COMPARISON OF SEVERAL PROCEDURES FOR PROCESS CONTROL

by

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1. Summary

The Average Run Length, which is defined as the average number of samples required before a decision to adjust the machine is reached, when it is operating at a specified quality level, is used to evaluate the operating characteristics of three different process control procedures--Shewhart type control charts, cumulative sum control charts, and signed sequential ranking procedures. In comparing their relative performance, the Average Run Lengths (ARL) required to stop the process when it is "in control" are set equal under the three procedures; the ARL's necessary to detect preassigned shifts in the process are then compared.

The ARL curve of the signed sequential ranking procedures indicates that this procedure is very efficient in detecting small changes in the process level (e.g. within 1.00σ). However, it is relatively insensitive to large deviations. It is particularly useful in cases where the underlying distribution of the process is not known, or is known not to be normal. On the other hand, if small changes in the process level are considered to be acceptable and hence can be tolerated, this procedure can lead to high costs due to unnecessary interruption of the process.

For small or large deviations, the behavior of the cu-sum scheme and the Shewhart scheme differ only slightly, although the CSCC is generally a little better. To detect an abrupt change of process level with moderate deviations from its target, it is advisable to use the cu-sum charts which are the best of the three in this region; on the other hand, to obtain a simple graphical schematization of a stable process, it may be convenient and economical to use the Shewhart type control charts.

2. Introduction

In a continuous manufacturing process, a problem of great interest to a quality control engineer is whether the process operates satisfactorily in accordance with the requirements that were specified in advance. There are numerous types of process control schemes, but the basic ideas are the same: repeated random samples are taken from the process, statistics are computed from the results, inferences are made about the process based on certain predetermined decision criteria, and finally, proper action is taken to make adjustments when needed.

In this paper we are concerned with an investigation of the operation of three different process control procedures; namely, Shewhart type control charts, cumulative sum control charts, and signed sequential ranking procedures. We are interested in comparing these procedures when a change in the process level, assumed to be the mean of the distribution, occurs. In the next three sections, a brief exposition of the mechanics of these three procedures is presented. A study of their relative performance will then be made.

A process is said to be in statistical control when the measurable characteristic of each item produced has the same probability distribution and these measurable characteristics are independent random variables. When a process is in statistical control, the parameters of the probability distribution usually can be estimated accurately. In applying a "control procedure" to a process, the procedure indicates when adjustment to the process should be made. When a process is "in control" at a satisfactory level, such adjustment should not be made. Hence, a measure of relative efficiency for a control procedure is the Average Run Length until process adjustment is called for when the process is in a state of statistical control (an incorrect decision is being made). When the process jumps "out of control," the Average Run Length required until the procedure calls for a machine adjustment is another measure of the relative efficiency of the control procedure (a correct decision is being made). In this paper, the three procedures are fixed so they have the same Average Run Length when the process is "in control." The procedures then will be compared when a shift in the mean occurs using the Average Run Length required for each procedure to detect the shift.

In order to facilitate our study we assume

1. The process can reasonably be approximated by a normal distribution. In addition, the homogeneity and randomness are maintained when the process is "in control." The signed sequential ranking procedure, as a nonparametric procedure, does not require this normality assumption. However, in order to evaluate and compare its operating characteristics with those of the other two process control procedures when the process

is not "in control," some distribution assumption is required. Normality is used because the other two procedures require such an assumption.

2. Sample size and sampling interval are fixed in such a way that both statistical and practical requirements are satisfied for the successful operation of process control procedures.

3. The variance of the process is known and remains constant even when the mean changes.

4. The statistic $\overline{X}_i$ (the sample average) is computed from the results of the i^{th} sampling and transformed to a normalized random variable Y_i through the relation $Y_i = \frac{\overline{X}_i - \mu_o}{\sigma/\sqrt{n}}$, where n is the sample size, μ_o is the process mean when the process is "in control," and σ is the standard deviation.

5. From now on, we deal with a sequence of random variables $Y_1, Y_2, \ldots,$ where Y_i's are independently identically distributed normal random variables with mean zero and variance one when the process is "in control."

3. Shewart Type Control Charts (SWCC)

The SWCC was originated by W. A. Shewhart in 1931 (The Economic Control of Manufactured Product, Van Nostrand, New York) and has been widely used in industry for some thirty years. The purpose of a SWCC is to provide a dynamic record of the process. This record will be used to differentiate the cause of variation in quality so that appropriate action may be taken. The overall SWCC procedures can be viewed as the application of repeated tests of the hypothesis $H_o: \mu = \mu_o$ versus the alternative $H_1: \mu \neq \mu_o$. We reject H_o and say the process is "out of control" whenever any individual sample statistic Y_i falls outside of the control limits. The limits, or so-called "action lines," depend on the size of the Type I error α. This error is the probability of having a point fall outside of the limits when the process is in a state of statistical control. In practice, the classical three-sigma limits are probably the most widely used critical values. This is equivalent to setting $\alpha = 0.00135$ in one direction, or 0.0027 in both directions. The basic idea can also be illustrated graphically by Figure 1.

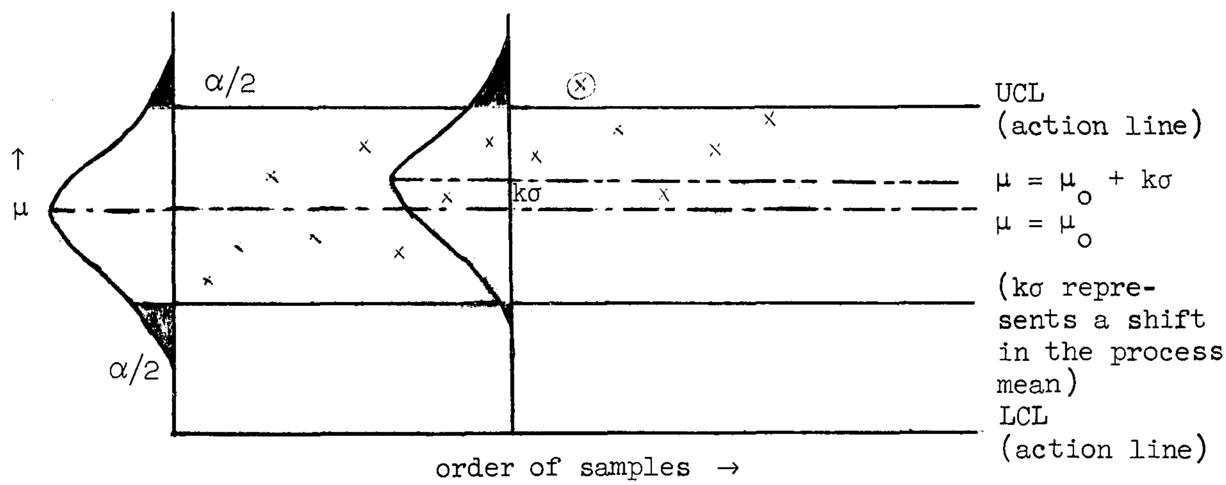

Figure 1: A SWCC Diagram.

As long as the points fall above the Lower Control Limit (LCL) and below the Upper Control Limit (UCL) the process is not adjusted. When a point falls outside these limits, as shown by the 10^{th} point in Figure 1, action is taken. If, as indicated in the figure, the process mean has shifted, this is a correct action. If the process is still in control at level μ_o, this is an incorrect action.

It is easily shown that the Average Run Length when the process is "in control" is given by $1/\alpha$ (see Section 5). For three-sigma limits, the Average Run Length is $1/0.0027 = 370$.

The behavior of a SWCC can be described by either an operating characteristic curve, which gives the risk β (the Type II error) of saying the process is "in control" (when actually the process is operating at a different level) or by the Average Run Length until the process is corrected when it is operating at this new level. These are related in that the Average Run Length is given by $1/(1 - \beta)$. Unfortunately, the OC curve does not help much in both the designing and operating of the SWCC scheme because the costs associated with the Type I error and the Type II error complicate the whole situation. For an interesting discussion of this aspect, see Duncan [1]. Therefore, we will use the Average Run Length criterion.

Possibly the principal advantages of the SWCC scheme are its simplicity in application and its value as a good visual presentation of the process. On the other hand, because the statistics are viewed independently and past data are not taken into account for current consideration, the SWCC is relatively insensitive to moderate changes in the mean value.

Various modifications and amendments have been made since the original proposal (e.g. some rules use runs of points or warning line schemes). However, the basic ideas are still the same.

4. Cumulative Sum Control Charts (CSCC)

In the last few years, the CSCC scheme has gained wide applicability because of its presumed efficiency relative to the SWCC in detecting a shift in process level. Most of the developmental work of the CSCC was done by E. S. Page [5], G. A. Barnard [7], K. W. Kemp [8], and P. L. Goldsmith and H. Whitfield [2].

In general, the procedure of the CSCC can be described as follows: paired data $\left\{m, \sum_{i=1}^{m} Y_i\right\}$ are plotted sequentially on the CSCC, where m designates the m^{th} sampling and $\sum_{i=1}^{m} Y_i$ designates the cumulative sum up to the m^{th} sampling. At each stage, a V-shaped mask will be placed over the chart, with the point 0 placed over the last point plotted on the chart. The line OP is horizontal (see Figure 2). If any one of the previously plotted points falls outside of the mask, the process is said to be "out of control." In addition, if the point lies below the lower boundary, it is regarded as an indication of an increase in the process level; if the point lies above the upper boundary, a decrease is indicated.

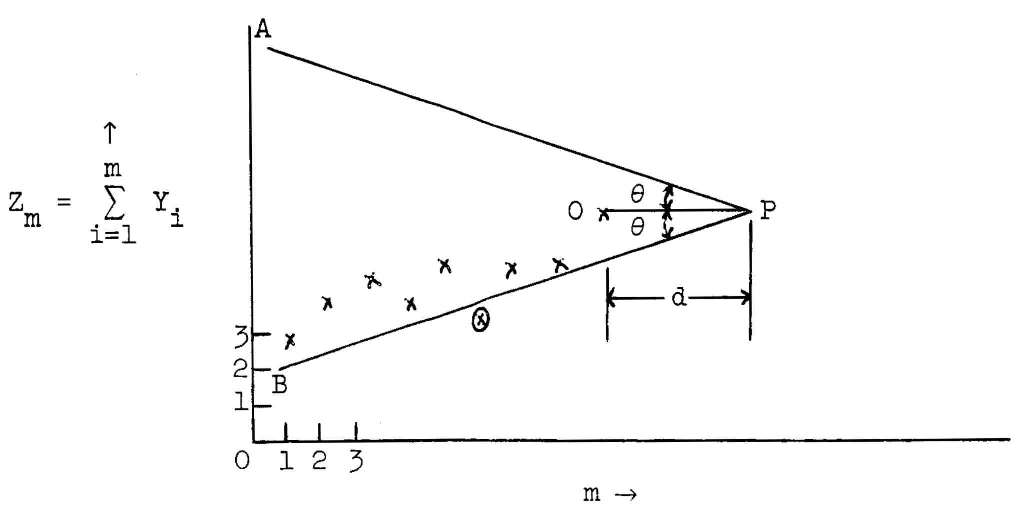

Figure 2: A CSCC Diagram.

The dimensions of the mask are completely specified by parameters d and θ (in Figure 2, note that $d = OP$ and $< APB = 2\theta$) provided that a unit scale for each coordinate is used. The behavior of the CSCC depends solely on the size of the mask. There are several ways of finding a suitable d and θ. One way is to try a variety of masks on historical records. A second way, as suggested by Goldsmith and Whitfield [2], is to consider the operating characteristics of various plans that are associated with different combinations of d's and θ's. In particular, they suggest using the Average Run Length. Two desired Average Run Lengths are specified as follows:

ARL_o - the Average Run Length needed to stop the process, when it is "in control";

ARL_1 - the Average Run Length needed to stop the process, when the current mean is $k\sigma$ off target μ_o.

The ARL for various combinations of d and θ can be obtained by simulation. The ARL curve, plotted as a function of process change, not only enables the statistician to assess the behavior of the V-shaped mask he plans to use, but also facilitates the design of suitable control schemes.

The third way to find d and θ, as suggested by Johnson and Leone [3], is to view the procedure as essentially a sequential test of hypothesis problem, with $H_o: \mu = \mu_o$ versus an alternative $H_1: \mu = \mu_o + k\sigma$. Using Wald's results leads to

$$d = \frac{2}{k^2} \ln \frac{1 - \beta}{\alpha}$$

$$\theta = \arctan (k/2)$$

This method suffers because of the difficulty in interpreting the meaning of α and β when the scheme is used as a process control procedure and not as a test of hypothesis.

5. Signed Sequential Ranking Procedures (SSRP)

There are situations in industry where process control is of critical importance for successful manufacturing operations, and where the underlying distribution is not known or, at least, cannot reasonably be assumed as normal. The problem of designing an effective process control scheme for this situation then arises. Obviously, it is not appropriate to use either the SWCC or the CSCC scheme because the choice of their related decision parameters depends on the normality assumption.[1] Recently, a process control scheme, called the signed sequential ranking procedure, was developed by E. A. Parent, Jr. [4]. This rather simple procedure can be used without any assumptions about the form of the underlying distribution. The aim of the SSRP is to detect when a change in the level of the process occurs no matter what distribution the underlying process comes from. The procedure is based on nonparametric statistics; specifically, the signs and relative sizes of the observations.

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[1]The fact that the sample means tend to normality because of the Central Limit Theorem is usually given as a justification for these procedures. The rate of convergence to normality may or may not be adequate, depending upon the form of the underlying distribution.

Suppose we are observing, sequentially, a sequence of independent random variables $Y_1, Y_2, \ldots$. The signed sequential rank T_n of Y_n relative to $Y_1, Y_2, \ldots, Y_n$ is the product of the rank of $|Y_n|$ among $|Y_1|, |Y_2|, \ldots, |Y_n|$ and the sign of Y_n (which is 1 if $Y_n \geq 0$ and -1 otherwise). Without loss of generality, the mean of the process is assumed to be zero. Based on the outcomes we form the new statistics $Z_n = T_n/n, \quad n = 1, 2, \ldots$.

The cumulative sums are $S_n = Z_1 + Z_2 + \cdots + Z_n$ and the decision rules are as follows:

if $b < S_n < a$ observe Y_{n+1} and compute S_{n+1};

if $S_n \leq b$ or $S_n \geq a$ stop the process and investigate.

Here it is assumed that $-\infty < b < 0 < a < \infty$ and observations are distributed according to $F(y)$, until a change takes place. $F(y)$ is a continuous distribution function with the property $F(-y) = F(0)[1 - F(y) + F(-y)]$ for all $y \geq 0$. This property is satisfied by symmetric (about 0) distributions, as well as by other distributions. The exact probability distribution for S_n is available through the following recurrence relation:

$$P(S_n = u) = P(S_{n-1} = u - Z_n) = \sum_y P(S_{n-1} = u - y)P(Z_n = y)$$

where y ranges over $-1, \; -\frac{n-1}{n}, \ldots, -\frac{1}{n}, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1$.

Let N be the smallest integral value for which S_n does not lie in the open interval (b, a). Then

$$P(N = n) = P(b < S_i < a \quad i = 1, 2, \dots, n-1, \; S_n \notin (b, a))$$

and $E(N) = \sum_{n=1}^{\infty} n\, P(N=n)$ gives the average number of observations, as a function of a, b, and $F(0)$, until the process stops. The process of carrying out these computations is rather tedious. By using some results of Wald from sequential analysis, the following approximations were obtained:

$$E(N) = \begin{cases} -3ab & F(0) = 1/2 \\ \dfrac{2b + 2(a - b)P(S_N > a)}{1 - 2F(0)} & F(0) \neq 1/2 \end{cases}$$

In the particular case of symmetric boundaries, $b = -a$, the expected number of observations needed to stop the process is:

$$E(N) = \begin{cases} 3a^2 & F(0) = 1/2 \\ \dfrac{2a(1 - e^{-ah} - \sinh(ah))(1 - \cosh(h))}{\sinh(ah)[\sinh(h) - h]} & F(0) \neq 1/2 , \end{cases}$$

where h depends on the value of $F(0)$. For detailed derivations, see Parent [4].

Figure 3 is a graph of $E(N)$, expressed as a function of $F(0)$.

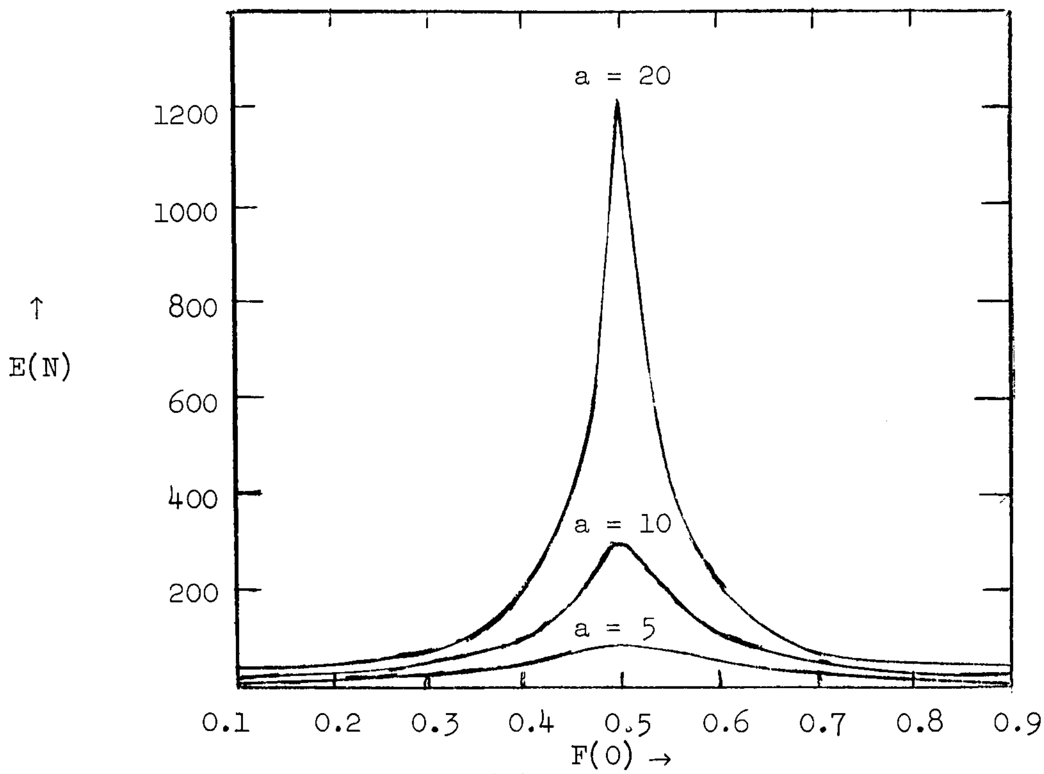

Figure 3: A graph of E(N) for the SSRP, expressed as a function of F(0).

6. Average Run Length (ARL)

We have chosen the Average Run Length as the criterion for comparing these three procedures. The Average Run Length is defined as the average number of samples required at a specified process level before a decision to adjust the machine is reached. In most cases the sample size is greater than one, so the ARL is actually the frequency of samples taken between the change in process level occurring and its detection. There are many variations of this terminology and its definition, but the idea behind each variation is essentially the same. This kind of criteria was introduced and has been extensively used by E. S. Page [5], Goldsmith and Whitfield [2], W. D. Ewan [6] for comparison of different process control schemes.

Since we are interested in the relative performance of the SWCC, CSCC, and SSRP, the parameters for these procedures are chosen so that the ARL's for these schemes are equal when the process is "in control." The ARL's for the three are then compared when preassigned changes in process level, which range from one to three standard deviations away from the target, occur.

Under the SWCC procedure we will stop the process when $|Y_i| > b$ for the first time, where Y_i's are normalized random variables as defined in Section 2 and b is the control limit. The size of b depends on the chosen Average Run Length. The size of b is also related to the size of the Type I error α, i.e.,

$$ARL_0 = 1/\alpha ,$$

and

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-b} e^{-\frac{1}{2}u^2} du + \frac{1}{\sqrt{2\pi}} \int_{b}^{\infty} e^{-\frac{1}{2}u^2} du = \alpha .$$

Thus, b can easily be found from tables of the cumulative normal distribution. Now, let P_{sw} be the probability of having a point fall outside the control limits on a single observation while the process is "in control" under the SWCC, i.e., $P_{sw} = \text{Prob}\ (|Y_i| > b)$. The number of trials until a point falls outside of the control limits for the first time then follows a geometric distribution with mean $E_{sw}(N) = 1/P_{sw}$

(where $E_{sw}(N)$ is the expected number of observations required to stop the process when it is still "in control"). Since $P_{sw} = \alpha$, we have $ARL_o = E_{sw}(N) = 1/\alpha$.

For a SWCC with a fixed ARL_o, the related ARL_1's under different process shifts (e.g. a deviation of k times the standard deviation from the mean), can be found through the following relation:

β = Probability of a point falling inside the control limits when the process level is at $\mu_1 \neq \mu_o$

= 1 - Probability of a point falling outside the limits when the process level is at $\mu_1 \neq \mu_o$

The probability of a point falling outside the control limits when the process level is at $\mu_o + k\sigma$ (here, we have $\sigma = 1$) is given by

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-(b+k)} e^{-\frac{1}{2}u^2} du + \frac{1}{\sqrt{2\pi}} \int_{b-k}^{\infty} e^{-\frac{1}{2}u^2} du = 1 - \beta .$$

As a result, using the same argument as above, we have

ARL_1 (with respect to a process change of k standard deviations off the target) $= \frac{1}{1 - \beta}$.

There is no analytical way to find either ARL_o or ARL_1 for the CSCC scheme except from a few empirical formulas which essentially are derived from simulated results. However, we note that a particular combination of d and θ specifies a CSCC plan, and hence, results in an ARL curve. Therefore, a complete knowledge of the ARL's associated with a CSCC plan is obtainable as long as a reliable simulation is used. Here we use the results obtained by Goldsmith and Whitfield [2]. In particular, we considered plans with ARL_o = 18, 30, 50, 110, 130, 150, 210, 320, 720, 1100, 1150, and 1500. Their evaluation was carried out by a Monte Carlo simulation on a Ferranti "Mercury" digital computer. A sequence of pseudo-random normal deviates was generated by a fast table look-up procedure. Each calculated ARL has a coefficient of variation less than 10%, an accuracy which is regarded as adequate for practical purposes.

Under the SSRP scheme, following the results of Section 5, we know that the expected number of observations required to stop the process, when it is "in control," is approximately $3a^2$. Now the decision parameters for the SWCC and SSRP can be found by equating the ARL_o's for the CSCC plans as listed in the last paragraph, i.e.,

$$ARL_o(CSCC) = ARL_o(SWCC) = ARL_o(SSRP)$$

and we know

$$ARL_o(SWCC) = 1/\alpha \ , \quad ARL(SSRP) = 3a^2 \ .$$

It follows that for the SWCC

$$\alpha = 1/ARL_o$$

and for the SSRP

$$a = \sqrt{ARL_o/3}$$

The ARL_1's for the SWCC were found by the method described previously, when various shifts in the process level occurred. To find ARL_1 for the SSRP scheme under various shifts in the process level, a Monte Carlo simulation model was built on a B5500 digital computer at Stanford University. A library program using a mixed congruential method was used to generate pseudo-random uniform deviates from $U[0, 1)$. These deviates are translated to normal deviates by a relation suggested by G. E. P. Box and M. E. Muller (Ann. Math. Stat., 29, 610-611). The simulation operates on the assumption that the process level is changed by an amount, k; that is, input data from $N(k, 1)$ will be fed into the simulation model to monitor the ARL_1's. It was found that each calculated ARL_1 had a coefficient of variation less than 10%. Thus, the results can be considered as satisfactory.

The ARL's for different plans (i.e. for different ARL_0's) under the SWCC, CSCC, and SSRP are tabulated in Table 1. The particular ARL curve associated with each plan, expressed as a function of process change, is plotted (Figure 4 to 15).

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The ARL's for different plans (i.e. for different ARL_0's) under the SWCC, CSCC, and SSRP are tabulated in Table 1. The particular ARL curve associated with each plan, expressed as a function of process change, is plotted (Figure 4 to 15).

Table 1: Average Run Lengths when process level has shifted by k (standard deviation) units.

ARL_o / schemes used / parameters used / k	(1) $ARL_o = 18$			(2) $ARL_o = 30$		
schemes used	SWCC	CSCC	SSRP	SWCC	CSCC	SSRP
parameters used	$b = 1.914$	$d = 1$, $\tan\theta = 0.50$	$a = 2.45$	$b = 2.127$	$d = 2$, $\tan\theta = 0.40$	$a = 3.17$
0.25	16.10	14.90	8.07	24.98	22.20	10.14
0.50	11.62	10.00	5.45	17.81	15.00	6.81
0.75	7.94	7.11	4.38	11.80	9.23	6.02
1.00	5.55	4.89	4.00	7.71	6.12	5.51
1.25	3.95	3.79	3.76	5.39	4.32	4.97
1.50	2.94	2.94	3.61	3.76	3.29	4.83
1.75	2.30	2.38	3.60	2.84	2.69	4.75
2.00	1.87	1.90	3.62	2.23	2.19	4.72
2.25	1.58	1.63	3.61	1.82	1.80	4.74
2.50	1.46	1.45	3.60	1.55	1.57	4.74
2.75	1.25	1.30	3.60	1.36	1.40	4.74
3.00	1.16	1.20	3.60	1.24	1.35	4.74

Table 1 (continued)

ARL_o	(3) $ARL_o = 50$			(4) $ARL_o = 110$		
schemes used	SWCC	CSCC	SSRP	SWCC	CSCC	SSRP
parameters used / k	$b = 2.33$	$d = 1$ $\tan\theta = 0.60$	$a = 4.09$	$b = 2.61$	$d = 5$ $\tan\theta = 0.30$	$a = 6.05$
0.25	42.20	37.60	12.46	89.35	61.90	17.74
0.50	27.70	23.40	9.34	57.50	27.10	13.72
0.75	17.50	13.20	7.88	31.80	14.20	11.51
1.00	10.92	7.64	7.02	18.60	8.10	10.75
1.25	7.13	5.38	6.53	11.50	6.07	10.04
1.50	4.92	3.75	6.75	7.48	4.72	10.03
1.75	3.56	2.86	6.46	5.14	3.78	10.00
2.00	2.69	2.29	6.46	3.69	3.37	9.95
2.25	2.14	1.58	6.43	2.78	2.89	9.96
2.50	1.76	1.57	6.43	2.19	2.41	9.96
2.75	1.51	1.50	6.43	1.79	2.12	9.96
3.00	1.34	1.41	6.43	1.53	2.00	9.96

Table 1 (continued)

ARL_o	(5) ARL_o = 130			(6) ARL_o = 150		
schemes used	SWCC	CSCC	SSRP	SWCC	CSCC	SSRP
parameters used / k	b = 2.66	d = 2 $\tan\theta$ = 0.50	a = 6.58	b = 2.72	d = 1 $\tan\theta$ = 0.70	a = 8.37
0.25	102.00	78.80	19.04	127.50	121.00	19.51
0.50	65.00	39.10	14.87	75.67	59.60	16.18
0.75	35.60	17.40	13.02	41.00	33.20	13.87
1.00	20.60	9.72	11.85	23.40	16.60	12.83
1.25	12.60	6.32	11.47	14.20	9.32	11.95
1.50	8.14	4.41	11.20	8.99	6.14	11.95
1.75	5.52	3.42	11.05	6.04	4.21	11.95
2.00	3.92	3.02	10.96	4.25	3.27	11.73
2.25	2.94	2.45	10.72	3.14	2.76	11.71
2.50	2.29	2.18	10.66	2.42	2.15	11.71
2.75	1.87	1.75	10.66	1.95	1.82	11.71
3.00	1.58	1.42	10.66	1.64	1.50	11.71

Table 1 (continued)

ARL_o / schemes used / parameters used / k	(7) $ARL_o = 210$			(8) $ARL_o = 320$		
	SWCC	CSCC	SSRP	SWCC	CSCC	SSRP
	$b = 2.82$	$d = 8$ $\tan\theta = 0.25$	$a = 8.37$	$b = 2.93$	$d = 2$ $\tan\theta = 0.55$	$a = 10.3$
0.25	196.00	93.30	23.77	270.00	138.00	28.92
0.50	98.00	29.40	18.37	133.25	43.40	23.06
0.75	52.10	14.10	16.54	75.45	19.90	20.57
1.00	29.12	8.78	15.24	37.30	10.00	19.45
1.25	16.90	6.22	14.52	21.50	6.35	18.35
1.50	10.70	5.00	14.62	13.08	5.98	18.21
1.75	7.04	4.12	14.67	8.41	4.51	18.20
2.00	4.85	3.52	14.27	5.68	3.68	18.16
2.25	3.51	3.00	14.22	4.03	3.26	17.78
2.50	2.67	2.64	14.22	3.00	2.76	17.78
2.75	2.12	2.39	14.22	2.34	2.41	17.78
3.00	1.75	2.30	14.22	1.89	2.24	17.78

Table 1 (continued)

ARL_o / schemes used / parameters used / k	(9) $ARL_o = 720$			(10) $ARL_o = 1100$		
schemes used	SWCC	CSCC	SSRP	SWCC	CSCC	SSRP
parameters used	$b = 3.18$	$d = 1$, $\tan\theta = 0.80$	$a = 15.5$	$b = 3.26$	$d = 8$, $\tan\theta = 0.30$	$a = 19.15$
0.25	588.24	515.00	43.18	769.98	381.00	53.07
0.50	270.21	253.00	34.63	345.23	68.00	42.87
0.75	133.34	91.80	31.11	166.67	23.90	38.72
1.00	68.50	35.40	28.87	84.00	12.70	36.56
1.25	37.30	17.40	28.46	49.61	8.54	35.75
1.50	21.50	9.62	28.36	25.50	6.83	34.58
1.75	13.09	6.42	28.10	15.28	5.44	34.62
2.00	8.42	4.20	27.63	8.78	4.49	34.50
2.25	5.68	3.15	27.42	6.43	5.92	34.46
2.50	4.03	2.52	27.42	4.49	3.32	34.46
2.75	3.00	2.27	27.42	3.28	3.20	34.46
3.00	2.34	2.10	27.42	2.52	3.00	34.46

Table 1 (continued)

ARL_o / schemes used / parameters used / k	(11) $ARL_o = 1150$			(12) $ARL_o = 1500$		
schemes used	SWCC	CSCC	SSRP	SWCC	CSCC	SSRP
parameters used / k	$b = 3.31$	$d = 2$ $\tan\theta = 0.60$	$a = 19.58$	$b = 3.36$	$d = 5$ $\tan\theta = 0.40$	$a = 22.38$
0.25	905.20	662.00	53.60	1112.00	695.00	61.07
0.50	400.00	172.00	44.30	476.00	119.00	49.95
0.75	199.28	55.70	39.33	222.00	37.80	45.29
1.00	96.32	21.90	37.16	110.00	16.00	42.60
1.25	50.82	12.80	36.17	57.50	9.32	42.22
1.50	28.50	7.14	35.40	31.80	6.94	41.26
1.75	16.84	5.28	35.60	18.60	5.48	41.04
2.00	10.52	4.00	35.24	11.50	4.54	40.80
2.25	6.92	3.27	35.34	7.49	3.72	40.76
2.50	4.18	2.92	35.34	5.14	3.35	40.77
2.75	3.48	2.38	35.34	3.78	3.00	40.76
3.00	2.64	1.99	35.34	2.79	2.76	40.76

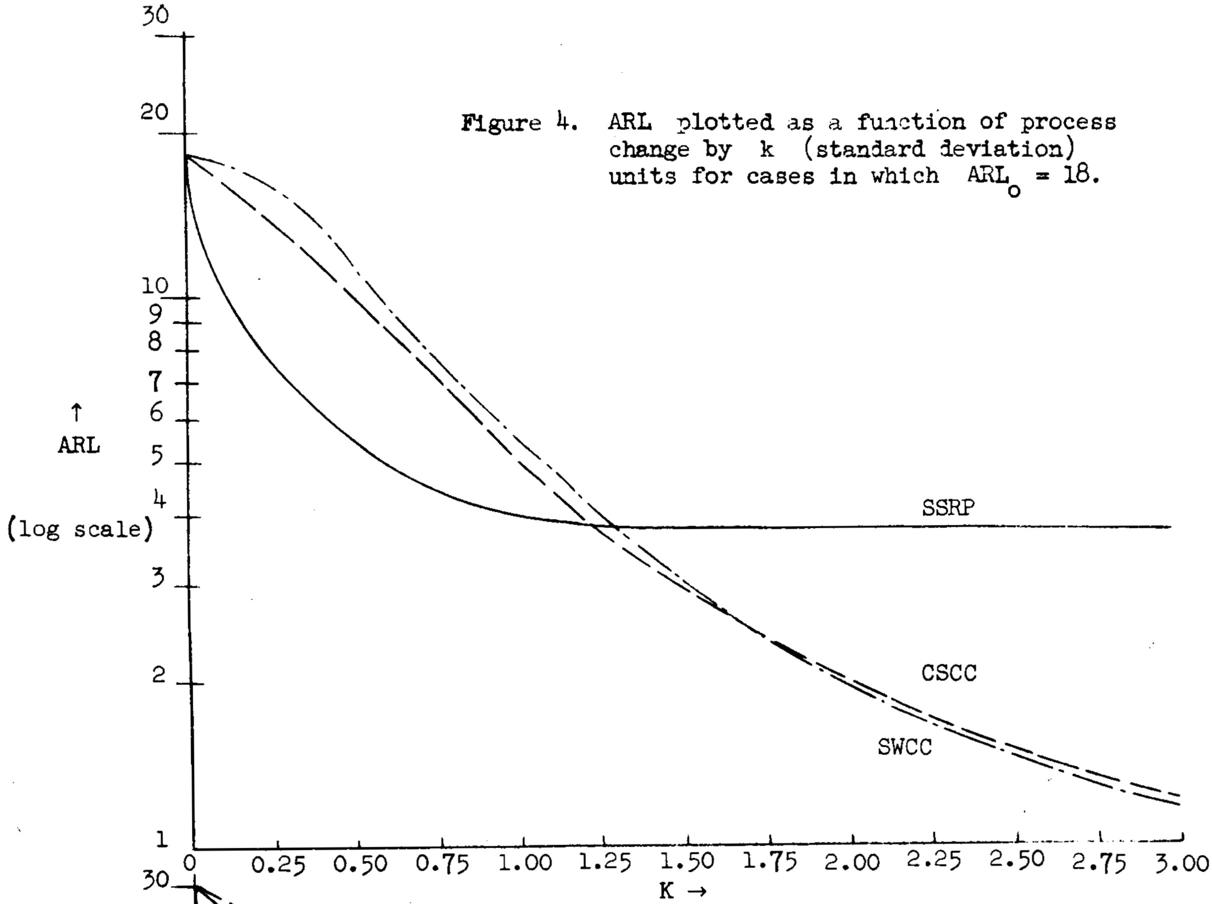


Figure 4. ARL plotted as a function of process change by k (standard deviation) units for cases in which $ARL_o = 18$.

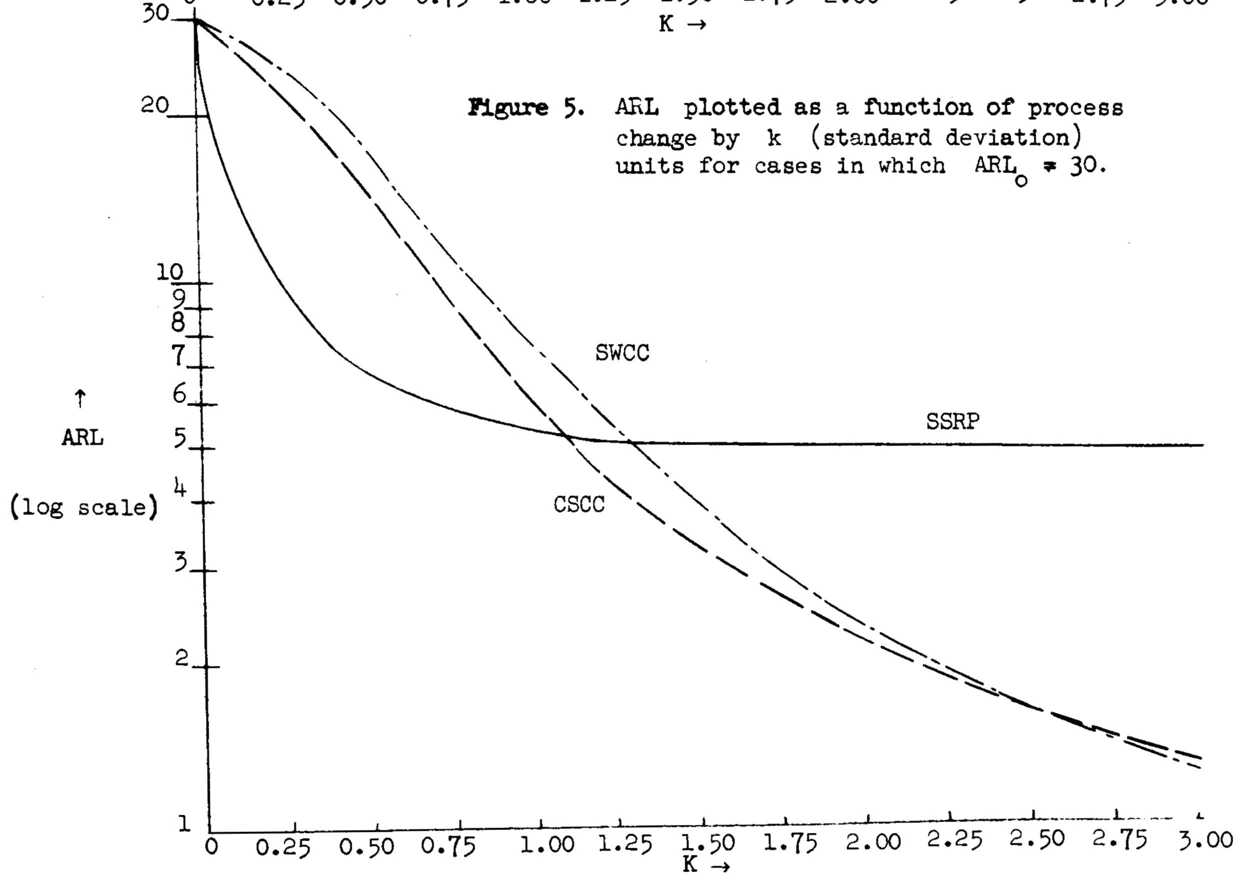
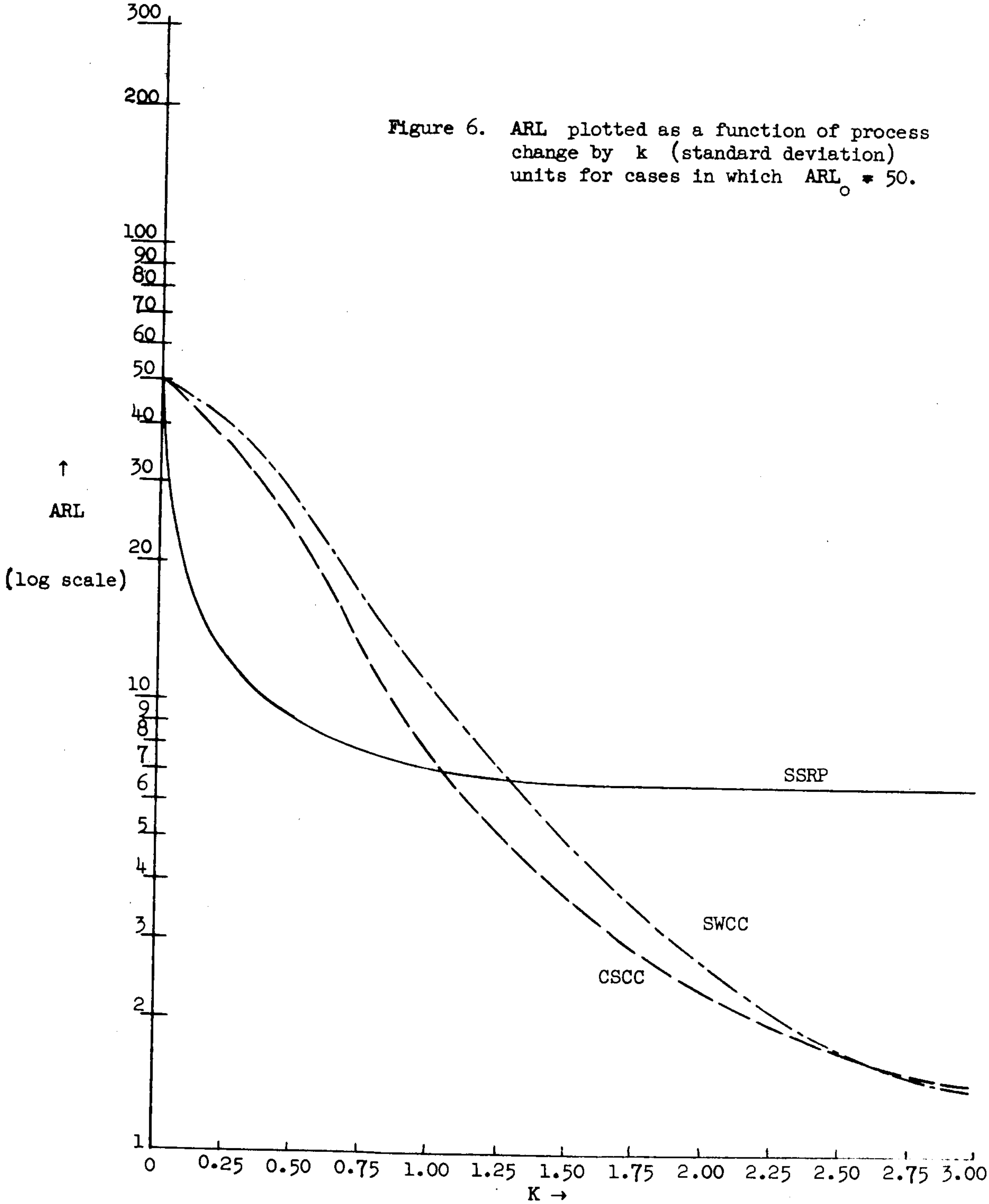


Figure 5. ARL plotted as a function of process change by k (standard deviation) units for cases in which $ARL_o = 30$.

Figure 6. ARL plotted as a function of process change by k (standard deviation) units for cases in which $ARL_0 = 50$.



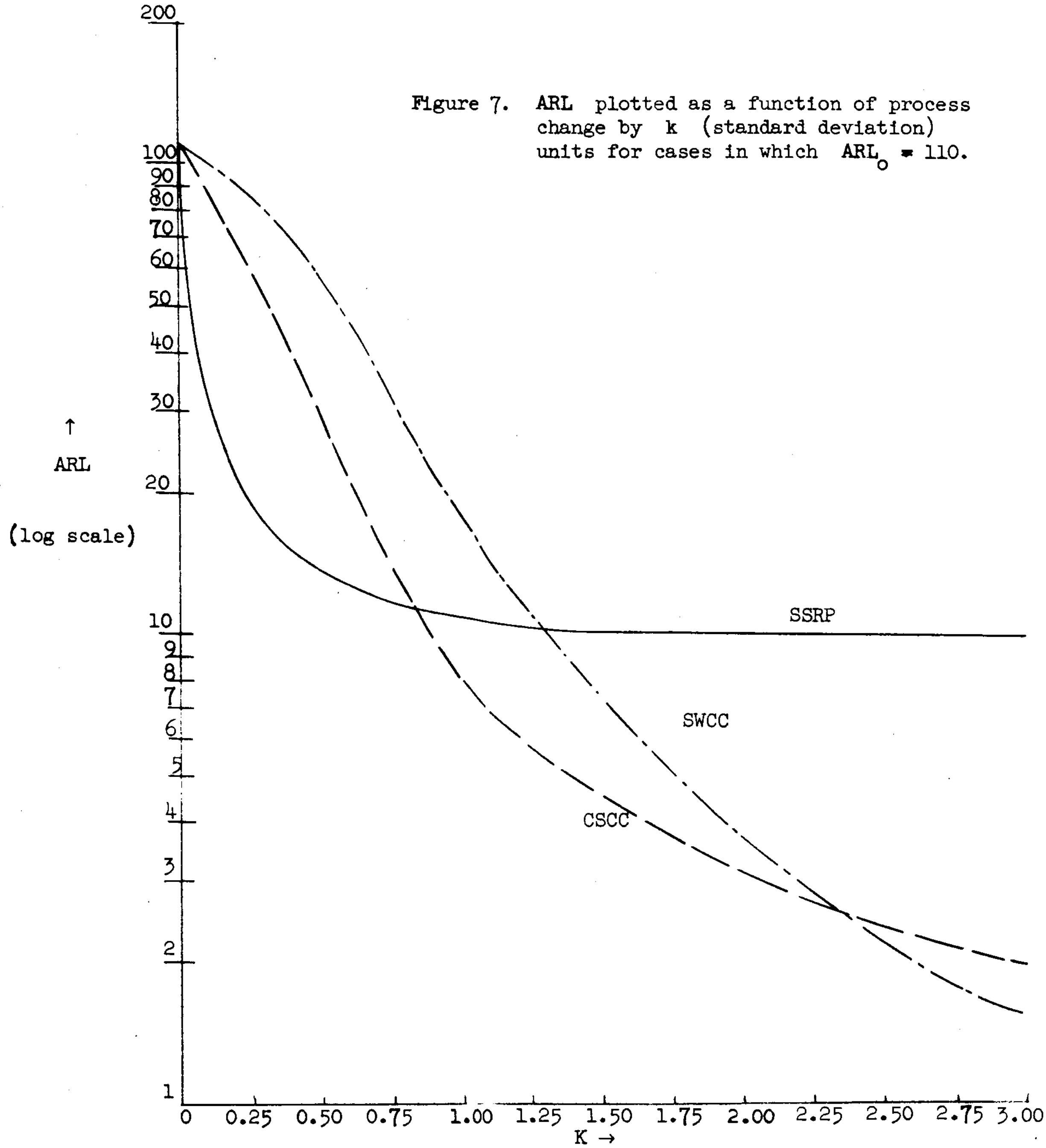


Figure 7. ARL plotted as a function of process change by k (standard deviation) units for cases in which $ARL_o = 110$.

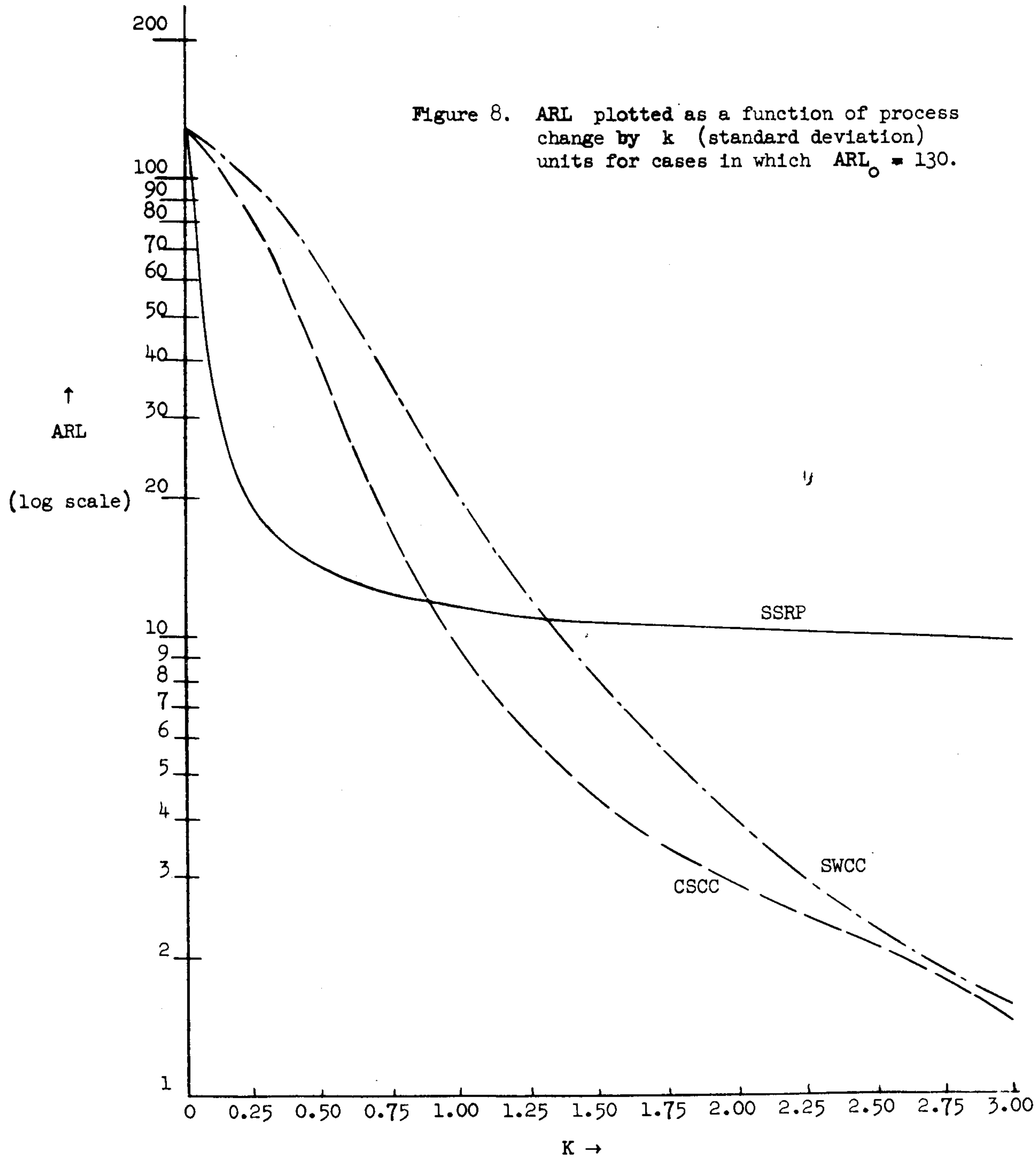


Figure 8. **ARL plotted as a function of process change by k (standard deviation) units for cases in which $ARL_0 = 130$.**

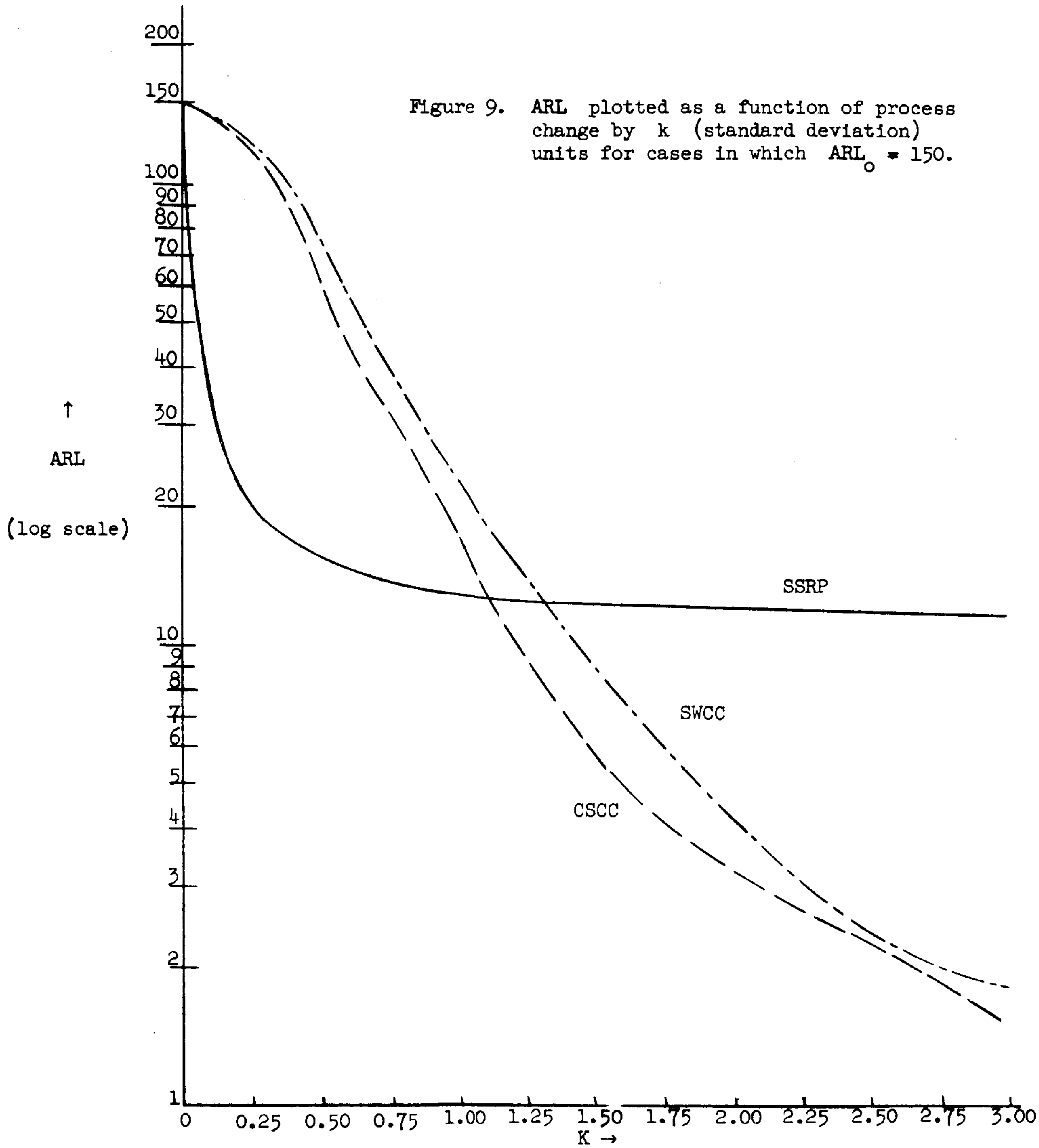


Figure 9. ARL plotted as a function of process change by k (standard deviation) units for cases in which $ARL_0 = 150$.

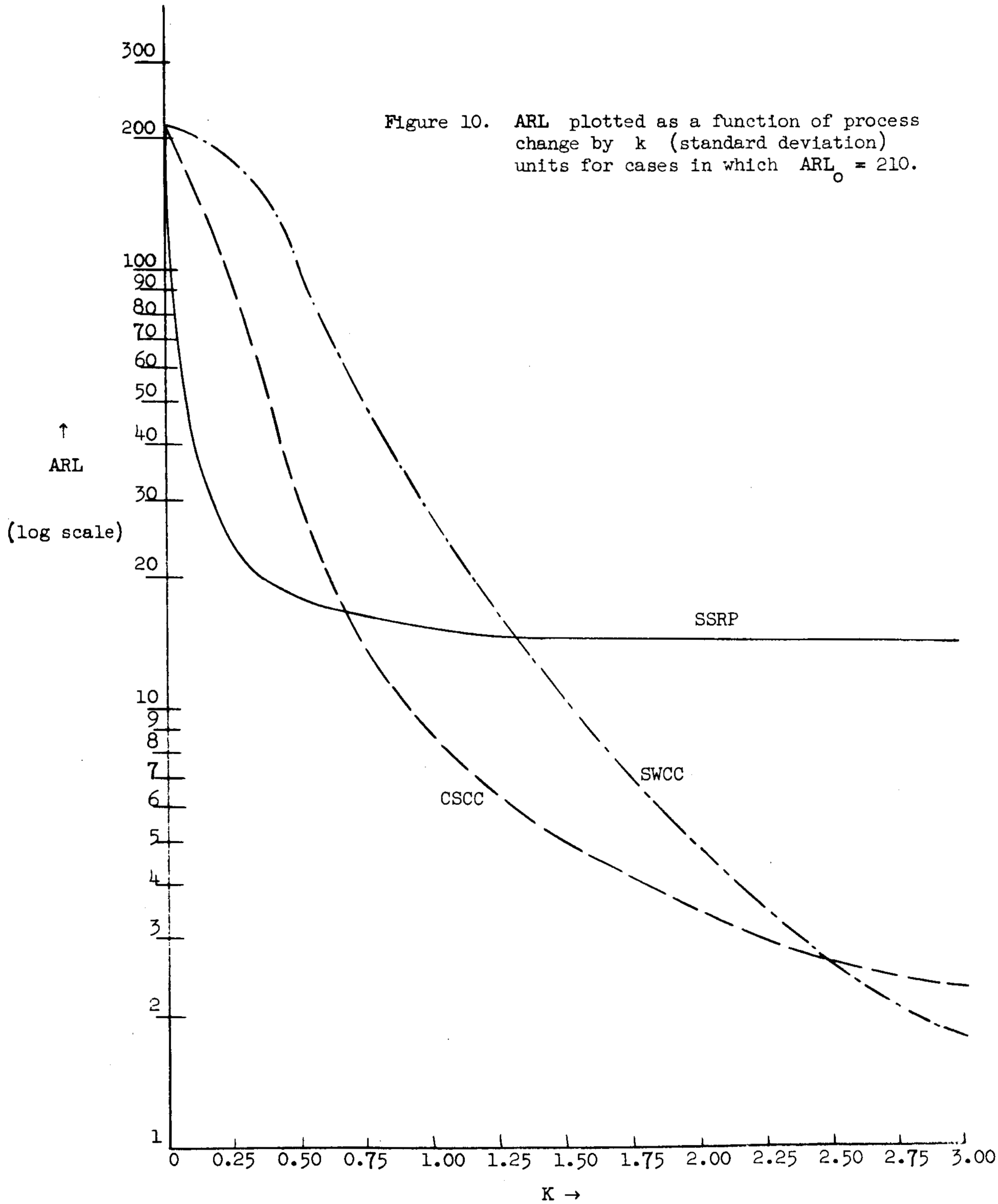


Figure 10. ARL plotted as a function of process change by k (standard deviation) units for cases in which ARL_o = 210.

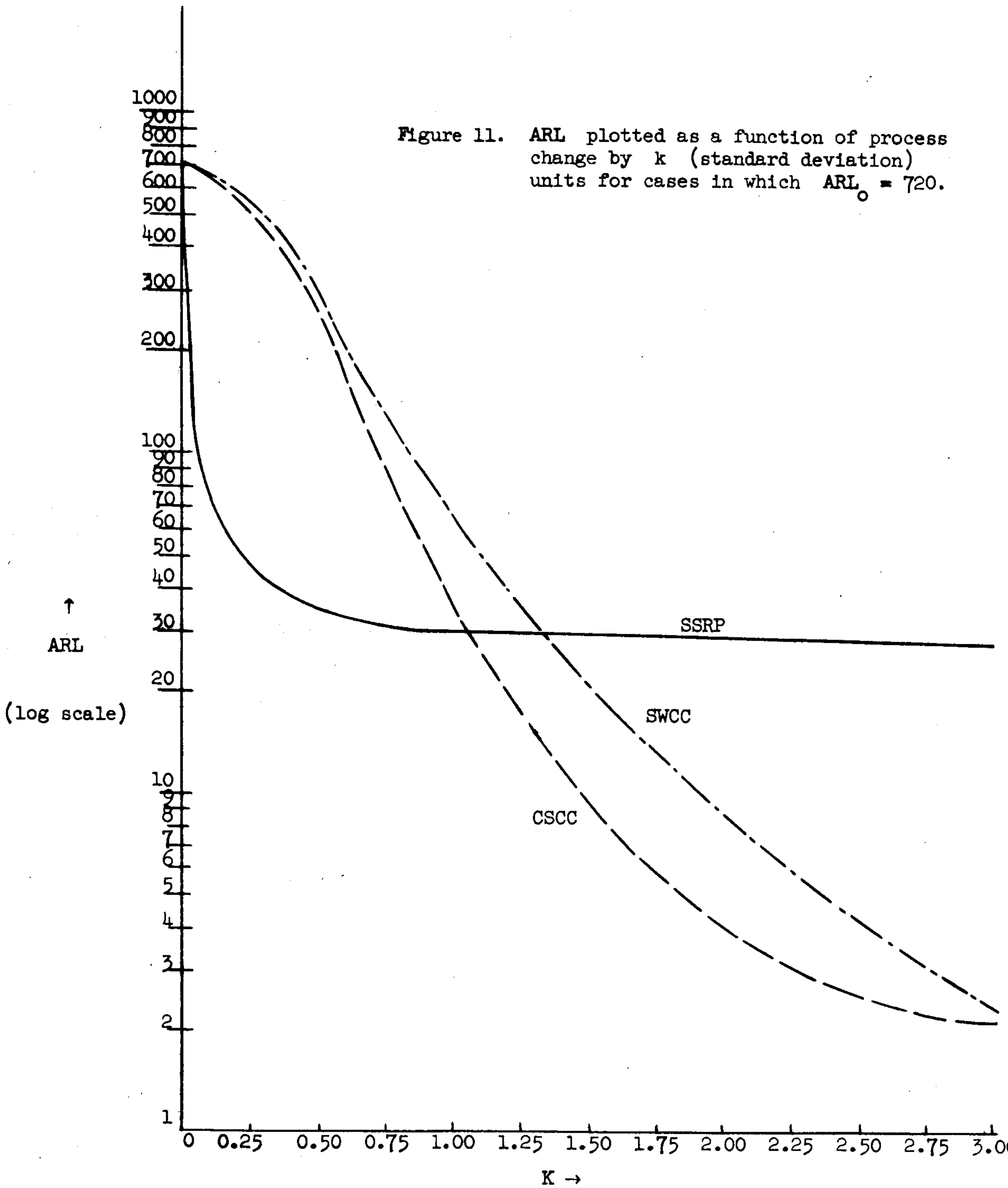


Figure 11. ARL plotted as a function of process change by k (standard deviation) units for cases in which $ARL_0 = 720$.

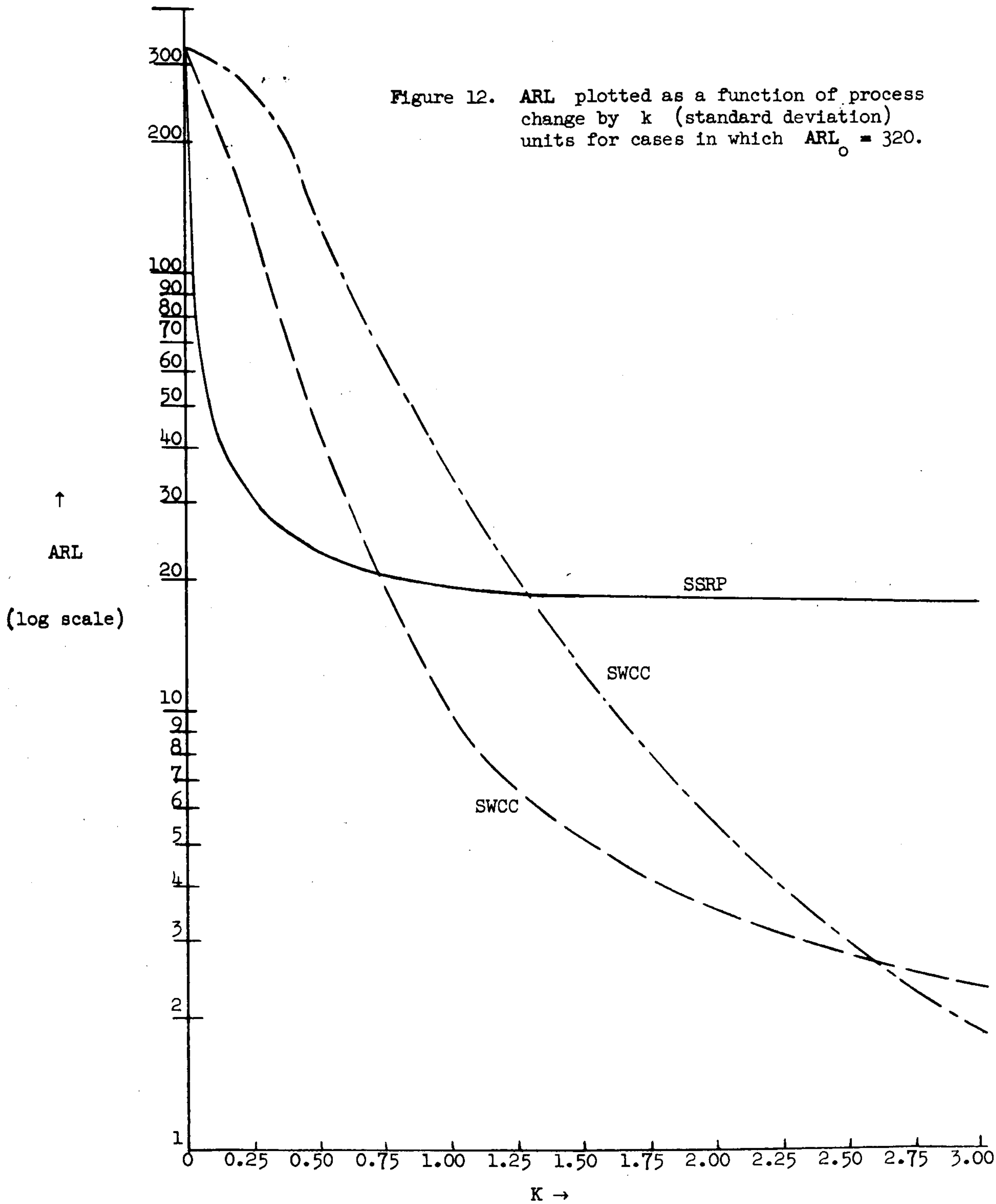


Figure 12. **ARL** plotted as a function of process change by k (standard deviation) units for cases in which **ARL**$_o$ = 320.

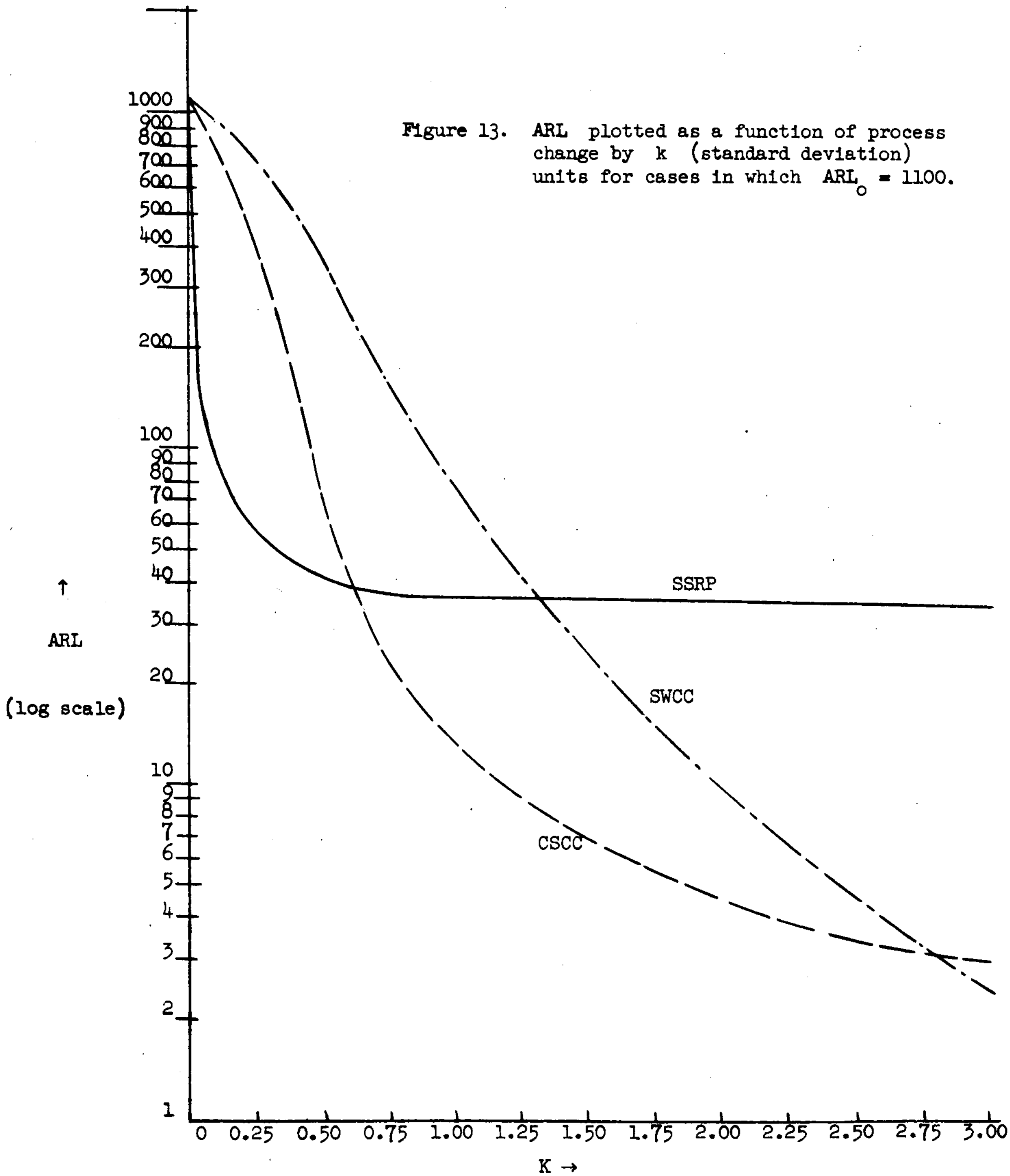


Figure 13. ARL plotted as a function of process change by k (standard deviation) units for cases in which $ARL_0 = 1100$.

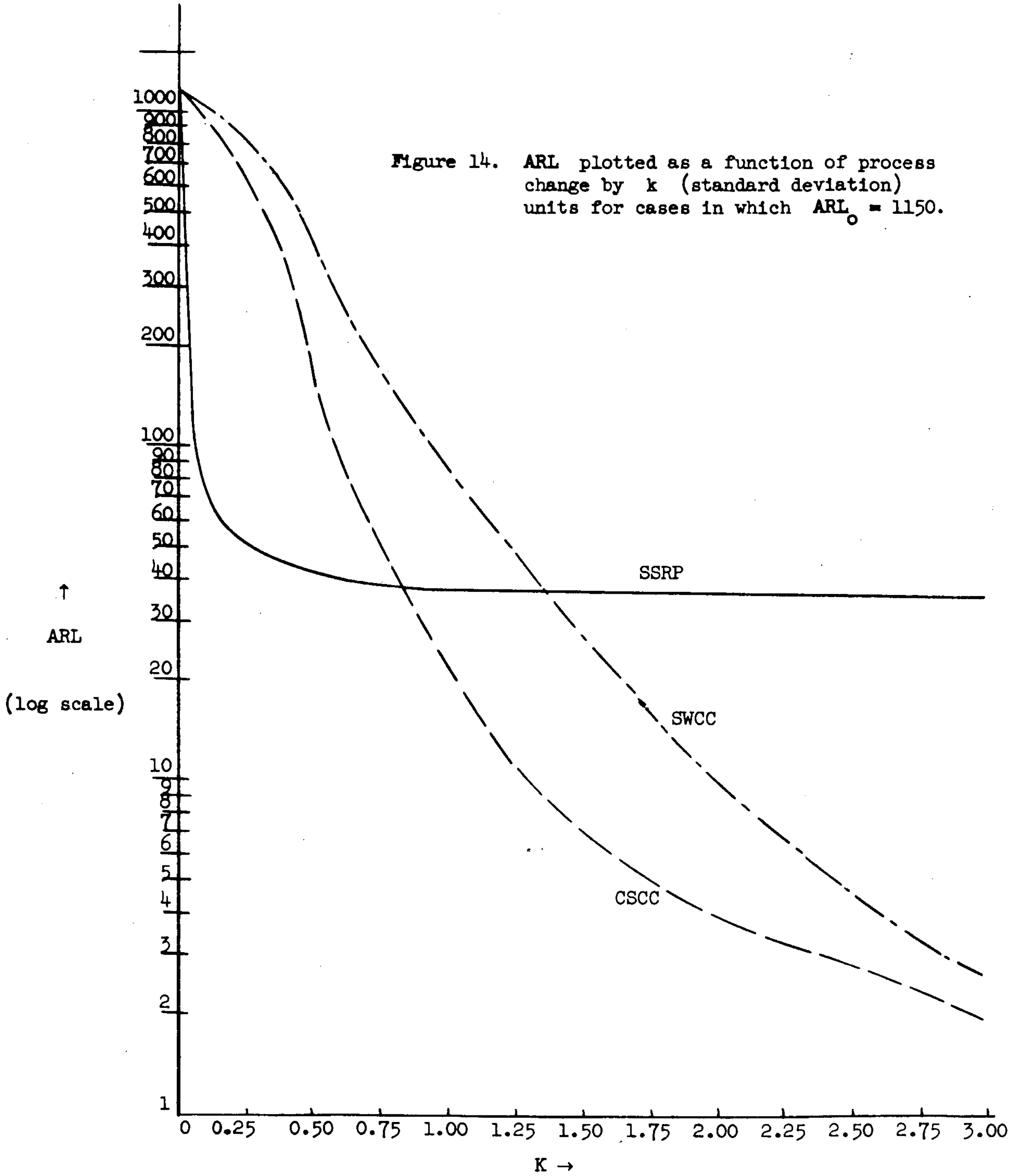


Figure 14. **ARL plotted as a function of process change by k (standard deviation) units for cases in which $ARL_0 = 1150$.**

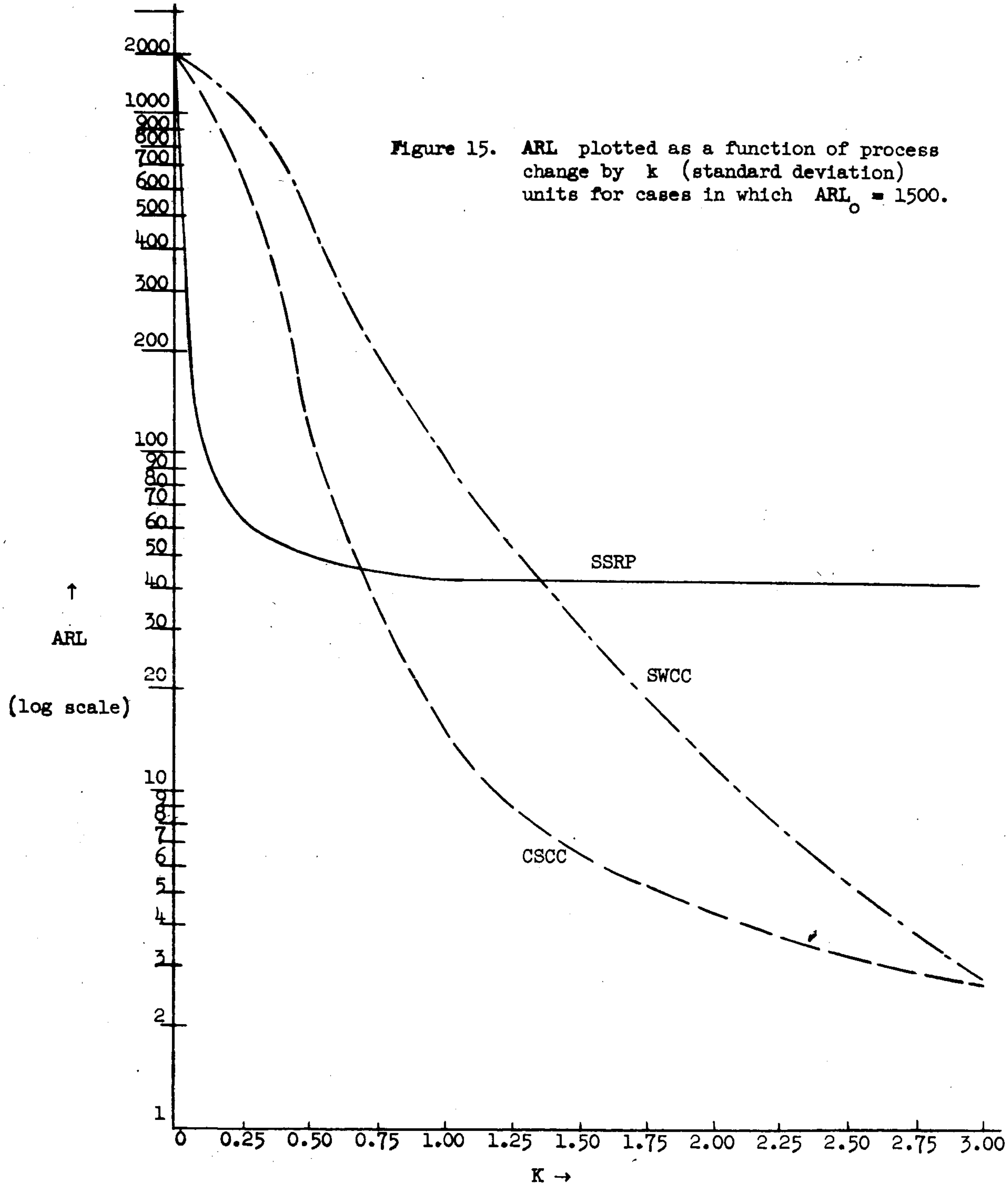


Figure 15. ARL plotted as a function of process change by k (standard deviation) units for cases in which $ARL_0 = 1500$.

7. Comparison of the Three Process Control Procedures

For a process where the measurable characteristic has a normal distribution and a shift in the process level is small (e.g. within a displacement of 0.5 standard deviation away from the mean), the Average Run Lengths of the SSRP scheme are considerably less than those of the other two schemes (this can be seen from the figures). Furthermore, the efficiencies of the SSRP relative to the SWCC and the CSCC increase as the ARL_o's increase. For example, in a plan with $ARL_o = 18$, using the CSCC scheme, it requires an average of 16 samples to detect a deviation of 0.25σ from the mean. This compares with 8 for the SSRP scheme. For a plan with $ARL_o = 1500$, however, it takes the CSCC almost 700 samples to make such a detection as compared with 60 for the SSRP. This is ten times as great as for SSRP. For the SWCC, with $ARL_o = 1500$, it takes about twenty times as many as are required for the SSRP.

For moderate changes of the process level, let us say in the range of $0.5\text{-}2.0\sigma$ from the mean, the ARL's of the SSRP scheme are generally less than those of the SWCC (if the change does not exceed 1.25σ). The SWCC becomes more and more efficient in detecting large deviations. Contrasted with the CSCC, the efficiency of the SSRP decreases as the deviation increases and becomes less efficient when deviations exceed about 1σ. The cross-over points of these schemes differ from plan to plan as the ARL_o changes. Generally speaking, the ARL's of the CSCC scheme have a "Z" shaped curve. The SWCC curve is almost linear with a negative 45^o slope. The SSRP curve looks like an exponential curve and rapidly becomes flat and almost parallel to the X-axis for large deviations. The flatness of the SSRP curve reflects its insensitivity (relative to the other two schemes) in case of large changes in the process level. Intuitively, this implies that there is a fixed number of observations that are required by the nonparametric procedure to detect a shift in the process level no matter what its magnitude may be.

Therefore, one can easily visualize the efficiency of the CSCC relative to the other two schemes. For a typical example, in a plan with ARL_o = 1100, on the average, 84 samples are required for the SWCC and 37 samples for the SSRP in order to detect a process change of 1.00σ away from its target. However, the CSCC needs only 13 samples to do the same job.

For a large deviation of process level (e.g. beyond 2σ), the ARL's of the CSCC and of SWCC differ slightly, although the CSCC is generally a little better. We can see this in the figures. However, for the SSRP scheme, it will, on the average, take ten times as many samples as do the other two.

In the region of small deviations, the steepness of the ARL curve of the SSRP scheme indicates that the average number of samples required to stop the process and make proper adjustment reduces drastically as process change increases. Now, if small changes in the process level are considered acceptable and thus can be tolerated, then, the cost of using the SSRP as a process control scheme will be comparatively high, because unnecessary interruptions may take place quite frequently.

At the very outset, for the sake of uniformity and simplicity of comparison, we assumed that the sample size and sampling interval for the three procedures were the same and they were chosen with the expectation of economical operation of process control. Unfortunately, it is rather difficult to choose such an "optimal" sample size and sampling interval (if there is any) because of the many complicated factors involved (i.e., cost of inspection, cost of not detecting an "untolerable" shift in the process level, cost of making unnecessary interruption of the process, the behavior of the manufacturing process itself). For a discussion of this aspect for the SWCC and the CSCC, see Ewan [6]. However, if homogenity and randomness of the process can be reasonably expected, then the principles suggested by Duncan [1] are helpful in making a satisfactory design. On the other hand, if the process is not homogeneous (i.e., within-sample dispersion is fairly large), then it is suggested that a small sample be taken frequently from the process. Under this kind of situation, it is wise to use the SSRP scheme, because now Y_i (that is, $\bar{X}_i$, or X_i, if sample size is one) do not necessarily follow the normal distribution so far as the Central Limit Theorem is concerned.

In summary then the SSRP is a good procedure to use when small deviations in the mean are expected, and the CSCC is a good procedure when moderate and large deviations are anticipated. On the other hand, when expected deviations are large the very simple SWCC is almost as good as CSCC, and can be recommended.

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This paper compares three procedures for controlling a process. The Shewhart Control Chart, the Cumulative Sum Chart, and the Signed Sequential Rank Procedure.

When the mean of a process shifts by a large amount, the Cumulative Sum Chart appears to be best. When deviations are small, the Signed Sequential Rank Procedure appears to be best.

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