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Forced Flexural Motion of a Rocket-Boosted Vehicle

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CONTENTS

FIGURES

ABSTRACT

13158

The equations of forced flexural motion are derived for a rocket-boosted vehicle controlled by vectored thrust. Planar motion is assumed. Motions involved in the system include flexural motion of the vehicle, fuel slosh, and rotation of the engine nozzle. The buckling effects of the tangential thrust component, and the longitudinal inertial and aerodynamic forces are ignored. The aerodynamic forces are derived from momentum theory. Forcing conditions which have been considered are those resulting from atmospheric motion, a sudden control change, or an initial disturbance.

Author

I. INTRODUCTION

This Memorandum presents a derivation of the equations of forced flexural motion of a rocket-boosted vehicle controlled by vectored thrust. The intent in this Memorandum is to present the development of the equations for forced motion of a rocket in a comprehensive derivation that does not include the detail that can be obtained from other references (e.g., Ref. 1, 2, 3, 4).

The vehicle is assumed to have no roll motion and to have isotropic inertial and aerodynamic characteristics around any lateral axis. With these assumptions, consideration of the three-dimensional motion of the vehicle is unnecessary. In this Memorandum, it is assumed that the motion of the vehicle is confined to a vertical plane.

It is assumed that the vehicle structure may be represented by an elastic beam having bending and shearing flexibility. Sloshing of fuel in the fuel tanks is represented by the motion of a simple pendulum attached to the beam (Ref. 5). The engine nozzle may be rotated by a command from the control system. Although the physical parameters of the vehicle and of the atmosphere are time-varying, it is considered here that they vary slowly compared with the motion being studied. Thus, it will be assumed that the physical parameters are constant, with the result that the equations of motion will have constant coefficients.

In the analysis that follows, the properties of the natural modes of flexural vibration of the beam with slosh pendulum and engine nozzle locked are developed. Expressions for the kinetic energy and elastic potential energy of the system are written considering a general planar motion involving the motion in the natural flexural modes, the rigid-body motion of the system, and the motion of the slosh pendulum and engine nozzle. Then, forcing conditions resulting from atmospheric motion, a sudden engine nozzle rotation, or an initial disturbance are expressed as generalized forces corresponding to the system coordinates, and the equations of motion for the system are written using Lagrange's equations. The momentum method is used in developing the expressions for the aerodynamic forces involved in the problem (Ref. 6, pp. 418–420). Finally, the equations of forced motion are written in a nondimensional form.

II. PROPERTIES OF THE NATURAL FLEXURAL MODES

The system to be analyzed is composed of an elastic beam, a slosh pendulum, and an engine nozzle. The beam has bending and shear flexibility. The slosh pendulum and nozzle are locked to the beam and are required to have the bending slope of the beam at the attachment points. The deflected system is shown in Fig. 1. The characteristic of the system and its motion are defined as follows:

EI = flexural rigidity of the beam

I_N = moment of inertia of nozzle around hinge point

$I_S = \mathfrak{M}_S l_S^2$ = moment of inertia of slosh pendulum around hinge point

$k'AG$ = shear rigidity of the beam

l_N = distance from hinge point to center of mass of nozzle

l_S = length of slosh pendulum

$\mathfrak{M}_N$ = mass of nozzle

$\mathfrak{M}_S$ = mass of slosh pendulum

r = radius of gyration of cross section of the beam

w_B = bending deflection of beam

w_S = shear deflection of beam

$w = w_B + w_S$ = total deflection of beam

μ = mass per unit length of the beam

Differentiation with respect to time and to position x will be denoted by a dot and a prime respectively.

Consider the equilibrium of a differential element of the beam as shown in Fig. 2, which defines the positive sense of the bending moment M and the shear force S in the beam. Also shown are the inertia force and moment acting on the element. The bending moment and shear force are related to the displacements by

$$M = EIw_B''$$
$$S = k'AGw_S'$$

Equilibrium of the element requires that the sum of forces and the sum of moments each vanish. Making use

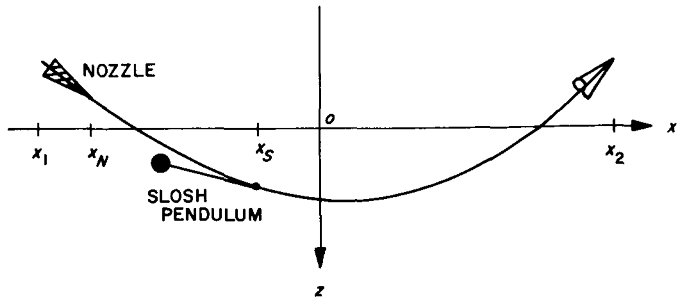

Fig. 1. Deflected elastic system—slosh pendulum and nozzle locked to beam

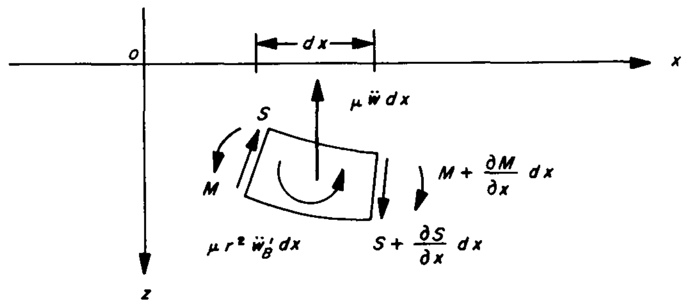

Fig. 2. Free-body diagram—differential element of beam

of the preceding equations, the two equations of motion of the beam result:

$$(EIw_B'')' + k'AGw_S' - \mu r^2 \ddot{w}_B' = 0$$
$$(k'AGw_S')' - \mu \ddot{w} = 0$$

The equations of motion above apply throughout the beam, except at the points of attachment of the slosh pendulum and the nozzle.

Consider the inertia force and moment transmitted to the beam by the slosh pendulum. Figure 3 shows the forces and moments acting on the slosh pendulum. Equilibrium requires that

$$F_S = -\mathfrak{M}_S [\ddot{w}(x_S) - l_S \ddot{w}_B'(x_S)]$$
$$M_S = \mathfrak{M}_S l_S \ddot{w}(x_S) - I_S \ddot{w}_B'(x_S)$$

The force and moment are applied in an opposite sense to the beam. Similarly, it may be shown that the force and moment applied to the beam by the nozzle are

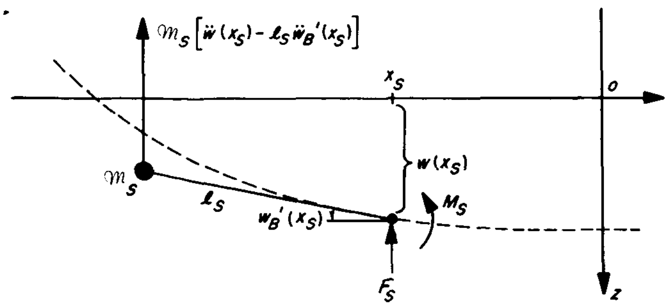

Fig. 3. Free-body diagram—slosh pendulum

$$F_N = -\mathcal{M}_N\left[\ddot{w}(x_N) - l_N\ddot{w}_B'(x_N)\right]$$
$$M_N = \mathcal{M}_N l_N \ddot{w}(x_N) - I_N\ddot{w}_B'(x_N)$$

The equations of motion represent the rate of change of the bending moment and the shear force along the beam. The rate of change of the bending moment and the shear force resulting from the slosh pendulum and nozzle may be written using the above equations and the concept of the Dirac delta function. If these results are added to the equations of motion, a set of equations may be written which apply throughout the beam, as follows:

$$(1) \qquad \begin{aligned} &(EIw_B'')' + k'AGw_S' - \mu r^2\ddot{w}_B' \\ &\quad + (\mathcal{M}_S l_S\ddot{w} - I_S\ddot{w}_B')\,\delta(x - x_S) \\ &\quad + (\mathcal{M}_N l_N\ddot{w} - I_N\ddot{w}_B')\,\delta(x - x_N) = 0 \end{aligned}$$

$$\begin{aligned} &(k'AGw_S')' - \mu\ddot{w} - \mathcal{M}_S(\ddot{w} - l_S\ddot{w}_B')\,\delta(x - x_S) \\ &\quad - \mathcal{M}_N(\ddot{w} - l_N\ddot{w}_B')\,\delta(x - x_N) = 0 \end{aligned}$$

The quantities δ are the delta functions.

It is well known that the solutions of the equations of motion will result in an infinite set of eigenvalues or natural frequencies ω, and a corresponding set of eigenfunctions or mode shapes. The mode shapes are completely described by the total displacement ϕ and the bending slope ψ. The equations of motion, Eq. (1), will be used next to derive some useful properties of the natural modes.

A motion of unit amplitude in the ith mode may be represented by

$$w = \phi_i\, e^{i\omega_i t}$$
$$w_S' = (\phi_i' - \psi_i)\, e^{i\omega_i t}$$
$$w_B' = \psi_i\, e^{i\omega_i t}$$

This motion must satisfy the equations of motion, Eq. (1), leading to

$$(2) \qquad \begin{aligned} &(EI\psi_i')' + k'AG(\phi_i' - \psi_i) + \mu r^2\omega_i^2\psi_i \\ &\quad - \omega_i^2(\mathcal{M}_S l_S\phi_i - I_S\psi_i)\,\delta(x - x_S) \\ &\quad - \omega_i^2(\mathcal{M}_N l_N\phi_i - I_N\psi_i)\,\delta(x - x_N) = 0 \end{aligned}$$

$$\begin{aligned} &[k'AG(\phi_i' - \psi_i)]' + \mu\omega_i^2\phi_i \\ &\quad + \mathcal{M}_S\omega_i^2(\phi_i - l_S\psi_i)\,\delta(x - x_S) \\ &\quad + \mathcal{M}_N\omega_i^2(\phi_i - l_N\psi_i)\,\delta(x - x_N) = 0 \end{aligned}$$

Multiplying Eq. (2) by ψ_j and ϕ_j, respectively, adding them together, and integrating over the beam results in

$$(3) \qquad \begin{aligned} &-\int_{x_1}^{x_2} EI\psi_i'\psi_j'\,dx - \int_{x_1}^{x_2} k'AG(\phi_i' - \psi_i)(\phi_j' - \psi_j)\,dx \\ &= -\omega_i^2\Big\{\int_{x_1}^{x_2}(\phi_i\phi_j + r^2\psi_i\psi_j)\,\mu\,dx \\ &\quad + [\mathcal{M}_S\phi_i(x_S)\,\phi_j(x_S) - \mathcal{M}_S l_S\psi_i(x_S)\,\phi_j(x_S) \\ &\quad - \mathcal{M}_S l_S\phi_i(x_S)\,\psi_j(x_S) + I_S\psi_i(x_S)\,\psi_j(x_S)] \\ &\quad + [\mathcal{M}_N\phi_i(x_N)\,\phi_j(x_N) - \mathcal{M}_N l_N\psi_i(x_N)\,\phi_j(x_N) \\ &\quad - \mathcal{M}_N l_N\phi_i(x_N)\,\psi_j(x_N) + I_N\psi_i(x_N)\,\psi_j(x_N)]\Big\} \end{aligned}$$

making use of the fact that the bending moment and shear force vanish at the ends of the beam. Similarly, a second equation may be written with the subscripts i and j interchanged. Subtracting the second equation from the first leads to

$$\begin{aligned} 0 = (\omega_j^2 - \omega_i^2)&\Big\{\int_{x_1}^{x_2}(\phi_i\phi_j + r^2\psi_i\psi_j)\,\mu dx \\ &+ [\mathcal{M}_S\phi_i(x_S)\,\phi_j(x_S) - \mathcal{M}_S l_S\psi_i(x_S)\,\phi_j(x_S) \\ &- \mathcal{M}_S l_S\phi_i(x_S)\,\psi_j(x_S) + I_S\psi_i(x_S)\,\psi_j(x_S)] \\ &+ [\mathcal{M}_N\phi_i(x_N)\,\phi_j(x_N) - \mathcal{M}_N l_N\psi_i(x_N)\,\phi_j(x_N) \\ &- \mathcal{M}_N l_N\phi_i(x_N)\,\psi_j(x_N) + I_N\psi_i(x_N)\,\psi_j(x_N)]\Big\} \end{aligned}$$

It can be anticipated that for $i \neq j$, $\omega_i \neq \omega_j$. Thus the term in braces must vanish for $i \neq j$, leading to the first orthogonality relationship:

$$(4) \qquad \begin{aligned} &\int_{x_1}^{x_2}(\phi_i\phi_j + r^2\psi_i\psi_j)\,\mu\,dx \\ &\quad + [\mathcal{M}_S\phi_i(x_S)\,\phi_j(x_S) - \mathcal{M}_S l_S\psi_i(x_S)\,\phi_j(x_S) \\ &\quad - \mathcal{M}_S l_S\phi_i(x_S)\,\psi_j(x_S) + I_S\psi_i(x_S)\,\psi_j(x_S)] \\ &\quad + [\mathcal{M}_N\phi_i(x_N)\,\phi_j(x_N) - \mathcal{M}_N l_N\psi_i(x_N)\,\phi_j(x_N) \\ &\quad - \mathcal{M}_N l_N\phi_i(x_N)\,\psi_j(x_N) + I_N\psi_i(x_N)\,\psi_j(x_N)] = 0 \\ &\qquad \text{for } i \neq j \end{aligned}$$

For the case $i = j$, define

$$(5) \qquad \mathfrak{M}_i = \int_{x_1}^{x_2} (\phi_i^2 + r^2\psi_i^2)\,\mu\,dx + [\mathfrak{M}_S\phi_i^2(x_S) - 2\mathfrak{M}_S l_S \phi_i(x_S)\,\psi_i(x_S) + I_S\psi_i^2(x_S)] + [\mathfrak{M}_N\phi_i^2(x_N) - 2\mathfrak{M}_N l_N \phi_i(x_N)\,\psi_i(x_N) + I_N\psi_i^2(x_N)]$$

The quantity $\mathfrak{M}_i$ will be referred to as the generalized mass of the ith mode.

Consider Eq. (3) again. If $i \neq j$, the right-hand side of this equation vanishes, resulting in the second orthogonality relationship:

$$(6) \qquad \int_{x_1}^{x_2} EI\psi_i'\psi_j'\,dx + \int_{x_1}^{x_2} k'AG(\phi_i' - \psi_i)(\phi_j' - \psi_j)\,dx = 0 \qquad \text{for } i \neq j$$

For $i = j$, making use of Eq. (5), Eq. (3) may be rewritten as follows:

$$(7) \qquad \int_{x_1}^{x_2} EI\psi_i'^2\,dx + \int_{x_1}^{x_2} k'AG(\phi_i' - \psi_i)^2\,dx = \mathfrak{M}_i\omega_i^2$$

Equations (6) and (7) will be useful later in writing the elastic potential energy of the system.

An additional useful relation may be obtained from Eq. (2). Multiplying the second equation of Eq. (2) by x, adding the result to the first equation of Eq. (2), and integrating over the beam leads to

$$-\omega_i^2\left\{\int_{x_1}^{x_2}(x\phi_i + r^2\psi_i)\,\mu\,dx + [\mathfrak{M}_S x_S\phi_i(x_S) - \mathfrak{M}_S l_S x_S\psi_i(x_S) - \mathfrak{M}_S l_S\phi_i(x_S) + I_S\psi_i(x_S)] + [\mathfrak{M}_N x_N\phi_i(x_N) - \mathfrak{M}_N l_N x_N\psi_i(x_N) - \mathfrak{M}_N l_N\phi_i(x_N) + I_N\psi_i(x_N)]\right\} = 0$$

Since $\omega_i \neq 0$ for any of the elastic modes, the term in the braces must vanish, resulting in

$$(8) \qquad \int_{x_1}^{x_2}(x\phi_i + r^2\psi_i)\,\mu\,dx + [\mathfrak{M}_S x_S\phi_i(x_S) - \mathfrak{M}_S l_S x_S\psi_i(x_S) - \mathfrak{M}_S l_S\phi_i(x_S) + I_S\psi_i(x_S)] + [\mathfrak{M}_N x_N\phi_i(x_N) - \mathfrak{M}_N l_N x_N\psi_i(x_N) - \mathfrak{M}_N l_N\phi_i(x_N) + I_N\psi_i(x_N)] = 0$$

The meaning of Eq. (8) is that the product of inertia of each of the modes with respect to the x, z axes is zero, and that the x, z axes are the principal axes.

Next, integrate the second equation of Eq. (2) over the beam, which yields

$$\omega_i^2\left\{\int_{x_1}^{x_2}\phi_i\,\mu\,dx + \mathfrak{M}_S[\phi_i(x_S) - l_S\psi_i(x_S)] + \mathfrak{M}_N[\phi_i(x_N) - l_N\psi_i(x_N)]\right\} = 0$$

Since $\omega_i \neq 0$ for any of the elastic modes, the term in the braces must vanish, leading to

$$(9) \qquad \int_{x_1}^{x_2}\phi_i\,\mu\,dx + \mathfrak{M}_S[\phi_i(x_S) - l_S\psi_i(x_S)] + \mathfrak{M}_N[\phi_i(x_N) - l_N\psi_i(x_N)] = 0$$

The meaning of Eq. (9) is that the center of mass of each of the modes lies on the x axis.

III. KINETIC AND POTENTIAL ENERGY

A. Frames of Reference

Consider the motion of the system, allowing the slosh pendulum and nozzle to move relative to the beam. The flexural motion of the beam and the relative motion of the slosh pendulum and nozzle will be measured relative to a frame of reference which moves with the system, as shown in Fig. 4. The flexural motion of the beam will be described in terms of the motion of the natural modes described in Sect. II. Thus, the origin o and the x, z axes of the coordinate system oxz represent the center of mass and the principal axes of the system with the slosh pendulum and nozzle locked.

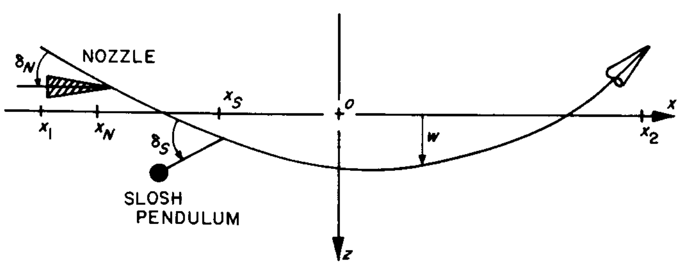

Fig. 4. Relationship of vehicle to moving frame of reference oxz

The flexural translation and rotation, w and w_B', of the elements of the beam and the relative rotation, δ_S and δ_N, of the slosh pendulum and nozzle, will be considered small quantities of the same order of magnitude.

In writing the equations of motion of the system, it will be necessary to consider the motion relative to an inertial frame of reference. The motion of the moving frame of reference will be considered relative to an Earth-fixed frame of reference, as shown in Fig. 5. The

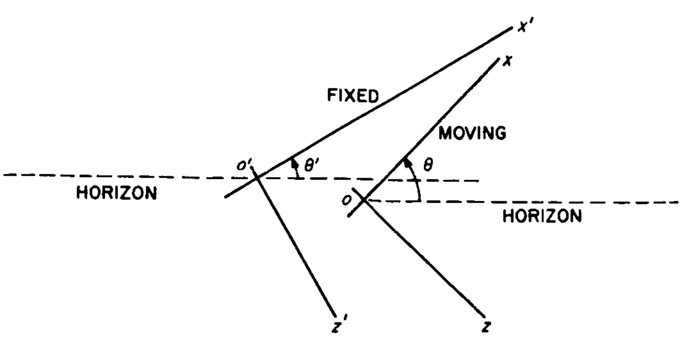

Fig. 5. Relationship of fixed and moving frames of reference o'x'z' and oxz

fixed coordinate system $o'x'z'$ will be considered to be inertial. The angles θ and θ' are angles of elevation relative to the local horizon. The motion of the moving coordinate system oxz is described by the position of the origin x_0', z_0' and the rotation $\varepsilon = \theta - \theta'$. It will be considered that the motion in the z direction z_0', and the angular motion ε are small quantities.

B. Kinetic Energy of the System

Consider the translational velocity of an element of the beam relative to the fixed frame of reference (see Fig. 6). The velocity components referred to the moving coordinates are

$$V_z = \dot{w} - x\dot{\theta} + \dot{x}_0'\varepsilon + \dot{z}_0'$$

$$V_x = w\dot{\theta} + \dot{x}_0'\left(1 - \frac{\varepsilon^2}{2}\right) - \dot{z}_0'\varepsilon$$

where terms no smaller than the product of small quantities are retained. In this equation, it is assumed that the elastic motion in the x direction resulting from bending of the beam may be neglected. The angular velocity of an element of the beam is given by $\theta - w_B'$.

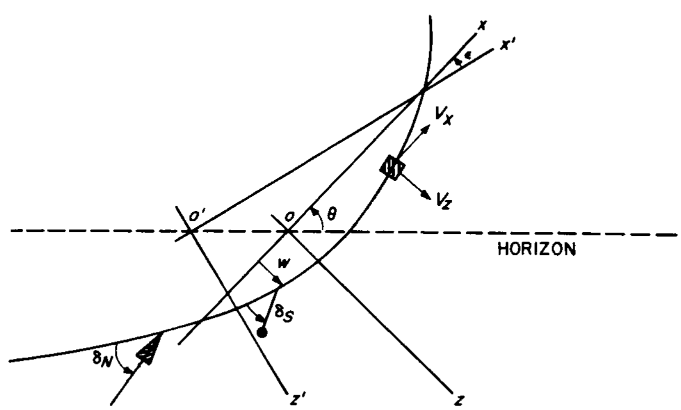

Fig. 6. Motion of a beam element

Similarly, the velocity components of the slosh mass and the nozzle center of mass are as follows:

$$V_{z_S} = \dot{w}(x_S) - l_S\dot{w}_B'(x_S) - (x_S - l_S)\dot{\theta} + l_S\dot{\delta}_S + \dot{x}_0'\varepsilon + \dot{z}_0'$$

$$V_{x_S} = l_S[\delta_S - w_B'(x_S)](\dot{\theta} + \dot{\delta}_S) - l_S\delta_S\dot{w}_B'(x_S) + w(x_S)\dot{\theta} + \dot{x}_0'\left(1 - \frac{\varepsilon^2}{2}\right) - \dot{z}_0'\varepsilon$$

$$V_{z_N} = \dot{w}(x_N) - l_N \dot{w}_B'(x_N) - (x_N - l_N)\,\dot{\theta} + l_N \dot{\delta}_N + \dot{x}_0'\varepsilon + \dot{z}_0'$$

$$V_{x_N} = l_N\,[\delta_N - w_B'(x_N)]\,(\dot{\theta} + \dot{\delta}_N) - l_N \delta_N \dot{w}_B'(x_N) + w(x_N)\,\dot{\theta} + \dot{x}_0'\left(1 - \frac{\varepsilon^2}{2}\right) - \dot{z}_0'\varepsilon$$

In these equations, terms involving $w_B'(x_S)\,\dot{w}_B'(x_S)$ and $w_B'(x_N)\,\dot{w}_B'(x_N)$, which represent elastic motion in the x direction resulting from beam bending, have been ignored. The angular velocity of the nozzle is given by $\dot{\theta} + \dot{\delta}_N - \dot{w}_B'(x_N)$.

The kinetic energy of the system may be written in the form

$$T = \frac{1}{2}\int_{x_1}^{x_2} [V_x^2 + V_z^2 + r^2(\dot{\theta} - \dot{w}_B')^2]\,\mu\,dx + \frac{1}{2}\mathcal{M}_S(V_{x_S}^2 + V_{z_S}^2) + \frac{1}{2}\mathcal{M}_N(V_{x_N}^2 + V_{z_N}^2) + \frac{1}{2}(I_N - \mathcal{M}_N l_N^2)\,[\dot{\theta} + \dot{\delta}_N - \dot{w}_B'(x_N)]^2 \tag{10}$$

The mass of the vehicle and the moment of inertia around the origin o are defined as follows:

$$\mathcal{M} = \int_{x_1}^{x_2} \mu\,dx + \mathcal{M}_S + \mathcal{M}_N$$

$$I = \int_{x_1}^{x_2} (x^2 + r^2)\,\mu\,dx + \mathcal{M}_S(x_S - l_S)^2 + \mathcal{M}_N(x_N - l_N)^2 + I_N - \mathcal{M}_N l_N^2$$

Further, the following relations may be written:

$$\int_{x_1}^{x_2} x\mu\,dx + \mathcal{M}_S(x_S - l_S) + \mathcal{M}_N(x_N - l_N) = 0$$

$$\int_{x_1}^{x_2} w\mu\,dx + \mathcal{M}_S[w(x_S) - l_S w_B'(x_S)] + \mathcal{M}_N[w(x_N) - l_N w_B'(x_N)] = 0$$

$$\int_{x_1}^{x_2} \dot{w}\mu\,dx + \mathcal{M}_S[\dot{w}(x_S) - l_S \dot{w}_B'(x_S)] + \mathcal{M}_N[\dot{w}(x_N) - l_N \dot{w}_B'(x_N)] = 0$$

$$\int_{x_1}^{x_2} (x\dot{w} + r^2\dot{w}_B')\,\mu\,dx + \mathcal{M}_S(x_S - l_S)\,[\dot{w}(x_S) - l_S\dot{w}_B'(x_S)] + \mathcal{M}_N(x_N - l_N)\,[\dot{w}(x_N) - l_N\dot{w}_B'(x_N)] + (I_N - \mathcal{M}_N l_N^2)\,\dot{w}_B'(x_N) = 0$$

The first two of these equations vanish, since the origin o is at the center of mass of the system with locked slosh pendulum and nozzle. The third equation represents the relative momentum in the z direction of the system with locked slosh pendulum and nozzle, and vanishes for the reason cited above. The fourth equation represents the relative angular momentum of the system with locked slosh pendulum and nozzle, and vanishes since the x, z axes are the principal axes. Making use of the definitions and equations above, the kinetic energy, Eq. (10), may be re-written as follows:

$$\begin{aligned}
T = \frac{1}{2}\Big\{ & \int_{x_1}^{x_2} (\dot{w}^2 + r^2\dot{w}_B'^2)\,\mu\,dx + \mathcal{M}_S[\dot{w}(x_S) - l_S\dot{w}_B'(x_S)]^2 \\
& + \mathcal{M}_N[\dot{w}(x_N) - l_N\dot{w}_B'(x_N)]^2 + (I_N - \mathcal{M}_N l_N^2)\,\dot{w}_B'^2(x_N)\Big\} \\
& + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}\mathcal{M}\dot{x}_0'^2 + \frac{1}{2}\mathcal{M}\dot{z}_0'^2 + \frac{1}{2}I_S\dot{\delta}_S^2 + \frac{1}{2}I_N\dot{\delta}_N^2 \\
& + \dot{\theta}\,[\dot{x}_0'(\mathcal{M}_S l_S \delta_S + \mathcal{M}_N l_N \delta_N) - \dot{\delta}_S(\mathcal{M}_S l_S x_S - I_S) \\
& \qquad - \dot{\delta}_N(\mathcal{M}_N l_N x_N - I_N)] \\
& + (\dot{x}_0'\varepsilon + \dot{z}_0')(\mathcal{M}_S l_S \dot{\delta}_S + \mathcal{M}_N l_N \dot{\delta}_N) \\
& + \dot{x}_0'\{\mathcal{M}_S l_S[(\delta_S - w_B'(x_S))\,\dot{\delta}_S - \delta_S\dot{w}_B'(x_S)] \\
& \qquad + \mathcal{M}_N l_N[(\delta_N - w_B'(x_N))\,\dot{\delta}_N - \delta_N\dot{w}_B'(x_N)]\} \\
& + \dot{\delta}_S[\mathcal{M}_S l_S\dot{w}(x_S) - I_S\dot{w}_B'(x_S)] \\
& \qquad + \dot{\delta}_N[\mathcal{M}_N l_N\dot{w}(x_N) - I_N\dot{w}_B'(x_N)]
\end{aligned}$$

The flexural motion is described in terms of the natural modes as follows:

$$w(x,t) = \sum_i \phi_i(x)\,q_i(t) \tag{11}$$

$$w_B'(x,t) = \sum_i \psi_i(x)\,q_i(t)$$

where q_i is the ith generalized coordinate of flexural motion. In terms of motion in the natural flexural modes, the kinetic energy may be written as follows:

$$\begin{aligned}
T = & \frac{1}{2}\sum_i \mathcal{M}_i\dot{q}_i^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}\mathcal{M}\dot{x}_0'^2 + \frac{1}{2}\mathcal{M}\dot{z}_0'^2 \\
& + \frac{1}{2}I_S\dot{\delta}_S^2 + \frac{1}{2}I_N\dot{\delta}_N^2 + \sum_i \dot{q}_i \\
& \times \{-\dot{x}_0'[\mathcal{M}_S l_S\psi_i(x_S)\,\delta_S + \mathcal{M}_N l_N\psi_i(x_N)\,\delta_N] \\
& \qquad + [\mathcal{M}_S l_S\phi_i(x_S) - I_S\psi_i(x_S)]\,\dot{\delta}_S \\
& \qquad + [\mathcal{M}_N l_N\phi_i(x_N) - I_N\psi_i(x_N)]\,\dot{\delta}_N\} \\
& + \sum_i q_i\dot{x}_0'[-\mathcal{M}_S l_S\psi_i(x_S)\,\dot{\delta}_S - \mathcal{M}_N l_N\psi(x_N)\,\dot{\delta}_N] \\
& + \dot{\theta}\,[\dot{x}_0'(\mathcal{M}_S l_S\delta_S + \mathcal{M}_N l_N\delta_N) \\
& \qquad - (\mathcal{M}_S l_S x_S - I_S)\,\dot{\delta}_S - (\mathcal{M}_N l_N x_N - I_N)\,\dot{\delta}_N] \\
& + (\dot{x}_0'\varepsilon + \dot{z}_0')(\mathcal{M}_S l_S\dot{\delta}_S + \mathcal{M}_N l_N\dot{\delta}_N) \\
& \qquad + \dot{x}_0'(\mathcal{M}_S l_S\delta_S\dot{\delta}_S + \mathcal{M}_N l_N\delta_N\dot{\delta}_N)
\end{aligned} \tag{12}$$

making use of the first orthogonality relation, Eq. (4), and the definition of generalized mass $\mathcal{M}_i$, Eq. (5).

Then the following derivatives, which will appear in the Lagrange equations of motion for the system, may be written:

$$(13)\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \mathcal{M}_i \ddot{q}_i + [\mathcal{M}_S l_S \phi_i(x_S) - I_S \psi_i(x_S)]\,\ddot{\delta}_S + [\mathcal{M}_N l_N \phi_i(x_N) - I_N \psi_i(x_N)]\,\ddot{\delta}_N - \mathcal{M}_S l_S \psi_i(x_S)\,\ddot{x}'_0 \delta_S - \mathcal{M}_N l_N \psi_i(x_N)\,\ddot{x}'_0 \delta_N$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}'_0}\right) = \mathcal{M}\ddot{x}'_0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}'_0}\right) = \mathcal{M}\ddot{z}'_0 + \mathcal{M}_S l_S \ddot{\delta}_S + \mathcal{M}_N l_N \ddot{\delta}_N$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = I\ddot{\theta} - (\mathcal{M}_S l_S x_S - I_S)\,\ddot{\delta}_S - (\mathcal{M}_N l_N x_N - I_N)\,\ddot{\delta}_N + \mathcal{M}_S l_S \ddot{x}'_0 \delta_S + \mathcal{M}_N l_N \ddot{x}'_0 \delta_N$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\delta}_S}\right) - \frac{\partial T}{\partial \delta_S} = I_S \ddot{\delta}_S + \sum_i [\mathcal{M}_S l_S \phi_i(x_S) - I_S \psi_i(x_S)]\,\ddot{q}_i - (\mathcal{M}_S l_S x_S - I_S)\,\ddot{\theta} - \sum_i \mathcal{M}_S l_S \psi_i(x_S)\,\ddot{x}'_0 q_i + \mathcal{M}_S l_S \ddot{z}'_0 + \mathcal{M}_S l_S \ddot{x}'_0 \delta_S$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\delta}_N}\right) - \frac{\partial T}{\partial \delta_N} = I_N \ddot{\delta}_N + \sum_i [\mathcal{M}_N l_N \phi_i(x_N) - I_N \psi_i(x_N)]\,\ddot{q}_i - (\mathcal{M}_N l_N x_N - I_N)\,\ddot{\theta} - \sum_i \mathcal{M}_N l_N \psi_i(x_N)\,\ddot{x}'_0 q_i + \mathcal{M}_N l_N \ddot{z}'_0 + \mathcal{M}_N l_N \ddot{x}'_0 \delta_N$$

In Eq. (13), the products of small quantities have been neglected. Further, ε has been set equal to zero, requiring the fixed and moving coordinates to coincide at the instant represented by the equations.

It is convenient to describe the acceleration of the moving origin in another form. The quantities which describe the motion of the origin o, shown in Fig. 7, are defined as follows:

V = velocity

θ = angle of elevation

α = angle of attack

γ = flight-path angle

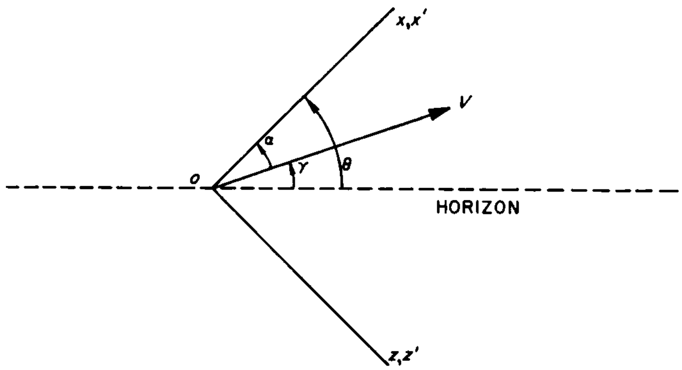

Fig. 7. Motion of the origin o of the moving coordinate system oxz

It is well known that the acceleration of o consists of a component $\dot{V}$ along V and a component $V\dot{\gamma}$ normal to V and in the upward direction. Then

$$(14)\quad \ddot{x}'_0 = \dot{V}$$
$$\ddot{z}'_0 = \dot{V}\alpha - V\dot{\gamma}$$

assuming α to be a small angle.

C. *Potential Energy of the System*

In terms of the displacements in bending and in shear, the flexural potential energy in the beam is

$$(15)\quad U = \frac{1}{2}\int_{x_1}^{x_2} EI w_B''^2\, dx + \frac{1}{2}\int_{x_1}^{x_2} k'AG(w' - w'_B)^2\, dx$$

Making use of Eq. (11), the flexural potential energy of the beam in terms of motion of the natural modes is

$$U = \frac{1}{2}\sum_i \sum_j q_i q_j \left[\int_{x_1}^{x_2} EI\psi'_i \psi'_j\, dx + \int_{x_1}^{x_2} k'AG(\phi'_i - \psi_i)(\phi'_j - \psi_j)\, dx\right]$$

With the aid of Eq. (6), the second orthogonality relationship, and of Eq. (7), the energy becomes

$$(16)\quad U = \frac{1}{2}\sum_i \mathcal{M}_i \omega_i^2 q_i^2$$

The following derivative, which will appear in the Lagrange equations of motion for the system, may be written:

$$(17)\quad \frac{\partial U}{\partial q_i} = \mathcal{M}_i \omega_i^2 q_i$$

IV. GENERALIZED FORCES

The generalized forces contained in the Lagrange equations of motion are defined as the virtual work done on the system by the external forces per unit virtual displacement. It will be necessary to consider the thrust force, aerodynamic forces, gravitational forces, the nozzle actuating moment, and damping forces.

A. Virtual Work

The thrust force acts at the point of attachment of the nozzle and consists of a component T tangent to the beam and a component $T\delta_N$ normal to the beam, as shown in Fig. 8. The component $T\delta_N$ results from the small rotation of the nozzle δ_N.

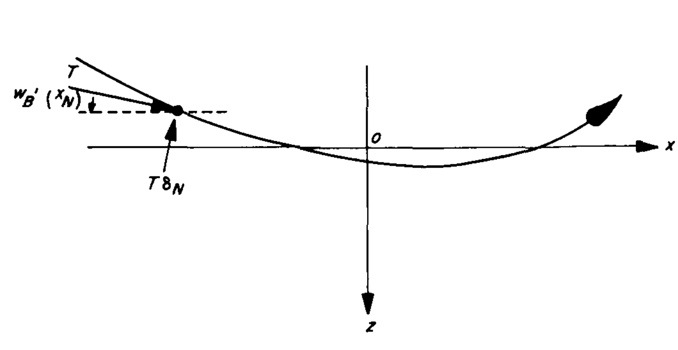

Fig. 8. Thrust-force components

Consider the virtual work done by the thrust force in a flexural virtual displacement $\delta w(x_N)$ of the point of attachment. It will be assumed here that the virtual work resulting from the tangential component T will be neglected. The effect of the above assumption is to ignore the buckling effect of a tangential compressive force on the beam, which is not likely to be important unless the vehicle acceleration is very high and the vehicle is very slender (Ref. 7). The virtual work done by the component $T\delta_N$ in an elastic virtual displacement $\delta w(x_N)$ is

$$\delta W = -T\delta_N \delta w(x_N) = -T\delta_N \sum_i \phi_i(x_N)\,\delta q_i$$

The virtual work done by the thrust force in a small virtual displacement of the moving origin o, given by $\delta x_0'$, $\delta z_0'$, is

$$\begin{aligned}\delta W &= T\delta x_0' + T\,[w_B'(x_N) - \delta_N]\,\delta z_0' \\ &= T\delta x_0' + T\,[\sum_i \psi_i(x_N)\,q_i - \delta_N]\,\delta z_0'\end{aligned}$$

Further, the virtual work done by the thrust force in a virtual rotation $\delta\theta$ is

$$\delta W = T\,[w(x_N) - x_N w_B'(x_N) + x_N\delta_N]\,\delta\theta$$

$$\delta W = T\,\{\sum_i [\phi_i(x_N) - x_N\psi_i(x_N)]\,q_i + x_N\delta_N\}\,\delta\theta$$

Consider the effect of the aerodynamic forces, represented here by a drag force D and a lift force L in the negative x and z directions respectively. The drag and lift forces per unit length will be represented by D' and L'. Further the aerodynamic moment around the origin o will be represented by M, considered positive in the sense of positive θ. The virtual work done by the aerodynamic forces in an elastic virtual displacement δw may be written as

$$\begin{aligned}\delta W &= -\int_{x_1}^{x_2} L'\delta w\,dx \\ &= -\sum_i \left[\int_{x_1}^{x_2} L'\phi_i\,dx\right]\delta q_i\end{aligned}$$

No virtual work will be done by the drag forces in an elastic virtual displacement consistent with the assumption that the elastic displacements in the x direction are zero. The virtual work done by the aerodynamic forces in a virtual displacement of the moving origin o is as follows:

$$\delta W = -D\delta x_0' - L\delta z_0'$$

where D and L are the total drag and lift forces. The virtual work done by the aerodynamic moment M in a virtual rotation is

$$\delta W = M\delta\theta$$

The virtual work done by the gravitational forces may be determined most easily by making use of previous results. Using D'Alembert's principle, it may be expected that the gravitational forces per unit mass $g \sin\theta$ and $g\cos\theta$ will have the same effect as the inertial forces per unit mass $\ddot{x}_0'$ and $-\ddot{z}_0'$. The terms involving $\ddot{x}_0'$ and $-\ddot{z}_0'$ in Eq. (13) represent the negative of the virtual work done by these inertial forces per unit virtual displacement. Making use of Eq. (13), the virtual work done by the gravitational forces may be written as

$$\begin{aligned}\delta W = {} & \sum_i [\mathcal{M}_S l_S \psi_i(x_S)\,\delta_S + \mathcal{M}_N l_N \psi_i(x_N)\,\delta_N]\,g\sin\theta\,\delta q_i - \mathcal{M}g\sin\theta\,\delta x_0' + \mathcal{M}g\cos\theta\,\delta z_0' - [\mathcal{M}_S l_S\delta_S + \mathcal{M}_N l_N \delta_N]\,g\sin\theta\,\delta\theta \\ & + \left\{\left[\sum_i \mathcal{M}_S l_S \psi_i(x_S)\,q_i - \mathcal{M}_S l_S \delta_S\right] g\sin\theta + \mathcal{M}_S l_S g\cos\theta\right\}\delta(\delta_S) \\ & + \left\{\left[\sum_i \mathcal{M}_N l_N \psi_i(x_N)\,q_i - \mathcal{M}_N l_N \delta_N\right] g\sin\theta + \mathcal{M}_N l_N g\cos\theta\right\}\delta(\delta_N)\end{aligned}$$

The nozzle actuation moment M_N is the moment exerted by the beam on the nozzle and will be considered positive in the direction of positive nozzle rotation δ_N. Because of the equal and opposite reaction on the beam, the virtual work done will vanish except for a virtual rotation of the nozzle. Thus the virtual work done by the nozzle actuation moment is

$$\delta W = M_N \delta(\delta_N)$$

Consider the effect of structural damping, slosh damping, and damping resulting from nozzle rotation. Aerodynamic damping effects are contained in the aerodynamic terms. If it is assumed that coupling of the elastic natural modes resulting from structural damping is negligible and, if the structural damping is represented in terms of an equivalent viscous damping, the virtual work done by structural damping may be written as

$$\delta W = -2 \sum_i \mathcal{M}_i \zeta_i \omega_i \dot{q}_i \delta q_i$$

In the above relation, the term ζ_i represents the damping ratio for the ith natural mode. Similarly the virtual work done by the slosh damping may be written as

$$\delta W = -2 I_S \zeta_S \left(\frac{g}{l_S}\right)^{1/2} \dot{\delta}_S \delta(\delta_S)$$

where ζ_S represents the damping ratio for slosh in a 1-g force field and $(g/l_S)^{1/2}$ represents the slosh natural frequency in the same condition. The virtual work done by the nozzle damping forces may be written as

$$\delta W = -C_N \dot{\delta}_N \delta(\delta_N)$$

where C_N represents the viscous damping moment per unit rotational velocity of the nozzle.

The virtual work done by all the forces considered may be obtained by summing all of the virtual work equations as follows:

B. Generalized Forces

The generalized forces are the coefficients of the virtual work expression, Eq. (18), and are expressed as

$$Q_i = \frac{\delta W}{\delta q_i} = -2\mathcal{M}_i \zeta_i \omega_i \dot{q}_i + \mathcal{M}_S l_S g \sin\theta\, \psi_i(x_S)\, \delta_S + [\mathcal{M}_N l_N g \sin\theta\, \psi_i(x_N) - T\phi_i(x_N)]\,\delta_N - \int_{x_1}^{x_2} L'\phi_i\, dx \tag{19}$$

$$Q_{x_0'} = \frac{\delta W}{\delta x_0'} = T - D - \mathcal{M} g \sin\theta$$

$$Q_{z_0'} = \frac{\delta W}{\delta z_0'} = T \sum_i \psi_i(x_N)\, q_i - T\delta_N - L + \mathcal{M} g \cos\theta$$

$$Q_\theta = \frac{\delta W}{\delta \theta} = T \sum_i [\phi_i(x_N) - x_N \psi_i(x_N)]\, q_i - \mathcal{M}_S l_S g \sin\theta\, \delta_S + [T x_N - \mathcal{M}_N l_N g \sin\theta]\, \delta_N + M$$

$$Q_{\delta_S} = \frac{\delta W}{\delta(\delta_S)} = \sum_i \mathcal{M}_S l_S g \sin\theta\, \psi_i(x_S)\, q_i - \mathcal{M}_S l_S g \sin\theta\, \delta_S - 2 I_S \zeta_S \left(\frac{g}{l_S}\right)^{1/2} \dot{\delta}_S + \mathcal{M}_S l_S g \cos\theta$$

$$Q_{\delta_N} = \frac{\delta W}{\delta(\delta_N)} = \sum_i \mathcal{M}_N l_N g \sin\theta\, \psi_i(x_N)\, q_i - \mathcal{M}_N l_N g \sin\theta\, \delta_N - C_N \dot{\delta}_N + \mathcal{M}_N l_N g \cos\theta + M_N$$

$$\begin{aligned} \delta W = {} & \sum_i \left\{ -\int_{x_1}^{x_2} L'\phi_i\, dx + \mathcal{M}_S l_S g \sin\theta\, \psi_i(x_S)\, \delta_S + [\mathcal{M}_N l_N g \sin\theta\, \psi_i(x_N) - T\phi_i(x_N)]\, \delta_N - 2\mathcal{M}_i \zeta_i \omega_i \dot{q}_i \right\} \delta q_i \\ & + [T - D - \mathcal{M} g \sin\theta]\, \delta x_0' + \left[T \sum_i \psi_i(x_N)\, q_i - T\delta_N - L + \mathcal{M} g \cos\theta \right] \delta z_0' \\ & + \left\{ T \sum_i [\phi_i(x_N) - x_N\psi_i(x_N)]\, q_i - \mathcal{M}_S l_S g \sin\theta\, \delta_S + [T x_N - \mathcal{M}_N l_N g \sin\theta]\, \delta_N + M \right\} \delta\theta \\ & + \left\{ \sum_i \mathcal{M}_S l_S g \sin\theta\, \psi_i(x_S)\, q_i - \mathcal{M}_S l_S g \sin\theta\, \delta_S + \mathcal{M}_S l_S g \cos\theta - 2 I_S \zeta_S \left(\frac{g}{l_S}\right)^{1/2} \dot{\delta}_S \right\} \delta(\delta_S) \\ & + \left\{ \sum_i \mathcal{M}_N l_N g \sin\theta\, \psi_i(x_N)\, q_i - \mathcal{M}_N l_N g \sin\theta\, \delta_N + \mathcal{M}_N l_N g \cos\theta + \mathcal{M}_N - C_N \dot{\delta}_N \right\} \delta(\delta_N) \end{aligned} \tag{18}$$

V. EQUATIONS OF FORCED MOTION

A. Lagrange's Equations of Motion

The Lagrange equations of motion for the system are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i \qquad i = 1, 2, \cdots \tag{20}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}'_0}\right) = Q_{x'_0}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}'_0}\right) = Q_{z'_0}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\delta}_S}\right) - \frac{\partial T}{\partial \delta_S} = Q_{\delta_S}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\delta}_N}\right) - \frac{\partial T}{\partial \delta_N} = Q_{\delta_N}$$

Making use of Eq. (13), (14), (17), and (19) the equations of motion for the system become

$$\begin{aligned}
&\mathcal{M}_i \ddot{q}_i + [\mathcal{M}_S l_S \phi_i(x_S) - I_S \psi_i(x_S)]\, \ddot{\delta}_S \\
&\quad + [\mathcal{M}_N l_N \phi_i(x_N) - I_N \psi_i(x_N)]\, \ddot{\delta}_N \\
&\quad + 2\mathcal{M}_i \zeta_{\omega_i} \omega_i \dot{q}_i + \mathcal{M}_i \omega_i^2 q_i \\
&\quad - \mathcal{M}_S l_S (\dot{V} + g \sin\theta)\, \psi_i(x_S)\, \delta_S \\
&\quad - [\mathcal{M}_N l_N (\dot{V} + g\sin\theta)\, \psi_i(x_N) - T\phi_i(x_N)]\, \delta_N \\
&\qquad + \int_{x_1}^{x_2} L' \phi_i \, dx = 0 \qquad i = 1, 2, \cdots
\end{aligned} \tag{21}$$

$$\mathcal{M}\dot{V} - T + D + \mathcal{M} g \sin\theta = 0$$

$$\begin{aligned}
&\mathcal{M}(\dot{V}\alpha - V\dot{\gamma}) + \mathcal{M}_S l_S \ddot{\delta}_S + \mathcal{M}_N l_N \ddot{\delta}_N \\
&\quad - T \sum_i \psi_i(x_N)\, q_i \\
&\qquad + T\delta_N - \mathcal{M} g \cos\theta + L = 0
\end{aligned}$$

$$\begin{aligned}
&I\ddot{\theta} - (\mathcal{M}_S l_S x_S - I_S)\, \ddot{\delta}_S - (\mathcal{M}_N l_N x_N - I_N)\, \ddot{\delta}_N \\
&\quad - T \sum_i [\phi_i(x_N) - x_N \psi_i(x_N)]\, q_i \\
&\quad + \mathcal{M}_S l_S (\dot{V} + g\sin\theta)\, \delta_S \\
&\qquad + [\mathcal{M}_N l_N (\dot{V} + g \sin\theta) - T x_N]\, \delta_N - M = 0
\end{aligned}$$

$$\begin{aligned}
&I_S \ddot{\delta}_S + \sum_i [\mathcal{M}_S l_S \phi_i(x_S) - I_S \psi_i(x_S)]\, \ddot{q}_i \\
&\quad - (\mathcal{M}_S l_S x_S - I_S)\, \ddot{\theta} \\
&\quad + 2 I_S \zeta_S \left(\frac{g}{l_S}\right)^{1/2} \dot{\delta}_S - \sum_i \mathcal{M}_S l_S (\dot{V} + g\sin\theta)\, \psi_i(x_S)\, q_i \\
&\quad + \mathcal{M}_S l_S (\dot{V} + g \sin\theta)\, \delta_S \\
&\qquad + \mathcal{M}_S l_S (\dot{V}\alpha - V\dot{\gamma} - g\cos\theta) = 0
\end{aligned}$$

$$\begin{aligned}
&I_N \ddot{\delta}_N + \sum_i [\mathcal{M}_N l_N \phi_i(x_N) - I_N \psi_i(x_N)]\, \ddot{q}_i \\
&\quad - (\mathcal{M}_N l_N x_N - I_N)\, \ddot{\theta} \\
&\quad + C_N \dot{\delta}_N - \sum_i \mathcal{M}_N l_N (\dot{V} + g\sin\theta)\, \psi_i(x_N)\, q_i \\
&\quad + \mathcal{M}_N l_N (\dot{V} + g\sin\theta)\, \delta_N \\
&\qquad + \mathcal{M}_N l_N (\dot{V}\alpha - V\dot{\gamma} - g\cos\theta) - M_N = 0
\end{aligned}$$

B. Equations of Forced Motion

It is assumed that the motion of the system consists of the sum of a forced motion and an undisturbed or trimmed motion. The motion of the system is represented by

$$\begin{aligned}
q_i &= q_{iT} + \bar{q}_i \\
V &= V_T + \bar{V} \\
\alpha &= \alpha_T + \bar{\alpha} \\
\theta &= \theta_T + \bar{\theta} \\
\gamma &= \gamma_T + \bar{\gamma} \\
\delta_S &= \delta_{ST} + \bar{\delta}_S \\
\delta_N &= \delta_{NT} + \bar{\delta}_N
\end{aligned} \tag{22}$$

where the subscript T represents the trimmed motion and the bar represents the forced motion. It is further assumed that the trimmed motion varies slowly with time and is essentially constant. The aerodynamic forces and nozzle actuating moment are represented by

$$\begin{aligned}
L &= L_T + \bar{L} \\
D &= D_T + \bar{D} \\
M &= M_T + \bar{M} \\
M_N &= M_{NT} + \bar{M}_N
\end{aligned} \tag{23}$$

The equations of motion, Eq. (21), for the trimmed motion are

$$\begin{aligned}
&\mathcal{M}_i\omega_i^2 q_{iT} - \mathcal{M}_S l_S(\dot V_T + g\sin\theta_T)\,\psi_i(x_S)\,\delta_{ST}\\
&\qquad - [\mathcal{M}_N l_N(\dot V_T + g\sin\theta_T)\,\psi_i(x_N) - T\phi_i(x_N)]\,\delta_{NT}\\
&\qquad\qquad + \int_{x_1}^{x_2} L'_T\phi_i\,dx = 0
\end{aligned}$$

$$\mathcal{M}\dot V_T - T + D_T + \mathcal{M}g\sin\theta_T = 0$$

$$\mathcal{M}\dot V_T\alpha_T - T\sum_i \psi_i(x_N)\,q_{iT} + T\delta_{NT} - \mathcal{M}g\cos\theta_T + L_T = 0$$

$$\begin{aligned}
&-T\sum_i[\phi_i(x_N) - x_N\psi_i(x_N)]\,q_{iT} + \mathcal{M}_S l_S(\dot V_T + g\sin\theta_T)\,\delta_{ST}\\
&\qquad + [\mathcal{M}_N l_N(\dot V_T + g\sin\theta_T) - Tx_N]\,\delta_{NT} - M_T = 0
\end{aligned}$$

$$\begin{aligned}
&-\sum_i \mathcal{M}_S l_S(\dot V_T + g\sin\theta_T)\,\psi_i(x_S)\,q_{iT}\\
&\qquad + \mathcal{M}_S l_S(\dot V_T + g\sin\theta_T)\,\delta_{ST}\\
&\qquad\qquad + \mathcal{M}_S l_S(\dot V_T\alpha_T - g\cos\theta_T) = 0
\end{aligned}$$

$$\begin{aligned}
&-\sum_i \mathcal{M}_N l_N(\dot V_T + g\sin\theta_T)\,\psi_i(x_N)\,q_{iT}\\
&\qquad + \mathcal{M}_N l_N(\dot V_T + g\sin\theta_T)\,\delta_{NT}\\
&\qquad\qquad + \mathcal{M}_N l_N(\dot V_T\alpha_T - g\cos\theta_T) - M_{NT} = 0
\end{aligned}$$

The equations of motion, Eq. (21), for the total motion are

$$\begin{aligned}
&\mathcal{M}_i\ddot{\bar q}_i + [\mathcal{M}_S l_S\phi_i(x_S) - I_S\psi_i(x_S)]\,\ddot{\bar\delta}_S\\
&\qquad + [\mathcal{M}_N l_N\phi_i(x_N) - I_N\psi_i(x_N)]\,\ddot{\bar\delta}_N\\
&\qquad + 2\mathcal{M}_i\zeta_i\omega_i\dot{\bar q}_i + \mathcal{M}_i\omega_i^2(q_{iT} + \bar q_i)\\
&\qquad - \mathcal{M}_S l_S[\dot V_T + \dot{\bar V} + g\sin(\theta_T + \bar\theta)]\,\psi_i(x_S)\,(\delta_{ST} + \bar\delta_S)\\
&\qquad - \{\mathcal{M}_N l_N[\dot V_T + \dot{\bar V} + g\sin(\theta_T + \bar\theta)]\,\psi_i(x_N) - T\phi_i(x_N)\}\\
&\qquad\qquad \times(\delta_{NT} + \bar\delta_N) + \int_{x_1}^{x_2}(L'_T + \bar L')\,\phi_i\,dx = 0
\end{aligned}$$

$$\mathcal{M}(\dot V_T + \dot{\bar V}) - T + D_T + \bar D + \mathcal{M}g\sin(\theta_T + \bar\theta) = 0$$

$$\begin{aligned}
&\mathcal{M}[(\dot V_T + \dot{\bar V})(\alpha_T + \bar\alpha) - (V_T + \bar V)\,\dot{\bar\gamma}] + \mathcal{M}_S l_S\ddot{\bar\delta}_S\\
&\qquad + \mathcal{M}_N l_N\ddot{\bar\delta}_N - T\sum_i \psi_i(x_N)\,(q_{iT} + \bar q_i) + T(\delta_{NT} + \bar\delta_N)\\
&\qquad\qquad - \mathcal{M}g\cos(\theta_T + \bar\theta) + L_T + \bar L = 0
\end{aligned}$$

$$\begin{aligned}
&I\ddot{\bar\theta} - (\mathcal{M}_S l_S x_S - I_S)\,\ddot{\bar\delta}_S - (\mathcal{M}_N l_N x_N - I_N)\,\ddot{\bar\delta}_N\\
&\qquad - T\sum_i[\phi_i(x_N) - x_N\psi_i(x_N)]\,(q_{iT} + \bar q_i)\\
&\qquad + \mathcal{M}_S l_S[\dot V_T + \dot{\bar V} + g\sin(\theta_T + \bar\theta)]\,(\delta_{ST} + \bar\delta_S)\\
&\qquad + \{\mathcal{M}_N l_N[\dot V_T + \dot{\bar V} + g\sin(\theta_T + \bar\theta)] - Tx_N\}\\
&\qquad\qquad \times(\delta_{NT} + \bar\delta_N) - M_T - \bar M = 0
\end{aligned}$$

$$\begin{aligned}
&I_S\ddot{\bar\delta}_S + \sum_i[\mathcal{M}_S l_S\phi_i(x_S) - I_S\psi_i(x_S)]\,\ddot{\bar q}_i - (\mathcal{M}_S l_S x_S - I_S)\,\ddot{\bar\theta}\\
&\qquad + 2I_S\zeta_S\left(\frac{g}{l_S}\right)^{1/2}\dot{\bar\delta}_S - \sum_i \mathcal{M}_S l_S[\dot V_T + \dot{\bar V} + g\sin(\theta_T + \bar\theta)]\\
&\qquad \times\psi_i(x_S)\,(q_{iT} + \bar q_i)\\
&\qquad + \mathcal{M}_S l_S[\dot V_T + \dot{\bar V} + g\sin(\theta_T + \bar\theta)]\,(\delta_{ST} + \bar\delta_S)\\
&\qquad + \mathcal{M}_S l_S[(\dot V_T + \dot{\bar V})(\alpha_T + \bar\alpha)\\
&\qquad\qquad - (V_T + \bar V)\,\dot{\bar\gamma} - g\cos(\theta_T + \bar\theta)] = 0
\end{aligned}$$

$$\begin{aligned}
&I_N\ddot{\bar\delta}_N + \sum_i[\mathcal{M}_N l_N\phi_i(x_N) - I_N\psi_i(x_N)]\,\ddot{\bar q}_i - (\mathcal{M}_N l_N x_N - I_N)\,\ddot{\bar\theta}\\
&\qquad + C_N\dot{\bar\delta}_N - \sum_i \mathcal{M}_N l_N[\dot V_T + \dot{\bar V} + g\sin(\theta_T + \bar\theta)]\\
&\qquad \times\psi_i(x_N)\,(q_{iT} + \bar q_i)\\
&\qquad + \mathcal{M}_N l_N[\dot V_T + \dot{\bar V} + g\sin(\theta_T + \bar\theta)]\,(\delta_{NT} + \bar\delta_N)\\
&\qquad + \mathcal{M}_N l_N[(\dot V_T + \dot{\bar V})(\alpha_T + \bar\alpha) - (V_T + \bar V)\,\dot{\bar\gamma}\\
&\qquad\qquad - g\cos(\theta_T + \bar\theta)] - M_{NT} - \bar M_N = 0
\end{aligned}$$

The trigonometric terms in the preceding equations may be expanded to

$$\sin(\theta_T + \bar\theta) = \sin\theta_T + \bar\theta\cos\theta_T$$

$$\cos(\theta_T + \bar\theta) = \cos\theta_T - \bar\theta\sin\theta_T$$

assuming $\bar\theta$ to be a small quantity.

If the equations for trimmed motion are subtracted from the equations for total motion, the following equations of forced motion result

$$\begin{aligned}
(24)\qquad &\mathcal{M}_i\ddot{\bar q}_i + [\mathcal{M}_S l_S\phi_i(x_S) - I_S\psi_i(x_S)]\,\ddot{\bar\delta}_S\\
&\qquad + [\mathcal{M}_N l_N\phi_i(x_N) - I_N\psi_i(x_N)]\,\ddot{\bar\delta}_N\\
&\qquad + 2\mathcal{M}_i\zeta_i\omega_i\dot{\bar q}_i + \mathcal{M}_i\omega_i^2\bar q_i\\
&\qquad - \mathcal{M}_S l_S(\dot V_T + g\sin\theta_T)\,\psi_i(x_S)\,\bar\delta_S\\
&\qquad - [\mathcal{M}_N l_N(\dot V_T + g\sin\theta_T)\,\psi_i(x_N) - T\phi_i(x_N)]\,\bar\delta_N\\
&\qquad\qquad + \int_{x_1}^{x_2}\bar L'\phi_i\,dx = 0
\end{aligned}$$

$$\mathcal{M}\dot{\bar V} + \bar D + \mathcal{M}g\,\bar\theta\cos\theta_T = 0$$

$$\begin{aligned}
&\mathcal{M}(\dot V_T\bar\alpha - V_T\dot{\bar\gamma}) + \mathcal{M}_S l_S\ddot{\bar\delta}_S\\
&\qquad + \mathcal{M}_N l_N\ddot{\bar\delta}_N - T\sum_i \psi_i(x_N)\,\bar q_i\\
&\qquad\qquad + T\bar\delta_N + \mathcal{M}g\sin\theta_T\bar\theta + \bar L = 0
\end{aligned}$$

$$\begin{aligned}
&I\ddot{\bar\theta} - (\mathcal{M}_S l_S x_S - I_S)\,\ddot{\bar\delta}_S - (\mathcal{M}_N l_N x_N - I_N)\,\ddot{\bar\delta}_N\\
&\qquad - T\sum_i[\phi_i(x_N) - x_N\psi_i(x_N)]\,\bar q_i\\
&\qquad + \mathcal{M}_S l_S(\dot V_T + g\sin\theta_T)\,\bar\delta_S\\
&\qquad\qquad + [\mathcal{M}_N l_N(\dot V_T + g\sin\theta_T) - Tx_N]\,\bar\delta_N - \bar M = 0
\end{aligned}$$

$$
\begin{aligned}
& I_S \ddot{\bar{\delta}}_S + \sum_i \left[\mathfrak{M}_S l_S \phi_i (x_S) - I_S \psi_i (x_S)\right] \ddot{\bar{q}}_i \\
& \quad - (\mathfrak{M}_S l_S x_S - I_S) \ddot{\bar{\theta}} + 2 I_S \zeta_S \left(\frac{g}{l_S}\right)^{1/2} \dot{\bar{\delta}}_S \\
& \quad - \sum_i \mathfrak{M}_S l_S (\dot{V}_T + g \sin \theta_T) \psi_i (x_S) \bar{q}_i \\
& \quad + \mathfrak{M}_S l_S (\dot{V}_T + g \sin \theta_T) \bar{\delta}_S \\
& \qquad + \mathfrak{M}_S l_S (\dot{V}_T \bar{\alpha} - V_T \dot{\bar{\gamma}} + g \sin \theta_T \bar{\theta}) = 0 \\
& I_N \ddot{\bar{\delta}}_N + \sum_i \left[\mathfrak{M}_N l_N \phi_i (x_N) - I_N \psi_i (x_N)\right] \ddot{\bar{q}}_i \\
& \quad - (\mathfrak{M}_N l_N x_N - I_N) \ddot{\bar{\theta}} + C_N \dot{\bar{\delta}}_N \\
& \quad - \sum_i \mathfrak{M}_N l_N (\dot{V}_T + g \sin \theta_T) \psi_i (x_N) \bar{q}_i \\
& \quad + \mathfrak{M}_N l_N (\dot{V}_T + g \sin \theta_T) \bar{\delta}_N \\
& \qquad + \mathfrak{M}_N l_N (\dot{V}_T \bar{\alpha} - V_T \dot{\bar{\gamma}} + g \sin \theta_T \bar{\theta}) - \bar{M}_N = 0
\end{aligned}
$$

In Eq. (24), terms involving the products of forced-motion variables have been ignored. Further, since the trim variables q_{iT}, α_T, δ_{ST}, and δ_{NT} may be expected to be small quantities, the products of these variables and the forced-motion variables have been ignored. In addition to Eq. (24), equations are needed for the aerodynamic forces and the nozzle actuating moment. The aerodynamic forces will be discussed in Sect. VI. The control-system equations needed to specify the nozzle actuating moment will not be discussed here.

C. *Forcing Conditions*

The equations of forced motion are suitable for an analysis of the forced motion resulting from disturbances involved in the launch and boosted flight of a rocket-boosted vehicle. For the important problems involving atmospheric motion such as gusts, turbulence, or wind shear, the forcing function would be contained in the aerodynamic lift and moment terms. For problems such as the determination of the forced motion resulting from a maneuver or a control-system malfunction, the forcing function would be contained in the nozzle actuating moment. Further, if the system is in motion at the instant of launch or stage separation, the resulting motion may be studied by considering the appropriate initial value problem.

VI. AERODYNAMIC FORCES

The momentum method will be used to determine the aerodynamic forces (Ref. 6, pp. 418–420). This method is applicable to small disturbances of slender vehicles at velocities up to low supersonic. The method is appropriate for the boost period of large rocket-boosted vehicles, which are often slender and which typically do not experience velocities beyond low supersonic during that part of the boost period where aerodynamic forces are important.

Consider an Earth-fixed frame of reference $o'x'z'$ (shown in Fig. 5, Sect. IIIA). The frame of reference $o'x'z'$ is also considered to be at rest relative to the air mass. The motion of the moving origin is given by the coordinates x_0' and z_0' and the relative orientation by $\varepsilon = \theta - \theta'$. It is assumed that z_0' and ε are small.

The method to be used is based on the assumption that the disturbed air flow is two-dimensional in planes normal to the direction of flight. Consider the air contained in the volume bounded by the planes x' and $x' + dx'$. It will be assumed that the disturbed flow is in the z' direction. The displacement of the beam in the z' direction at x' is given by

$$z' = z_0' - \varepsilon x + w$$

where

$$x = x' - x_0'$$

Thus, the velocity of the beam in the z' direction at x' is

$$\dot{z}' = \dot{z}_0' - \dot{\varepsilon} x - \varepsilon \dot{x} + \dot{w} + w' \dot{x}$$

But

$$\dot{z}_0' = V(\alpha - \varepsilon)$$

$$\dot{\varepsilon} = \dot{\theta}$$

$$\dot{x} = -\dot{x}_0' = -V$$

Making use of these equations, the velocity of the beam in the z' direction at x' becomes

$$\dot{z}' = V(\alpha - w') - \dot{\theta} x + \dot{w}$$

Assume that the momentum of the air contained in the differential volume may be written as

$$dI_{z'} = \rho A dx' [V(\alpha - w') - \dot{\theta} x + \dot{w}]$$

where ρA represents the virtual mass per unit length of a cylinder having the same area and shape as the vehicle at the point considered. The force per unit length applied to the air mass will be equal to the time rate of change of the momentum per unit length. Thus, the lift per unit length applied to the vehicle may be written as

$$\begin{aligned}(25) \qquad L' &= \frac{d}{dt}\left(\frac{dI_{z'}}{dx'}\right) \\ &= \rho V^2 [-A'\alpha + (Aw')'] \\ &\quad + \rho V \{[(Ax)' + A]\dot{\theta} - [(A\dot{w})' + A\dot{w}']\} \\ &\quad + \rho A[-V(\dot{\theta} - \dot{\alpha}) - x\ddot{\theta} + \ddot{w}]\end{aligned}$$

The last line of Eq. (25) represents the product of the virtual mass per unit length and the lateral acceleration of the vehicle. It will be assumed that the density of the air is small compared with the density of the vehicle and the last line of the equation will be ignored. If the elastic deflections are written in terms of the natural modes, Eq. (25) may be written as

$$\begin{aligned}(26) \qquad L' &= \rho V^2 \left[-A'\alpha + \sum_i (A\phi_i')' q_i\right] \\ &\quad + \rho V \left\{[(Ax)' + A]\dot{\theta} - \sum_i [(A\phi_i)' + A\phi_i']\dot{q}_i\right\}\end{aligned}$$

In order to account for atmospheric motion such as gusts, turbulence, or wind shear, the quantity α will be replaced by $\alpha + \alpha_w$, where α_w is the angle of attack resulting from atmospheric motion.

The total lift L may be determined by integrating the lift per unit length L' over the beam, as follows:

$$\begin{aligned}(27) \qquad L &= \rho V^2 \Big\{[A(x_1) - A(x_2)](\alpha + \alpha_w) \\ &\quad + \sum_i [A(x_2)\phi_i'(x_2) - A(x_1)\phi_i'(x_1)] q_i\Big\} \\ &\quad + \rho V \Bigg\{\left[A(x_2)x_2 - A(x_1)x_1 + \int_{x_1}^{x_2} A\,dx\right]\dot{\theta} \\ &\quad - \sum_i \Bigg[A(x_2)\phi_i(x_2) - A(x_1)\phi_i(x_1) \\ &\quad + \int_{x_1}^{x_2} A\phi_i'\,dx\Bigg]\dot{q}_i\Bigg\}\end{aligned}$$

Similarly, the total moment M, positive in the θ direction, becomes

$$\begin{aligned}(28) \qquad M &= \int_{x_1}^{x_2} xL'\,dx \\ &= \rho V^2 \Bigg\{-\left[\int_{x_1}^{x_2} A'x\,dx\right](\alpha + \alpha_w) \\ &\quad + \sum_i \left[\int_{x_1}^{x_2} (A\phi_i')' x\,dx\right] q_i\Bigg\} \\ &\quad + \rho V \Bigg\{\left[\int_{x_1}^{x_2} [(Ax)' + A]\,x\,dx\right]\dot{\theta} \\ &\quad - \sum_i \left[\int_{x_1}^{x_2} [(A\phi_i)' + A\phi_i']\,x\,dx\right]\dot{q}_i\Bigg\}\end{aligned}$$

Further, the following integral, which represents the generalized forces acting on the elastic motions, may be evaluated:

$$\begin{aligned}(29) \qquad &\int_{x_1}^{x_2} L'\phi_i\,dx \\ &= \rho V^2 \Bigg\{-\left[\int_{x_1}^{x_2} A'\phi_i\,dx\right](\alpha + \alpha_w) \\ &\quad + \sum_j \int_{x_1}^{x_2} (A\phi_j')'\phi_i\,dx\Bigg] q_j\Bigg\} \\ &\quad + \rho V \Bigg\{\left[\int_{x_1}^{x_2} [(Ax)' + A]\phi_i\,dx\right]\dot{\theta} \\ &\quad - \sum_j \left[\int_{x_1}^{x_2} [(A\phi_j)' + A\phi_j']\phi_i\,dx\right]\dot{q}_j\Bigg\}\end{aligned}$$

Equations (27), (28), and (29) may be used to determine the aerodynamic forces corresponding to the total motion, the sum of a forced motion and a trimmed motion, defined by Eq. (22). Similarly, the aerodynamic forces corresponding to the trimmed motion may be determined. The difference between the total and trimmed aerodynamic forces represents the aerodynamic forces resulting from the forced motion. Thus, the aerodynamic forces appearing in the equations of forced motion, Eq. (24), may be written as follows:

$$(30)\qquad \bar{L} = \rho V_T^2 \Big\{ [A(x_1) - A(x_2)](\bar{\alpha} + \alpha_w) + \sum_i [A(x_2)\phi_i'(x_2) - A(x_1)\phi_i'(x_1)]\bar{q}_i \Big\}$$
$$+ \rho V_T \Big\{ \Big[A(x_2)x_2 - A(x_1)x_1 + \int_{x_1}^{x_2} A\,dx \Big] \dot{\bar{\theta}} - \sum_i \Big[A(x_2)\phi_i(x_2) - A(x_1)\phi_i(x_1) + \int_{x_1}^{x_2} A\phi_i'\,dx \Big] \dot{\bar{q}}_i \Big\}$$

$$\bar{M} = \rho V_T^2 \Big\{ -\Big[\int_{x_1}^{x_2} A'x\,dx\Big](\alpha + \alpha_w) + \sum_i \Big[\int_{x_1}^{x_2} (A\phi_i')'x\,dx\Big]\bar{q}_i \Big\}$$
$$+ \rho V_T \Big\{ \Big[\int_{x_1}^{x_2} [(Ax)' + A]x\,dx\Big]\dot{\bar{\theta}} - \sum_i \Big[\int_{x_1}^{x_2} [(A\phi_i)' + A\phi_i']x\,dx\Big]\dot{\bar{q}}_i \Big\}$$

$$\int_{x_1}^{x_2} \bar{L}'\phi_i\,dx = \rho V_T^2 \Big\{ -\Big[\int_{x_1}^{x_2} A'\phi_i\,dx\Big](\bar{\alpha} + \alpha_w) + \sum_j \Big[\int_{x_1}^{x_2} (A\phi_j')'\phi_i\,dx\Big]\bar{q}_j \Big\}$$
$$+ \rho V_T \Big\{ \Big[\int_{x_1}^{x_2} [(Ax)' + A]\phi_i\,dx\Big]\dot{\bar{\theta}} - \sum_j \Big[\int_{x_1}^{x_2} [(A\phi_j)' + A\phi_j']\phi_i\,dx\Big]\dot{\bar{q}}_j \Big\}$$

In Eq. (30), products involving the forced-motion variables $\bar{V}$, $\bar{\alpha}$, $\bar{q}_i$, $\dot{\bar{\theta}}$, $\dot{\bar{q}}_i$, and $\bar{\theta}$ and the trimmed variables α_T, q_{iT}, θ_T, q_{iT}, and θ_T have been ignored.

The aerodynamic forces are customarily expressed in a form involving nondimensional coefficients. Define nondimensional time τ as

$$(31)\qquad \tau = \frac{V_T}{l_r}t$$

where l_r is a reference length. Then the aerodynamic quantities, Eq. (30), may be written as follows:

$$(32)\qquad \bar{L} = \frac{1}{2}\rho V_T^2 \cdot A_r \left[C_{L\alpha}(\bar{\alpha} + \alpha_w) + \sum_i C_{Lq_i}\frac{\bar{q}_i}{l_r} + C_{L\dot{\theta}}\frac{d\bar{\theta}}{d\tau} + \sum_i C_{L\dot{q}_i}\frac{d\frac{\bar{q}_i}{l_r}}{d\tau} \right]$$

$$\bar{M} = \frac{1}{2}\rho V_T^2 \cdot A_r l_r \left[C_{M\alpha}(\bar{\alpha} + \alpha_w) + \sum_i C_{Mq_i}\frac{\bar{q}_i}{l_r} + C_{M\dot{\theta}}\frac{d\bar{\theta}}{d\tau} + \sum_i C_{M\dot{q}_i}\frac{d\frac{\bar{q}_i}{l_r}}{d\tau} \right]$$

$$\int_{x_1}^{x_2} \bar{L}'\phi_i\,dx = \frac{1}{2}\rho V_T^2 \cdot A_r \left[C_{q_i\alpha}(\bar{\alpha} + \alpha_w) + \sum_j C_{q_iq_j}\frac{\bar{q}_j}{l_r} + C_{q_i\dot{\theta}}\frac{d\bar{\theta}}{d\tau} + \sum_j C_{q_i\dot{q}_j}\frac{d\left(\frac{\bar{q}_i}{l_r}\right)}{d\tau} \right]$$

where A_r is a reference area and the aerodynamic coefficients are

$$(33)\qquad C_{L\alpha} = \frac{2}{A_r}[A(x_1) - A(x_2)]$$

$$C_{Lq_i} = \frac{2l_r}{A_r}[A(x_2)\phi_i'(x_2) - A(x_1)\phi_i'(x_1)]$$

$$C_{L\dot{\theta}} = \frac{2}{A_r l_r}\left[A(x_2)x_2 - A(x_1)x_1 + \int_{x_1}^{x_2} A\,dx \right]$$

$$C_{L\dot{q}_i} = \frac{2}{A_r}\left[-A(x_2)\phi_i(x_2) + A(x_1)\phi_i(x_1) - \int_{x_1}^{x_2} A\phi_i'\,dx \right]$$

$$C_{M\alpha} = -\frac{2}{A_r l_r}\int_{x_1}^{x_2} A'x\,dx$$

$$C_{Mq_i} = \frac{2}{A_r}\int_{x_1}^{x_2} (A\phi_i')'x\,dx$$

$$C_{M\dot{\theta}} = \frac{2}{A_r l_r^2}\int_{x_1}^{x_2} [(Ax') + A]x\,dx$$

$$C_{M\dot{q}_i} = -\frac{2}{A_r l_r}\int_{x_1}^{x_2} [(A\phi_i)' + A\phi_i']x\,dx$$

$$C_{q_i\alpha} = -\frac{2}{A_r}\int_{x_1}^{x_2} A'\phi_i\,dx$$

$$C_{q_iq_j} = \frac{2l_r}{A_r}\int_{x_1}^{x_2} (A\phi_j')'\,\phi_i\,dx$$

$$C_{q_i\dot{\theta}} = \frac{2}{A_r l_r}\int_{x_1}^{x_2} [(Ax)' + A]\,\phi_i\,dx$$

$$C_{q_i\dot{q}_j} = -\frac{2}{A_r}\int_{x_1}^{x_2} [(A\phi_j)' + A\phi_j']\,\phi_i\,dx$$

Generally, some of the terms in the aerodynamic coefficients, Eq. (33), are negligible. The terms involving the angle of attack $(\bar{\alpha} + \alpha_w)$ are usually important in all three equations in Eq. (32). The terms involving the pitch rate $(d\bar{\theta}/d\tau)$ will usually be negligible except for the term in the moment equation. The terms involving the elastic motion will often be negligible, but may be important if the vehicle is sufficiently flexible. In the third equation, the terms involving coupling of the elastic modes, $i \neq j$, are often unimportant.

VII. NONDIMENSIONAL EQUATIONS

It is usually convenient to nondimensionalize equations of motion involving aerodynamic forces. The equations for forced motion, Eq. (24), will be written in nondimensional form, making use of the equations for the aerodynamic forces, Eq. (32). The nondimensional time τ is defined by Eq. (31). The derivatives with respect to τ will be designated by the operational notation

$$D = \frac{d}{d\tau}$$

$$D^2 = \frac{d^2}{d\tau^2}$$

The equations for forced motion may be written in the form

$$(34)\qquad \sum_j (\mathcal{M}_{ij}D^2 + C_{ij}D + K_{ij})\frac{\bar{q}_i}{l_r} + C_{i\theta}D\bar{\theta} + (\mathcal{M}_{iS}D^2 + K_{iS})\,\bar{\delta}_S + (\mathcal{M}_{iN}D^2 + K_{iN})\,\bar{\delta}_N + K_{i\alpha}\bar{\alpha} = -K_{i\alpha}\alpha_w \qquad i = 1, 2, 3, \cdots$$

$$(-\mathcal{M}_{\gamma\gamma}D + K_{\gamma\gamma})\,\bar{\gamma} + \sum_i (C_{\gamma i}D + K_{\gamma i})\frac{\bar{q}_i}{l_r} + C_{\gamma\theta}D\bar{\theta} + \mathcal{M}_{\gamma S}D^2\bar{\delta}_S + (\mathcal{M}_{\gamma N}D^2 + K_{\gamma N})\,\bar{\delta}_N + K_{\gamma\alpha}\bar{\alpha} = K_{\gamma\alpha_w}\alpha_w$$

$$(\mathcal{M}_{\theta\theta}D^2 + C_{\theta\theta}D)\,\bar{\theta} + \sum_i (C_{\theta i}D + K_{\theta i})\frac{\bar{q}_i}{l_r} + (\mathcal{M}_{\theta S}D^2 + K_{\theta S})\,\bar{\delta}_S + (\mathcal{M}_{\theta N}D^2 + K_{\theta N})\,\bar{\delta}_N + K_{\theta\alpha}\bar{\alpha} = -K_{\theta\alpha}\alpha_w$$

$$(\mathcal{M}_{SS}D^2 + C_{SS}D + K_{SS})\,\bar{\delta}_S + \sum_i (\mathcal{M}_{Si}D^2 + K_{Si})\frac{\bar{q}_i}{l_r} + M_{S\theta}D^2\bar{\theta} + (-\mathcal{M}_{S\gamma}D + K_{S\gamma})\,\bar{\gamma} + K_{S\alpha}\bar{\alpha} = 0$$

$$(\mathcal{M}_{NN}D^2 + C_{NN}D + K_{NN})\,\bar{\delta}_N + \sum_i (\mathcal{M}_{Ni}D^2 + K_{Ni})\frac{\bar{q}_i}{l_r} + \mathcal{M}_{N\theta}D^2\bar{\theta} + (-\mathcal{M}_{N\gamma}D + K_{N\gamma})\,\bar{\gamma} + K_{N\alpha}\bar{\alpha} = \bar{\mathbf{M}}_N$$

The equation for forced longitudinal motion, the second equation of Eq. (24), has not been repeated in Eq. (34) since it is not needed in determining the flexural motion. The nondimensional coefficients are defined as follows:

$$(35)\qquad \mathcal{M}_{ii} = \frac{\mathcal{M}_i}{\mathcal{M}_r}$$

$$\mathcal{M}_{ij} = 0 \qquad \text{for } i \neq j$$

$$\mathcal{M}_{iS} = \mathcal{M}_{Si} = \frac{\mathcal{M}_S}{\mathcal{M}_r}\frac{l_S}{l_r}\phi_i(x_S) - \frac{I_S}{\mathcal{M}_r l_r^2}\,[l_r\psi_i(x_S)]$$

$$\mathcal{M}_{iN} = \mathcal{M}_{Ni} = \frac{\mathcal{M}_N}{\mathcal{M}_r}\frac{l_N}{l_r}\phi_i(x_N) - \frac{I_N}{\mathcal{M}_r l_r^2}\,[l_r\psi_i(x_N)]$$

$$\mathcal{M}_{\gamma\gamma} = \frac{\mathcal{M}}{\mathcal{M}_r}$$

$$\mathcal{M}_{\gamma S} = \mathcal{M}_{S\gamma} = \frac{\mathcal{M}_S}{\mathcal{M}_r}\frac{l_S}{l_r}$$

$$\mathcal{M}_{\gamma N} = \mathcal{M}_{N\gamma} = \frac{\mathcal{M}_N}{\mathcal{M}_r}\frac{l_N}{l_r}$$

$$\mathcal{M}_{\theta\theta} = \frac{I}{\mathcal{M}_r l_r^2}$$

$$\mathcal{M}_{\theta S} = \mathcal{M}_{S\theta} = -\frac{\mathcal{M}_S}{\mathcal{M}_r}\frac{l_S}{l_r}\frac{x_S}{l_r} + \frac{I_S}{\mathcal{M}_r l_r^2}$$

$$\mathcal{M}_{\theta N} = \mathcal{M}_{N\theta} = -\frac{\mathcal{M}_N}{\mathcal{M}_r}\frac{l_N}{l_r}\frac{x_N}{l_r} + \frac{I_N}{\mathcal{M}_r l_r^2}$$

$$\mathcal{M}_{SS} = \frac{I_S}{\mathcal{M}_r l_r^2}$$

$$\mathcal{M}_{NN} = \frac{I_N}{\mathcal{M}_r l_r^2}$$

$$C_{ii} = 2\zeta_i \frac{\mathcal{M}_i}{\mathcal{M}_r}\left(\frac{\omega_i l_r}{V_T}\right) + \rho\frac{A_r l_r}{2\mathcal{M}_r} C_{q_i \dot{q}_i}$$

$$C_{ij} = \rho\frac{A_r l_r}{2\mathcal{M}_r} C_{q_i \dot{q}_j} \qquad \text{for } i \neq j$$

$$C_{i\theta} = \rho\frac{A_r l_r}{2\mathcal{M}_r} C_{q_i \dot{\theta}}$$

$$C_{\gamma i} = \rho\frac{A_r l_r}{2\mathcal{M}_r} C_{L q_i}$$

$$C_{\gamma\theta} = \rho\frac{A_r l_r}{2\mathcal{M}_r} C_{L\dot{\theta}}$$

$$C_{\theta\theta} = -\rho\frac{A_r l_r}{2\mathcal{M}_r} C_{M\dot{\theta}}$$

$$C_{\theta i} = -\rho\frac{A_r l_r}{2\mathcal{M}_r} C_{M\dot{q}_i}$$

$$C_{SS} = 2\zeta_S \frac{I_S}{\mathcal{M}_r l_r^2}\left(\frac{g l_r}{V_T^2}\cdot\frac{l_r}{l_S}\right)^{1/2}$$

$$C_{NN} = \frac{C_N}{\mathcal{M}_r l_r V_r}$$

$$K_{ii} = \frac{\mathcal{M}_i}{\mathcal{M}_r}\left(\frac{\omega_i l_r}{V_T}\right)^2 + \rho\frac{A_r l_r}{2\mathcal{M}_r} C_{q_i q_i}$$

$$K_{ij} = \rho\frac{A_r l_r}{2\mathcal{M}_r} C_{q_i q_j}$$

$$K_{iS} = K_{Si} = \frac{\mathcal{M}_S}{\mathcal{M}_r}\frac{l_S}{l_r}\left(\rho\frac{A_r l_r}{2\mathcal{M}} C_{D_0} - \frac{T l_r}{\mathcal{M} V_T^2}\right) \times [l_r \psi_i (x_S)]$$

$$K_{iN} = \frac{\mathcal{M}_N}{\mathcal{M}_r}\frac{l_N}{l_r}\left(\rho\frac{A_r l_r}{2\mathcal{M}} C_{D_0} - \frac{T l_r}{\mathcal{M} V_T^2}\right)[l_r \psi_i (x_N)] + \frac{T l_r}{\mathcal{M}_r V_T^2}\phi_i (x_N)$$

$$K_{i\alpha} = \rho\frac{A_r l_r}{2\mathcal{M}_r} C_{q_i \alpha}$$

$$K_{\gamma\gamma} = \frac{\mathcal{M}}{\mathcal{M}_r}\frac{g l_r}{V_T^2}\sin\theta_T$$

$$K_{\gamma i} = \rho\frac{A_r l_r}{2\mathcal{M}_r} C_{L q_i} - \frac{T l_r}{\mathcal{M}_r V_T^2}[l_r \psi_i (x_N)]$$

$$K_{\gamma N} = \frac{T l_r}{\mathcal{M}_r V_T^2}$$

$$K_{\gamma\alpha} = \rho\frac{A_r l_r}{2\mathcal{M}_r}\left(C_{L\alpha} - C_{D_0}\right) + \frac{T l_r}{\mathcal{M}_r V_T^2}$$

$$K_{\gamma\alpha_w} = -\rho\frac{A_r l_r}{2\mathcal{M}_r} C_{L\alpha}$$

$$K_{\theta i} = -\rho\frac{A_r l_r}{2\mathcal{M}_r} C_{M q_i} - \frac{T l_r}{\mathcal{M}_r V_T^2}[\phi_i (x_N) - x_N \psi_i (x_N)]$$

$$K_{\theta S} = \frac{\mathcal{M}_S}{\mathcal{M}_r}\frac{l_S}{l_r}\left(\frac{T l_r}{\mathcal{M} V_T^2} - \rho\frac{A_r l_r}{2\mathcal{M}} C_{D_0}\right)$$

$$K_{\theta N} = \frac{\mathcal{M}_N}{\mathcal{M}_r}\frac{l_N}{l_r}\left(\frac{T l_r}{\mathcal{M} V_T^2} - \rho\frac{A_r l_r}{2\mathcal{M}} C_{D_0}\right) - \frac{T l_r}{\mathcal{M}_r V_T^2}\frac{x_N}{l_r}$$

$$K_{\theta\alpha} = -\rho\frac{A_r l_r}{2\mathcal{M}_r} C_{M\alpha}$$

$$K_{SS} = \frac{\mathcal{M}_S}{\mathcal{M}_r}\frac{l_S}{l_r}\left(\frac{T l_r}{\mathcal{M} V_T^2} - \rho\frac{A_r l_r}{2\mathcal{M}} C_{D_0}\right)$$

$$K_{S\gamma} = \frac{\mathcal{M}_S}{\mathcal{M}_r}\frac{l_S}{l_r}\frac{g l_r}{V_T^2}\sin\theta_T$$

$$K_{S\alpha} = \frac{\mathcal{M}_S}{\mathcal{M}_r}\frac{l_S}{l_r}\left(\frac{T l_r}{\mathcal{M} V_T^2} - \rho\frac{A_r l_r}{2\mathcal{M}} C_{D_0}\right)$$

$$K_{NN} = \frac{\mathcal{M}_N}{\mathcal{M}_r}\frac{l_N}{l_r}\left(\frac{T l_r}{\mathcal{M} V_T^2} - \rho\frac{A_r l_r}{2\mathcal{M}} C_{D_0}\right)$$

$$K_{Ni} = \frac{\mathcal{M}_N}{\mathcal{M}_r}\frac{l_N}{l_r}\left(\rho\frac{A_r l_r}{2\mathcal{M}} C_{D_0} - \frac{T l_r}{\mathcal{M} V_T^2}\right)[l_r \psi_i (x_N)]$$

$$K_{N\gamma} = \frac{\mathcal{M}_N}{\mathcal{M}_r}\frac{l_N}{l_r}\frac{g l_r}{V_T^2}\sin\theta_T$$

$$K_{N\alpha} = \frac{\mathcal{M}_N}{\mathcal{M}_r}\frac{l_N}{l_r}\left(\frac{T l_r}{\mathcal{M} V_T^2} - \rho\frac{A_r l_r}{2\mathcal{M}} C_{D_0}\right)$$

$$\bar{\mathsf{M}}_N = \frac{\bar{M}_N}{\mathcal{M}_r V_T^2}$$

In Eq. (35), $\mathcal{M}_r$, A_r, and l_r are a reference mass, area, and length, respectively.

VIII. DISCUSSION

The determination of the forced flexural motion of the vehicle under consideration requires the solution of the equations for forced motion of the system, Eq. (34). These coupled equations describe the motions of the natural modes of flexural vibrations of the vehicle, the translational and pitching motions of the vehicle, and the motions of the engine nozzle and liquid fuel. In addition to the given equations, control-system equations are needed to specify the engine nozzle actuating moment.

Given the solution for the motion of the generalized coordinates of flexural motion q_i, the flexural displacements w and the rotations w'_B may be determined by summing the motions in all the natural flexural modes, as given by Eq. (11). With a complete knowledge of the flexural motion, the bending moments and shear forces in the vehicle may also be determined.

Forcing conditions which may be considered include atmospheric motion such as gusts, turbulence, or wind shear, thrust-vectoring resulting from control-system commands or malfunctions, and an initial disturbance. The effect of atmospheric motion is contained in the term α_w, the angle of attack resulting from atmospheric motion. The effect of control-system commands is contained in the term $\bar{M}_N$, the nondimensional nozzle actuating moment. Ignition or separation transients may be treated as initial value problems.

The natural modes and frequencies of flexural motion required in developing the equations of motion may be determined by any of the standard methods for the analysis of beam motion (Ref. 6, Chap. 4). Information needed for the approximate representation of the fuel motion by means of a simple pendulum is available in the literature (Ref. 5). The needed aerodynamic coefficients were developed in this Memorandum using the momentum method (Ref. 6, pp. 418–20). However, the coefficients may be determined experimentally or by another appropriate analytical method.

NOMENCLATURE

C_N damping coefficient, nozzle motion

D total drag, negative x direction

D' drag per unit length, negative x direction

EI flexural rigidity of beam

I moment of inertia, vehicle, around c.g.

I_N moment of inertia, nozzle, around hinge

I_S moment of inertia, slosh pendulum, around hinge

$k'AG$ shear rigidity of beam

l_N length, nozzle hinge to nozzle c.g.

l_S length, slosh pendulum

l_r reference length

L total lift, negative z direction

L' lift per unit length, negative z direction

$\mathfrak{M}$ mass of vehicle

$\mathfrak{M}_i$ generalized mass, ith flexural mode

$\mathfrak{M}_N$ mass, engine nozzle

$\mathfrak{M}_S$ mass, slosh pendulum

$\mathfrak{M}_r$ reference mass

M total aerodynamic moment, θ direction

M_N nozzle actuating moment

q_i generalized coordinate, ith flexural mode

Q_i generalized force, ith flexural mode

$Q_{x_0'}$ generalized force, rigid-body motion, x direction

$Q_{z_0'}$ generalized force, rigid-body motion, z direction

$Q_{\delta N}$ generalized force, engine nozzle

$Q_{\delta S}$ generalized force, slosh pendulum

Q_θ generalized force, pitching motion

r radius of gyration of beam cross section around lateral axis

T thrust force

V speed of vehicle

w total deflection in flexure

w_B bending deflection

w_S shearing deflection

x_1 location, aft end of vehicle

x_2 location, forward end of vehicle

x_N location, nozzle hinge

x_S location, slosh pendulum hinge

α angle of attack

α_w angle of attack resulting from atmospheric motion

γ trajectory angle above horizon

δ_N deflection angle, engine nozzle

δ_S deflection angle, slosh pendulum

ζ_i damping ratio, ith flexural mode

ζ_S damping ratio, slosh pendulum, 1-g force field

θ elevation angle above horizon

μ mass per unit length of beam

ρ air mass density

ρA air virtual mass per unit length

τ $V_T t/l_r$, nondimensional time

ϕ_i deflection shape, ith flexural mode

ψ_i bending slope shape, ith flexural mode

ω_i natural frequency, ith flexural mode

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