Cross-Correlation Functions of Spherical Waves Propagating Through a Slab Containing Anisotropic Irregularities

by

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Abstract

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In general, satellites are moving at such a fast velocity that the ionosphere as well as the imbedded irregularities can be considered as frozen. The interest is then in cross-correlating signals received at two or more receivers as the satellite speeds across the sky. This report is a study of the cross-correlations between two spherical waves passing through a slab containing anisotropic irregularities.

The cross-correlation functions are derived as functions of the distance between the transmitters. A condition for maximum correlation is obtained and is found to correspond to the case when two rays intersect in the slab, thus proving the often intuitively assumed condition for maximum cross-correlation. This maximum correlation function ρ_M is then expressed as a function of the distance between the receivers, x. For large values of x, ρ_M varies as $1/x$. The results indicate possibilities of determining the height and slab thickness of the ionospheric irregularities from satellite scintillation data.

Contents

I. Introduction

This report is a study of the cross-correlation of satellite signals through a slab containing small anisotropic irregularities. The cross-correlation is defined as the correlation between signals at one receiver when the satellite is at one position and signals at another receiver when the satellite is at a new position. In spaced-receiver experiments, one often uses the observed correlations to determine the height, thickness and other parameters of the ionospheric irregularities. Intuitively, the correlation should be a maximum when the two rays intersect in the slab of irregularities. One of the purposes of this report is to prove mathematically that this indeed is the case. We follow closely Yeh's derivation of auto-correlation functions (Yeh, 1962) to formulate the cross-correlation functions as functions of the distance d between the two satellite positions. The value of d which makes the cross-correlation function a maximum is denoted by d_M. From the geometry of the problem (Fig. 1), it can be shown that this d_M corresponds to the case when the two rays intersect in the slab. The maximum value of the correlation is then expressed as a function of x, the distance between the two receivers, and is found to agree with the result derived by McClure and Swenson (1964).

II. Derivation of the Cross-Correlation Functions

Assume the slab is characterized by a refractive index:

$$n(\vec{x}) = \langle n \rangle [1 + \epsilon \mu(\vec{x})] \tag{1}$$

where $\langle n \rangle$ is a constant, $\mu(\vec{x})$ is a random variable of position and ϵ is a small constant.

The geometry of the problem is shown in Fig. 1. The slab is assumed to be extended to infinity in x and y directions.

The origin is taken at 0, at the level where the satellite is moving. a, b, c, respectively represent the distance between the satellite height and the top of the slab, the thickness of the slab, and the distance between the bottom of the slab to the level where the receivers are located. $A_1(x_T, 0, 0)$ is the first position of the satellite, $A_2(x_T-d, 0, 0)$ is the other one, while d measures the distance between these two positions. The receivers are at $B_1(-x/2, 0, h)$ and $B_2(x/2, 0, h)$ respectively. $S(x', y', z')$ represents the position of an arbitrary scatterer in the slab. r_1, r_1' and R_1 are the distances between the satellite A_1 and receiver B_1, between A_1 and the scatterer, and between B_1 and the scatterer respectively. If we assume harmonic time variation, $\exp(i\omega t)$, and all lengths are normalized by the wave number k defined by:

$$k = \frac{\omega \langle n \rangle}{c} \tag{2}$$

then the signal from A_1 to B_1 can be expressed as, assuming small fluctuation per wave length, (Karavainikov, 1957; Yeh, 1962)

$$\psi(\vec{x}_1) = \frac{kA(\vec{x}_1)}{r_1} e^{-i[r_1 + Q_1(\vec{x}_1)]} \tag{3}$$

where

$$Q_1 = \frac{r_1 \epsilon}{2\pi} \int_{v'} \frac{\mu(\vec{x}_1)}{r_1' R_1} \sin(r_1' + R_1 - r_1) \, d^3 \vec{x}_1' \tag{4}$$

$$S_1 = \log \frac{A}{A_0} = \frac{r_1 \epsilon}{2\pi} \int_{v'} \frac{\mu(\vec{x}_1)}{r_1' R_1} \cos(r_1' + R_1 - r_1) \, d^3 \vec{x}_1' \tag{5}$$

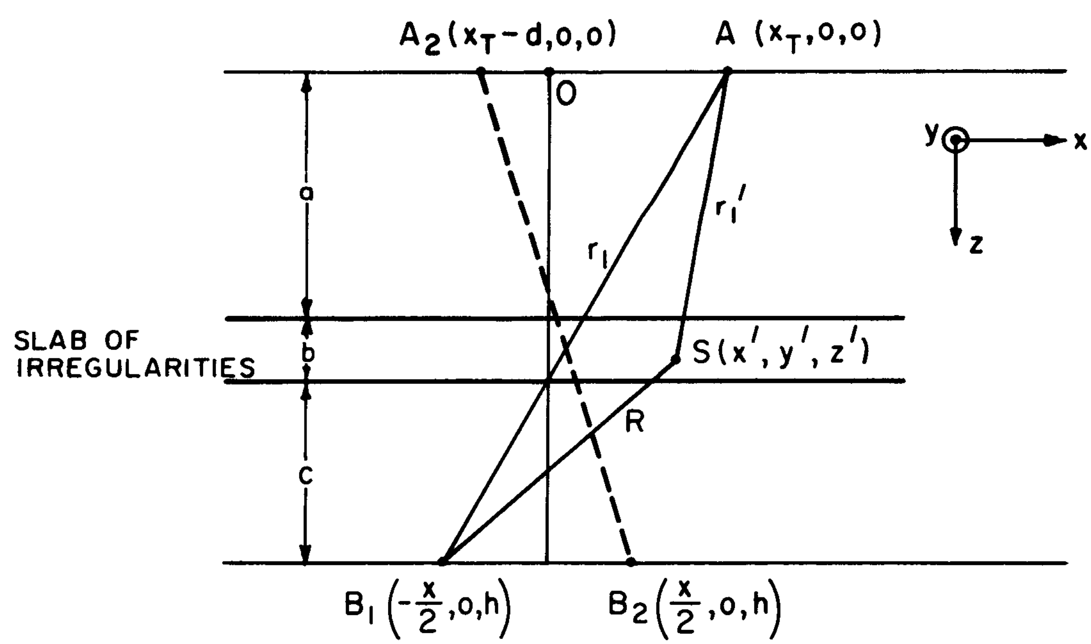

Fig. 1. Geometry of the Problem.

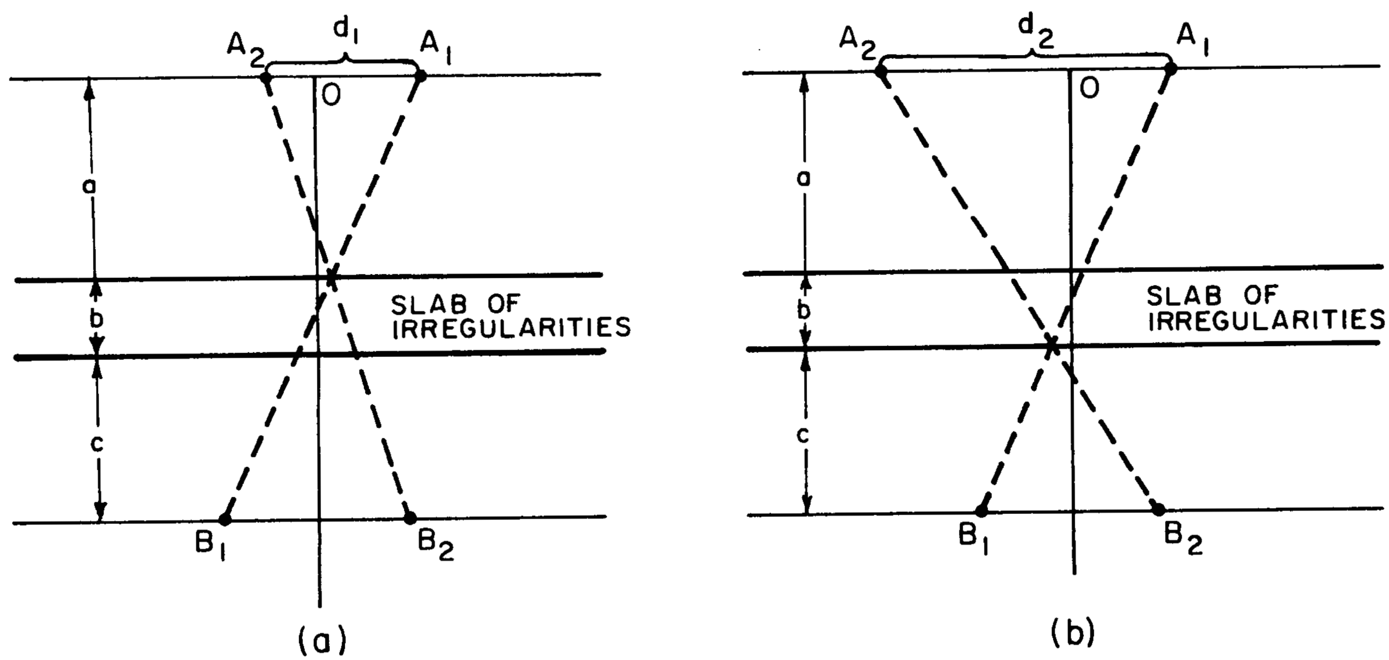

Fig. 2. Two Extreme Cases.

Here Q_1 is the phase departure from original spherical wave and S_1 is the logarithmic amplitude. The integrations are carried out in the whole region of the irregularities. Note that $r_1' + R_1 - r_1$ represents the phase difference between the scattered path and the direct path.

Similar expressions can be written for A_2 and B_2. The normalized cross-correlation functions are then defined by:

$$\rho_Q = \frac{\langle Q_1 Q_2 \rangle}{\langle Q^2 \rangle} = \frac{r_1 r_2 \epsilon^2}{\langle Q^2 \rangle 4\pi^2} \int_{v'}\int_{v'} \frac{\langle \mu(\vec{x}_1')\mu(\vec{x}_2') \rangle}{r_1' \, r_2' \, R_1 \, R_2} \sin(r_1'+R_1-r_1)\sin(r_2'+R_2-r_2)\, d^3\vec{x}_1' \, d^3\vec{x}_2' \tag{6}$$

$$\rho_S = \frac{\langle S_1 S_2 \rangle}{\langle S^2 \rangle} = \frac{r_1 r_2 \epsilon^2}{4\pi^2 \langle S^2 \rangle} \int_{v'}\int_{v'} \frac{\langle \mu(\vec{x}_1')\mu(\vec{x}_2') \rangle}{r_1' \, r_2' \, R_1 \, R_2} \cos(r_1'+R_1-r_1)\cos(r_2'+R_2-r_2)\, d^3\vec{x}_1' \, d^3\vec{x}_2' \tag{7}$$

where $\langle Q^2 \rangle$ and $\langle S^2 \rangle$ are auto-correlation functions for phase and amplitude respectively (Yeh, 1962).

We shall derive the correlation functions as functions of d. We see from Figs. 2a, 2b that d_1 and d_2 are the distances for the two extreme cases in which the rays intersect at the top and the bottom of the slab respectively. For any d such that $d_1 \leq d \leq d_2$, the rays will intersect in the slab. From Fig. 2, we have:

$$d_1 = ax/(h-a) \tag{8}$$

$$d_2 = (h-c)x/c \tag{9}$$

We shall prove that the correlation is maximum for values of d in such a range. Making the usual assumption that the characteristic scales of the irregularities are much larger than a wavelength, (Chernov, 1961; Karavainikov, 1957; Tatarski, 1961) we can approximate the distances in the integrals (6) and (7) as follows. When appearing in the phase factor,

$$r_1' + R_1 - r_1 \cong \frac{1}{2\zeta_1'} \left\{ y_1'^2 + [(x_1'-x_T) + z_1'(x_T+x/2)/h]^2 \right\} \tag{10}$$

$$r_2' + R_2 - r_2 \cong \frac{1}{2\zeta_2'} \left\{ y_2'^2 + [(x_2'-x_T+d) - z_2'(x/2-x_T+d)/h]^2 \right\} \tag{11}$$

$$\zeta_1' = z_1'(h-z_1')/h \qquad \zeta_2' = z_2'(h-z_2')/h \tag{12}$$

and when appearing in the denominator,

$$r' = z', \qquad R = h - z', \qquad r = h \tag{13}$$

Let us define

$$I_3 = \pi(\langle Q_1 Q_2\rangle + \langle S_1 S_2\rangle)/\epsilon^2\langle\mu^2\rangle \tag{14}$$

$$I_4 = \pi(\langle Q_1 Q_2\rangle - \langle S_1 S_2\rangle)/\epsilon^2\langle\mu^2\rangle \tag{15}$$

and the normalized correlation function of the medium by

$$\rho_\mu(x') = \frac{\langle\mu(x_1')\mu(x_2')\rangle}{\langle\mu^2\rangle} \tag{16}$$

We have then, from equations (6), (7), and (10) to (15),

$$I_3 = \frac{1}{\pi} \iint \frac{\rho_\mu(\vec{x}')d^3\vec{x}_1' d^3\vec{x}_2'}{4\zeta_1' \zeta_2'} \cos\left\{ \frac{y_1'^2 + [(x_1'-x_T)+(x_T+x/2)z_1'/h]^2}{2\zeta_1'} - \frac{y_2'^2 + [(x_2' - x_T+d) - (x/2-x_T+d)\, z_2'/h]^2}{2\zeta_2'} \right\} \tag{17}$$

$$I_4 = \frac{1}{\pi} \iint \frac{\rho_\mu(\vec{x}')d^3\vec{x}_1'd^3\vec{x}_2'}{4\zeta_1'\zeta_2'} \cos\left\{\frac{y_1'^2 + [(x_1'-x_T)+(x_T+x/2)z_1'/h]^2}{2\zeta_1'}\right.$$

$$\left. + \frac{y_2'^2 + [(x_2' - x_T+d) - (x/2 - x_T + d)z_2'/h]^2}{2\zeta_2'}\right\} \quad (18)$$

Following the usual procedure of changing to relative and center of mass coordinates:

$$x' = x_2' - x_1', \quad y' = y_2' - y_1', \quad z' = z_2' - z_1' \quad \text{relative coordinates} \quad (19)$$

$$\alpha' = \frac{1}{2}(x_1' + x_2'), \quad \beta' = \frac{1}{2}(y_1' + y_2'), \quad \gamma' = \frac{1}{2}(z_1' + z_2') \quad \text{center of mass coordinates} \quad (20)$$

We can carry out the α', and β' integration in (16) and (17):

$$I_3 = \iint \frac{\rho_\mu(\vec{x}')d^3\vec{x}'d\gamma'}{2(\zeta_1' + \zeta_2')} \sin \frac{1}{2(\zeta_1'-\zeta_2')} \left\{y'^2 + [(x+d)\gamma-(x_T-d/2)z'-(x'+d)h]^2/h^2\right\} \quad (21)$$

$$I_4 = \iint \frac{\rho_\mu(\vec{x}')d^3\vec{x}'d\gamma'}{2(\zeta_1' + \zeta_2')} \sin \frac{1}{2(\zeta_1'+\zeta_2')} \left\{y'^2 + [(x+d)\gamma-(x_T-d/2)z'-(x'+d)h]^2/h^2\right\} \quad (22)$$

Now, we shall introduce an auto-correlation function for the medium of the form:

$$\rho_\mu(x') = \exp\left\{-(x'^2/\ell_x^2 + y'^2/\ell_y^2 + z'^2/\ell_z^2)\right\}$$

Note that in general $\ell_x \neq \ell_y \neq \ell_z$, and irregularities are anisotropic.

Substituting (23) into (21) and (22), we can integrate with respect to x' and y', and obtain:

$$I_3 = \mathrm{Im}\,\pi \int_a^{a+b} d\gamma' \int_{z=-b}^{b} dz' \frac{\exp\left\{-\dfrac{[(x+d)\gamma'/h-(x_T-d/2)z'/h-d]^2}{\ell_x^2 + i2z'(2\gamma'/h-1)} - \dfrac{z'^2}{\ell_z^2}\right\}}{\sqrt{[2z'(2\gamma'/h-1)/\ell_y^2-i][2z'(2\gamma'/h-1)/\ell_x^2-i]}} \tag{24}$$

$$I_4 = \mathrm{Im}\,\pi \int_a^{a+b} d\gamma' \int_{-b}^{b} dz' \frac{\exp\left\{-\dfrac{[(x+d)\gamma'/h-(x_T-d/2)z'/h-d]^2}{\ell_x^2[i\,D_x-iz'^2/h\,\ell_x^2+1]} - \dfrac{z'^2}{\ell_z^2}\right\}}{\sqrt{(D_y-z'^2/h\ell_y^2-i)(D_x-z'^2/h\;{}_x^2-i)}} \tag{25}$$

where $D_x = 4\gamma'(h-\gamma')/h\ell_x^2$, $D_y = 4\gamma'(h-\gamma')/h\ell_y^2$ (26)

are the equivalent wave numbers. Im represents the imaginary part. For z' integration, we note that since $b \gg \ell_z$, the contribution of the integral mainly comes from $z' \leq \ell_z$, the limit can then be taken as $-\infty$ to $+\infty$. Also since $\ell \gg 1$, terms like $2z'(2\gamma/h-1)/\ell_y^2$, $z'^2/h\ell_z^2$ in the integrand can be neglected compared to unity. With these simplifications, we have

$$I_3 = \frac{\pi^2}{2}\frac{\ell_x\ell_z h}{x+d}\left\{\mathrm{erf}\left[\frac{x+d}{h}\alpha a - \alpha d + \frac{x+d}{h}\alpha b\right] - \mathrm{erf}\left[\frac{x+d}{h}\alpha a - \alpha d\right]\right\} \tag{27}$$

where
$$\alpha = \frac{h}{[\ell_x^2 h^2 + \ell_z^2(x_T-d/2)^2]^{1/2}} \tag{28}$$

$$I_4 = \mathrm{Im}\,\pi^{3/2}\int_a^{a+b} \frac{i\ell_z\ell_x\beta}{(1+iD_y)^{1/2}}\, e^{-\beta^2[(x+d)\gamma'/h-d]^2}\, d\gamma' \tag{29}$$

where $$\beta = \frac{h}{[h^2\ell_x^2(1 + iD_x) + \ell_z^2(x_T - d/2)^2]^{1/2}} \qquad (30)$$

Assume the slab is thin as compared to the distance between satellite and the top of the slab. In the slab, the values of D's will not change much. We can represent them by some average values $\bar{D}$'s. Then, equation (29) can be integrated, treating D and β as constant, we obtain finally:

$$I_4 = \mathrm{Im}\,\frac{\pi^2}{2}\,\frac{i}{(1+i\bar{D}_y)^{1/2}}\,\frac{\ell_x\ell_z h}{x+d}\left[\mathrm{erf}\left(\frac{x+d}{h}\beta a - \beta d + \frac{x+d}{h}\beta b\right) - \mathrm{erf}\left(\frac{x+d}{h}\beta a - \beta d\right)\right] \qquad (31)$$

The cross-correlation functions can be then written as, from (14) and (15):

$$\rho_Q(d) = \frac{\epsilon^2\langle\mu^2\rangle}{2\pi\langle Q^2\rangle}(I_3 + I_4) \qquad (32)$$

$$\rho_S(d) = \frac{\epsilon^2\langle\mu^2\rangle}{2\pi\langle S^2\rangle}(I_3 - I_4) \qquad (33)$$

where I_3 and I_4 are given by (27) and (31) respectively. For very thin slab, further approximations can be made in the expressions for I_3 and I_4. We consider the following two cases, using Yeh's (1962) results for $\langle S^2\rangle$ and $\langle Q^2\rangle$.

(1) D's >> 1, Fraunhober region, for $xb/h\ell_x < 1$,

$$\rho_Q(d) = \rho_S(d) \simeq \alpha\ell_x[1-\alpha^2ab(x+d)^2/h^2+\alpha^2b(x+d)d/h]\cdot e^{-\alpha^2[ax/h\,-\,(1-a/h)d]^2} \qquad (34)$$

(2) D's << 1, Fresnel region, for $xb/h\ell_x < 1$,

$$\rho_Q(d) \cong a\ell_x[1-a^2ab(x+d)^2/h^2 + a^2b(x+d)d/h]\cdot e^{-a^2[ax/h-(1-a/h)d]^2} \tag{35}$$

$$\rho_S(d) = e^{-\delta^2/\ell_x^2}\left\{1+\Theta\delta-\frac{1}{\Delta}[4\bar{D}_x(1+\Theta\delta)(3\bar{D}_x+\bar{D}_y-\bar{D}_x\delta^2/\ell_x^2)\delta^2/\ell_x^2+\Theta\delta(-12\bar{D}_x^2-4\bar{D}_x\bar{D}_y+8\delta^2/\ell_x^2)]\right\} \tag{36}$$

where

$$\Theta = (x+d)b/h\ell_x^2$$

$$\delta = d(1-a/h)-ax/h \tag{37}$$

$$\Delta = 3\bar{D}_x^2 + 3\bar{D}_y^2 + 2\bar{D}_x\bar{D}_y$$

In the case of isotropic irregularities, $\ell_x = \ell_y = \ell$, $\bar{D}_x = \bar{D}_y$. ρ_S is reduced to:

$$\rho_S(d) = [(1-2\delta^2/\ell^2+\delta^4/2\ell^4) + \frac{3b(x+d)}{h\ell}\frac{\delta}{\ell}(1-\delta^2/\ell^2+\delta^4/6\ell^4)]\,e^{-\delta^2/\ell^2} \tag{38}$$

III. Maximum Correlations

If we differentiate equations (34) and (35) with respect to d and set the result equal to zero, we have the equation for d_M, corresponding to the maximum (or minimum) values of correlation functions. After some algebraic manipulations, the equation becomes:

$$(1-\frac{aba^2x^2}{h^2})\frac{ax}{h} + \frac{1-2a/h}{1-a/h}\frac{bx}{2h} + d_M[\frac{b}{h} - (1-\frac{a}{h})(1-\frac{aba^2x^2}{h^2}) + \frac{ax}{h}(\frac{a^2bx}{h} - \frac{2a^2abx}{h^2})] +$$

$$+ [\frac{ax}{h}(\frac{a^2b}{h} - \frac{aba^2}{h^2}) - (1-\frac{a}{h})(\frac{a^2bx}{h} - \frac{2a^2abx}{h^2})]\,d_M^2 - (1-\frac{a}{h})(\frac{a^2b}{h} - \frac{aba^2}{h^2})\,d_M^3 = 0 \tag{39}$$

Here we have assumed that α is independent of d. From (28) we can see that this is a fairly reasonable approximation. In most cases,

$$\alpha \sim \frac{1}{\ell_x}$$

To solve equation (39), we notice that as a result of our assumption of thin slab, terms containing (b/h) are small compared to unity. We can use perturbation method to solve for d_M as a series expansion of (b/h). Neglecting all (b/h) terms, we get first:

$$ax/h - d_{M0}(1-a/h) = 0 \tag{40}$$

Therefore $$d_{M0} = ax/(h-a) = d_1 \tag{41}$$

where d_1 is given by equation (8). Next, let $d_M = d_{M0} + \lambda$, substitute this into equation (39), solve for λ to the order of (b/h), we have:

$$\lambda = \frac{1}{h-(a+b)} \cdot \frac{bx}{2} \frac{1}{(1-a/h)} \tag{42}$$

Therefore $$d_M = \frac{ax}{h-a} + \frac{bx}{2c} \frac{1}{(1-a/h)}$$

$$= \frac{1}{2} \left[\frac{ax}{h-a} + \frac{a+b}{c}x\right]$$

$$= \frac{1}{2} (d_1 + d_2) \tag{43}$$

Corresponding to this d_M, the point of intersection for the two rays is: (from Fig. 2)

$$z_M = a + \frac{b(b+c)}{b + 2c} \simeq a + b/2$$

which is the center of the slab.

It can be shown that this d_M does indeed correspond to the maximum value of correlation functions by calculating $(\partial^2\rho/\partial^2 d)$ at d_M. Therefore, we have obtained the result that the cross-correlation is maximum when the two rays intersect at the center of the slab. This maximum value of the correlation function is:

$$\rho_M = \frac{\sqrt{\pi}}{2} \frac{(h-a)\,\ell_x}{xb(1+b/2c)} \left[\mathrm{erf}\left(\frac{bx}{2c\ell_x}\right) - \mathrm{erf}\left(\frac{bx}{2c\ell_x} - \frac{bx}{\ell_x} \cdot \frac{1+b/2c}{h-a}\right)\right] \tag{44}$$

We notice that for $x = 0$, $\rho_M = 1$, as it should be. For large values of x, ρ_M is proportional to $1/x$, since the difference between the two error functions approaches a constant **2** approximately.

The above discussion is for both ρ_Q and ρ_S in the case $D \gg 1$ and also for ρ_Q when $D \ll 1$. To find the maximum of the amplitude correlation function ρ_S in the case when $D \ll 1$, we differentiate equation (38) with respect to d, and set the result equal to zero, we obtain:

$$-\frac{6\delta}{\ell^2} + \frac{6\delta^3}{\ell^4} + \frac{\delta^5}{\ell^6} + \frac{3b(x+d)}{h\ell^2}\left[1 - \frac{5\delta^2}{\ell^2} + \frac{17\delta^4}{6\ell^4} - \frac{\delta^6}{3\ell^6}\right] + \frac{3b}{h\ell^2}\left(\delta - \frac{\delta^3}{\ell^2} + \frac{\delta^5}{6\ell^4}\right) \frac{1}{(1-a/h)} = 0 \tag{45}$$

where δ is given in equation (37).

Again, if we first neglect terms in (b/h), we have one root of the equation, $\delta = 0$, which corresponds to:

$$d_{M0} = \frac{ax}{h-a} = d_1 \tag{46}$$

Now if we let

$$d = d_0 + \lambda', \qquad \delta = (1-a/h)\,\lambda' \tag{47}$$

Substituting them into equation (45), solve for λ' to the first order of (b/h), we have:

$$\lambda' = \frac{1}{2} \frac{bx}{h-(a+b)} \frac{1}{(1-a/h)} \tag{48}$$

Exactly the same as equation (42). Therefore, for the case $D \ll 1$, the maximum of the amplitude cross-correlation function ρ_S also occurs when the two rays intersect at the middle of the slab.

IV. Results and Graphs

The cross-correlation functions are plotted in Figs. 3-6 against d, for several values of x. The maximum value of the correlation function is also plotted as a function of x. We have used the following general satellite experiments data: (McClure, Swenson, 1964)

h = 1000 km a = 650 km c = 300 km b = 50 km ℓ = 1 km.

V. Conclusion

We have derived the cross-correlation functions for spherical waves propagating through a slab with anisotropic irregularities as functions of distance between the two source points. We have proved, for a thin slab, that the correlation is a maximum when the two rays cross at the middle of the slab. This fact can be of some practical use. For example, we can determine the height and slab thickness of the ionospheric irregularities from satellite scintillation data (McClure and Swenson, 1964).

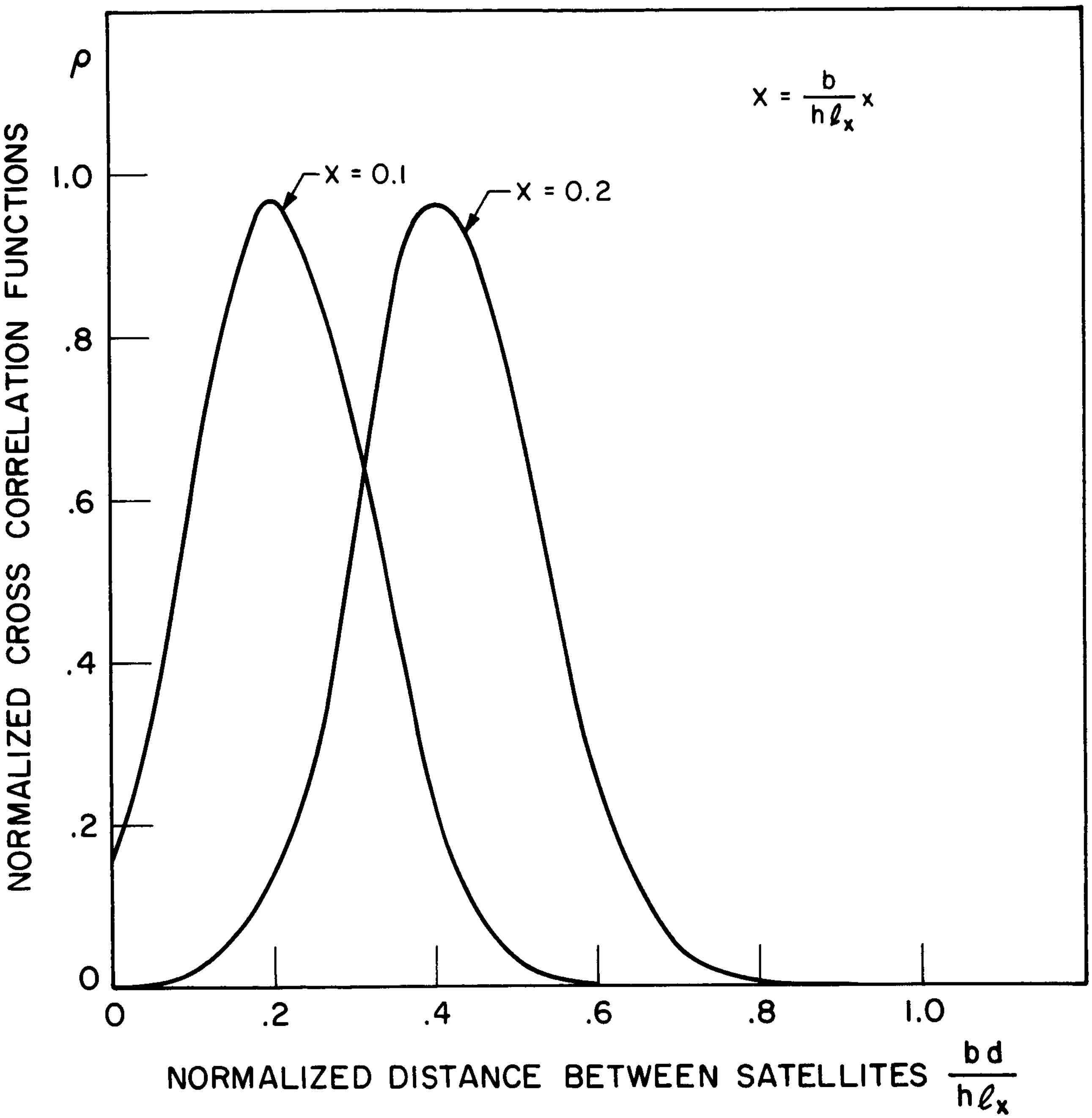

Fig. 3. Amplitude and Phase Correlation Functions for $D \gg 1$ and Phase Correlation Function for $D \ll 1$.

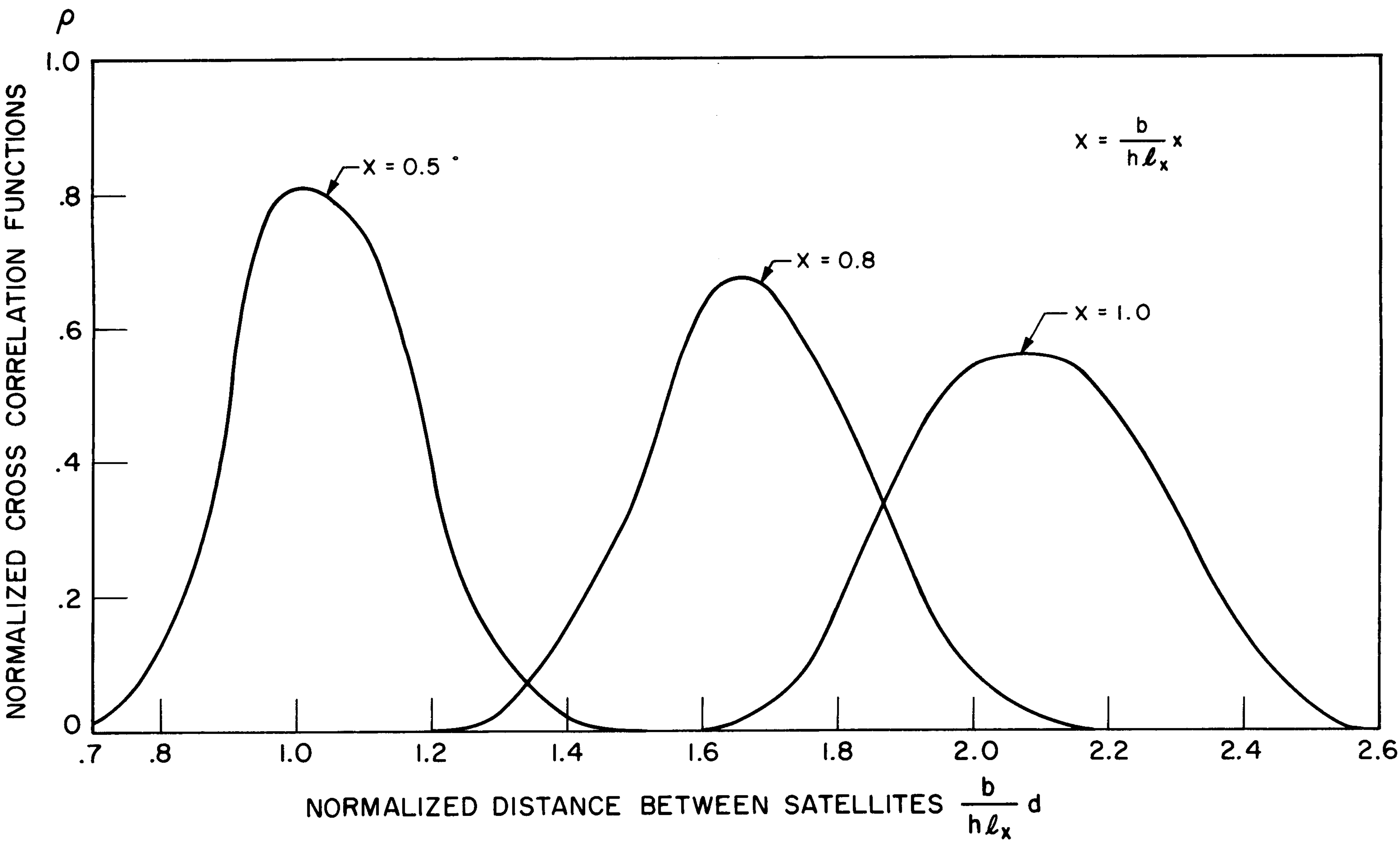

Fig. 4. Amplitude and Phase Correlation Functions for $D \gg 1$ and Phase Correlation Function for $D \ll 1$.

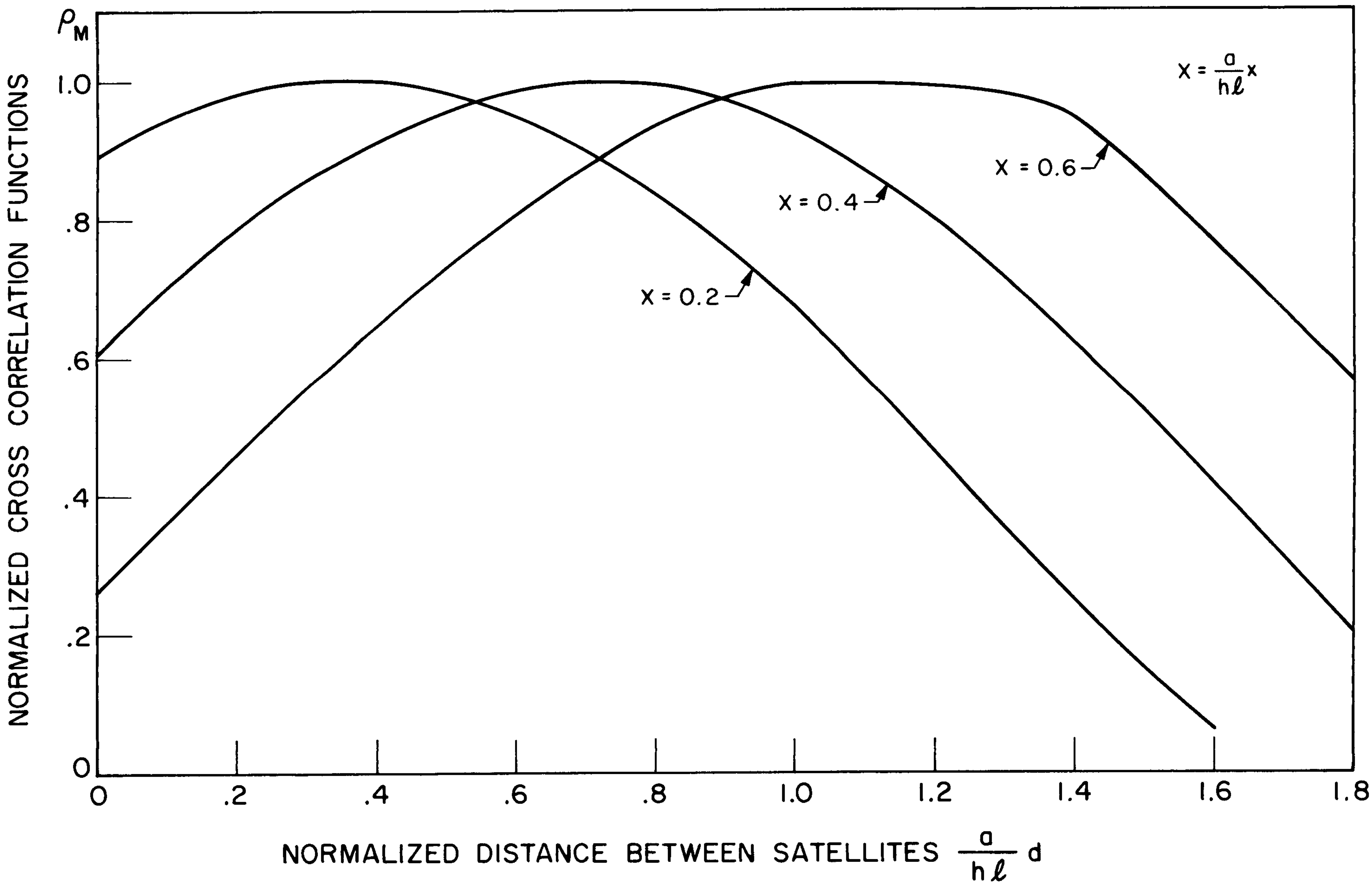

Fig. 5. Amplitude Correlation Functions for D << 1.

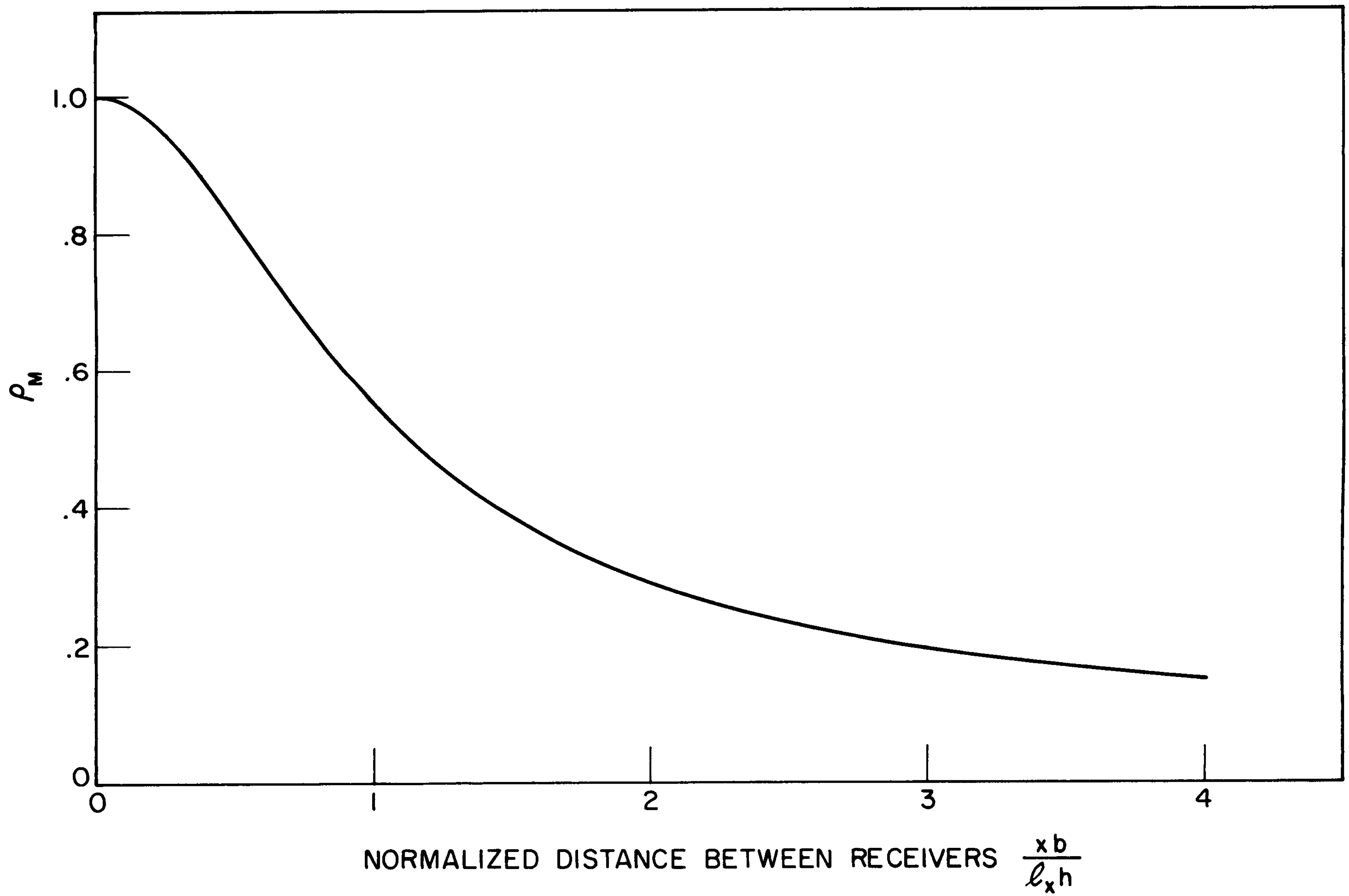

Fig. 6. The Maximum Normalized Cross-Correlation Function.

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