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THE RADIAL HEAT EQUATION AND LAPLACE TRANSFORMS

by Louis R. Bragg

Prepared under Grant No. NsG-544 *by*
CASE INSTITUTE OF TECHNOLOGY
Cleveland, Ohio

for

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1. Introduction. Let μ be a real parameter and let Δ_μ denote the radial Laplacian operator $\Delta_\mu \equiv D_r^2 + \frac{\mu-1}{r} D_r$. In some recent papers, the author [1] and D. T. Haimo [4] have been concerned with the study of expansions of solutions of the radial heat equation

$$(1.1) \qquad \frac{\partial}{\partial t} u(r,t) = \Delta_\mu u(r,t)$$

in terms of the radial heat polynomials $\{R_j^\mu(r,t)\}_{j=0}^\infty$ and their Appell transforms $\{\tilde{R}_j^\mu(r,t)\}_{j=0}^\infty$ when $\mu > 1$ (the author's notations differ). The elements of these sets are defined by

$$(1.2) \qquad \begin{cases} \text{(a)} \quad R_j^\mu(r,t) = j!(4t)^j L_j^{(\frac{\mu}{2}-1)}(-r^2/4t) \\ \text{(b)} \quad \tilde{R}_j^\mu(r,t) = t^{-2j} S_\mu(r,t) R_j^\mu(r,-t) \end{cases}$$

in which $L_j^{(\frac{\mu}{2}-1)}(x)$ is the generalized Laguerre polynomial of degree j and index $\frac{\mu}{2}-1$ and $S_\mu(r,t)$ is the source solution $(4\pi t)^{-\mu/2} e^{-r^2/4t}$. Underlying the development of these expansion theorems is a pair of integral representations of solutions of (1.1). The first representation yields a solution of (1.1) subject to the initial condition $u(r,0) = \varphi(r)$ and is given by

$$(1.3) \qquad u(r,t) = \int_0^\infty K_\mu(r,\xi;t)\varphi(\xi)d\xi$$

in which

$$(1.4) \qquad K_\mu(r,\xi;t) = \frac{1}{2t} r^{1-\mu/2} \xi^{\mu/2} e^{-(r^2+\xi^2)/4t} I_{\frac{\mu}{2}-1}\left(\frac{r\xi}{2t}\right)$$

and $I_{\frac{\mu}{2}-1}$ denotes the modified Bessel function of index $\frac{\mu}{2}-1$. The second representation for a solution of (1.1) for large $t (t > \sigma \geqq 0)$ is given by

$$u(r,t) = \int_0^\infty \zeta_\mu(r,\xi;t)\psi(\xi)d\xi \tag{1.5}$$

in which

$$\zeta_\mu(t,\xi;t) = (2\pi)^{-\mu/2} r^{1-\mu/2} \xi^{\mu/2} J_{\frac{\mu}{2}-1}(r\xi)e^{-\xi^2 t} \tag{1.6}$$

and $\varphi(\xi)$ is an entire function of growth $(1,\sigma)$ in ξ^2. Although these representations are quite useful in many situations, the involvement of the Bessel functions in these integrals generally lead to complications in most applications. Moreover, a number of results of theoretical significance are not immediately evident from (1.3) or (1.5).

In this paper, we develop representations alternative to (1.3) and (1.5) that involve Laplace transforms and their inverses. The importance of these alternate forms lie in the fact that (i) the elements of distribution theory can be more readily fitted into the study of solutions and properties of solutions of (1.1) and (ii) the extensive literature and tables relating to Laplace transforms can be brought to bear on the solutions of (1.1) in applications. We obtain in § 3, for instance, the result that the radial function $\widetilde{R}_j^\mu(r,t)$ is a solution of (1.1) defined by (1.3) that corresponds to a point distribution involving $r^{2-\mu}\delta^{(j)}(r^2)$. This permits another characterization for expansions of solutions of (1.1) in terms of the set $\{\widetilde{R}_j^\mu(r,t)\}_{j=0}^\infty$. Finally, we note that inverse Laplace transforms often lead to convolution type integrals, some of which diverge in the ordinary sense. One can, nevertheless, attach meanings to such integrals through the use of finite and logarithmic parts of divergent integrals [2]. We give an illustrative example of one such integral in § 4.

2. The Laplace Transform Representations. In this section, we state and develop the integral representations alternative to (1.3) and (1.5). The proof of only the first of these will be undertaken since the second result follows by a similar line of reasoning. It will also follow that if suitable data is supplied on the positive time axis, rather than on the r-axis, then we can obtain a solution of (1.1) by calculating the inverse Laplace transform of a function involving this data.

Theorem 2.1. Let $u(r,t)$ be the solution of (1.1) defined by (1.3) that corresponds to the initial data $u(r,0) = \varphi(r)$. Let

$$T_\mu(p,t) = \int_0^\infty e^{-\{\frac{1}{4t} - \frac{1}{p}\}x} x^{\mu/2-1} \varphi(x^{1/2})dx \,, \tag{2.1}$$

Then

$$u(r,t) = \pi^{\mu/2} S_\mu(r,t)\left(\frac{r^2}{16t^2}\right)^{1-\mu/2} \mathcal{L}_p^{-1}\left\{\frac{1}{p^{\mu/2}} T_\mu(p,t)\right\} \tag{2.2}$$

in which the variable in this inverse Laplace transform is replaced by $r^2/16t^2$.

Proof. If we select $a = r^2/16t^2$ and make use of (1.3) and (1.4), we have

$$W(r,t) = (4t)^{\mu/2} e^{r^2/4t} (r^2/16t^2)^{\mu/2-1} u(r,t)$$

$$= 2\int_0^\infty \varphi(\xi)\xi^{\mu/2} e^{-\xi^2/4t} a^{\mu/4-1/2} I_{\frac{\mu}{2}-1}(2\sqrt{a\xi^2})d\xi$$

$$= \int_0^\infty \varphi(x^{1/2}) x^{\mu/2-1} e^{-x/4t} \left\{\frac{a}{x}\right\}^{\mu/4-1/2} I_{\frac{\mu}{2}-1}(2\sqrt{ax})dx \,,$$

this last following from the change of variables $\xi^2 = x$.

Then

$$
(2.3)\qquad
\begin{aligned}
\mathcal{L}_p\{W(r,t)\} &= \int_0^\infty e^{-pa} W(r,t)\,da \\
&= \int_0^\infty \varphi(x^{1/2}) x^{\mu/2-1} e^{-x/4t} \left\{ \int_0^\infty e^{-pa} \left[\frac{a}{x}\right]^{\frac{\mu}{4}-\frac{1}{2}} I_{\frac{\mu}{2}-1}(2\sqrt{ax})\,da \right\} dx \\
&= \int_0^\infty \varphi(x^{1/2}) x^{\mu/2-1} e^{-x/4t} \frac{1}{p^{\mu/2}} e^{x/p} dx \\
&= p^{-\mu/2} T_\mu(p,t) \quad .
\end{aligned}
$$

Then solving for $u(r,t)$, we obtain Theorem 2.1. The validity of the interchange of the orders of integration from the second to the third member in (2.3) follows by absolute integrability while the value of the bracketed term in the third member of (2.3) is tabulated in [3].

Corollary 2.1. Let $u(r,t)$ be a solution of (1.1) that corresponds to $u(0,t) = f(t)$ and suppose that this $u(r,t)$ can be represented by (1.3) for some $\varphi(r)$. Then $u(r,t)$ has the representation

$$
(2.4)\qquad u(r,t) = e^{-r^2/4t}(r^2/16t^2)^{1-\mu/2} \mathcal{L}_p^{-1}\left\{\frac{\Gamma(\mu/2)}{(p-4t)^{\mu/2}} f\left(\frac{pt}{p-4t}\right)\right\}
$$

in which the variable in this inverse Laplace transform is replaced by $r^2/16t^2$.

Proof. We need only identify the expression $T_\mu(p,t)$ in (2.1) for this. It follows from our hypotheses, (1.3), and (1.4) that for $s > 0$,

$$
(2.5)\qquad
\begin{aligned}
f(s) = u(0,s) = \lim_{r\to 0} u(r,s) &= \frac{1}{2^{\mu-1} s^{\mu/2} \Gamma(\frac{\mu}{2})} \int_0^\infty \xi^{\mu-1} e^{-\xi^2/4s} \varphi(\xi)\,d\xi \\
&= \frac{1}{2^{\mu} s^{\mu/2} \Gamma(\frac{\mu}{2})} \int_0^\infty x^{\mu/2-1} e^{-x/4s} \varphi(x^{1/2})\,dx \ ,
\end{aligned}
$$

this last step following from the change of variables $\xi^2 = x$. We now make the identification $\frac{1}{4s} = \frac{1}{4t} - \frac{1}{p}$ so that $s = pt/(p-4t)$. Upon substituting this into the first and last members of (2.5), we get

$$T_\mu(p,t) = (4pt)^{\mu/2}\Gamma(\mu/2)(p-4t)^{-\mu/2}f(\frac{pt}{p-4t})$$

The stated result follows upon substituting this into (2.2).

Theorem 2.2. Let $u(r,t)$ be a solution of (1.1) defined by the integral representation (1.5). Let

$$\tilde{T}_\mu(p,t) = \int_0^\infty e^{-\{t + \frac{1}{p}\}x} x^{\mu/2-1}\psi(x^{1/2})dx \ . \tag{2.6}$$

Then

$$u(r,t) = \frac{r^{2-\mu}}{4\pi^{\mu/2}} \mathcal{L}_p^{-1}\{p^{-\mu/2}\, \tilde{T}_\mu(p,t)\} \tag{2.7}$$

in which the variable in this inverse transform is replaced by $r^2/4$.

3. Generalized Functions. We now relate the above theorems to distribution theory in so far as it applies to the expansions of solutions of (1.1). Our first result relates the radial function $\tilde{R}_j^\mu(r,t)$ to the distribution $r^{2-\mu}\delta^{(j)}(r^2)$ while the second result relates $R_j^\mu(r,t)$ to this same distribution. Our final result of this section gives a distribution characterization for expansions of solutions of (1.1) in terms of the set $\{\tilde{R}_j^\mu(r,t)\}_{j=0}^\infty$.

Theorem 3.1. Let the initial data $\varphi(r)$ in Theorem 2.1 be selected to be $r^{2-\mu}\delta^{(j)}(r^2)$ where $\delta(x)$ denotes the usual delta distribution. Then

$$u(r,t) = \frac{(-1)^j\pi^{\mu/2}}{4^{2j}\Gamma(j + \frac{\mu}{2})} \tilde{R}_j^\mu(r,t) \ . \tag{3.1}$$

Proof. If we use the fact that $\int_0^\infty e^{-ax}\delta^{(j)}(x)dx = a^j$ holds for the delta function, then we get

$$p^{-\mu/2}T_\mu(p,t) = (4t)^{-j}(p-4t)^j p^{-(\frac{\mu}{2}+j)} .$$

But $(p-4t)^j = \mathcal{L}_p\{e^{4at}\delta^{(j)}(a)\}$ and $p^{-(j+\frac{\mu}{2})} = \mathcal{L}_p\left\{\frac{a^{j+\frac{\mu}{2}-1}}{\Gamma(j+\mu/2)}\right\}$.

Using the convolution theorem for Laplace transforms along with the properties of the Dirac distribution, we obtain

$$\mathcal{L}_p^{-1}\{p^{-\mu/2}T_\mu(p,t)\}$$

$$= \frac{1}{(4t)^j\Gamma(j+\frac{\mu}{2})}\int_0^{r^2/16t^2}\left(\frac{r^2}{16t^2}-\xi\right)^{j+\frac{\mu}{2}-1} e^{4\xi t}\delta^{(j)}(\xi)d\xi$$

$$= \frac{(-1)^j}{(4t)^j\Gamma(j+\frac{\mu}{2})} D_\xi^j\left\{e^{4\xi t}\left(\frac{r^2}{16t^2}-\xi\right)^{j+\mu/2-1}\right\}_{\xi=0}$$

$$= \frac{e^{r^2/4t}}{(4t)^{j+\mu/2-1}\Gamma(j+\frac{\mu}{2})} D_\sigma^j\left\{e^{-\sigma}\sigma^{j+\frac{\mu}{2}-1}\right\}_{\sigma=r^2/4t} ,$$

this last following from the change of variables $\sigma = \frac{r^2}{4t} - 4\xi t$. If we now make use of the Rodrigues' formula for the Laguerre polynomials (p.84 of [5]) and the definitions (1.2), it follows that the $u(r,t)$ defined by (2.2) for this $\varphi(r)$ is given by (3.1).

By similar reasoning, we can prove

Theorem 3.2. Let the function $\psi(r)$ in Theorem 2.2 be selected to be $\psi(r) = r^{2-\mu}\delta^{(j)}(r^2)$. Then

$$u(r,t) = \frac{(-1)^j e^{\mu\pi i}}{(4\pi)^{\mu/2}\Gamma(j+\frac{\mu}{2})4^j} R_j^\mu(r,t) . \tag{3.2}$$

On the basis of Theorem 3.1, Theorems 5.3 and 5.4 of [1], and Stirling's formula, we have the following characterization for expansions in terms of the set $\{\widetilde{R}_j^\mu(r,t)\}_{j=0}^\infty$.

Theorem 3.3. Let $\{a_j\}_{j=0}^{\infty}$ be a number sequence satisfying the condition $\limsup_{j \to \infty} |a_j|^{1/j} = 4\sigma$ and let $\varphi(r) = r^{2-\mu} \sum_{j=0}^{\infty} a_j \delta^{(j)}(r^2)$ in Theorem 2.1. Then the corresponding function $u(r,t)$ given by

$$u(r,t) = \pi^{\mu/2} \sum_{j=0}^{\infty} \frac{(-1)^j a_j}{4^{2j}\Gamma(j+\frac{\mu}{2})} \tilde{R}_j^{\mu}(r,t)$$

converges absolutely to a solution of (1.1) in the half-plane $t > \sigma \geqq 0$.

Thus, the generalized function $\varphi(r)$, in this case, acts as a potential function concentrated at the origin that gives a meaningful effect only after σ units of time have elapsed. From Theorem 3.2 and the results in § 5 of [1], an analogous theorem can be formulated for expansions in terms of the radial heat polynomials in the strip $|t| < \sigma$ by regarding the $\psi(r)$ in Theorem 2.2 as a potential function concentrated at $r = 0$ in the half plane $t > \sigma$. We omit its formal statement here, however.

4. Finite and Logarithmic Parts. In determining the inverse Laplace transforms in (2.2), say, it is natural, in many situations, to make use of the convolution integral and in a variety of ways. It is clear from (2.1) that, in general, $\lim_{p \to \infty} T_\mu(p,t) \neq 0$ while $\lim_{p \to \infty} p^{-\mu/2} T_\mu(p,t) = 0$ and $\mu > 0$. In this case, one can write $p^{-\mu/2}T_\mu(p,t) = \{p^{-(\frac{\mu}{2}-\epsilon)}\} \cdot \{p^{-\epsilon}T_\mu(p,t)\}$ for some ϵ with $0 < \epsilon \leqq \mu/2$. With this decomposition and the abbreviation $a = r^2/16t^2$, it follows by the convolution theorem that

$$\mathcal{L}_p^{-1}\{p^{\mu/2}T_\mu(p,t)\} = \frac{1}{\Gamma(\frac{\mu}{2}-\epsilon)} a^{\frac{\mu}{2}-1-\epsilon} * \mathcal{L}_p^{-1}\{p^{-\epsilon}T_\mu(p,t)\} . \tag{4.1}$$

The choice of ϵ will, of course, depend upon μ and $T_\mu(p,t)$ as well a the particular form of the inverse transform one seeks. If, however, $\mu \leqq 0$, any integral of the form (4.1) diverges in the ordinary sense. It is still possible to associate meanings with such integrals by the use of the finite and logarithmic parts (pf and pℓ) of divergent integrals. This procedure thus gives us a method for continuing solutions. For notations and basic definitions connected with this method, the reader is referred to [2].

We illustrate the method for only one example, namely $\varphi(r) = r^{2-\mu}$. For this, $T_\mu(p,t) = 4pt(p-4t)^{-1}$ and $p^{-\mu/2}T_\mu(p,t) = (4t)p^{-(\frac{\mu}{2}-1)}(p-4t)^{-1}$. If $\mu = 2$, this transform can be inverted directly to give $\mathcal{L}_p^{-1}\{p^{-\mu/2}T_\mu(p,t)\} = 4te^{4at}$. If $\mu > 2$, we can write

$$(4.2) \qquad \mathcal{L}_p^{-1}\{p^{-\mu/2}T_\mu(p,t)\} = \frac{4t}{\Gamma(\frac{\mu}{2}-1)} \{a^{\mu/2-2} * e^{4at}\} .$$

Although this integral is improper if $2 < \mu < 4$, it converges nonetheless. Suppose, however, that $\mu \leqq 2$ in (4.2). Then that integral is meaningless in the ordinary sense. For the purpose of attaching a meaning to this integral, we examine separately the cases μ not an even integer and μ an even integer. It is convenient to first rewrite the solution $u(r,t)$ given by (2.2) and (4.2) for $\mu > 2$ by making the change of variables $4a\xi = \sigma$. Then

$$(4.3) \qquad u(r,t) = \frac{r^{2-\mu}}{\Gamma(\frac{\mu}{2}-1)} \int_0^{r^2/4t} \sigma^{\mu/2-2}e^{-\sigma}d\sigma.$$

Case (i) <u>μ not an even integer</u>. Then the solution of (1.1) is given by

$$u(r,t) = \frac{r^{2-\mu}}{\Gamma(\frac{\mu}{2}-1)} \operatorname{pf}\left\{\int_0^{r^2/4t} \sigma^{\mu/2-2}e^{-\sigma}d\sigma\right\}.$$

Through an integration by parts, we can calculate $u(r,t)$ explicitly to be

$$u(r,t) = \sum_{j=0}^{m} \frac{r^{2-\mu}(r^2/4t)^{\mu/2-1+j} e^{-r^2/4t}}{\Gamma(\mu/2+j)}$$

$$+ \frac{r^{2-\mu}}{\Gamma(\frac{\mu}{2}+m)} \int_0^{r^2/4t} \sigma^{\mu/2-1+m} e^{-\sigma} d\sigma$$

where m is the least positive integer such that $\mu/2 + m > 0$. Clearly, $\lim_{t \to 0+} u(r,t) = r^{2-\mu}$ for this.

Case (ii) <u>μ and even integer</u>. In this situation, a solution may be defined by

$$u_\mu(r,t) = \begin{cases} C_2 \, \mathrm{p}\ell.\left\{\int_0^{r^2/4t} \sigma^{-1} e^{-\sigma} d\sigma\right\} & \text{if } \mu = 2 \\ C_{2m} r^{2-2m} \mathrm{pf}\left\{\int_0^{r^2/4t} \sigma^{m-2} e^{-\sigma} d\sigma\right\} & \text{if } \mu = 2m,\ m \le 0. \end{cases}$$

The constants C_{2j} here are so selected that $\lim_{t \to 0+} u(r,t) = r^{2-2j}$ and replace the reciprocal of the gamma function in (4.3) which vanishes. Using the technique for evaluating these, we finally obtain explicitly,

$$u(r,t) = \begin{cases} \frac{1}{\Gamma'(1)}\left\{e^{-r^2/4t} \ln(r^2/4t) + \int_0^{r^2/4t} (\ln\sigma) e^{-\sigma} d\sigma\right\}, \quad \mu = 2 \\ \frac{(-1)^{-m+3}(-m+1)!}{\Gamma'(1)} \left\{ \sum_{j=1}^{-m+1} \frac{(-1)^j (\frac{r^2}{4t})^{-[m+2-j]} r^{2-2m} e^{-r^2/4t}}{(-m+1)(-m)\cdots(-m+2-j)} \right. \\ \left. + \frac{(-1)^{-m+3} r^{2-2m}}{(-m+1)!} e^{-r^2/4t} \ln(r^2/4t) + \frac{(-1)^{-m+3} r^{2-2m}}{(-m+1)!} \int_0^{r^2/4t} (\ln\sigma) e^{-\sigma} d\sigma \right\} \\ \mu = 2m,\ m \le 0. \end{cases}$$

REFERENCES

1. L. R. Bragg, "The Radial Heat Polynomials and Related Functions," Trans. Amer. Math. Soc. (To appear).
2. F. J. Bureau, "Divergent Integrals and Partial Differential Equations," Comm. Pure Appl. Math 8, 143-202, (1955).
3. R. V. Churchill, Modern Operational Mathematics in Engineering, McGraw-Hill Co., New York, 1944.
4. D. T. Haimo, "Functions with the Huygens Property," Bull. Amer. Math. Soc., Vol 71, no. 3, May, 1965.
5. W. Magnus and F. Oberhettinger, Formulas and Theorems for Functions of Mathematical Physics, Chelsea Pub. Co., New York, 1949.

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