25, II-4.35 and II-4.42 equation 4.21 can be written, in La Place transform notation as

$$s^2 \left[B\right]\left\{Q_n\right\} + \left[A\right]\left\{Q_n\right\} + s \left[D\right]\left\{Q_n\right\} = \left[k\right]\left\{\delta_{an}\right\} + \left\{\frac{P_m}{M}\right\} \qquad \text{II-4.43}$$

II-4.5 INPUT FREQUENCIES

The undamped and undriven motion that results when a single generalized coordinate Q_m is given a magnitude of unity and all other generalized coordinates are made zero, has been defined as an input mode. Since the system is linear the damped driven modes represented by equation II-4.43 will be linear combinations of these input modes. For studying the frequency shift of the system, it is necessary to know the natural frequencies of these input modes. For a single coordinate undamped and undriven mode, the $\left[D\right]$, $\left[k\right]$ and $\left\{\frac{P_m}{M}\right\}$ matrices become zero and the $\left[B\right]$ and $\left[A\right]$ matrices are 1 x 1 matrices. Equation II-4.43 becomes

$$s^2 B_{mm} Q_m + A_{mm} Q_m = 0$$

$$\left(s^2 B_{mm} + A_{mm}\right) Q_m = 0$$

then

$$s^2 = -\frac{A_{mm}}{B_{mm}}$$

$$\omega_{input} = \left[\frac{A_{mm}}{B_{mm}}\right]^{1/2}$$

but

$$A_{mm} = A_m$$

$$B_{mm} = B_m - \left[\frac{I}{M} w_m^2 + v_m^2\right] + \sum_k \frac{\Delta I_k}{M} \lambda_{km}^2$$

$$\omega_{INPUT} = \left[\frac{A_m}{B_m - \frac{I}{M} w_m^2 + v_m^2 + \sum_k \frac{\Delta I_k}{M} \lambda_{km}^2}\right]^{1/2} \qquad \text{II-4.44}$$

The slosh mass, m_j, has an associated sloshing frequency of ω_j calculated by the sloshing program. This frequency is equal to the square root of $\frac{k_j}{m_j}$ and is the natural frequency of the spring mass system when attached to a stationary body. In the slosh input mode the spring mass is attached to a moving missile body and the input natural frequency as given by equation II-4.44 will be higher than the square root of $\frac{k_j}{m_j}$. Likewise the engine input frequency calculated from equation II-4.44 will be higher than the square root of $\frac{K_{ae} + K_{ne}}{I_{Ge}}$. In the case of the input bending modes the input frequency is equal to the inputed modal frequency unless inertia corrections have been made. The addition of ΔI's will produce higher input bending frequencies.

II-5 EXTERNAL FORCES

The external forces are assumed to result from the thrust of the gimballed engines, T_{Ge}, the thrust of the fixed engines, T_{Ff}, and the aerodynamic forces P_a. The external forces will be expressed by components along the $\bar{i}$ - $\bar{j}$ axes in the form

$$P = I\,P_x + \bar{j}\,P_y$$

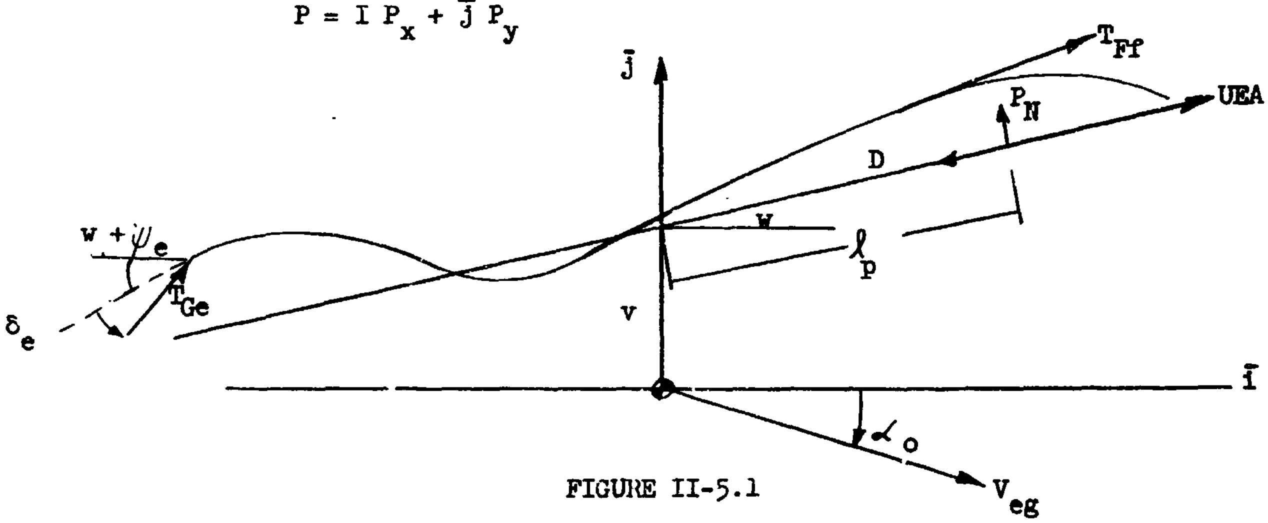


FIGURE II-5.1

The external forces due to the gimballed engine e are

$$P_{xe} = T_{Ge} \cos (w + \psi_e + \delta_e) \approx T_{Ge}$$

$$P_{ye} = T_{Ge} \sin (w + \psi_e + \delta_e) \approx T_{Ge} (w + \psi_e + \delta_e)$$

$$x_{pe} = x_e \qquad \text{II-5.1}$$

$$y_{pe} = v + x_e w + u_e$$

The external forces due to the fixed engine f are

$$P_{xf} = T_{Ff} \cos (w + \psi_{Ff}) \approx T_{Ff}$$

$$P_{yf} = T_{Ff} \sin (w + \psi_{Ff}) \approx T_{Ff} (w + \psi_{Ff})$$

$$x_{pf} = x_{Ff} \qquad \text{II-5.2}$$

$$y_{pf} = v + x_{Ff} w + u_{Ff}$$

The aerodynamic forces are obtained from the pressure distribution on the surface of the missile body. The net aerodynamic force on an element of the missile dx_h in length, measured along the UEA, is found by integrating the pressure over the external surface of that element. This net force is resolved into components dP_N normal to the UEA and dP_D along the UEA. These forces are defined by

$$dP_N = \alpha C_{Nx} \, dx \, q \, S$$

$$dP_D = dD + \alpha \, C_{Ax} \, dx \, q \, S$$

where the axial or drag forces are measured positive aft and

$\alpha = (\alpha_o + w)$ = angle of attack of the UEA

q = dynamic pressure

S = reference area

C_{Nx} = distributed non-dimensional force coefficient

C_{Ax} = distributed non-dimensional force coefficient

dD = drag force independent of angle of attack

Defining

$$N_{\alpha x} = C_{Nx} \, q \, S$$

$$A_{\alpha x} = C_{Ax} \, q \, S$$

then

$$dP_N = \alpha \, N_{\alpha x} \, dx$$

$$dP_D = dD + \alpha \, A_{\alpha x} \, dx$$

The distributed normal and axial force coefficients $N_{\alpha x}$ and $A_{\alpha x}$ are functions of x_h and are the force per unit length per radian of angle of attack.

For the purpose of this analysis, it is assumed that axial force is not a function of the angle of attack ($A_{\alpha x} = 0$) and

$$dP_D = dD$$

$$P_D = \int_{TM} dD = D$$

where D is the drag force in pounds and is considered a constant at a given time of flight.

It is further assumed that the normal force distribution is not a function of elastic deformation and is independent of the local bending slope.

The external aero forces due to drag are then

$$P_{xD} = - D \cos w \approx -D$$

$$P_{yD} = - D \sin w \approx -Dw$$

$$x_{pD} = \ell_p \qquad \text{II-5.3}$$

$$y_{pD} = v + \ell_p w$$

where ℓp is the x_h coordinate of the center of pressure defined below. The external aero forces due to normal force are then

$$dP_{xN} = - dP_N \sin w \approx 0$$

$$dP_{yN} = dP_N \cos w \approx dP_N$$

$$x_{pN} = x_h \qquad \text{II-5.4}$$

$$y_{pN} = v + x_h w + u$$

The total normal force P_N is then

$$P_N = \int_{TM} dP_N = \alpha \int_{TM} N_{\alpha x} dx = \alpha\ N_\alpha$$

where N_α is the total normal force coefficient

Taking moments of the aero forces about the origin on the UEA

$$dM_A = x_h \, dP_N$$

$$M_A = \int_{TM} x_h \, dP_N = \alpha \int_{TM} x_h \, N_{\alpha x} \, dx = N_\alpha \alpha \ell p$$

where ℓp is the location of the center of pressure

$$\ell p = \frac{1}{N_\alpha} \int_{TM} x_h \, N_{\alpha x} \, dx$$

Then for the aerodynamic forces

$$\frac{P_{xa}}{M} = - \frac{D}{M}$$

$$\frac{P_{ya}}{M} = \frac{N_\alpha}{M} \alpha - \frac{Dw}{M}$$

$$P_{ya} \, x_p = \int_{TM} dP_N \, x_h - Dw \, \ell p = (N_\alpha \alpha - Dw) \ell p$$

$$P_{xa} \, y_p = -D(v + \ell_p w)$$

II-5.1 FORCING FUNCTION FOR AXIAL ACCELERATION EQUATION

The axial acceleration equation II-2.24 requires the forcing function $\frac{1}{M} \sum_P P_x$

$$\frac{1}{M} \sum_P P_x = \sum_e \frac{T_{Ge}}{M} + \sum_f \frac{T_{Ff}}{M} - \frac{D}{M}$$

$$\frac{1}{M} \sum_P P_x = \frac{T_T - D}{M} = \beta_x \qquad \text{II-5.5}$$

where $T_T = \sum_e T_{Ge} + \sum_f T_{Ff}$ = total thrust

II-5.2 FORCING FUNCTION FOR NORMAL ACCELERATION EQUATION

The forcing function associated with normal acceleration equation II-2.25 is $\frac{1}{M}\sum_{p} P_y$

$$\frac{1}{M}\sum_{p} P_y = \sum_{e} \frac{T_{Ge}}{M}(w + \psi_e + \delta_e) + \sum_{f} \frac{T_{Ff}}{M}(w + \psi_{Ff}) + \frac{N_\alpha}{M}\alpha - \frac{Dw}{M}$$

$$\frac{1}{M}\sum_{p} P_y = \frac{N_\alpha}{M}\alpha + w\beta_x + \sum_{i}\left[\sum_{e}\frac{T_{Ge}}{M}\lambda_{ei} + \sum_{f}\frac{T_{Ff}}{M}\lambda_{Ffi}\right] b_i + \sum_{e}\frac{T_{Ge}}{M}\delta_e$$

Define $\left[\sum_{e}\frac{T_{Ge}}{M}\lambda_{ei} + \sum_{f}\frac{T_{Ff}}{M}\lambda_{Ffi}\right] = \alpha_{1i}$

or $\left[\sum_{e}\frac{T_{Ge}}{M}\lambda_{en} + \sum_{f}\frac{T_{Ff}}{M}\lambda_{Ffn}\right] = \alpha_{1n}$

α_{1n} is a constant for each bending mode and is zero when n not equal to i.

Then

$$\frac{1}{M}\sum_{p} P_y = \frac{N_\alpha}{M}\alpha + \left\{w_n\beta_x + \alpha_{1n} + \frac{T_{Gn}}{M}\right\}_T \left\{Q_n\right\} \qquad \text{II-5.6}$$

$$\frac{1}{M}\sum_{p} P_y = \frac{N_\alpha}{M}\alpha + \left\{a_{yn}\right\}_T \left\{Q_n\right\} \qquad \text{II-5.7}$$

where

$$a_{yn} = \beta_x w_n + \alpha_{1n} + \frac{T_{Gn}}{M} \qquad \text{II-5.8}$$

where $\frac{T_{Gn}}{M}$ is zero for all modes except engine modes

II-5.3 FORCING FUNCTION FOR MOMENT EQUATION

The forcing function for the moment equation II-2.21 is the net moment of the external forces about the system center of gravity,

$$\sum_{p}\left[P_y x_p - P_x y_p\right]$$

$$\sum_{p}\left[P_y x_p - P_x y_p\right] = \sum_{p}\left[P_y x_p\right] - \sum_{p}\left[P_x y_p\right] \qquad \text{II-5.9}$$

The total external moment will be found in two parts, the moment due to y components and the moment due to x components of the forces.

$$\sum_p P_y x_p = \sum_e T_{Ge} (w + \psi_e + \delta_e)\, x_e + \sum_f T_{Ff} (w + \psi_{Ff}) + (N_\alpha \alpha - Dw) \ell_p$$

dividing by total mass M

$$\frac{1}{M} \sum_p P_y x_p = \frac{N_\alpha \ell_p}{M} \alpha + w \left[\sum_e \frac{T_{Ge}}{M} x_e + \sum_f \frac{T_{Ff}}{M} x_f - \frac{D \ell_p}{M} \right] +$$

$$+ \sum_i \left[\sum_e \frac{T_{Ge}}{M} x_e \lambda_{ei} + \sum_f \frac{T_{Ff}}{M} x_f \lambda_{Ffi} \right] b_i + \sum_e \frac{T_{Ge}}{M} x_e \delta_e$$

Define $\left[\sum_e \frac{T_{Ge}}{M} x_e + \sum_f \frac{T_{Ff}}{M} x_f - \frac{D_p}{M} \right] = \beta_T = \text{constant}$ II-5.10

$$\left[\sum_e \frac{T_{Ge}}{M} x_e \lambda_{ei} + \sum_f \frac{T_{Ff}}{M} x_f \lambda_{Ffi} \right] = \alpha_{2i}$$

or $\left[\sum_e \frac{T_{Ge}}{M} x_e \lambda_{en} + \sum_f \frac{T_{Ff}}{M} x_f \lambda_{Ffn} \right] = \alpha_{2n}$

α_{2n} is a constant for each bending mode and is zero when n not equal to i

then

$$\frac{1}{M} \sum_p P_y x_p = \frac{N_\alpha \ell_p}{M} \alpha + \left\{ \beta_T w_n + \alpha_{2n} + \frac{T_{Gn}}{M} x_n \right\}_T \left\{ Q_n \right\} \qquad \text{II-5.11}$$

for the x components the mom^nt is

$$\sum_p P_x y_p = \sum_e T_{Ge} (v + x_e w + u_e) + \sum_f T_{Ff} (v + x_f w + u_f) - D(v + \ell_p w)$$

dividing by M

$$\frac{1}{M} \sum_p P_x y_p = \beta_x v + \beta_T w + \sum_i \left[\sum_e \frac{T_{Ge}}{M} \phi_{ei} + \sum_f \frac{T_{Ff}}{M} \phi_{Ffi} \right] b_i$$

Define $\left[\sum_e \frac{T_{Ge}}{M} \phi_{ei} + \sum_f \frac{T_{Ff}}{M} \phi_{Ffi}\right] = \alpha_{3i}$

or $\left[\sum_e \frac{T_{Ge}}{M} \phi_{en} + \sum_f \frac{T_{Ff}}{M} \phi_{Ffn}\right] = \alpha_{3n}$ II-5.12

α_{3n} is a constant for each bending mode and is zero when n not equal to i.

Therefore,

$$\frac{1}{M} \sum_p P_x y_p = \left\{\beta_x v_n + \beta_T w_n + \alpha_{3n}\right\}_T \left\{Q_n\right\} \quad \text{II-5.13}$$

From equation II-5.9, II-5.11, and II-5.13 the total external moment is

$$\sum_p P_y x_p - P_x y_p = N_\alpha \ell_p \alpha + M \left\{\frac{T_{Gn}}{M} x_n - \beta_x v_n + \alpha_{2n} - \alpha_{3n}\right\}_T \left\{Q_n\right\}$$

dividing by I

$$\frac{1}{I} \sum_p \left[P_y x_p - P_x y_p\right] = \frac{N_\alpha \ell_p}{I} \alpha + \frac{M}{I} \left\{\frac{T_{Gn}}{M} x_n - \beta_x v_n + \alpha_{2n} - \alpha_{3n}\right\}_T \left\{Q_n\right\} \quad \text{II-5.14}$$

$$\frac{N_\alpha \ell_p}{I} = \mu_\alpha \quad \text{II-5.15}$$

Let $\frac{M}{I} \left(\frac{T_{Gn}}{M} x_n - \beta_x v_n + \alpha_{2n} - \alpha_{3n}\right) = u_n$ II-5.16

$$\frac{1}{I} \sum_p \left[P_y x_p - P_x y_p\right] = \mu_\alpha \alpha + \left\{\mu_n\right\}_T \left\{Q_n\right\} \quad \text{II-5.17}$$

II-5.4 FORCING FUNCTION FOR m^{th} MODAL EQUATION

The forcing function associated with the m^{th} generalized coordinate is given by equation II-2.22 as

$$\frac{P_m}{M} \sum_p \left[\frac{P_x}{M} \frac{\partial x_p}{\partial Q_m} + \frac{P_y}{M} \frac{\partial y_p}{\partial Q_m}\right]$$

Since x_{pe}, x_{pf}, and x_{pa} as given by equations II-5.1, II-5.2, and II-5.4, are all constants

$$\frac{\partial x_p}{\partial Q_m} = 0$$

for all coordinates and all driving forces result from components of force in the y direction. Therefore,

$$\frac{P_m}{M} = \frac{1}{M} \sum_P P_y \frac{\partial y_p}{\partial Q_m}$$

$$\frac{P_m}{M} = \sum_e \frac{P_{ye}}{M} \frac{\partial}{\partial Q_m} (v + x_e w + u_e) + \sum_f \frac{P_{yf}}{M} \frac{\partial}{\partial Q_m} (v + x_{Ff} w + u_f)$$

$$- \frac{Dw}{M} \frac{\partial}{\partial Q_m} (\quad + \ell_p w) + \frac{\alpha}{M} \int_{TM} \left[N_{\alpha x} \, dx \frac{\partial}{\partial Q_m} (v + x_h w + u) \right]$$

$$\frac{P_m}{M} = \sum_e \frac{P_{ye}}{M} (v_m + x_e w_m + \phi_{em}) + \sum_f \frac{P_{yf}}{M} (v_m + x_{Ff} w_m + \phi_{Ffm})$$

$$- \frac{Dw}{M} (v_m + \ell_p w_m) + v_m \frac{N_\alpha}{M} \alpha + w_m \ell_p \frac{N_\alpha}{M} \alpha + \frac{\alpha}{M} \int_{TM} \phi_m N_{\alpha x} \, dx$$

$$\frac{P_m}{M} = v_m \sum_P \frac{P_y}{M} + w_m \sum_P \frac{P_y}{M} x_p + \frac{\alpha}{M} \int_{TM} \phi_m N_{\alpha x} \, dx + \sum_e \frac{P_{ye}}{M} \phi_{em} + \sum_f \frac{P_{yf}}{M} \phi_{Ffm} \qquad \text{II-5.18}$$

$\sum_P \frac{P_y}{M}$ is the sum of the forces in the y direction divided by M and is given by equation II-5.7. $\sum_P \frac{P_y}{M} x_p$ is the external moment due to the y components divided by M and is given by equation II-5.11. Letting the last two terms of equation II-5.18 equal X_m

Defining $N_{\alpha Dm} = \int_{TM} \phi_m \, N_{\alpha x} \, dx$

$$\frac{P_m}{M} = \frac{N_\alpha}{M}(v_m + \ell_p w_m)\,\alpha + \frac{N_{\alpha Dm}}{M}\alpha + \left\{ v_m a_{yn} + \beta_T w_m w_n + w_m \alpha_{2n} + w_m \frac{T_{Gn}}{M} x_n \right\} \left\{ Q_n \right\} + X_m \qquad \text{II-5.19}$$

where

$$X_m = \sum_e \frac{P_{ye}}{M}\phi_{em} + \sum_f \frac{P_{yf}}{M}\phi_{Ffm}$$

$$X_m = \sum_e \frac{T_{Ge}}{M}(w + \psi_e + \delta_e)\,\phi_{em} + \sum_f \frac{T_{Ff}}{M}(w + \psi_{Ff})\,\phi_{Fm}$$

$$X_m = w\,\alpha_{3m} + \sum_i \left[\sum_e \frac{T_{Ge}}{M}\lambda_{ei}\,\phi_{em} + \sum_f \frac{T_{Ff}}{M}\lambda_{Ffi}\,\phi_{Ffm} \right] b_i + \sum_n \frac{T_{Gn}}{M}\phi_{nm}\,Q_n$$

Define $\left[\sum_e \frac{T_{Ge}}{M}\lambda_{en}\,\phi_{em} + \sum_f \frac{T_{Ff}}{M}\lambda_{Ffn}\,\phi_{Ffm} \right] = \alpha_{4mn}$ II-5.20

α_{4mn} is a constant for each combination of m and n and is zero unless both m and n represent bending modes.

then

$$X_m = \left\{ w_n\,\alpha_{3m} + \alpha_{4mn} + \frac{T_{Gn}}{M}\phi_{em} \right\}_T \left\{ Q_n \right\}$$

then from II-5.19

$$\frac{P_m}{M} = \frac{N_\alpha}{M}(v_m + \ell_p w_m)\,\alpha + \frac{N_{\alpha Dm}}{M}\alpha + \left\{ v_m a_{yn} + \beta_T w_m w_n + w_m \alpha_{2n} + w_m \frac{T_{Gn}}{M} x_n + w_n \alpha_{3m} + \alpha_{4mn} + \frac{T_{Gn}}{M}\phi_{nm} \right\}_T \left\{ Q_n \right\} \qquad \text{II-5.21}$$

$$\left\{ \frac{P_m}{M} \right\} = \left\{ J_m \right\} \alpha + \left[P \right] \left\{ Q_n \right\} \qquad \text{II-5.22}$$

where

$$J_m = \frac{N_\alpha}{M}(v_m + \ell_p w_m) + \frac{N_{\alpha Dm}}{M} \qquad \text{II-5.23}$$

$$P_{mn} = v_m a_{yn} + \beta_T w_m w_n + w_m \alpha_{2n} + w_m \frac{T_{Gn}}{M} x_n + w_n \alpha_{3m} + \alpha_{4mn} + \frac{T_{Gn}}{M} \phi_{nm} \qquad \text{II-5.24}$$

II-6 DYNAMIC EQUATIONS IN GENERALIZED COORDINATES

Combining the results of sections II-2, II-4, and II-5, the dynamic equations representing the motion of the displaced or perturbed missile can be written in terms of generalized coordinates.

II-6.1 AXIAL ACCELERATION EQUATION

The axial acceleration along the nominal flight path and along the $\bar{i}$ axes are given by equations II-2.19 and II-2.24 as

$$\ddot{Q}_H = -V_{cg}(\dot{\theta} + \dot{\alpha}_o)\sin(\theta + \alpha_o) + \dot{V}_{cg} \cos(\theta + \alpha_o) = \cos\theta \sum_P \frac{P_x}{M} + \sin\theta \sum_P \frac{P_y}{M} - g\cos\gamma$$

$$a_x = -V_{cg}(\dot{\theta} + \dot{\alpha}_o)\sin\alpha_o + \dot{V}_{cg}\cos\alpha_o = \sum_P \frac{P_x}{M} - \cos\theta \, g \cos\gamma$$

For small angles and dropping non-linear terms these accelerations become

$$\ddot{Q}_H = \dot{V}_{cg} = \sum_P \frac{P_x}{M} - g\cos\gamma$$

$$a_x = \dot{V}_{cg} = \sum_P \frac{P_x}{M} - g\cos\gamma$$

from equation II-5.5

$$\ddot{Q}_H = a_x = \dot{V}_{cg} = \beta_x - g\cos\gamma = \text{a constant} \qquad \text{II-6.1}$$

Therefore for the linear system under consideration the acceleration along the nominal flight path is equal to the acceleration along the $\bar{i}$ axis which is a constant.

II-6.2 NORMAL ACCELERATION EQUATION

The normal accelerations perpendicular to the nominal flight path and perpendicular to the $\bar{i}$ axis are given by equations II-2.20 and II-2.25 as

$$\ddot{Q}_L = -V_{cg}(\dot{\theta} + \dot{\alpha}_o)\cos(\theta + \alpha_o) - \dot{V}_{cg} \sin(\theta + \alpha_o) = \cos\theta \sum_P \frac{P_y}{M} - \sin\theta \sum_P \frac{P_x}{M}$$

$$a_y = -V_{cg}(\dot{\theta} + \dot{\alpha}_o) \cos\alpha_o - \dot{V}_{cg} \sin\alpha_o = \sum_P \frac{P_y}{M} - \sin\theta\, g \cos\gamma$$

For small angles and dropping non-linear terms, where V_{cg} is a constant, these accelerations become

$$\ddot{Q}_L = - V_{cg}(\dot{\theta} + \dot{\alpha}_o) - V_{cg} (\theta + \alpha_o) = \sum_P \frac{P_y}{M} - \theta \sum_P \frac{P_x}{M}$$

$$a_y = - V_{cg} (\dot{\theta} + \dot{\alpha}_o) - \dot{V}_{cg}\, \alpha_o = \sum_P \frac{P_y}{M} - \theta\, g \cos\gamma$$

from II-5.5 and II-5.6

$$\sum_P \frac{P_x}{M} = \beta_x$$

$$\sum_P \frac{P_y}{M} = \frac{N_\alpha}{M}\alpha + \left\{a_{yn}\right\}^T \left\{Q_n\right\}$$

$$\ddot{Q}_L = -V_{cg}(\dot{\theta} + \dot{\alpha}_o) - \dot{V}_{cg}(\theta + \alpha_o) = \frac{N_\alpha}{M}\alpha - \theta\beta_x + \left\{a_{yn}\right\}^T \left\{Q_n\right\} \qquad \text{II-6.2}$$

$$a_y = -V_{cg}(\dot{\theta} + \dot{\alpha}_o) - \dot{V}_{cg}\, \alpha_o = \frac{N_\alpha}{M}\alpha - \theta\, g \cos\gamma + \left\{a_{yn}\right\}^T \left\{Q_n\right\} \qquad \text{II-6.3}$$

These equations show that the accelerations normal to the nominal flight path and normal to the $\bar{i}$ axis are not equal in the linear system but

$$a_y = \ddot{Q}_L + \theta \dot{V}_{cg} = \ddot{Q}_L + a_x \theta \qquad \text{II-6.4}$$

In addition equation II-6.2 shows that the velocity of the system center of gravity normal to the nominal flight path is given by

$$\dot{Q}_L = -V_{cg} (\theta + \alpha_o) \qquad \text{II-6.5}$$

By equations II-6.2 and II-6.3 the normal accelerations can be expressed in terms of either α_o or α and the remaining coordinates, where α is the angle of attack of the UEA and α_o is the angle of attack of the $\bar{i}$ axis, therefore

$$\alpha = \alpha_o + w = \alpha_o + \left\{w_n\right\}_T \left\{Q_n\right\} \qquad \text{II-6.6}$$

Since α is the more convenient variable, α_o is replaced by $(\alpha - w)$ in equations II-6.2 and II-6.3. In addition $\dot{V}_{cg}$ is replaced by the constant $a_x = \beta_x - g \cos \gamma$. Then in terms of the LaPlace operator, s, these equations become

$$\ddot{Q}_L = -(V_{cg}s + a_x)(\theta + \alpha) + (V_{cg}s + a_x) \left\{w_n\right\}_T \left\{Q_n\right\} = \frac{N_\alpha}{M} \alpha - \theta\beta_x + \left\{a_{yn}\right\}_T \left\{Q_n\right\} \qquad \text{II-6.7}$$

$$a_y = -V_{cg}s\,\theta - (V_{cg}s + a_x)\alpha + (V_{cg}s + a_x)\left\{w_n\right\}_T \left\{Q_n\right\} = \frac{N_\alpha}{M}\alpha - \theta\, g \cos \gamma + \left\{a_{yn}\right\}_T \left\{Q_n\right\} \qquad \text{II-6.8}$$

Using either equation II-6.7 or II-6.8 the dynamic equation in terms of the variable α is

$$(V_{cg}s + \frac{N_\alpha}{M} + a_x)\alpha + (V_{cg}s - g \cos \gamma)\theta + \left\{a_{yn} - (V_{cg}s + a_x)\, w_n\right\}_T \left\{Q_n\right\} = 0 \qquad \text{II-6.9}$$

Since it is convenient to know the normal acceleration in the system, it is desirable to retain either $\ddot{Q}_L$ or a_y and convert either equation II-6.7 or II-6.8 into two equations retaining α as a dependent variable. In the previous analysis of reference 1, a_y the acceleration along the $\bar{j}$ axis was used. However, in this present analysis $\ddot{Q}_L$, the acceleration normal to the nominal trajectory, will be retained. Equation II-6.7 then becomes

$$\ddot{Q}_L = -(V_{cg}s + a_x)\,(\theta + \alpha) + (V_{cg}s + a_x) \left\{w_n\right\}_T \left\{Q_n\right\} \qquad \text{II-6.10}$$

$$\ddot{Q}_L = \frac{N_\alpha}{M} \alpha - \beta_x\theta + \left\{a_{yn}\right\}_T \left\{Q_n\right\} \qquad \text{II-6.11}$$

Equation II-6.10 is the so called normal force equation and II-6.11 is the normal acceleration equation. The normal force equation can then be written as

$$(V_{cg}s + a_x)\alpha + \ddot{Q}_L + (V_{cg}s + a_x)\,\Theta - s\left\{N_{vn}\right\}_T\left\{Q_n\right\} - \left\{N_{an}\right\}_T\left\{Q_n\right\} = 0 \qquad \text{II-6.12}$$

where $N_{vn} = V_{cg}\,w_n$

$N_{an} = a_x\,w_n$

The normal acceleration equation can be written as

$$-\frac{N_\alpha}{M}\alpha + \ddot{Q}_L + \beta_x\,\Theta - \left\{a_{yn}\right\}_T\left\{Q_n\right\} = 0 \qquad \text{II-6.13}$$

II-6.3 MOMENT EQUATION

The moment equation II-2.21 is

$$\frac{d}{dt}\,(I\dot{\Theta}) = -\sum_P\left[P_y x_P - P_x y_P\right]$$

For small angles and a linear system

$$\ddot{\Theta} = -\frac{1}{I}\sum_P\left[P_y x_P - P_x y_P\right]$$

From equation II-5.17

$$\ddot{\Theta} = -\mu_\alpha\,\alpha - \left\{\mu_n\right\}_T\left\{Q_n\right\}$$

in terms of the LaPlace operator the moment equation becomes

$$\mu_\alpha\alpha + s^2\Theta + \left\{\mu_n\right\}_T\left\{Q_n\right\} = 0 \qquad \text{II-6.14}$$

II-6.4 MODAL EQUATIONS

From equation II-4.43 the modal equations can be written as

$$s^2\left[B\right]\left\{Q_n\right\} + \left[A\right]\left\{Q_n\right\} + s\left[D\right]\left\{Q_n\right\} = \left[k\right]\left\{\delta_{an}\right\} + \left\{\frac{P_m}{M}\right\}$$

From equation II-5.21

$$\left\{\frac{P_m}{M}\right\} = \left\{J_m\right\} \alpha + \left[P\right]\left\{Q_n\right\}$$

Finally the modal equations become

$$s^2 \left[B\right]\left\{Q_n\right\} + s \left[D\right]\left\{Q_n\right\} + \left[A\right]\left\{Q_n\right\} = \left[k\right]\left\{\delta_{an}\right\} + \left\{J_m\right\}\alpha + \left[P\right]\left\{Q_n\right\} \qquad \text{II-6.15}$$

II-6.5 SUMMARY OF EQUATIONS

The dynamic equations derived for a missile configuration considering flexible body, sloshing, and engine dynamics are:

Axial Acceleration equation II-6.1

$$\dot{V}_{cg} = a_x = \beta_x - g \cos \gamma \qquad \text{II-6.16}$$

Normal Force Equation II-6.12

$$(V_{cg}\, s + a_x)\,\alpha + \ddot{Q}_L + (V_{cg} s + a_x)\,\theta - s \left\{N_{vn}\right\}_T \left\{Q_n\right\} - \left\{N_{an}\right\}_T \left\{Q_n\right\} = 0 \qquad \text{II-6.17}$$

Normal Acceleration Equation II-6.13

$$-\frac{N_\alpha}{M}\alpha + \ddot{Q}_L + \beta_x\theta - \left\{a_{yn}\right\}_T \left\{Q_n\right\} = 0 \qquad \text{II-6.18}$$

Normal Equation II-6.14

$$\mu_\alpha\alpha + s^2\theta + \left\{\mu_n\right\}_T \left\{Q_n\right\} = 0 \qquad \text{II-6.19}$$

Modal Equations II-6.15

$$s^2 \left[B\right]\left\{Q_n\right\} + s \left[D\right]\left\{Q_n\right\} + \left[A\right]\left\{Q_n\right\} = \left[k\right]\left\{\delta_{am}\right\} + \left[J_m\right]\alpha + \left[P\right]\left\{Q_n\right\} \qquad \text{II-6.20}$$

where n has values from 1 thru N

As pointed out in section II-4, the system has N + 3 degrees of freedom. The motion of the system is described by the above N + 4 equations in the N + 4 variables α, a_x, θ, Q_L and N values of Q_n. However, the variable

α can be expressed as a linear combination of $\ddot{Q}_L$, θ and the Q_n's by equation II-6.18, therefore, α is a dependent variable and is retained along with equation II-6.18 as a matter of convenience as discussed in section II-6.2. The N + 3 coordinates which define the motion of the lineary system are then $\dot{V}_{cg}$, $\ddot{Q}_L$, θ and the N values of Q_n. For the linear system the degree of freedom defined by $\ddot{Q}_H = \dot{V}_{cg} = a_x =$ constant, equation II-6.16, is uncoupled from the other coordinates and consists of a constant acceleration along the Q_H axis. Since the force along the Q_H axes is constant, it is independent of Q_H and $\dot{Q}_H$ and there are no spring or damping forces along the axes and the value of Q_H and $\dot{Q}_H$ at time t are arbitrary. The constant, a_x, has been substituted for $\ddot{Q}_H$ or $\dot{V}_{cg}$ and V_{cg} has been arbitrarily assigned to $\dot{Q}_H$ in the normal force equation II-6.17, therefore, the axial translational mode of motion has been eliminated from the system of equations. Elimination of equation II-6.16 reduces the system to N + 2 degrees of freedom equivalent to defining the missile motion with respect to a set of axis translating along the Q_H axes with velocity V_{cg}. The L inertial axes is then coincident with the P axes and the coordinate Q_H is zero.

II-7 MISSILE DYNAMICS IN GENERALIZED COORDINATES

The dynamic equations given in section II-6 together with an engine hydraulic equations and an instrument equation, are the data required for the missile dynamics block of figure I-2 . The missile dynamics represents the transfer function from delta command, δ_c, to the gyro outputs, θ_P and θ_R. Figure II-7.1 contains a block diagram showing the effective flow of the data in the dynamic equations. It is assumed that in a system containing P gimballed engines, the command signals for each engine or group of engines may be different. The engine hydraulic equations which give the engine actuator position δ_{ae} corresponding to a given delta command, can be of any form but will usually appear in the form

$$(K_1 s + K_2)\ \delta_e + (K_3 s + K_4)\ \delta_{ae} + \delta_{ce} = 0 \qquad \text{II-7.1}$$

requiring a feedback of engine positions δ_e.

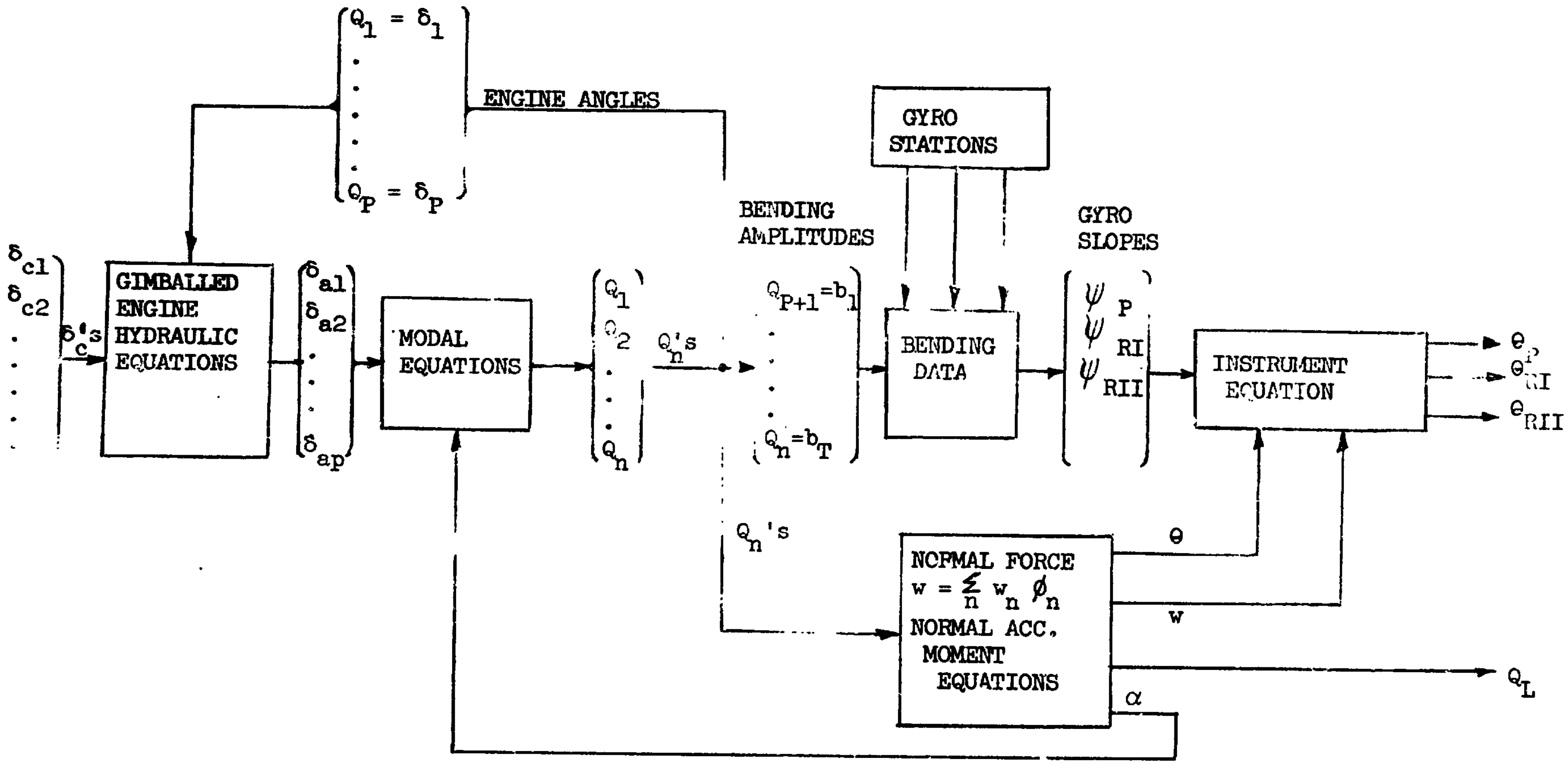


FIGURE II - 7.1

MISSILE DYNAMICS IN GENERALIZED COORDINATES

The engine actuator position, and the angle of attack feedback drive the modal equations II-6.15 producing the modal coordinates Q_n. These coordinates in turn drive the normal acceleration and moment equations together with the equation

$$w = \sum_n w_n Q_n$$

to produce the rigid body rotation of the $\bar{i} - \bar{j}$ axes, θ, the rigid body rotation of the UEA with respect to the $\bar{i} - \bar{j}$ axes, w, and the angle of attack α. In addition the engine deflections are fedback to the engine hydraulic equations and the bending coordinates are combined with the bending data to produce the bending slopes at each gyro location. These bending slopes are combined with the rigid body rotations to produce the gyro signals.

$$\theta_P = \theta - w - \psi_P$$

$$\theta_{RI} = \theta - w - \psi_{RI} \qquad \text{II-7.2}$$

$$\theta_{RII} = \theta - w - \psi_{RII}$$

$$\theta_R = \theta_{RI} + a_R \theta_{RII}$$

It is assumed that one position and two rate gyros are used and the net rate signal is a ratioed sum of the two rate gyros.

That portion of the system matrix representing missile dynamics is shown in Figure II-7.2. The first row of the matrix represents the normal force equation II-6.12. The second row is the normal acceleration equation II-6.13, and the third row is the moment equation II-6.14. The next N rows are the N modal equations II-6.15. The system shown is assumed to have three gimballed engines, therefore, columns 11, 12, and 13 represent the three engine actuator positions. Row 11 and column 14 intorduce the rigid body rotation with respect to the $\bar{i} - \bar{j}$ axes were

$$w = \{w_n\}_T \{Q_n\}$$

FIGURE II - 7.2

SYSTEM MATRIX IN GENERALIZED COORD

	1	2	3	4	5	6	7	8	9	10	11
	α	$\ddot{Q}_L$	θ	Q_1	Q_2	Q_3	....	Q_n	....	Q_N	δ_{a1}
1	$V_{cg}\, s + a_x$	1	$V_{cg}\, s + a_x$	$-s\, N_{v1} - N_{a1}$	$-s\, N_{v2} - N_{a2}$	$-s\, N_{v3} - N_{a3}$	....	$-s\, N_{vn} - N_{an}$	....	$-s\, N_{vN} - N_{aN}$	
2	$-\frac{N_\alpha}{M}$	1	β_x	$-a_{y1}$	$-a_{y2}$	$-a_{y3}$	....	$-a_{yn}$	....	$-a_{yN}$	
3	μ_α		s^2	μ_1	μ_2	μ_3	....	μ_n	....	μ_N	
4	$-J_1$										$-k_1$
5	$-J_2$										
6	$-J_3$										
7	$\vdots$						$s^2 [B] + s [D] + [A] - [P]$				
8	$-J_n$										
9	$\vdots$										
10	$-J_N$										
11				w_1	w_2	w_3	....	w_n	....	w_N	
12			-1	λ_{P1}	λ_{P2}	λ_{P3}	....	λ_{Pn}	....	λ_{PN}	
13			-1	$(\lambda_{RI})_1$	$(\lambda_{RI})_2$	$(\lambda_{RI})_3$	....	$(\lambda_{RI})_n$	....	$(\lambda_{RI})_N$	
14			-1	$(\lambda_{RII})_1$	$(\lambda_{RII})_2$	$(\lambda_{RII})_3$	....	$(\lambda_{RII})_n$	....	$(\lambda_{RII})_N$	
15											
16			$K_1 s + K_2$								$K_2 s + K_3$
17				$K_4 s + K_5$							
18											

ES

12	13	14	15	16	17	18	19	
δ_{a2}	δ_{a3}	w	θ_P	θ_{RI}	$\theta_{R\,II}$	θ_R	δ_c	**EQUATION**
								II - 6.12
								II - 6.13
								II - 6.14
$-k_2$								
	$-k_3$							
								II - 6.15
		-1						$w=\{w_n\}_r\{Q_n\}$
		1	1					
		1		1				II - 7.2
		1			1			
				a_R	1	-1		
							1	
K_6s+K_7							1	II - 7.1

The next four rows and columns introduce the instrument equations II-7.2 while the last three rows represent the engine hydraulic equations II-7.1. For the system shown it is assumed that a single command signal, column 19 controls the actuators of the first two gimballed engines and the third engine is restrained by making its command signal zero.

The input to the missile dynamics is δ_c of column 19 while the output is the gyro signals of columns 15 and 18. The system matrix is completed by adding rows and columns representing the autopilot which operates on the gyro signals to produce the command signal.

The matrix representing missile dynamics, as shown in Figure II-7.2, will have a number of columns one greater than the number of rows. Adding an additional row to the matrix by placing unity in the delta command column, δ_c , will make the input δ_c zero and the output arbitrary. The values of s which make the determinant of the resulting square matrix zero represent the natural frequencies of the missile dynamic system, and are the open loop poles of the system due to missile dynamics. The eigenvector associated with each pole or natural frequency represents the mode shape for that pole or frequency expressed as a linear combination of the input modes.

III-1 COMBINED MODE REPRESENTATION

In Section II the dynamic equations were derived for a missile configuration considering flexible body, sloshing, and engine dynamics. These equations of motion are:

Normal Force Equation

$$(V_{cg}s + a_x)\alpha + \ddot{Q}_L + (V_{cg}s + a_x)\theta - s\left\{N_{vn}\right\}_T\left\{Q_n\right\} - \left\{N_{an}\right\}_T\left\{Q_n\right\} = 0 \qquad \text{III-1.1}$$

Normal Acceleration Equation

$$-\frac{N_\alpha}{M}\alpha + \ddot{Q}_L + \beta_x\theta - \left\{a_{yn}\right\}_T\left\{Q_n\right\} = 0 \qquad \text{III-1.2}$$

Moment Equation

$$\mu_\alpha\alpha + s^2\theta + \left\{\mu_n\right\}_T\left\{Q_n\right\} = 0 \qquad \text{III-1.3}$$

Modal Equations

$$s^2\left[B\right]\left\{Q_n\right\} + s\left[D\right]\left\{Q_n\right\} + \left[A\right]\left\{Q_n\right\} = \left[k\right]\left\{\delta_{am}\right\} + \left\{J_m\right\}\alpha + \left[P\right]\left\{Q_n\right\} \qquad \text{III-1.4}$$

where n has values from 1 through N.

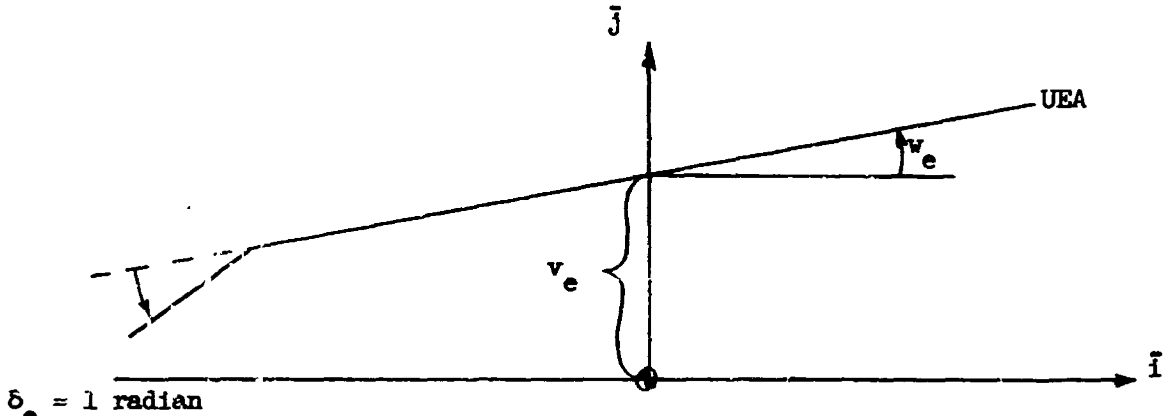
As noted in Section II-6.5, the system has N+2 degrees of freedom with respect to a translating set of inertial axes. The motion of the system is described by the above N+3 equations in the variables α, θ, $\ddot{Q}_L$ and the N values of Q_n, where $\ddot{Q}_K$ and Equation III-1.2 are a dependent variable and equation. The variables θ, and the Q_n's are generalized coordinates and α is a function of θ, the Q_n's and the rate of change of the generalized coordinate Q_L.

$$\alpha = -\left[\theta + \frac{\dot{Q}_L}{V_{cg}} - \left\{v_n\right\}_T\left\{Q_n\right\}\right] \qquad \text{III-1.5}$$

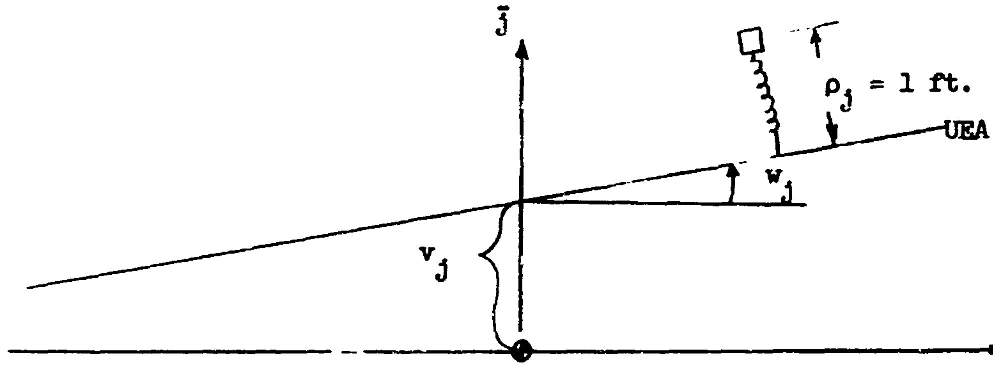
The generalized coordinate Q_L is the displacement of the missile center of graivty normal to the nominal flight path. The generalized coordinate θ is the rotation of the body axes, $\bar{i}$-$\bar{j}$, and the N coordinates, Q_n define the motion of the missile with respect to the $\bar{i}$-$\bar{j}$ axes. The displacement of the center of gravity along the nominal flight path has been eliminated from the equations by assuming the reference axes to be translating along the flight path at a velocity V_{cg}.

For each of the N coordinates Q_n there is a so-called input mode as defined in Section II-4. Each input mode is the amplitude of motion resulting when the corresponding coordinate Q_n has an amplitude of unity and all other coordinates are zero. The input modes when Q_n represents an engine deflection, a sloshing deflection, and a bending deflection are shown in Figure III-1. The input engine modes are normalized to an engine deflection of 1 radian and the motion consists of a rigid body translation, v_e, and a rigid body rotation, ψ_e, of the UEA. The input slosh modes are normalized to a slosh mass deflection of 1 foot and the motion consists of a rigid body translation, v_j, and a rigid body rotation, w_j, of the UEA. The input bending modes are normalized to a mass equal to the total mass of the missile for motion with respect to the UEA. The rigid body translation v_i is zero and the rigid body rotation, w_i, exists only when inertia corrections have been made. Since a bending coordinate, b_i, of unity represents motion equivalent to a normal mode, b_i is measured at the point where the deflection is unity. The modal deflections and slopes are in ft. per ft. and radians per ft. since b_i is in feet.

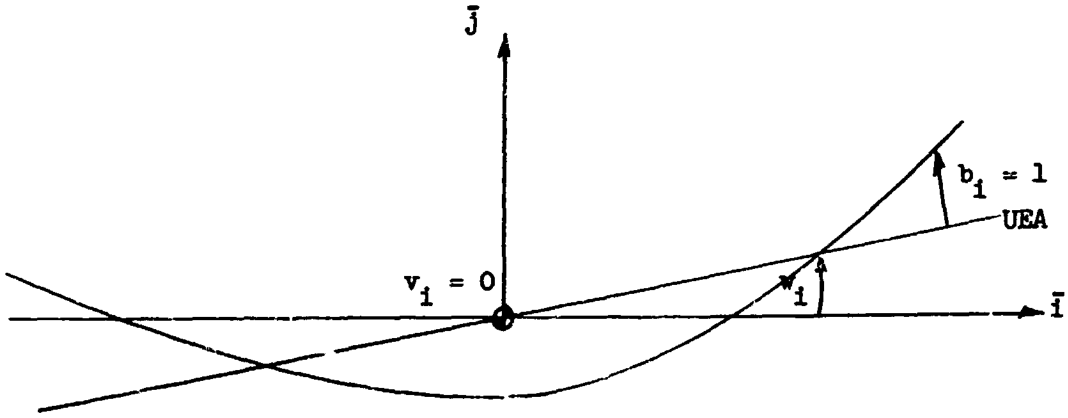
As noted in Section II-4 the modes defined as input modes are not true modes of the system since each can only exist if all other coordinates are held fixed. However, since the system is linear, superposition is valid and any free or driven motion of the system is a linear combination of the motions associated with the input modes and may be defined in terms of normal coordinates which are a linear combination of the input coordinates Q_n. It was also noted in Section II-7 that the modes of oscillation associated with the open loop natural frequencies or poles are a linear combination of the input modes. Likewise, the modes of oscillation associated with the closed loop natural frequencies or roots of the complete autopilot configuration are a linear combination of the input modes. Therefore, the motion associated with the closed loop roots is a linear combination of the motion associated with the open loop poles. From the above it can be seen that



INPUT ENGINE MODE



INPUT SLOSH MODE



INPUT BENDING MODE

FIGURE III-1
INPUT MODES

the modes for the closed loop roots, or the final motion of the system can be expressed in terms of a linear combination of any set of modes and their coordinates which are in turn a linear combination of the input modes. These new modes will be referred to in this analysis as "combined modes." It may be seen that the modes corresponding to the open loop poles of the system are in effect combined modes each with a normal coordinate which is a linear combination of the input coordinates Q_n. The final motion of the system corresponding to the closed loop roots may then be expressed in terms of a linear combination of these normal coordinates.

Combined mode representation in connection with missile dynamic equations consists of defining a set of orthogonal modes in terms of a linear combination of the so-called input modes defined in Section II-4 and shown in Figure III-1. Replacing the input modes by this set of orthogonal combined modes produces a set of missile dynamic equaitons in a form which is independent of missile configuration with regard to propellant tank and engine arrangement. By using this representation the final stability equations retain the same form regardless of the missile system configuration. Also, as noted in Section II-3.2, the highest input bending frequency should be above the highest engine or slosh mode frequency in order to reduce truncation errors due to the use of a finite number of bending modes. This criterian usually results in the use of from three to five bending modes. The use of additional modes increases the size of the system matrix and requires additonal computer time to obtain the solutions. With the use of combined mode representation, a larger number of bending modes may be used in the calculation of the combined modes and the dropping of the higher frequency orthogonal combined modes from the system equations will tend to reduce truncation errors.

III-2 SELECTION OF COMBINED MODES

In Section III-1 the principle of superposition in a linear system was used to show that the motion of the missile may be described in terms of a linear combination of a set of combined modes and the dynamic equations can be expressed in terms of their normal coordinates which in turn are a linear combination of the coordinates Q_n. Let the normal coordinates of the combined modes be represented by q_a, a equals 1 through N. Combined mode representation consists of making a linear coordinate transformation from the generalized coordinates Q_n to the normal coordinates q_a and defining the motion in terms of the combined modes instead of the input modes. Considering the modal equation represented by III-1.4,

$$s^2 \left[B\right]\left\{Q_n\right\} + s \left[D\right]\left\{Q_n\right\} + \left[A\right]\left\{Q_n\right\} = \left[k\right]\left\{\delta_{am}\right\} + \left\{J_m\right\} \alpha + \left[P\right]\left\{Q_n\right\}$$

The left side of the equation is a function of only the Q_n's and when equated to zero represents a eigenvalue problem, the solution of which gives the natural frequencies and mode shapes for the system when the actuator position, aerodynamic forces, and thrust forces are zero. The right side of the equation then represents the driving forces for these new combined modes. The matrix $\left[P\right]\left\{Q_n\right\}$ represents the spring forces resulting from engine thrust and is also only a function of the Q_n's. This term may be transferred to the left side of the equation and the combined modes then become the undriven damped modes with engine thrust forces present. Likewise, the aerodynamic forces $\left\{J_m\right\} \alpha$ can be made functions of the Q_n's only by using Equations III-1.1, III-1.2, and III-1.3 to express α as a function of Q_n's. Transferring this term to the left side of the equation will result in combined modes for undriven, damped modes with engine thrust and aerodynamic forces present. The only driving force remaining on the right side is $\left[k\right]\left\{\delta_{am}\right\}$ due to engine actuator positions. These mode shapes and frequencies require only the addition of the engine hydraulic equation to yield the system open loop poles and associated mode shapes. Calculation of the combined modes with the loop opened in this manner would require a considerable expenditure of computer time approaching that required to obtain open loop poles. The use of combined modes is only justified if the modes can be obtained quickly wi[illegible] a small expenditure of computer time. This criterian requires that the aerodynamic forces $\left\{J_m\right\} \alpha$ and the thrust spring forces $\left[P\right]\left\{Q_n\right\}$ remain on the right side of Equation III-1.4 as driving forces. In addition, the damping forces represented by s $\left[D\right]\left\{Q_n\right\}$ make the constants which express the combined modes as a linear combination of the input modes, complex numbers which greatly increases the required computer time. The more suitable form of Equation III-1.4 for combined mode calculations is then

$$s^2 \left[B\right]\left\{Q_n\right\} + \left[A\right]\left\{Q_n\right\} = -s\left[D\right]\left\{Q_n\right\} + \left[k\right]\left\{\delta_{am}\right\} + \left\{J_m\right\}\alpha + \left[P\right]\left\{Q_n\right\} \qquad \text{III-2.1}$$

In this form the homogeneous equation

$$s^2 \left[B\right]\left\{Q_n\right\} + \left[A\right]\left\{Q_n\right\} = 0 \qquad \text{III-2.2}$$

may be solved by the $AX = \lambda BX$ computer subroutine (Reference 6). This subroutine is extremely fast but requires that the $\left[A\right]$ and $\left[B\right]$ matrix be symmetric and that the $\left[B\right]$ matrix be positive and definite. The solution of Equation III-2.2 will give the undamped natural frequencies of the system when the actuator positions, the aerodynamic forces, and the thrust forces are all zero. The $\left[P\right]$ matrix representing the thrust spring forces cannot be placed on the left side of the equation since it is not a symmetric matrix. In the previous analysis, Reference 1, the symmetric components of the $\left[P\right]$ matrix were placed on the left side and included in the combined modes; however, for this analysis all of the effects of the thrust forces are considered driving forces.

III-3 COORDINATE TRANSFORMATION

Solution of Equation III-2.2 will result in N values of s^2 where $s^2 = -\Omega^2$ and N associated eigenvectors $\left\{e_{na}\right\}$. Where:

Ω_a = natural frequency of the a^{th} combined mode

$\left\{e_{na}\right\}$ = a column matrix of the values of $\left\{Q_n\right\}$ obtained from III-2.2 when $s^2 = -\Omega_a^2$.

Then e_{na} is the amount of Q_n present in the a^{th} combined mode. The eigenvectors $\left\{e_{na}\right\}$ obtained from the subroutine are normalized such that

$$\left\{e_{na}\right\}_T \left[B\right]\left\{e_{na}\right\} = 1 \qquad \text{III-3.1}$$

and since the modes are orthogonal

$$\{e_{na}\}_T [B]\{e_{nb}\} = 0 \qquad \text{III-3.2}$$

Let

$$[E] = \left[\{e_{n1}\}\{e_{n2}\}\cdot\cdot\cdot\cdot\{e_{na}\}\cdot\cdot\cdot\cdot\{e_{nN}\}\right] = [e_{na}] \qquad \text{III-3.3}$$

when $[E]$ is a N x N square matrix, the columns of which are the eigenvectors found in the solution of Equation III-2.2. This $[E]$ matrix is defined as the normalized modal matrix of Equation III-2.2 From III-3.1, III-3.2, and III-3.3

$$[E]_T [B][E] = [I] \qquad \text{III-3.4}$$

Since the values of B and A in Equation III-2.2 are the masses and spring constants divided by the total mass of the system, Equations III-3.1 and III-3.4 are equivalent to normalizing the combined modes to a mass equal to the total mass. Since $\{e_{na}\}$ is the solution of III-2.2 when $s^2 = -\Omega_a^2$

$$[A]\{e_{na}\} = \Omega_a^2 [B]\{e_{na}\} \text{ from which } [A][E] = [B][E][\Omega^2] \qquad \text{III-3.5}$$

where $[\Omega^2]$ is a diagonal matrix of the values of Ω_a^2, premultiplying III-3.5 by $[E]_T$

$$[E]_T [A][E] = [E]_T [B][E][\Omega^2]$$

using III-3.4

$$[E]_T [A][E] = [\Omega^2] \qquad \text{III-3.6}$$

Since e_{na} is the amount of Q_n present in the a^{th} combined mode and q_a is the amount of the a^{th} combined mode present,

$$Q_n = \sum_a e_{na}\, q_a = \{n^{th} \text{ row of } [E]\}\{q_a\}$$

then

$$\{Q_n\} = [E]\{q_a\} \qquad \text{III-3.7}$$

Substituting this coordinate transformation into Equation III-1.4

$$s^2 \left[B\right]\left[E\right]\left\{q_a\right\} + \left[A\right]\left[E\right]\left\{q_a\right\} = -s \left[D\right]\left[E\right]\left\{q_a\right\} + \left[k\right]\left\{\delta_{am}\right\} + \left\{J_m\right\} \alpha + \left[P\right]\left[E\right]\left\{q_a\right\}$$

premultiplying by $\left[E\right]_T$

$$s^2 \left[E\right]_T \left[B\right]\left[E\right]\left\{q_a\right\} + \left[E\right]_T \left[A\right]\left[E\right]\left\{q_a\right\} = -s \left[E\right]_T \left[D\right]\left[E\right]\left\{q_a\right\} + \left[E\right]_T \left[k\right]\left\{\delta_{am}\right\}$$
$$+ \left[E\right]_T \left\{J_m\right\}\alpha + \left[E\right]_T \left[P\right]\left[E\right]\left\{q_a\right\}$$

from III-2.6 and III-2.8

$$s^2\left[I\right]\left\{q_a\right\} + \left[\Omega^2\right]\left\{q_a\right\} = -s \left[E\right]_T \left[D\right]\left[E\right]\left\{q_a\right\} + \left[E\right]_T \left[k\right]\left\{\delta_{am}\right\} + \left[E\right]_T \left\{J_m\right\} \alpha$$
$$+ \left[E\right]_T \left[P\right]\left[E\right]\left\{q_a\right\}$$

$$\left[s^2 + \Omega_a^2\right]\left\{q_a\right\} = -s \left[E\right]_T \left[D\right]\left[E\right]\left\{q_a\right\} + \left[E\right]_T \left[k\right]\left\{\delta_{am}\right\} + \left[E\right]_T \left\{J_m\right\} \alpha$$
$$+ \left[E\right]_T \left[P\right]\left[E\right]\left\{q_a\right\} \qquad \text{III-3.8}$$

where

$\left[s^2 + \Omega_a^2\right]$ is a diagonal matrix = $(s^2 + \Omega_a^2) \left[I\right]$

Let $\left[D_c\right] = \left[E\right]_T \left[D\right]\left[E\right] = \left[D_{ab}\right]$ = damping matrix in III-3.9

normal coordinates where D_{ab} is 1/M times the damping force associated with the q_a coordinate due to unit velocity of the q_b coordinate.

Let $\left[k_c\right] = \left[E\right]_T \left[k\right] = \left[k_{ab}\right]$ III-3.10

where k_{ab} is 1/M times the driving force for the a^{th} combined mode due to unit deflection of the b^{th} engine actuator.

Let $\left\{J_{ca}\right\} = \left[E\right]_T \left\{J_m\right\}$ III-3.11

where J_{ca} is 1/M times the driving force for the a^{th} combined mode due to a unit angle of attack.

Let $\left[P_c\right] = \left[E\right]_T \left[P\right]\left[E\right] = \left[P_{ab}\right]$ III-3.12

where P_{ab} is 1/M times the thrust spring force associated with the a^{th} normal coordinate due to unit deflection of the b^{th} normal coordinate. Then the modal equation in normal coordinates becomes

$$\left[\left[I\right] s^2 + \left[\Omega^2\right] + s \left[D_c\right] - \left[P_c\right]\right]\left\{q_a\right\} = \left[k_c\right]\left\{\delta_{am}\right\} + \left\{J_{ca}\right\}\alpha \qquad \text{III-3.13}$$

III-4 DYNAMIC EQUATIONS IN NORMAL COORDINATES

Substituting the coordinate transformation $\left\{Q_n\right\} = \left[E\right]\left\{q_a\right\}$ into the normal force equation III-1.1

$$(V_{cg}\, s + a_x)\alpha + \ddot{Q}_L + (V_{cg}\, s + a_x)\theta - s\left\{N_{vn}\right\}_T \left[E\right]\left\{q_a\right\} - \left\{N_{an}\right\}_T \left[E\right]\left\{q_a\right\} = 0$$

or

$$(V_{cg}\, s + a_x)\alpha + \ddot{Q}_L + (V_{cg}\, s + a_x)\theta - s\left\{N_{cva}\right\}_T \left\{q_a\right\} - \left\{N_{caa}\right\}_T \left\{q_a\right\} = 0 \qquad \text{III-4.1}$$

where

$$\left\{N_{cva}\right\} = \left[E\right]_T \left\{N_{vn}\right\} \qquad \text{III-4.2}$$

$$\left\{N_{caa}\right\} = \left[E\right]_T \left\{N_{an}\right\} \qquad \text{III-4.3}$$

The normal acceleration equation III-1.2 becomes

$$-\frac{N_\alpha}{M}\alpha + \ddot{Q}_L + \beta_x\,\theta - \left\{a_{yn}\right\}_T \left[E\right]\left\{q_a\right\} = 0$$

$$-\frac{N_\alpha}{M}\alpha + \ddot{Q}_L + \beta_x\,\theta - \left\{a_{yca}\right\}_T \left\{q_a\right\} = 0 \qquad \text{III-4.4}$$

where

$$\left\{a_{yca}\right\} = \left[E\right]_T \left\{a_{yn}\right\} \qquad \text{III-4.5}$$

The moment equation III-1.3 becomes:

$$\mu_\alpha \alpha + s^2 \theta + \left\{\mu_n\right\}_T \left[E\right] \left\{q_a\right\} = 0$$

$$\mu_\alpha \alpha + s^2 \theta + \left\{\mu_{ca}\right\}_T \left\{q_a\right\} = 0 \qquad \text{III-4.6}$$

where

$$\left\{\mu_{ca}\right\} = \left[E\right]_T \left\{\mu_n\right\} \qquad \text{III-4.7}$$

the modal equation is given by III-3.13.

III-5 COMBINED MODES

The combined modes selected in Section III-2 are defined by the solutions to Equation III-2.2

$$s^2 \left[B\right]\left\{Q_n\right\} + \left[A\right]\left\{Q_n\right\} = 0$$

The resulting N values of Ω_a and the corresponding N eigenvectors $\left\{e_{na}\right\}$ represent the N combined mode frequencies and mode shapes. The order of numbering the combined modes is arbitrary and is a function of the manner in which the roots of Equation III-2.2 are numbered. Since the equation is to be solved by means of the AX = λBX computer subroutine, it is convenient to order the roots and therefore the combined modes in the order which the computer subroutine finds the roots. This process is not arbitrary since in a loosely coupled system the routine tends to find roots in an order corresponding to the frequencies represented by the data along the main diagonals of the A and B matrices. As a result, the first combined mode will usually have a frequency close to the first input engine frequency and will contain the largest first engine amplitude. Likewise, the second combined mode will have a frequency close to and contain large second engine amplitudes, etc. This correspondence between the input mode frequencies and the combined mode frequencies has resulted in a tendency to name the combined in the same order as the input modes, first engine mode, first slosh mode, second slosh mode, etc. However, it should be noted that this correspondence of input and output frequencies may not hold for tightly coupled modes such as two slosh or engine modes with input frequencies close together. In fact, a rerun of the program with slight changes in parameters may cause two such modes to exchange position in the output. This process has no effect upon the equations or their solutions once the combined modes are numbered but does lead to some confusion if the combined modes are associated with a corresponding input mode.

As noted in Section III-3, the eigenvector $\{e_{na}\}$ is the solution to Equation III-2.2 when $s^2 = -\Omega_a^2$. Therefore, e_{na} is the amount of Q_n present in the a^{th} combined mode, and letting $\{Q_n\}$ equal $\{e_{na}\}$ defines the a^{th} combined mode.

From Equation III-3.3

$$[E] = \left[\{e_{n1}\} \{e_{n2}\} \{e_{n3}\} \cdots \{e_{na}\} \cdots \{e_{nN}\} \right]$$

and

$$[E]_T = \begin{bmatrix} \{e_{n1}\}_T \\ \{e_{n2}\}_T \\ \{e_{na}\}_T \\ \{e_{nN}\}_T \end{bmatrix}$$

the a^{th} column of the modal matrix $[E]$ gives the composition of the a^{th} combined mode in terms of the normalized input modes. Likewise, the a^{th} row of $[E]_T$ also gives the composition of the a^{th} combined mode.

It was noted in Section II-5.2 that the deflections and slopes of an input bending mode could be multiplied by -1 thereby defining a new input bending mode and shifting the phase of the finial b_i by 180 degrees. In order to be consistent, a positive input bending mode was defined as that phase of the mode which results in a positive deflection at the aft end of the missile. Likewise, the positive input engine and slosh modes were defined in Section III-1 Figure III-1 as that phase of the motion resulting from positive engine and slosh deflections. Since e_{na} is the amount of the n^{th} input mode present in the a^{th} combined mode, a value of e_{na} of +.7 for example indicates that the mode shape of the a^{th} combined mode contains .7

times the normal n^{th} input mode. Defining the opposite phase of the input mode would result in an e_{na} of -.7 and the resulting combined mode would be the same. Therefore, it may be seen that the definition of a positive phase for the input modes is not required in order to determine the combined modes or to perform a stability analysis but has been done in order to provide a consistent basis upon which to compare the pahse relationship between the input modes for different missile configurations or for different times of flight. It may also be seen that the phase of the a^{th} combined mode and therefore the phase of q_a can be changed by 180 degrees if the a^{th} column of $\left[E\right]$ or the a^{th} row of $\left[E\right]_T$ is multiplied by a factor of -1.0. It is then desirable to define a positive phase of the combined modes or in effect a positive phase of each q_a. Since the engine actuator positions represent the external driving forces for the dynamic equations, and the first inputed engine is usually the main control engine, the deflection of the first input engine provides a convenient common reference for the combined modes. The positive phase of each combined mode is then taken as that phase for which the first engine deflection is positive. This phase relationship is accomplished by multiplying the columns of the normalized modal matrix $\left[E\right]$ by plus or minus unity such that all of the elements of the first row of the matrix are positive numbers. The a^{th} combined mode then represents the motion with respect to the i-j axes when the normal coordinate q_a is equal to + 1.0.

Referring to Figure III-5.1

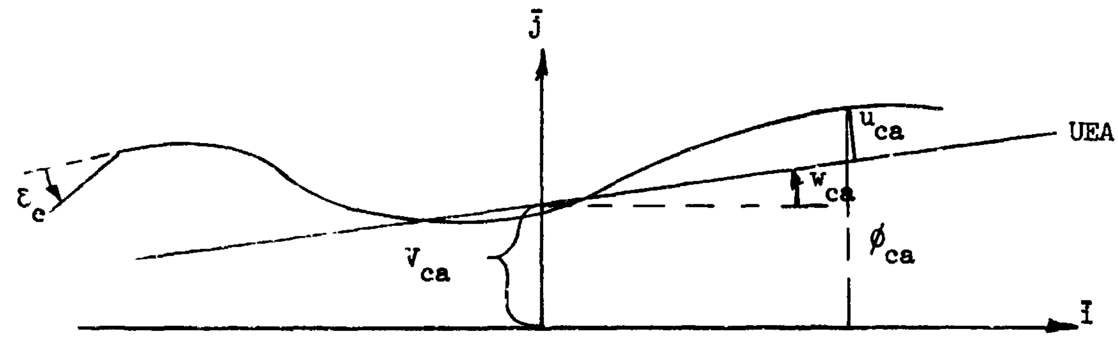


Figure III-5.1

The deflection of the a^{th} combined mode at station $\hat{\xi}$ is

$$\phi_{c\hat{\xi}a} = v_{ca} + \frac{(\xi_{cg} - \hat{\xi})}{12} w_{ca} + u_{c\hat{\xi}a} \qquad \text{III-5.1}$$

The total slope of the a^{th} combined mode at station $\hat{\xi}$ is

$$\phi'_{c\hat{\xi}a} = w_{ca} + u'_{c\hat{\xi}a} \qquad \text{III-5.2}$$

The bending slope of the a^{th} combined mode at station $\hat{\xi}$ is

$$\lambda_{c\hat{\xi}a} = w_{ca} + \psi_{c\hat{\xi}a} \qquad \text{III-5.3}$$

From Equation II-4.8

$$v = \{v_m\}_T \{Q_n\}$$

$$v_{ca} = \{v_m\}_T \{e_{na}\} \qquad \text{III-5.4}$$

or $$\{v_{ca}\} = [E]_T \{v_m\} \qquad \text{III-5.5}$$

From Equation II-4.18

$$w = \{w_m\}_T \{Q_n\}$$

$$w_{ca} = \{w_m\}_T \{e_{na}\} \qquad \text{III-5.6}$$

or $$\{w_{ca}\} = [E]_T \{w_m\} \qquad \text{III-5.7}$$

from II-3.6

$$u_{\hat{\xi}} = \sum_i \phi_{\hat{\xi}i} b_i = \{\phi_{\hat{\zeta}n}\}_T \{Q_n\} \qquad \text{III-5.8}$$

$$u'_{\hat{\xi}} = \sum_i \phi'_i b_i = \{\phi'_{\hat{\zeta}n}\}_T \{Q_n\} \qquad \text{III-5.9}$$

$$\psi = \sum_i \lambda_i b_i = \{\lambda_n\}_T \{Q_n\} \qquad \text{III-5.10}$$

$$u_{c\xi a} = \left\{\phi_{\xi n}\right\}_T \left\{e_{na}\right\} \qquad \text{III-5.11}$$

$$u'_{c\xi a} = \left\{\phi'_{\xi n}\right\}_T \left\{e_{na}\right\} \qquad \text{III-5.12}$$

$$\psi_{c\xi a} = \left\{\lambda_{\xi n}\right\}_T \left\{e_{na}\right\} \qquad \text{III-5.13}$$

Equations III-5.1, III-5.2, and III-5.3 become

$$\phi_{c\xi a} = \left\{v_m + \frac{\xi_{cg} - \xi}{12} w_m + \phi_{\xi n}\right\}_T \left\{e_{na}\right\} \qquad \text{III-5.14}$$

$$\phi'_{c\xi a} = \left\{w_m + \phi'_{\xi n}\right\}_T \left\{e_{na}\right\} \qquad \text{III-5.15}$$

$$\lambda_{c\xi a} = \left\{w_m + \lambda_{\xi n}\right\}_T \left\{e_{na}\right\} \qquad \text{III-5.16}$$

where m = n.

From Figure III-5.1 and the above equations it may be seen that where the input bending modes are the natural frequencies and mode shapes for the system with engines and sloshing masses locked to the beam, the combined modes are the natural frequencies and mode shapes for the same system with the engines and sloghing masses free and restrained by their respective springs and inertia corrections are applied to the beam.

The magnitude of the engine deflections and sloshing deflections associated with each combined mode can be obtained from the modal matrix $\left[E\right] = \left[e_{na}\right]$ since e_{na} is a amplitude of the n^{th} generalized coordinate in the a^{th} combined mode.

In combined mode representation the motion of the system is described in terms of the normal combined mode coordinates q_a. However, a feedback of engine deflections is required for the engine hydraulic equations, therefore, each engine deflection must be expressed in terms of the normal coordinates q_a. From Equation III-3.7

$$\{Q_n\} = [E]\{q_a\}$$

$$Q_n = \{n^{th} \text{ row of } [E]\}\{q_a\}$$

$$\delta_e = \{e^{th} \text{ row of } [E]\}\{q_a\}$$

Defining the e^{th} row of $[E]$ as $\{\Delta_{ea}\}$ III-5.17

then

$$e_a = \{\Delta_{ea}\}_T \{q_a\} \qquad \text{III-5.18}$$

III-6 MISSILE DYNAMICS IN NORMAL COORDINATES

In Section II-2, the dynamic equations were derived for a system of masses in terms of the translational motion of the system center of gravity, the rotation of a set of $\bar{i}$ - $\bar{j}$ body axes, and the motion of the masses with respect to the $\bar{i}$ - $\bar{j}$ body axes. The system of masses was defined in terms of a missile configuration in section II-3, and the motion with respect to the $\bar{i}$ - $\bar{j}$ axes was derived in section II-4 in terms of a set of N generalized coordinates, Q_n. The generalized coordinates used where those for which the motion was statically uncoupled. Missile dynamics in terms of these generalized coordinates is presented in section II-7.

The motion of the system with respect to the $\bar{i}$ - $\bar{j}$ axes has been redefined in section III in terms of the normal modes of the undamped, undriven system and the associated normal coordinates. The dynamic equations for the system transformed into normal coordinates are given in section III-4. These equations together with an engine hydraulic equation and an instrument equation are the data required for the missile dynamics block of Figure I - 2. The missile dynamics represents the transfer function from delta command, δ_c, to the gyro outputs, θ_P and θ_R. Figure III-6.1 contains a block diagram showing the effective flow of the data in the dynamic equations in normal coordinates. It is assumed that in a system containing P gimballed engines, the command signals for each engine or group of engines may be different. The engine hydraulic equations will again be of the form of equation II-7.1 requiring a feedback of engine positions δ_e. The engine actuator positions together with the angle of attack feedback drive the modal equations III-3.13 producing the normal coordinates q_a. These coordinates in turn drive the

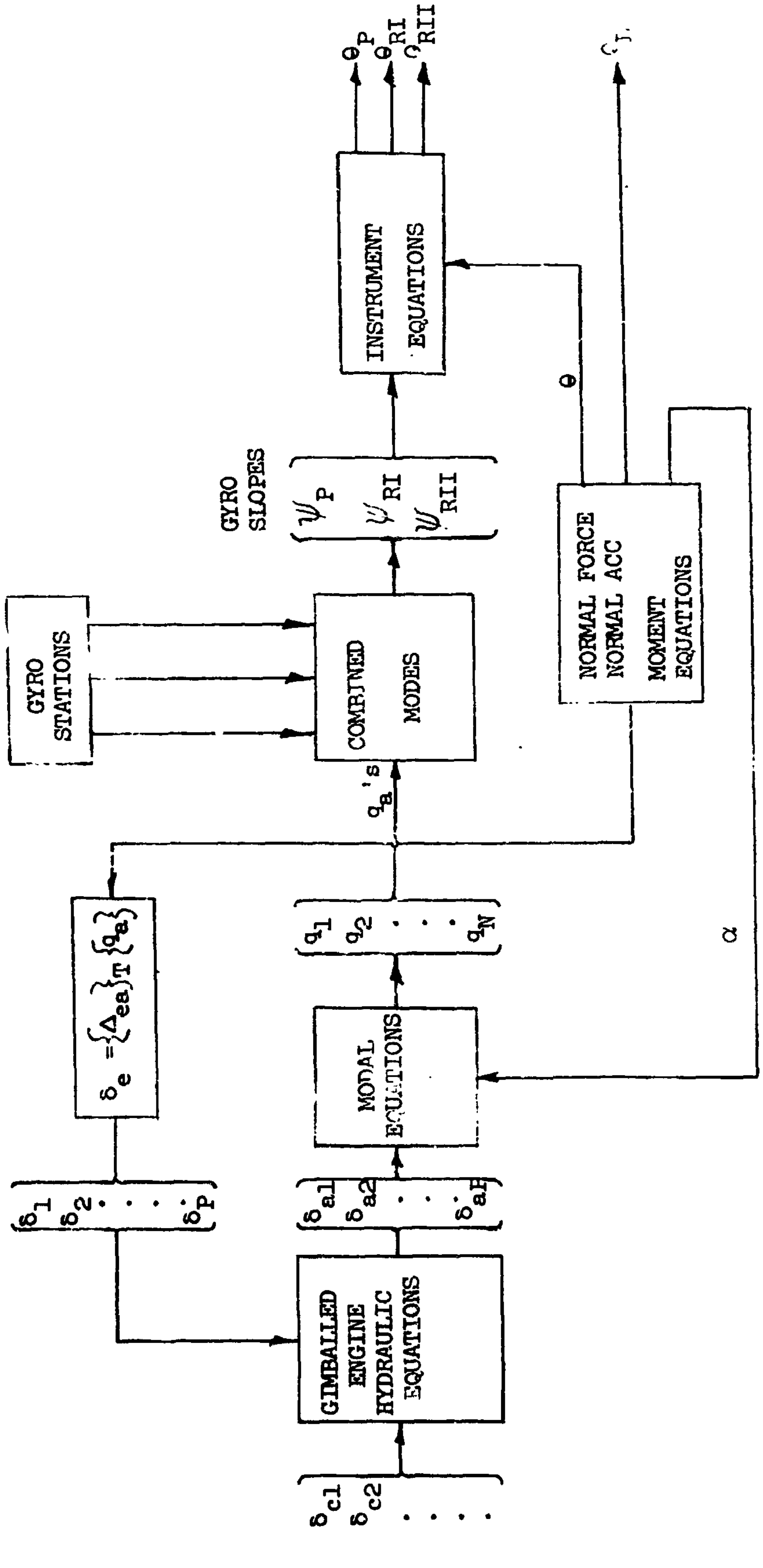


FIGURE III - 6.1
MISSILE DYNAMICS IN NORMAL COORDINATES

normal force, normal acceleration and moment equations to produce the rigid body rotation of the $\bar{i}$ - $\bar{j}$ axes, θ, and the angle of attack α. In addition the normal coordinates operate with equation III-5.18 to produce the engine deflection to be fedback to the engine hydraulic equations and are combined with the combined mode bending data to produce the bending slopes at each gyro location. These bending slopes are combined with the rigid body rotations to produce the gyro signals. Since the rigid body rotation due to motion with respect to the $\bar{i}$ - $\bar{j}$ axes, w, is included in the combined mode slopes this term does not appear in the instrument equations when in normal coordinates. Equations II-7.2 then become

$$\theta_P = \theta - \psi_P$$

$$\theta_{RI} = \theta - \psi_{RI}$$

$$\theta_{RII} = \theta - \psi_{RII} \qquad \text{III-6.1}$$

$$\theta_R = \theta_{RI} + a_R\, \theta_{RII}$$

That portion of the system matrix representing missile dynamics in normal coordinates is shown in Figure III-6.2. The first row of the matrix represents the normal force equation III-4.1. The second row is the normal acceleration equation III-4.4, and the third row is the moment equation III-4.6. The next N rows are the N modal equations III-3.13. The system shown is assumed to have three gimballed engines, therefore, columns 11, 12 and 13 and rows 11, 12 and 13 represent the engine angles found from equation III-5.18 while columns 14, 15 and 16 represent the three engine actuators. The next four rows and columns introduce the instrument equations III-6.1 while the last three rows represent the engine hydraulic equations II-7.1. For the system shown it is assumed that a single command signal, column 21, controls the actuators of the first two gimballed engines and the third engine is restrained by making its command signal zero.

The input to the missile dynamics is δ_c of column 21 while the output is the gyro signals of columns 17 and 20. The system matrix is completed by adding rows and columns representing the autopilot which operates on the gyro signals to produce the command signal.

FIGURE III - 6.2 SYSTEM MATRIX IN NORMAL COORDIN

	1	2	3	4	5	6	7	8	9	10	11
	α	$\ddot{Q}_L$	θ	q_1	q_2	q_3	....	q_a	....	q_N	δ_1
1	$V_{cg}\,s + a_x$	1	$V_{cg}\,s + a_x$	$-sN_{cv1}$ $-N_{ca1}$	$-sN_{cv2}$ $-N_{ca2}$	$-sN_{cv3}$ $-N_{ca3}$	....	$-sN_{cva}$ $-N_{caa}$	....	$-sN_{cvN}$ $-N_{caN}$	
2	$-\frac{N_\alpha}{M}$	1	β_x	$-a_{yc1}$	$-a_{yc2}$	$-a_{yc3}$	....	$-a_{yca}$	....	$-a_{ycN}$	
3	μ_α		s^2	μ_{c1}	μ_{c2}	μ_{c3}	....	μ_{ca}	....	μ_{cN}	
4	$-J_{c1}$										
5	$-J_{c2}$										
6	$-J_{c3}$										
7	$\vdots$			$s^2\left[I\right] + \left[\Omega^2\right] - s\left[D_c\right] - \left[P_c\right]$							
8	$-J_{ca}$										
9	$\vdots$										
10	$-J_{cN}$										
11				$-\Delta_{11}$	$-\Delta_{12}$	$-\Delta_{13}$	....	$-\Delta_{1a}$	....	$-\Delta_{1N}$	1
12				$-\Delta_{21}$	$-\Delta_{22}$	$-\Delta_{23}$	....	$-\Delta_{2a}$	....	$-\Delta_{2N}$	
13				$-\Delta_{31}$	$-\Delta_{32}$	$-\Delta_{33}$	....	$-\Delta_{3a}$	....	$-\Delta_{3N}$	
14			-1	λ_{P1}	λ_{P2}	λ_{P3}	....	λ_{Pa}	....	λ_{PN}	
15			-1	$\left(\lambda_{RI}\right)_1$	$\left(\lambda_{RI}\right)_2$	$\left(\lambda_{RI}\right)_3$	....	$\left(\lambda_{RI}\right)_a$	....	$\left(\lambda_{RI}\right)_N$	
16			-1	$\left(\lambda_{RII}\right)_1$	$\left(\lambda_{RII}\right)_2$	$\left(\lambda_{RII}\right)_3$	....	$\left(\lambda_{RII}\right)_a$	....	$\left(\lambda_{RII}\right)_N$	
17											
18											$K_1 s+K_2$
19											
20											

12	13	14	15	16	17	18	19	20	21	
δ_2	δ_3	δ_{a1}	δ_{a2}	δ_{a3}	θ_P	θ_{RI}	$\theta_{R\ II}$	θ_R	δ_c	**EQUATION**
										III - 4.3
										III - 4.4
										III - 4.7
		$-k_{11}$	$-k_{12}$	$-k_{13}$						
		$-k_{21}$	$-k_{22}$	$-k_{23}$						
		$-k_{31}$	$-k_{32}$	$-k_{33}$						
		$\vdots$	$\vdots$	$\vdots$						III - 3.13
		$-k_{a1}$	$-k_{a2}$	$-k_{a3}$						
		$\vdots$	$\vdots$	$\vdots$						
		$-k_{N1}$	$-k_{N2}$	$-k_{N3}$						
1										III - 5.18
	1									
					1					
						1				
							1			III - 6.1
						a_R	1	-1		
		K_2s+K_3							1	
K_5s+K_6			K_7s+K_8						1	II - 7.1

The matrix representing missile dynamics, as shown in figure III-6.2, will have a number of columns one greater than the number of rows. Adding an additional row to the matrix by placing unity in the delta command column, δ_c, will make the input δ_c zero and the output arbitrary. The values of s which make the determinate of the resulting square matrix zero represent the natural frequencies of the missile dynamic system, and are the open loop poles of the system due to missile dynamics, the eigenvectors associated with each pole or natural frequencies represents the mode shape for that pole or frequency expressed as a linear combination of the combined modes.

GLOSSARY

a - subscript denoting a^{th} combined mode
a_r - coefficient used in combining rate gyro outputs
a_x - acceleration of c.g. along $\bar{i}$ axis
a_y - acceleration of c.g.along $\bar{j}$ axis
b - subscript denoting bending
b_i - generalized bending coordiante for i^{th} bending mode
c - subscript denoting combined mode data
e - subscript denoting e^{th} gimballed engine
f - subscript denoting f^{th} fixed engine
g - acceleration of gravity
h - subscript referring to the UEA
i - subscript denoting i^{th} bending mode
$\bar{i}$ - rotating reference axis
j - subscript denoting j^{th} sloshing mass
$\bar{j}$ - rotating reference axis
k - subscript denoting k^{th} inertia correction
k_I - fluid inertia correction factor
k_i - equivalent bending spring constant
k_j - slosh spring constant
$[k]$ - spring matrix
ℓ_{Ge} - distance from gimbal to center of gravity of e^{th} gimballed engine
ℓp - distance from c.g. to center of pressure
m - subscript denoting m^{th} generalized coordinate
m_f - total fluid mass in tank
m_j - equivalent slosh mass
n - subscript denoting n^{th} generalized coordinate
p - subscript denoting p^{th} gimballed engine
q_a - normal coordinate for a^{th} combined mode
r - radius of gyration of element of mass dm
r - subscript denoting r^{th} slosh mass
s - LaPlace operator
t - time
t - subscript denoting t^{th} bending mode

(Cont'd) GLOSSARY

u - translational deflection of beam element at x_h due to elastic deformation

u' - total slope at x_h due to elastic deformation

v - displacement of UEA along $\bar{j}$ axis

w - rotation of UEA in $\bar{i}$-$\bar{j}$ plane

x - coordinate along $\bar{i}$ axis

y - coordinate along $\bar{j}$ axis

z_e - coordinate on e^{th} gimballed engine measured aft from gimbal

$[A]$ - normalized spring matrix

$[B]$ - normalized mass matrix

D - total aerodynamic drag

$[D]$ - damping matrix

D_{Ge} - gimbal damping for e^{th} engine (ft-lbs per radian)

D_{sj} - damping for j^{th} slosh mass (lbs. per ft/sec.)

D_{bi} - damping j i^{th} bending mode (lbs. per ft./sec.)

E - subscript denoting a point on a gimballed engine

F - number of fixed engines

F - subscript denoting a fixed engine

F_D - dissipation function

H - inertial axis

I - mass moment of inertia of system about its c.g.

I_{Ge} - mass moment of inertia of e^{th} engine about its gimbal point

$[J]$ - matrix representing aerodynamic forces

K - number of inertia corrections

K_{ae} - spring constant for e^{th} engine

L - inertial axis

L_T - length of fluid tank

M - total mass of system

M_i - normalized mass of i^{th} bending mode

M_{Ge} - mass of e^{th} gimballed engine

$M_{\delta e}$ - moment at gimbal for e^{th} gimballed engine

N - number of degrees of freedom with respect to $\bar{i}$-$\bar{j}$ axes

(Cont'd) GLOSSARY

N_α - normal force coefficient (lbs per radian)

$N_{\alpha PD}$ - distributed normal force coefficient

N_{vn} - normal force due to n^{th} coordiante as a result of the velocity

N_{an} - normal force due to n^{th} coordiante as a result of aerodynamics

P - axis parallel to L axis but translating with c.g.

P - number of gimballed engines

P - external force when used with subscript

Q - generalized coordinate when used with subscript

R - axis parallel to H axis but translating with c.g.

R - number of fluid tanks

R_T - fluid tank radius

T - number of bending modes

T_{Ge} - thrust of e^{th} gimballed engine

T_{Ff} - thrust of f^{th} fixed engine

T_T - total thrust of all engines

T - used as subscript to denote the transpose of a matrix

V_{cg} - velocity of center of mass at time t

V_G - potential energy due to gravity

V_F - internal potential energy

α - angle of attack of UEA

α_o - angle of attack of $\bar{I}$ axis

β_α - thrust constant $\dfrac{T_T - D}{M}$

γ - angle between local vertical and nominal trajectory

δ_e - deflection of e^{th} gimballed engine

δ_{ae} - position of actuator on e^{th} gimballed engine

θ - rotation of $\bar{I}$-$\bar{J}$ axis in inertial space

θ_P - position gyro angle

θ_R - rate gyro angle

Δ_{ea} - engine angle of e^{th} engine due to a^{th} combined mode

Ω - pertebation of V_{cg}

Ω_a - frequency of a^{th} combined mode

(Cont'd) GLOSSARY

$[\Omega^2]$ - frequency matrix

Λ - rotation of mass element dm with respect to the R axis

Ψ - rotation of mass element dm with respect to the R axis

ψ - rotation of mass element dm with respect to the UEA

$\mathcal{M}$ - momentum

$\mathcal{T}$ - Kinetic energy

ΔI - inertia correction

ρ - slosh amplitude in feet

ζ - damping ratio

ϕ - modal deflection

ϕ' - total modal slope

λ - modal bending slope

ω - input modal frequency

μ_α - rotational acceleration per unit angle of attack

u_n - rotational acceleration per unit Q_n

σ - real part of root

UEA - undeformed elastic axis

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