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# RATIONAL APPROXIMATIONS TO THE SOLUTION OF THE SECOND ORDER RICATTI EQUATION

by

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## PREFACE

This report covers research initiated by Headquarters, National Aeronautics and Space Administration on Contract NASA Hq. R&D 80X0108(64), 10-74-740-124-08-06-11, PR 10-2487, "Nonlinear Dynamics of Thin Shell Structures." The research work upon which this report is based was accomplished at Midwest Research Institute with Mr. Howard Wolko as project monitor.

This report covers work conducted from 15 January 1965 to 14 October 1965.

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25 October 1965

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## I. INTRODUCTION

In a previous work Merkes and Scott [1] constructed continued fraction solutions to the first order Ricatti equation by using a sequence of linear fractional transformations. Fair [2] utilized the $\tau$-method, see the paper by Luke [3], to develop main diagonal Padé approximations to the solution of the first order Ricatti equation with rational coefficients. Rational approximations are advantageous to study the behavior of the solutions in a global sense. That is, they are useful for evaluation of functional values in the complex plane including zeros and poles.

In this report we develop continued fraction (and hence rational) approximations to the solution of a second order nonlinear equation which includes as special cases the equations treated in [1] and [2]. These approximations are obtained by using a sequence of linear transformations which leave the differential equation invariant, see Davis [4], and are presented in Section II. For an application, in Section III, the algorithm is applied to obtain approximations to Painlevé's first and second transcendents.

## II. DEVELOPMENT OF THE RATIONAL APPROXIMATIONS

Consider the generalized second order Ricatti equation

$$(A_o+B_oy)y'' + (C_o+D_oy)y' - 2B_o(y')^2 + E_o + F_oy + G_oy^2 + H_oy^3 = 0 ,$$
$$y(0) = \alpha_o , \tag{1}$$

where the coefficients in (1) have Taylor's series expansions of the form

$$A_o = x^2 \sum_{k=0}^{\infty} a_kx^k , \; B_o = x^2 \sum_{k=0}^{\infty} b_kx^k , \; C_o = x \sum_{k=0}^{\infty} c_kx^k ,$$

$$D_o = x \sum_{k=0}^{\infty} d_kx^k , \; E_o = \sum_{k=0}^{\infty} e_kx^k \;\; e_o \neq 0 , \; F_o = \sum_{k=0}^{\infty} f_kx^k \;\; f_o \neq 0 ,$$

$$G_o = x \sum_{k=0}^{\infty} g_kx^k \text{ and } H_o = x \sum_{k=0}^{\infty} h_kx^k . \tag{2}$$

We further assume that the solution of (1) has a power series expansion of the form

$$y = \alpha_o + \sum_{k=1}^{\infty} \beta_k x^k \ . \tag{3}$$

Note that (2) and (3) together with (1) uniquely determine $\alpha_o$ and $\beta_1$ . We also require that the coefficients in (3) have the property that

$$\Delta_p = \begin{vmatrix} \alpha_o & \beta_1 \ldots\ldots & \beta_p \\ \beta_1 & \beta_2 \ldots\ldots & \beta_{p+1} \\ \vdots & \vdots & \ddots \\ \beta_p & \beta_{p+1} & \beta_{2p} \end{vmatrix} \neq 0 \ , \ p = 0,1,2\ldots \quad ,$$

and

$$\Gamma_{2p+1} = \begin{vmatrix} \beta_1 & \beta_2 \ldots\ldots & \beta_{p+1} \\ \beta_2 & \beta_3 \ldots\ldots & \beta_{p+2} \\ \vdots & \vdots & \ddots \\ \beta_{p+1} & \beta_{p+2} & \beta_{2p+1} \end{vmatrix} \neq 0 \ , \ p = 0,1,2\ldots \quad . \tag{4}$$

Then (4) insures that $y$ has a continued fraction expansion of the form

$$y = \cfrac{\alpha_o}{1 + \cfrac{\alpha_1 x}{1 + \cfrac{\alpha_2 x}{1 + \ddots}}} \quad . \tag{5}$$

A transformation of the type

$$y = \frac{m(x) + n(x)y^*}{p(x) + q(x)y^*}$$

where $m$ , $n$ , $p$ and $q$ are polynomials in $x$ may be necessary to bring the differential equation into the required form. We suppose that this has already been done. See [4] for the results of applying transformations of this type to (1). We give an example in Section III.

The even approximants of (5) are the main diagonal Padé approximations to $y$ which have the following properties. Let

$$y_n = \frac{P_n}{Q_n} = \frac{\sum_{k=0}^{n} p_{n,k}x^k}{\sum_{k=0}^{n} q_{n,k}x^k} \tag{6}$$

be the $n^{th}$ order main diagonal Padé approximation to $y$ . If $Q_n$ is formally divided into $P_n$ , the resulting power series agrees with the power series representation of $y$ for the first $(2n+1)$ terms. The polynomials $P_n$ and $Q_n$ both satisfy the recurrence relation

$$P_n = \left[1 + (\alpha_{2n-1}+\alpha_{2n})x\right]P_{n-1} - \alpha_{2n-1}\alpha_{2n-2}x^2P_{n-2} ,$$

$$P_o = \alpha_o , \; P_1 = \alpha_o(1+\alpha_2 x) , \; Q_o = 1 \text{ and } Q_1 = 1 + (\alpha_1+\alpha_2)x . \tag{7}$$

Thus, rational approximations to the solution of (1) are immediately forthcoming if the values $\alpha_1,\alpha_2...$ can be computed. These values can be obtained by utilizing a sequence of linear fractional transformations. Let

$$y = y_o , \; y_n = \alpha_n(1+xy_{n+1})^{-1} , n \geq 0 . \tag{8}$$

Repeated application of (8) to (1) and division by $\alpha_n x$ at each step yields

$$(A_{n+1} + B_{n+1}y_{n+1})y''_{n+1} + (C_{n+1} + D_{n+1}y_{n+1})y'_{n+1} - 2B_{n+1}(y'_{n+1})^2$$
$$+ E_{n+1} + F_{n+1}y_{n+1} + G_{n+1}y^2_{n+1} + H_{n+1}y^3_{n+1} = 0 \quad , \tag{9}$$

where

$$\begin{aligned} A_{n+1} &= - A_n - \alpha_n B_n \ , \\ B_{n+1} &= - xA_n \ , \\ C_{n+1} &= - 2x^{-1}(A_n + \alpha_n B_n) - C_n - \alpha_n D_n \ , \\ D_{n+1} &= 2A_n - xC_n \ , \\ E_{n+1} &= x^{-1}(\alpha_n^{-1}E_n + F_n + \alpha_n G_n + \alpha_n^2 H_n) \ , \\ F_{n+1} &= - x^{-1}(C_n + \alpha_n D_n) + 3\alpha_n^{-1}E_n + 2F_n + \alpha_n G_n \ , \\ G_{n+1} &= 2x^{-1}A_n - C_n + 3\alpha_n^{-1}xE_n + xF_n \ , \end{aligned} \tag{10}$$

and

$$H_{n+1} = \alpha_n^{-1}x^2E_n \ , \quad n = 0,1,2,\ldots \quad .$$

It is easily shown that

$$A_{n+1}(0) = B_{n+1}(0) = C_{n+1}(0) = D_{n+1}(0) = G_{n+1}(0) = H_{n+1}(0) = 0 \quad ,$$

$E_{n+1}$ and $F_{n+1}$ are defined at $x = 0$ and

$$y_{n+1}(0) = \alpha_{n+1} = - \frac{E_{n+1}(0)}{F_{n+1}(0)} \quad , \quad n = 0,1,2,\ldots \quad . \tag{11}$$

Thus the values $\alpha_k$ appearing in (5) can be computed recursively.

Proof of the convergence of the approximants in (6) in the general case seems elusive especially since the values of $\alpha_k$ are not in general known in closed form. However, in the special case of the first order Ricatti equation, some convergence proofs are available, see [1] and [2]. Heuristically one can imply convergence of the approximants in (6) by comparing the values of the $n^{th}$ and $(n+1)^{st}$ approximations. In practice this works very well, and in the cases investigated the actual error of the $n^{th}$ approximant is the same order of magnitude as the difference, $y_{n+1}(x) - y_n(x)$ .

## III. EXAMPLES

We consider two examples which exhibit the utility of these approximations when used to approximate both the function and its poles.

Painlevé's first and second transcendents are defined by the differential equations

$$u'' - 6u^2 - \lambda x = 0 \ , \ u(0) = 1 \ , \ u'(0) = 0 \ , \tag{12}$$

and

$$v'' - 2v^3 - xv - \mu = 0 \ , \ v(0) = 1 \ , \ v'(0) = 0 \ , \tag{13}$$

respectively. In what follows, $\lambda = \mu = 1.0$ .

To cast (12) and (13) into the required form of (1), we set $u = 1 + 3x^2\bar{u}$ and $v = 1 + 1.5x^2\bar{v}$ in which case (12) and (13) become

$$3x^2\bar{u}'' + 12x\bar{u}' + (6 - 36x^2)\bar{u} - 54x^4\bar{u}^2 - (6 + x) = 0 \ , \ \bar{u}(0) = 1 \tag{14}$$

and

$$3x^2\bar{v}'' + 10x\bar{v}' + (5 - 18x^2 - 6x^3)\bar{v} - 27x^4\bar{v}^2 - 13.5x^6\bar{v}^3$$
$$- (5 + 4x) = 0 \ , \ \bar{v}(0) = 1 \ . \tag{15}$$

Now $u$ has a pole of the second order at $x = 1.2067$ and $v$ has a simple pole at $x = 1.1577$. This behavior manifests itself in Tables I and II below in which $\bar{u}_6$ and $\bar{v}_6$ are the sixth order main diagonal Padé approximations to $\bar{u}$ and $\bar{v}$ obtained using the algorithm of Section II. We have

$$u_6 = 1 + 3x^2\bar{u}_6 \quad \text{and} \quad v_6 = 1 + 1.5x^2\bar{v}_6 \ .$$

TABLE I

| $x$ | $u(x)$ | $u_6(x)$ |
|---|---|---|
| 0.0 | 1.0000 | 1.0000 |
| 0.1 | 1.0305 | 1.0305 |
| 0.2 | 1.1264 | 1.1264 |
| 0.3 | 1.3015 | 1.3015 |
| 0.4 | 1.5831 | 1.5831 |
| 0.5 | 2.0228 | 2.0228 |
| 0.6 | 2.7212 | 2.7212 |
| 0.7 | 3.8909 | 3.8909 |
| 0.8 | 6.0383 | 6.0383 |
| 0.9 | 10.6226 | 10.6223 |
| 1.0 | 23.3936 | 23.3860 |
| 1.1 | 87.7732 | 87.3769 |

TABLE II

| $x$ | $v(x)$ | $v_6(x)$ |
|---|---|---|
| 0.0 | 1.0000 | 1.0000 |
| 0.1 | 1.0152 | 1.0152 |
| 0.2 | 1.0626 | 1.0626 |
| 0.3 | 1.1464 | 1.1464 |
| 0.4 | 1.2742 | 1.2742 |
| 0.5 | 1.4592 | 1.4592 |
| 0.6 | 1.7254 | 1.7254 |
| 0.7 | 2.1184 | 2.1184 |
| 0.8 | 2.7369 | 2.7369 |
| 0.9 | 3.8344 | 3.8343 |
| 1.0 | 6.3110 | 6.3104 |

The values of $u(x)$ and $v(x)$ were taken from a paper by Simon [5] who used (12) and (13) as examples in a study of a numerical integration technique for the solution of initial value problems in ordinary differential equations.

The poles of smallest magnitude of $u_6$ and $v_6$ are $1.2058 \pm i.0134$ and $1.1578$, respectively, so that one would expect that the poles of $u$ and $v$ could be computed to any desired degree of accuracy using higher order approximations. Indeed this is the case when the approximations converge.

## REFERENCES


1. Merkes, E. P., and Scott, W. T., "Continued Fraction Solutions of the Ricatti Equation," J. Math. Anal. Appl., 4, 309-327 (1962).

2. Fair, W., "Padé Approximations to the Solution of the Ricatti Equation," Math. Comp., v. 18, No. 88, 627-634 (1964).

3. Luke, Y. L., "The Padé Table and the $\tau$-Method," J. Math. and Phys., v. 37, 110-127 (1958).

4. Davis, H. T., Introduction to Nonlinear Differential and Integral Equations, Dover, New York, Ch. 8 (1962).

5. Simon, W. E., "Numerical Technique for Solution and Error Estimate for the Initial Value Problem," Math. Comp., v. 18, No. 91, 387-393 (1965).