TECHNICAL NOTE R-164

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ANALYSIS OF TWO-DIMENSIONAL, UNSTEADY FLOW IN A PROPELLANT TANK UNDER LOW GRAVITY BY FINITE DIFFERENCE METHODS

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by Carl T. K. Young

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BROWN ENGINEERING COMPANY, INC.

HUNTSVILLE, ALABAMA

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ANALYSIS OF TWO-DIMENSIONAL, UNSTEADY FLOW IN A PROPELLANT TANK UNDER LOW GRAVITY BY FINITE DIFFERENCE METHODS

October 1965

Prepared For

PROPULSION DIVISION
P&VE LABORATORY
GEORGE C. MARSHALL SPACE FLIGHT CENTER

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ABSTRACT

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This report describes a numerical procedure for solving a two-dimensional, unsteady flow problem. The fluid is in a tank, has a free surface, a periodic source (to be terminated) on the side of the tank, and is subject to a near zero gravitational field. The flow is assumed to be incompressible and irrotational so that the problem reduces to a boundary value problem governed by the Laplace equation.

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TABLE OF CONTENTS

	Page
INTRODUCTION	1
NUMERICAL ANALYSIS	3
Problem Descriptions	3
Finite-Difference Method	4
Calculation of Flow Field	10
Calculation Procedures	14
CONCLUSIONS	15
REFERENCES	16

LIST OF SYMBOLS

a	Mesh size, ft
d	Depth of slot, ft
f	Fraction of mesh size
g	Effective gravity, ft/sec^2
H	Liquid height in static equilibrium, ft
h_o	Average liquid height under static equilibrium condition, ft
T	Surface tension, lbf/ft
t	Time, sec
u	x-component velocity, ft/sec
u_o	Source velocity, ft/sec
v	y-component velocity, ft/sec
W	Width of the tank, ft
x	x-coordinate
y	y-coordinate
Δ	Difference
η	Perturbation of liquid height, ft
θ	Contact angle, degree
ρ	Density, $slug/ft^3$
ϕ	Velocity potential, ft^2/sec
ω	Relaxation factor

LIST OF SYMBOLS (Continued)

Superscripts

k	Number of iterations
s	Point on the free surface

Subscripts

i	Index for x-coordinate
j	Index for y-coordinate
n	Index for t-coordinate

INTRODUCTION

The study of dynamic behavior of liquids in greatly reduced gravitational fields has become a subject of intense interest in recent years as space technology has progressed. Knowledge of the frequency of the oscillatory motion of fuel in a large rocket propellant tank can help the designer to prevent its being in resonance with the motion of the vehicle or of its control system, thus avoiding such undesirable effects as dynamic instability. The behavior of the liquid-vapor interface is important to the engine restart operation after a prolonged coasting period, as the liquid and vapor tend to mix together when the gravity force is almost absent. Other applications can be found in the life support system, fuel cell system, etc., in a space vehicle.

The present study is concerned with the dynamic behavior of the free surface, and the flow field in general, of liquid hydrogen in the propellant tank of the S-IVB rocket stage during flight after the main fuel supply has been cut off. A water hammer effect is created by such a sudden cutoff operation and becomes the periodic source of the tank. During this stage of flight the gravitational field has been reduced to a magnitude of about 10^{-5} g and surface tension can no longer be neglected when considering the behavior of the free surface. For such a problem one is required to solve the full Navier-Stokes equations in two dimensions with a moving boundary, the so-called Stefan's problem. Aside from the additional complications arising from the presence of a moving boundary, the analytical or numerical solutions of Navier-Stokes equations present tremendous difficulties. Most of the attempts in the past have failed in the obtaining of a numerical solution to the full Navier-Stokes equations. Only in a few problems with simple boundary conditions has limited success been achieved. Therefore, in order to obtain a more practical solution in the present problem, some simplifications must be made.

The first simplification is to assume that the viscous effects can be neglected and that the flow is irrotational. The irrotationality condition is used with the continuity equation to obtain Laplace's equation in terms of the velocity potential. Satterlee and Reynolds[6] have found that in the case of a circular cylindrical tank the inviscid analysis gives a natural frequency of oscillation only a few percent lower than actual. The assumption of potential flow is therefore useful and practical, and there is no reason to believe that it cannot be applied to the two-dimensional case.

Next, the concept of small perturbations is introduced, i.e., it is assumed that the free surface can be expressed as the sum of the static equilibrium surface and a small perturbation. This has the advantage of simplifying the free surface conditions and fixing the mesh points on the free surface when finite-difference techniques are employed. Also, fluid properties, including the density and surface tension, are considered constant.

Even with such simplifications, an analytical solution of the present problem is not in sight and numerical solutions must be attempted. The following chapter outlines the procedure for the solution of this problem by finite-difference methods.

NUMERICAL ANALYSIS

PROBLEM DESCRIPTIONS

The assumption of an incompressible, potential flow leads to the following boundary value problem.

Governing Differential Equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{in the region R}$$

Boundary Conditions

On Solid Wall

$$\frac{\partial \phi}{\partial n} = 0 \qquad \text{n normal to the wall}$$

At the Source

$$\frac{\partial \phi}{\partial x} = u_o(t)$$

Dynamic Surface Condition

$$\left(\frac{\partial \phi}{\partial t}\right)_{y=H} + \left(\frac{\partial^2 \phi}{\partial t\, \partial y}\right)_{y=H} \eta(x,t) + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right]_{y=H}$$

$$+ g\eta - \frac{T}{\rho}\left\{\frac{1}{\left[1+\left(\frac{dH}{dx}\right)^2\right]^{3/2}}\frac{\partial^2 \eta}{\partial x^2} - \frac{3\,\frac{dH}{dx}\,\frac{d^2H}{dx^2}}{\left[1+\left(\frac{dH}{dx}\right)^2\right]^{5/2}}\frac{\partial \eta}{\partial x}\right\} \simeq 0 \qquad (1)$$

Kinematic Surface Condition

$$\frac{\partial \eta}{\partial t} - \left(\frac{\partial \phi}{\partial y}\right)_{y=H} - \left(\frac{\partial^2 \phi}{\partial y^2}\right)_{y=H} \eta$$

$$+ \left[\left(\frac{\partial \phi}{\partial x}\right)_{y=H} + \left(\frac{\partial^2 \phi}{\partial x\, \partial y}\right)_{y=H} \eta\right]\left(\frac{dH}{dx} + \frac{\partial \eta}{\partial x}\right) = 0 \qquad (2)$$

Contact Angle Condition (No Hysteresis)

$$\left(\frac{\partial\eta}{\partial x}\right)_{x=0,\ W} = 0$$

Initial Conditions

$$u(x, y, 0) = 0$$

$$v(x, y, 0) = 0$$

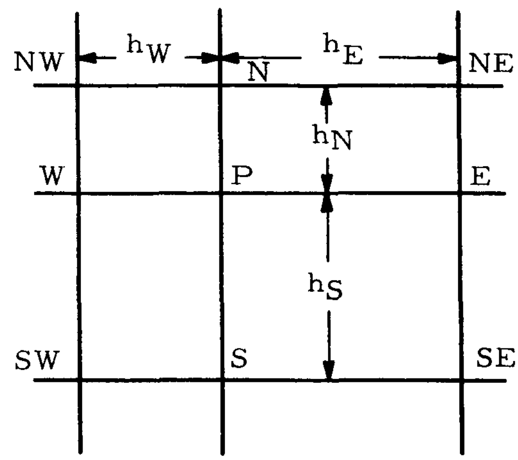
$$\phi(x, y, 0) = 0$$

$$\eta(x, 0) = 0$$

The equations for the surface conditions are derived in Reference 2. The fluid is assumed to be in static equilibrium initially, and the source velocity $u_o(t)$ is a periodic function.

FINITE-DIFFERENCE METHOD

The continuous derivatives in the differential equations above are replaced by discrete finite differences as follows:



$$\left(\frac{\partial\phi}{\partial x}\right)_P = \frac{h_W}{h_E\,(h_E + h_W)}\ \phi_E + \frac{h_E - h_W}{h_E\, h_W}\ \phi_P - \frac{h_E}{h_W\,(h_E + h_W)}\ \phi_W$$

$$\left(\frac{\partial\phi}{\partial y}\right)_P = \frac{h_S}{h_N\,(h_S + h_N)}\ \phi_N + \frac{h_N - h_S}{h_N\, h_S}\ \phi_P - \frac{h_N}{h_S\,(h_N + h_S)}\ \phi_S$$

$$\left(\frac{\partial^2 \phi}{\partial x^2}\right)_P = 2\left[\frac{\phi_E}{h_E\,(h_E + h_W)} - \frac{\phi_P}{h_E\,h_W} + \frac{\phi_W}{h_W\,(h_E + h_W)}\right]$$

$$\left(\frac{\partial^2 \phi}{\partial y^2}\right)_P = 2\left[\frac{\phi_N}{h_N\,(h_N + h_S)} - \frac{\phi_P}{h_N\,h_S} + \frac{\phi_S}{h_S\,(h_S + h_N)}\right]$$

$$\left(\frac{\partial^2 \phi}{\partial x\,\partial y}\right)_P = \frac{h_W}{h_E\,(h_E + h_W)}\left[\frac{h_S}{h_N\,(h_S + h_N)}\,\phi_{NE} + \frac{h_N - h_S}{h_N\,h_S}\,\phi_E - \frac{h_S}{h_S\,(h_N + h_S)}\,\phi_{SE}\right]$$

$$+ \frac{h_E - h_W}{h_E\,h_W}\left[\frac{h_S}{h_N\,(h_S + h_N)}\,\phi_N + \frac{h_N - h_S}{h_N\,h_S}\,\phi_P - \frac{h_N}{h_S\,(h_N + h_S)}\,\phi_S\right]$$

$$- \frac{h_E}{h_W\,(h_E + h_W)}\left[\frac{h_S}{h_N\,(h_S + h_N)}\,\phi_{NW} + \frac{h_N - h_S}{h_N\,h_S}\,\phi_W - \frac{h_N}{h_S\,(h_N + h_S)}\,\phi_{SW}\right]$$

For equal mesh sizes, the truncation errors in each of these formulas would be $O(h^2)$ as $h \to 0$, as compared to $O(h)$ for different mesh sizes. This shows the great advantage of using equally spaced nets.

A network of mesh points is laid over the flow field as shown in Figure 1. Choice of mesh size a depends on considerations of truncation errors, storage capacity, running time of the computer, and programming ease. The mesh size must be chosen such that both the width of the tank and the depth of the slot $(h_2 - h_1)$ are integral multiples of the mesh size. Ordinarily this is impossible without making the mesh size intolerably small; however, with slight adjustment of either or both of W and d, the mesh size may be chosen with relative ease. The position of the slot should also be adjusted, if necessary, to meet certain programming requirements. For all practical purposes these adjustments can be made without appreciably affecting the flow characteristics. The mesh size is then the same throughout most of the flow field. Adjacent to the free surface mesh points are selected at the intersections of the static equilibrium surface and the vertical lines, therefore the mesh sizes are different.

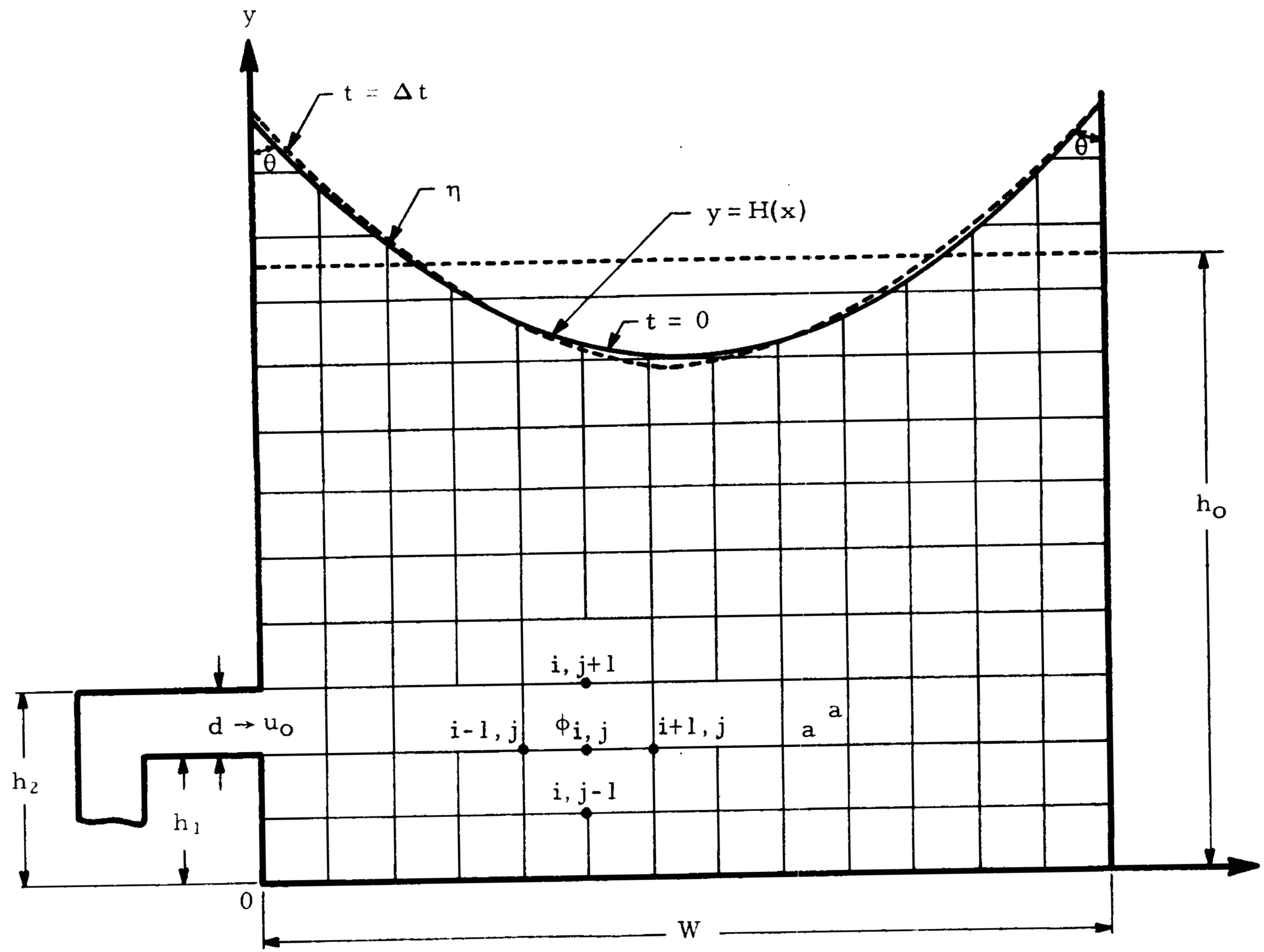


Figure 1. Layout of the Mesh Points

At each of the regular interior mesh points the Laplace equation is replaced by a finite-difference equation using the five point formula.

$$\phi_{i,j,n+1} = \frac{1}{4} (\phi_{i-1,j,n+1} + \phi_{i,j-1,n+1} + \phi_{i+1,j,n+1} + \phi_{i,j+1,n+1}) \tag{3}$$

For mesh points on the solid wall the Neumann boundary condition $\partial\phi/\partial n = 0$ can be satisfied by locating dummy points outside the wall at the images of the points immediately inside the wall, so that

On the left wall,

$$\phi_{1,j,n+1} = \frac{1}{4} (2\, \phi_{2,j,n+1} + \phi_{1,j-1,n+1} + \phi_{1,j+1,n+1}) \tag{4}$$

On the right wall,

$$\phi_{I,j,n+1} = \frac{1}{4} (2\, \phi_{I-1,j,n+1} + \phi_{I,j-1,n+1} + \phi_{I,j+1,n+1}) \tag{5}$$

On the bottom of the tank,

$$\phi_{i,1,n+1} = \frac{1}{4} (2\, \phi_{i,2,n+1} + \phi_{i-1,1,n+1} + \phi_{i+1,1,n+1}) \tag{6}$$

At the two corners,

$$\phi_{1,1,n+1} = \frac{1}{2} (\phi_{1,2,n+1} + \phi_{2,1,n+1}) \tag{7}$$

$$\phi_{I,1,n+1} = \frac{1}{2} (\phi_{I-1,1,n+1} + \phi_{I,2,n+1}) \tag{8}$$

At the source, since $\dfrac{\partial\phi}{\partial x} = u_o(t)$,

$$\phi_{1,j,n+1} = \frac{1}{4} [2\, \phi_{2,j,n+1} + \phi_{1,j-1,n+1} + \phi_{1,j+1,n+1} - 2a\, u_o(t)] \tag{9}$$

On the free surface,

$$\phi^s_{i,j,n+1} = \frac{\Delta t_2 (\Delta t_1 + \Delta t_2)}{\Delta t_1} \Bigg\{ \frac{\Delta t_2}{\Delta t_1 (\Delta t_1 + \Delta t_2)} \phi^s_{i,j,n-1} - \frac{\Delta t_2 - \Delta t_1}{\Delta t_1 \Delta t_2} \phi^s_{i,j,n}$$

$$- \frac{1}{2} (u^s_{i,j,n})^2 - \frac{1}{2} (v^s_{i,j,n})^2 - g\, \eta_{i,n} - \left(\frac{\partial v^s}{\partial t}\right)_{i,j,n} \eta_{i,n}$$

$$+ \frac{T}{\rho \left[1 + \left(\frac{dH}{dx}\right)^2\right]^{3/2}} \left(\frac{\partial^2 \eta}{\partial x^2}\right)_{i,n}$$

$$- \frac{3T \frac{dH}{dx} \frac{d^2H}{dx^2}}{\rho \left[1 + \left(\frac{dH}{dx}\right)^2\right]^{5/2}} \left(\frac{\partial \eta}{\partial x}\right)_{i,n} \Bigg\} \tag{10}$$

$$\eta_{i,n+1} = \frac{\Delta t_2 (\Delta t_1 + \Delta t_2)}{\Delta t_1} \Bigg\{ \frac{\Delta t_2}{\Delta t_1 (\Delta t_1 + \Delta t_2)} \eta_{i,n-1} - \frac{\Delta t_2 - \Delta t_1}{\Delta t_1 \Delta t_2} \eta_{i,n}$$

$$+ v_{i,j,n} + \left(\frac{\partial v^s}{\partial y}\right)_{i,j,n} \eta_{i,n}$$

$$- \left[u^s_{i,j,n} + \left(\frac{\partial u^s}{\partial y}\right)_{i,j,n} \eta_{i,n}\right] \left(\frac{dH}{dx} + \frac{\partial \eta}{\partial x}\right)_{i,n} \Bigg\} \tag{11}$$

where velocities on the surface $u^s_{i,j,n}$ and $v^s_{i,j,n}$ can be obtained by a procedure to be described later. (Actually they are velocities at points with coordinates corresponding to the static equilibrium surface.) For $\left(\frac{\partial v^s}{\partial t}\right)_{i,j,n}$ a backward difference is used.

$$\left(\frac{\partial v^s}{\partial t}\right)_{i,j,n} = \frac{v^s_{i,j,n} - v^s_{i,j,n-1}}{\Delta t_1}$$

The terms $\frac{dH}{dx}$ and $\frac{d^2H}{dx^2}$ are obtained by modifying the appropriate equations of Reference 3. They are

$$\frac{dH}{dx} = \pm \left\{ \frac{1}{\left[1 - \frac{H - H_o}{W}\left(\frac{B_o}{2}\,\frac{H + H_o}{W} + 2\cos\theta - \frac{B_o\,h_o}{W}\right)\right]^2} - 1 \right\}^{\frac{1}{2}}$$

where

$H_o = H$ when $x = W/2$

$B_o = \rho\, g\, W^2/T$ = Bond number.

The sign for $\frac{dH}{dx}$ depends on the value of x. For $x > W/2$, $\frac{dH}{dx}$ is positive; $\frac{dH}{dx}$ is negative for $x < W/2$.

$$\frac{d^2H}{dx^2} = \frac{\frac{1}{W}\left(\frac{B_o\,H}{W} + 2\cos\theta - \frac{B_o\,h_o}{W}\right)}{\left[1 - \frac{H - H_o}{W}\left(\frac{H + H_o}{W}\,\frac{B_o}{2} + 2\cos\theta - \frac{B_o\,h_o}{W}\right)\right]^3}$$

where

$$h_o = \frac{1}{W}\int_0^W H\,dx \;.$$

Also,

$$\left(\frac{\partial^2\eta}{\partial x^2}\right)_{i,\,n} = \frac{\eta_{i+1,\,n} - 2\eta_{i,\,n} + \eta_{i-1,\,n}}{a^2}$$

$$\left(\frac{\partial\eta}{\partial x}\right)_{i,\,n} = \frac{\eta_{i+1,\,n} - \eta_{i-1,\,n}}{2a}$$

$$\left(\frac{\partial^2\eta}{\partial x^2}\right)_{x=0,\,W} = \frac{-2\cot\theta}{a}$$

Thus, $\phi^s_{i,j,n+1}$ can be calculated by Equation 10. Using the first two terms of a Taylor series expansion,

$$\left(\frac{\partial v^s}{\partial y}\right)_{i,j,n} = \left(\frac{\partial v}{\partial y}\right)_{i,j-1,n} + \Delta y_2 \left(\frac{\partial^2 v}{\partial y^2}\right)_{i,j-1,n}$$

$$= \frac{\Delta y_1}{\Delta y_2 (\Delta y_1 + \Delta y_2)} v^s_{i,j,n} + \frac{\Delta y_2 - \Delta y_1}{\Delta y_1 \Delta y_2} v_{i,j-1,n}$$

$$- \frac{\Delta y_2}{\Delta y_1 (\Delta y_1 + \Delta y_2)} v_{i,j-2,n}$$

$$+ 2 \left[\frac{v^s_{i,j,n}}{\Delta y_2 (\Delta y_1 + \Delta y_2)} - \frac{v_{i,j-1,n}}{\Delta y_1 \Delta y_2} + \frac{v_{i,j-2,n}}{\Delta y_1 (\Delta y_1 + \Delta y_2)}\right] \Delta y_2$$

where

$$\Delta y_1 = y_{i,j-1} - y_{i,j-2}$$

$$\Delta y_2 = y_{i,j} - y_{i,j-1}$$

$\left(\frac{\partial u^s}{\partial y}\right)$ can be calculated similarly.

CALCULATION OF FLOW FIELD

In a typical case there are at least several hundred mesh points in the flow field and there are an equal number of equations to be solved simultaneously at any time level. Since the resulting matrix is sparse, the best way to solve these equations is by an iterative method. The iterative scheme to be used is the successive over-relaxation method or the extrapolated Liebmann method when applied to Laplace's equation, which has the form

$$\phi^{k+1}_{i,j,n+1} = \frac{\omega}{4}\left(\phi^{k}_{i+1,j,n+1} + \phi^{k+1}_{i-1,j,n+1} + \phi^{k}_{i,j+1,n+1} + \phi^{k+1}_{i,j-1,n+1}\right) - (\omega - 1)\,\phi^{k}_{i,j,n+1} .$$

The relaxation factor, ω, lies somewhere between 1 and 2. The optimum value of ω, which gives the highest convergence rate, is given by Young's[4] formula

$$\omega_b = 1 + \frac{\lambda}{\left[1 + (1 - \lambda)^{\frac{1}{2}}\right]^2}$$

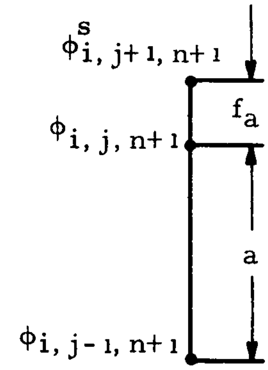
where λ is the spectral radius of the Gauss-Seidel method. For the case of Laplace's equation and for a rectangle with sides Ra and Sa, where a is the mesh size and R and S are integers. λ is given by

$$\lambda = \frac{1}{4} \left(\cos \frac{\pi}{R} + \cos \frac{\pi}{S}\right)^2 .$$

Since the configuration under consideration is not rectangular, it is suggested that one use either a rectangle which contains the given region or one which has approximately the same area and proportions. These formulas have been found to agree very well with the results of test runs, referred to later, for which the liquid surface was assumed to be flat.

The successive over-relaxation method has been proven to be superior over the Gauss-Siedel method in terms of convergence rate. This is so when applied to the five point formula, provided that the ordering of the points is properly chosen. Experience shows that by iterating column wise from left to right with arrows pointing upward in each column, the highest convergence rate can be obtained.

It should be pointed out, however, that in obtaining mesh points immediately below the free surface, it is usually undesirable to apply the modified five-point formula since the mesh size ratios in such cases may be so large that the iterations may converge very slowly and sometimes may even diverge. A different method is needed. The following formula is the so-called "interpolation of degree one":



$$\phi_{i,\,j,\,n+1} = \frac{f\,\phi_{i,\,j-1,\,n+1} + \phi^{s}_{i,\,j+1,\,n+1}}{1+f} \qquad (12)$$

The truncation errors in such interpolations are $O(a^2)$, compared with $O(a)$ in the modified five-point formula and $O(a^2)$ in the five-point formula. This shows the advantage in using the interpolation formula. Note also that Equation 12 is of positive type with diagonal dominance, which is excellent for numerical solutions.

After the velocity potentials, ϕ's, have been calculated for every mesh point on the new time level, the velocities will be computed. The following procedure is to be followed:

$$u_{i,\,j,\,n+1} = \left(\frac{\partial\phi}{\partial x}\right)_{i,\,j,\,n+1}$$

$$v_{i,\,j,\,n+1} = \left(\frac{\partial\phi}{\partial y}\right)_{i,\,j,\,n+1}$$

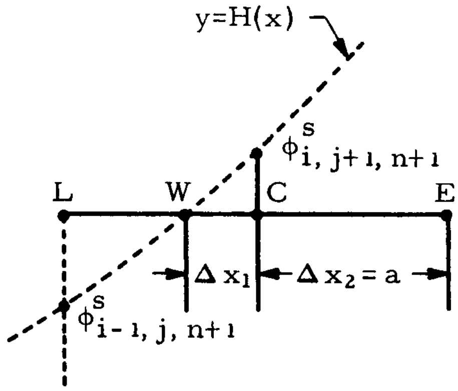
To find $\left(\frac{\partial\phi}{\partial x}\right)_{i,\,j,\,n+1}$, it is necessary to test whether $(x_{i,\,j} + a,\ y_{i,\,j})$ or $(x_{i,\,j} - a,\ y_{i,\,j})$ is a point outside the surface. If one of them is such a point, the value of ϕ at the surface point intersected by the horizontal line $y = (j-1)\,a$ is obtained through linear interpolation of the two nearest surface points. This new surface point is then used for the calculation of $\left(\frac{\partial\phi}{\partial x}\right)_{i,\,j,\,n+1}$. For example, as shown below, it is intended to find u_c.

Point L, with coordinates $(x_C - a,\ y_C)$, is located outside the surface. Thus,

$$\phi_E = \phi_{i+1,\,j,\,n+1}$$

$$\phi_C = \phi_{i,\,j,\,n+1}$$

$$\phi_W^s = \frac{\Delta x_1\, \phi^s_{i-1,\,j,\,n+1} + (a - \Delta x_1)\, \phi^s_{i,\,j+1,\,n+1}}{a}$$



Then

$$u_{i,\,j,\,n+1} = \left(\frac{\partial \phi}{\partial x}\right)_{i,\,j,\,n+1} = \frac{\Delta x_1}{\Delta x_2\,(\Delta x_1 + \Delta x_2)}\, \phi_E + \frac{\Delta x_2 - \Delta x_1}{\Delta x_1\, \Delta x_2}\, \phi_C$$

$$- \frac{\Delta x_2}{\Delta x_1\,(\Delta x_1 + \Delta x_2)}\, \phi_W^s \quad .$$

Also,

$$v_{i,\,j,\,n+1} = \left(\frac{\partial \phi}{\partial y}\right)_{i,\,j,\,n+1} = \frac{\Delta y_1}{\Delta y_2\,(\Delta y_1 + \Delta y_2)}\, \phi_{i,\,j+1,\,n+1}$$

$$+ \frac{\Delta y_2 - \Delta y_1}{\Delta y_1\, \Delta y_2}\, \phi_{i,\,j,\,n+1} - \frac{\Delta y_2}{\Delta y_1\,(\Delta y_1 + \Delta y_2)}\, \phi_{i,\,j-1,\,n+1} \quad .$$

For velocities on the free surface, apply a Taylor series expansion to first degree

$$u^s_{i,j,n+1} = u_{i,j-1,n+1} + \Delta y_2 \left(\frac{\partial u}{\partial y}\right)_{i,j-1,n+1}$$

$$= u_{i,j-1,n+1} + \Delta y_2 \left[\frac{\Delta y_1}{\Delta y_2(\Delta y_1 + \Delta y_2)} u^s_{i,j,n+1} + \frac{\Delta y_2 - \Delta y_1}{\Delta y_1 \Delta y_2} u_{i,j-1,n+1} - \frac{\Delta y_2}{\Delta y_1(\Delta y_1 + \Delta y_2)} u_{i,j-2,n+1}\right]$$

$$u^s_{i,j,n+1} = \frac{\Delta y_1 + \Delta y_2}{\Delta y_2} \left[\frac{\Delta y_2}{\Delta y_1} u_{i,j-1,n+1} - \frac{\Delta y_2^2}{\Delta y_1(\Delta y_1 + \Delta y_2)} u_{i,j-2,n+1}\right]$$

Similarly,

$$v^s_{i,j,n+1} = \frac{\Delta y_1 + \Delta y_2}{\Delta y_2} \left[\frac{\Delta y_2}{\Delta y_1} v_{i,j-1,n+1} - \frac{\Delta y_2^2}{\Delta y_1(\Delta y_1 + \Delta y_2)} v_{i,j-2,n+1}\right]$$

CALCULATION PROCEDURES

Before proceeding to calculations for points on a new time level, the time increment must be selected. The value of the increment depends on the stability and convergence requirements, and is mostly determined by experience. Surface conditions are then obtained by Equations 10 and 11, and values of ϕ in the flow field are approximated by extrapolations through the corresponding points on the two previous time levels. Then the iteration process starts. First, values of ϕ for points immediately below the free surface are obtained by Equation 12. Next, mesh points are revised column wise from left to right, with the direction in each column pointing upward, according to Equations 3 through 9. These two procedures are carried out alternatively until the maximum error in ϕ at any point drops below the previously set tolerance, usually around 10^{-6}, and iteration is terminated. Velocities are then computed, thus completing the calculations of the flow field on a new time level. A new time increment is again selected, and the whole procedure is repeated.

CONCLUSIONS

Several test runs have been made to determine the validity of Young's formula for predicting the value of the optimum relaxation factor. Assuming a flat surface with constant ϕ across the width of the tank, the flow field is laid with 44×38 mesh points. The best value for ω was found to be around 1.94 which agrees well with the value of 1.92 given by Young's formula. With the value of 1.94 for ω, it took approximately 200 iterations to converge to 10^{-6} in ϕ for the first time increment and only a few iterations for subsequent time levels with the same time increment.

Running time on the UNIVAC 1107 computer was less than 0.5 sec per sweep, which is tolerable since on the average it does not require many sweeps for convergence.

As was shown previously, an explicit finite-difference scheme was employed to solve the differential equations for the free surface. This could cause instability if time increments were not properly chosen. With the presence of the mixed derivative and nonlinear terms, there is no known theoretical method to obtain the stability criteria for such a difference scheme. The stability criteria are to be determined only by numerical experiments. Since central time differences were used, it is conceivable that the difference scheme may always be unstable no matter how small the time increment is, as in the study performed by Richardson[8]. If this should happen, either a forward time difference or DuFort and Frankel difference scheme would have to be used. It is unknown, however, what effects the mixed derivative and nonlinear terms will have on these difference schemes.

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13. ABSTRACT

This report describes a numerical procedure for solving a two-dimensional, unsteady flow problem. The fluid is in a tank, has a free surface, a periodic source (to be terminated) on the side of the tank, and is subject to a near zero gravitational field. The flow is assumed to be incompressible and irrotational so that the problem reduces to a boundary value problem governed by the Laplace equation.

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