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AEROTHERMOCHEMICAL ANALYSIS OF NON-OSCILLATORY AND OSCILLATORY CHARACTERISTICS OF LIQUID BIPROPELLANT ROCKET MOTORS: DROPLET EVAPORATION, WAVE EQUATIONS, MOTOR INSTRUMENTATION, NON-OSCILLATORY PROGRAM COMPUTATION

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Prepared For

Jet Propulsion Laboratory
Pasadena, California

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I. DROPLET VAPORIZATION: HEAT TRANSFER LIMITED MODEL

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A model is presented for the determination of the vaporization rate of a liquid fuel droplet. The governing equation is the heat conduction equation with the appropriate boundary and initial conditions, which include an energy balance at the surface of the droplet. An approximate solution to this problem is found utilizing a heat balance integral and an assumed temperature distribution. Transient and steady-state solutions are thereby obtained and applied to a particular combustion problem. The solution is compared to a diffusion limited analysis and is found to be in agreement.

Finally, the method is used to determine the additional mass release which is due to the passage of a longitudinal wave over the evaporation droplet.

This work was done by Bernard Grossman as partial fulfillment of the requirements for the Master's Degree in Astronautics.

This section is included as a self-contained unit.

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LIST OF SYMBOLS

A	area of the droplet
a	initial droplet radius
c_p	specific heat at constant pressure
h	film coefficient
k	thermal conductivity
q	Laplace variable
q_n	heat transfer to the film
q_e	heat transfer to the droplet
q_r	heat of reaction
q_t	total heat transfer from the flame
q_v	heat use to vaporize the surface liquid
r	radial coordinate
r_s	radius of the vaporizing droplet
T	temperature
$\overline{T}_d$	mean droplet temperature
T_f	flame temperature
T_i	initial or injection temperature
T_v	vaporization temperature
t	time
u	transformed variable, rT or $r(T_f - T)$
w	mass evaporization rate

α	thermal diffusivity
γ	non-dimensional heat transfer parameter,
δ	arbitrary function of time
η	non-dimensional droplet radius r_s/a
λ	heat of vaporization
μ	non-dimensional heat transfer parameter
ρ	density of the liquid
τ	non-dimensional time
φ	additional mass evaporated due to unsteady heat input, $\Delta m/m_o$

A. INTRODUCTION

The problem of combustion and combustion instability in liquid propellant rocket motors has put much emphasis on the determination of heterogeneous, multi-phase burning mechanisms. Propellant vaporization is an important phenomenon occurring during the combustion process. Thus, considerable research has been devoted to solving problems concerned with the evaporation of fuel droplets.[1-3]

This analysis is used to predict the effect of heat transfer on the vaporization rate of a fuel droplet. The model presented is not constrained to have a diffusion limited evaporation rate and has the versatility of treating time dependent heat inputs.

The problem is formulated in two parts: the initial heat-up, and the evaporation analyses. The initial heat-up solution is provided only to generate the functional form of the assumed temperature profile to be used in the vaporization analysis. The evaporation solution is developed by using an integral technique on the basic heat conduction equation as developed by Goodman[4] and adapted by Parisse[5] to a spherical body.

The vaporization solution is then applied to a particular rocket motor and the vaporization rate history with chamber location is determined. Finally a technique is developed which considers the effect of a continuous, longitudinal wave on the mass evaporation rate of a liquid fuel droplet in a rocket motor.

B. ANALYSIS

The transient vaporization of a spherical liquid fuel droplet is considered. The model of a burning droplet consists of: (a) a spherical liquid fuel droplet; (b) a thin film comprised of fuel vapor and combustion gases; and (c) a spherical flame shroud. A schematic diagram is shown in Figure (1). The liquid fuel droplet is assumed to have constant thermal properties, and to remain spherical throughout the entire evaporation process. In the vapor film region the diffusing vapors are heated to the flame temperature. When the vaporized fuel reaches the flame boundary, it is assumed that it mixes with the oxidizer and reacts.

The problem of the initial heat-up of the droplet is now developed. Initially, the liquid droplet is at the injection temperature T_i. For this temperature, and the fuel under consideration, the saturation pressure is small. Under these conditions the amount of mass evaporated from the droplet is also small and therefore neglected. In view of this assumption a heat-up condition without mass evaporation results. Thus the droplet can be considered a heat sink. The heat conduction equation, given below, describes the transfer of thermal energy within the droplet:

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} \right) \qquad 0 \le r \le a \qquad (1)$$

with the inital condition

$$T(r,0) = T_i \qquad (2)$$

and the boundary conditions

$$-K\left(\frac{\partial T}{\partial r}\right)_{a,t} = h\left[T_f - T(a,t)\right] \qquad (3)$$

$$T(0,t) \text{ is finite }, \qquad (4)$$

where α is the thermal diffusivity, K is the thermal conductivity, a is the initial radius of the droplet and h is the film coefficient.

A solution to this linear, transient heat conduction problem is found in Carslaw and Jaeger[6]. The solution which is given in series form does not converge for short periods of time. Hence another solution to this problem, which is more adaptable to short periods of time, was sought. Carslaw and Jaegar[7] outline a general method of applying Laplace transforms to transient heat conduction problems in order to obtain solutions which are suitable for short times. Parisse[5] applies this method to a problem in transient heat conduction with spherical symmetry. Utilizing this method, the heat conduction equation is transformed to a more convenient form by the substitution $u = r\,(T_f - T)$.

Applying Laplace transforms to the resulting system, the subsidiary equation becomes

$$\frac{d^2\bar{u}}{dr^2} - q^2\bar{u} = -\left(\frac{T_f - T_i}{\alpha}\right) r \qquad (5)$$

$$\frac{d\bar{U}}{dr} + \frac{\mu}{a}\bar{U} = 0 \qquad \text{at } r = a, \tag{6}$$

$$\bar{U} = 0 \qquad \text{at } r = 0 \tag{7}$$

where $q^2 = p/\mu$ and $\overline{U}$ is the Laplace transform of u .

The solution to this system of equations in series form is

$$\bar{U} = \left(\frac{T_f - T_i}{p}\right) r - \frac{(1+\mu)(T_f - T_i)}{p} \sum_{n=0}^{\infty} (-1)^n \frac{(q - \mu/a)^n}{(q+\mu/a)^{n+1}} \left\{ e^{-q[(2n+1)a-r]} - e^{-q[(2n+1)a+r]} \right\} \tag{8}$$

For small values of time, the series in equation (8) can be truncated after the n = 0 term, since the inclusion of additional terms in the series produces terms of a smaller order of magnitude. The inverse transformation of the resulting equation yields the desired initial heat-up solution

$$\begin{aligned} \frac{T - T_i}{T_f - T_i} = \frac{a}{r}\left(\frac{1+\mu}{\mu}\right) \Bigg\{ & \operatorname{erfc}\frac{(a-r)}{2\sqrt{\alpha t}} - \operatorname{erfc}\frac{(a+r)}{2\sqrt{\alpha t}} \\ & - \exp\left[\frac{\mu}{a}(a-r) + \frac{\mu^2}{a^2}\alpha t\right] \operatorname{erfc}\left[\frac{a-r}{2\sqrt{\alpha t}} + \frac{\mu}{a}\sqrt{\alpha t}\right] \\ & + \exp\left[\frac{\mu}{a}(a+r) + \frac{\mu^2}{a^2}\alpha t\right] \operatorname{erfc}\left[\frac{a+r}{2\sqrt{\alpha t}} + \frac{\mu}{a}\sqrt{\alpha t}\right] \Bigg\} \end{aligned} \tag{9}$$

As the temperature of the droplet increases, the mass evaporation becomes substantial and must be explicitly included in the boundary conditions of the vaporization solution.

Referring to Figure (2), the total heat transfer q_t, from the surrounding gases goes to heating the liquid droplet q_l, vaporizing the surface liquid q_v, and heating the ensuing gases to the flame temperature, q_h.

$$q_t = q_l + q_v + q_h \tag{10}$$

Choosing a convective model for the energy transport from the flame to droplet, equation (10) becomes

$$h(T_f - T_v) = K\left(\frac{\partial T}{\partial r}\right)_{r_s,t} + \frac{W}{A}\lambda + \frac{W}{A}C_p(T_f - T_v) \tag{11}$$

where λ is the heat of vaporization, w is the mass evaporation rate, A is the surface area of the droplet, and $r_s(t)$ is the coordinate which locates the position of the outer surface at any time t. The conservation of mass equation in differential form is

$$W = -4\pi r_s^2 \rho \frac{dr_s}{dt} = -A\rho \frac{dr_s}{dt} \tag{12}$$

where ρ_l is the density of the liquid. Employing this result equation

(11) becomes

$$k\left(\frac{\partial T}{\partial r}\right)_{r_s,t} = h(T_f - T_v) + \rho\left[\lambda + c_p(T_f - T_v)\right]\frac{dr_s}{dt} \tag{13}$$

For the vaporization process, the heat conduction equation applies for the interval $0 \leq r \leq r_s(t)$. One boundary condition at the surface of the droplet is

$$T\ (r_{s,t}) = T_v \tag{14}$$

Although the vaporization temperature T_v, will vary with time, its appearance in the following analysis will be as the difference $T_f - T_v$ which within the approximations made in this analysis, remains constant. Now the time variable is defined to be zero when $r_s = a$. Hence from a consideration of the continuity of heat input for the heat-up and vaporization analysis

$$\frac{dr_s}{dt} = 0 \quad \text{at} \quad t = 0 \tag{15}$$

The initial condition determined by the heat-up analysis is

$$T\ (r, 0) = T_o\ (r) \tag{16}$$

The system of equations (1), (13), (14), (15), (16) is non-linear due to the presence of $r_s(t)$. Parisse^6 outlined a method of applying Goodman's heat balance integral to the ablation of a solid sphere. As a result of the assumptions made in this analysis, the method of solution of the problem is similar.

Equation (1) is converted to a more convenient form by the transformation u = rT, whereupon

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial r^2} \qquad 0 \le r \le r_s \qquad (17)$$

Integrating equation (16) over the volume of the liquid

$$\frac{d}{dt}\int_0^{r_s} u\,dr - u(r_s,t)\frac{dr_s}{dt} = \alpha\left[\left(\frac{\partial u}{\partial r}\right)_{r_s,t} - \left(\frac{\partial u}{\partial r}\right)_{r_s,t}\right]$$

or

$$\frac{d}{dt}\int_0^{r_s} Tr\,dr - r_s T(r_s,t)\frac{dr_s}{dt} = \alpha\left[r_s\left(\frac{\partial T}{\partial r}\right)_{r_s,t} + T(r_s,t) - T(0,t)\right] \qquad (19)$$

On the basis of the functional form of the initial heat-up solution, the radial time dependant temperature profile is assumed to be of the form

$$\frac{T - T_i}{T_v - T_i} = \exp\left[-\frac{(r_s - r)}{\delta(t)}\right] \qquad (20)$$

where $\delta(t)$ is arbitrary. The function $\delta(t)$ can be evaluated from equation (19)

$$K\left(\frac{\partial T}{\partial r}\right)_{r_s,t} = \frac{T_v - T_i}{\delta(t)} = h(T_f - T_v) + \rho\left[\lambda + c_p(T_f - T_v)\right]\frac{dr_s}{dt} \qquad (21)$$

Equation (20) becomes

$$\frac{T-T_i}{T_v-T_i} = \exp\left\{-\left[\frac{h(T_f-T_i)+\rho[\lambda+C_p(T_f-T_i)]\frac{dr_s}{dt}}{k(T_v-T_i)}\right](r_s-r)\right\} \tag{22}$$

This temperature profile satisfies conditions (14) and (15), and equation (16) approximately. A plot of this and the exact profile obtained from the heat-up solution, evaluated at a given time is shown in Figure (3).

Substituting equation (22) into equation (19) the following second order, ordinary differential equation in non-dimensional form is obtained.

$$\ddot{\eta} = -\frac{(\gamma+\dot{\eta})^2}{\phi^2\left[\eta(\gamma+\dot{\eta})-2\phi\right]}\left\{\phi\left[(\gamma+\dot{\eta})-\phi\dot{\eta}\right]+\eta(\gamma+\dot{\eta})\left[\gamma+(1+\phi)\eta\right]\right\} \tag{23}$$

where

$$\tau = \frac{\alpha t}{a^2}$$

$$\eta(\tau) = r_s/a$$

$$\dot{\eta}(\tau) = \frac{d\eta}{d\tau}$$

$$\ddot{\eta}(\tau) = \frac{d^2\eta}{d\tau^2}$$

$$\gamma = \frac{ah(T_f - T_v)}{\rho\alpha\left[\lambda + c_p(T_f - T_v)\right]} \tag{24}$$

$$\phi = \frac{k(T_v - T_i)}{\rho\alpha\left[\lambda + c_p(T_f - T_v)\right]} \tag{25}$$

A steady state vaporization rate is obtained by setting $\ddot{\eta} = 0$, which results in

$$\dot{\eta}_{ss} = -\gamma \tag{26}$$

$$\eta_{ss} = 1 - \gamma\tau \tag{27}$$

C. RESULTS

The results of this analysis is applied to the determination of the vaporization rate of a liquid fuel droplet moving through a rocket motor. The chamber is divided into a number of sections where the thermodynamic properties are specified and assumed constants for the interval. Thus, once the variation of film coefficient, flame temperature and mean droplet velocity with chamber location are known, the burning rate and droplet history can be determined using the preceding analysis. The steady-state solution, equations (26) and (27), is used as an approximation, or the more exact equation (23) can be integrated numerically. The following scheme can be used.

$$\eta(\tau_i) = \eta(\tau_{i-1}) + \dot{\eta}(\tau_{i-1})\Delta\tau + \ddot{\eta}(\tau_{i-1})\frac{(\Delta\tau)^2}{2} \qquad (28)$$

$$\dot{\eta}(\tau_i) = \dot{\eta}(\tau_{i-1}) + \ddot{\eta}(\tau_{i-1})\Delta\tau \qquad (29)$$

The values of $\eta(\tau_i)$ and $\dot{\eta}(\tau_i)$ can be substituted into equation (23) to obtain $\ddot{\eta}(\tau_i)$. Initially, $\eta(0) = 1$ and $\dot{\eta}(0) = 0$.

The steady-state and transient vaporization rate solutions are applied step-wise to a particular rocket motor for the purpose of comparison. The steady-state vaporization rate is found to be in agreement with the transient evaporation rate and for the particular fuel considered, the variation is within 2%. The droplet vaporization rate is shown as a function of chamber location in Figure (4). For the remainder of this discussion, the steady-state solution to equation (23) is used.

The vaporization rate of a fuel droplet in a rocket motor determined in this paper is compared to the vaporization rate calculated from the method given in Burstein et al.[1] The ballistics of 50/50 Aerozine-Nitrogen Tetroxide propellants are determined. The thermodynamic properties calculated in Burstein's analysis is used here and a plot of some of these is shown in Figures (5) and (6). For the propellant combination considered, a modification of the energy balance, equation (13), is made. Since the dissociation of hydrazine is exothermic, a heat of reaction q_r must be included in the heat source term. Thus equation (13) becomes

$$h(T_f - T_i) = K\left(\frac{\partial T}{\partial r}\right)_{r_s,t} - \rho\left[\lambda + C_p(T_f - T_v) + q_r\right]\frac{dr_s}{dt} \qquad (30)$$

The mean value of the vaporization temperature, T_v is found by integrating over the energy distribution in the liquid

$$\overline{T}_d = \frac{1}{r_s}\int_0^{r_s} T\,dr \qquad (31)$$

where $\overline{T}_d$ is a mean droplet temperature and T is given in equation (22). Thus equation (31) is integrated to yield T_v.

Now with these modifications, the two methods are used to calculate the variation of the vaporization rate and droplet radius with chamber location. The results of the two analyses are given in Figures (7) and (8). The technique given in this paper is within 20% of that used in Reference 1.

Combustion instability is a measure of the energy accumulated in a cavity and is dependent upon the interaction of waves with energy sources.[9] As a wave passes over an evaporating fuel droplet, additional energy can be released which can couple to the wave and drive it. For the purpose of this analysis a step function heat input which decays exponentially with time is considered to represent the energy transport to the droplet. Hence

$$q_t = \bar{q}\, e^{-4t/t^*} + h(T_f - T_v) \tag{32}$$

where $\bar{q}$ is the additional heat input and t^* is the time constant of the exponential.

Substituting equation (32) into equations (29) and (19) and performing the required mathematical manipulations yields the following second-order differential equation for the evaporation rate due to a time dependent heat input

$$\ddot{\eta} = \frac{-\psi^2}{\phi^2[\eta\psi - 2\phi]}\left\{\eta\psi^2 + \phi\psi + \phi\dot{\eta}[\eta\psi - \phi]\right\} + \bar{\gamma}\beta e^{-\beta\tau} \tag{33}$$

where

$$\psi = \bar{\gamma} e^{-\beta\tau} + \gamma + \dot{\eta} \tag{34}$$

$$\bar{\gamma} = \frac{\bar{q}a}{\rho\alpha[\lambda + c_p(T_f - T_v)]} \tag{35}$$

$$\beta = 4a^2/\alpha t^* \tag{36}$$

Equation (33) possesses similar characteristics as equation (23) and may be solved using the same numerical integration, equations (28) and (29).

This solution is now applied stop-wise to the rocket motor which has been previously described. The unsteady heat input is applied at each chamber section and hence the effect of chamber location on the additional burning rate is found. Mass conservation, equation (12) is used to yield the additional mass evaporation due to the unsteady heat input. For a $\overline{q}$ of 20% and time constants $t^* = 1, 0.1,$ and 0.01 milliseconds, the additional mass evaporation with chamber section is calculated and plotted in Figure (9).

D. CONCLUSION

A heat transfer analysis for the determination of the vaporization rate of a liquid fuel droplet has been presented. Steady-state and transient vaporization rate solutions were obtained. The vaporization rate given herein was compared to that calculated using Burstein's[1] analysis, with a diffusion limited model. Both solutions were found to be in agreement and the vaporization rates differed by a maximum of 20%.

It has also been shown that an amount of additional mass can be released due to the passage of a wave over an evaporation droplet and that favorable positions in a rocket chamber exist which support this mass release.

Finally, it may be concluded that a working analysis which can determine the propellant vaporization rates of a liquid fuel droplet with steady and unsteady heat inputs has been presented.

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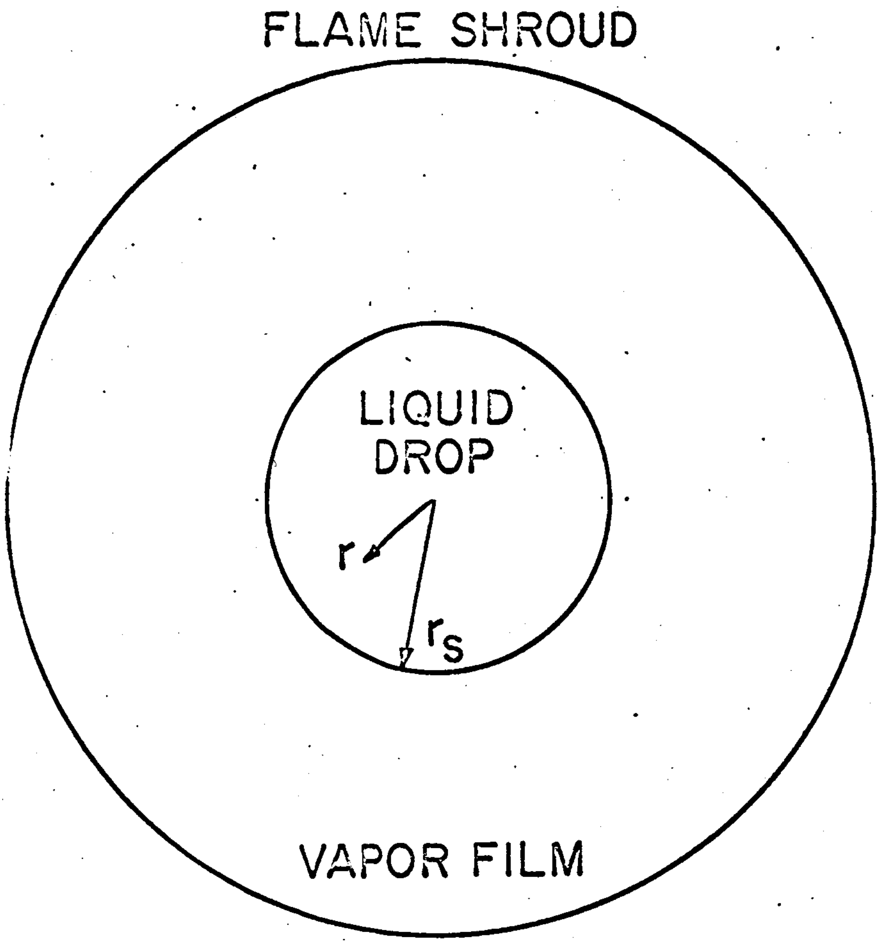


FIG. I COMBUSTION MODEL

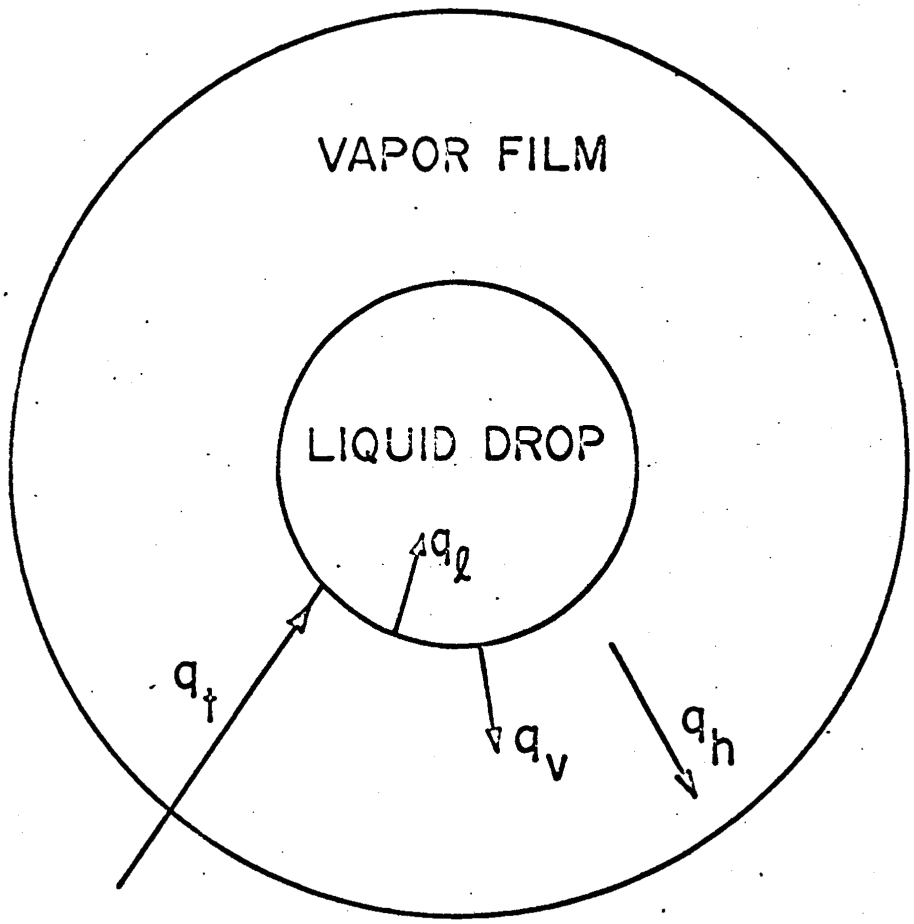


FIG.2 HEAT TRANSFER TO LIQUID DROP

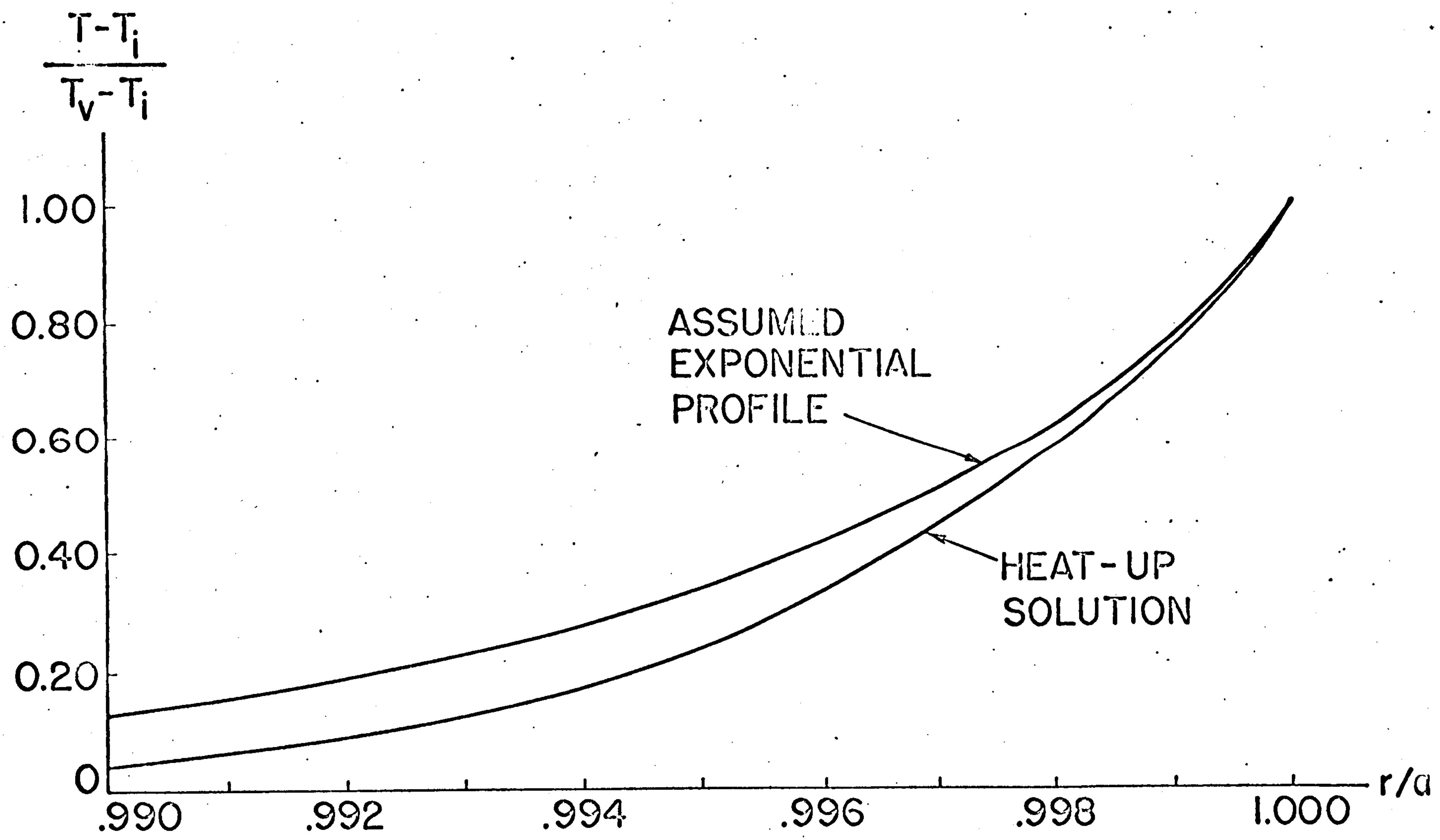


FIG. 3 RADIAL TEMPERATURE DISTRIBUTION WHEN DROPLET FIRST BEGINS TO EVAPORATE

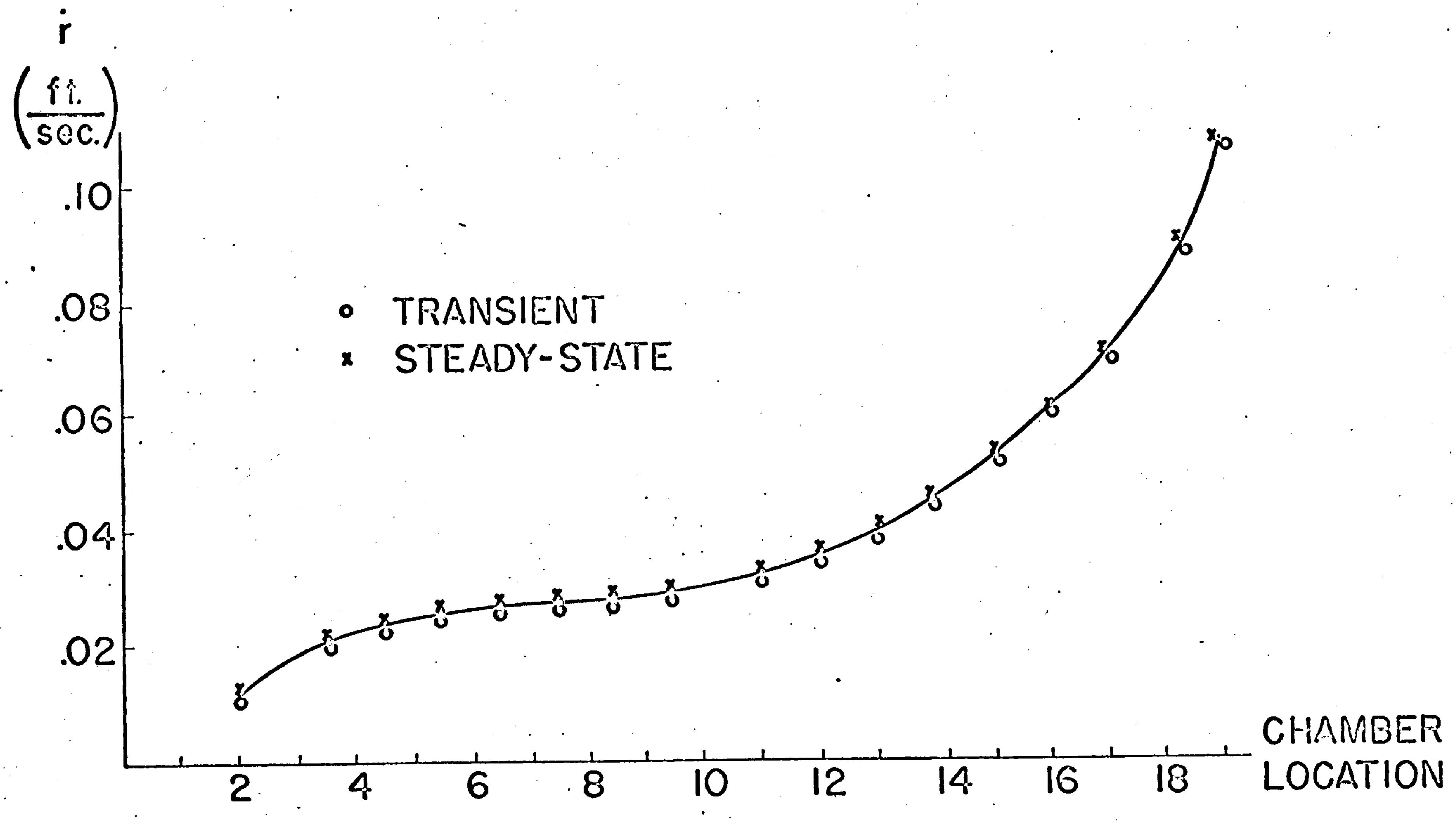


FIG. 4 TRANSIENT AND STEADY-STATE VAPORIZATION RATES

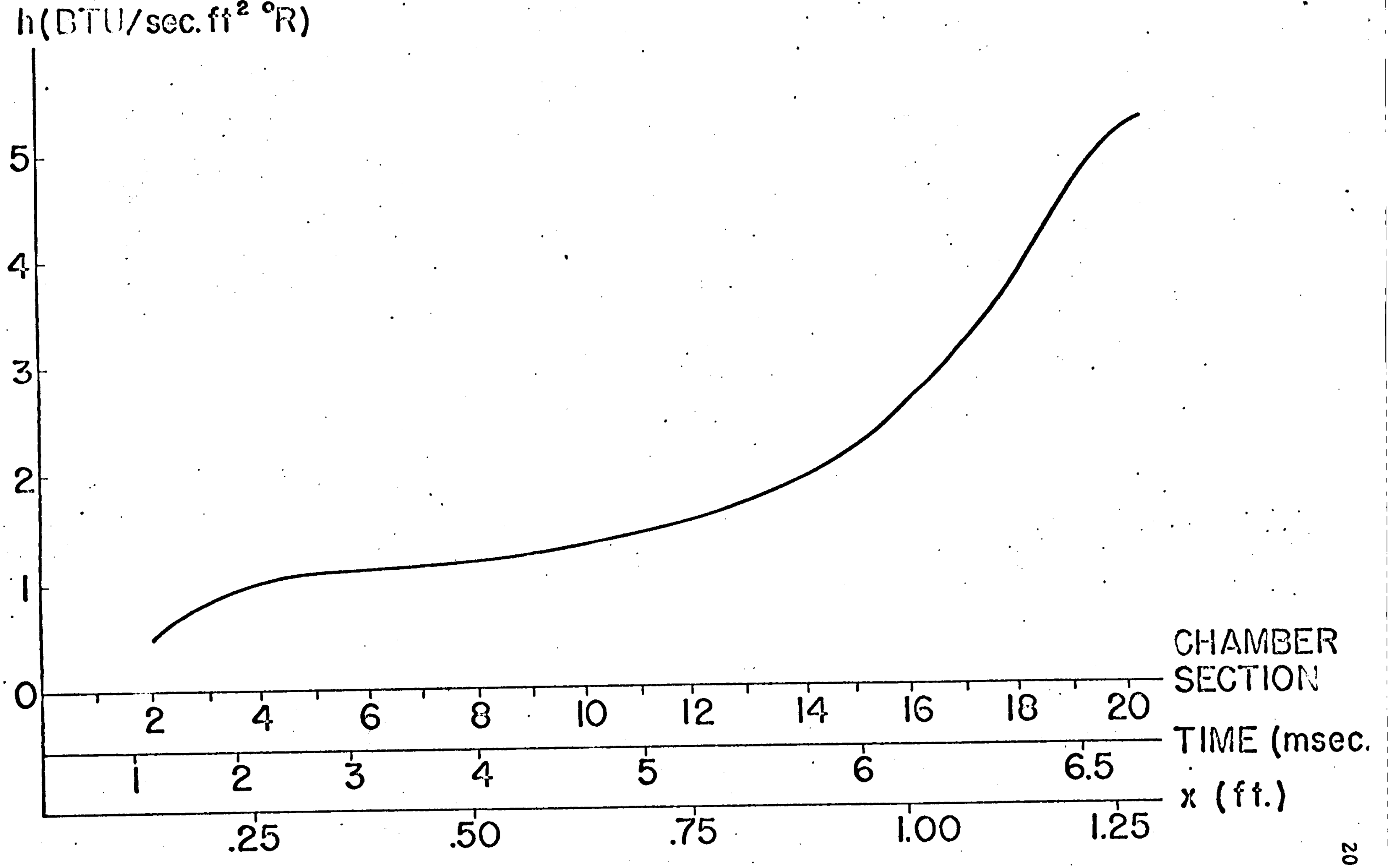


FIG. 5 FILM COEFFICIENT GRADIENT 50/50 AEROZINE

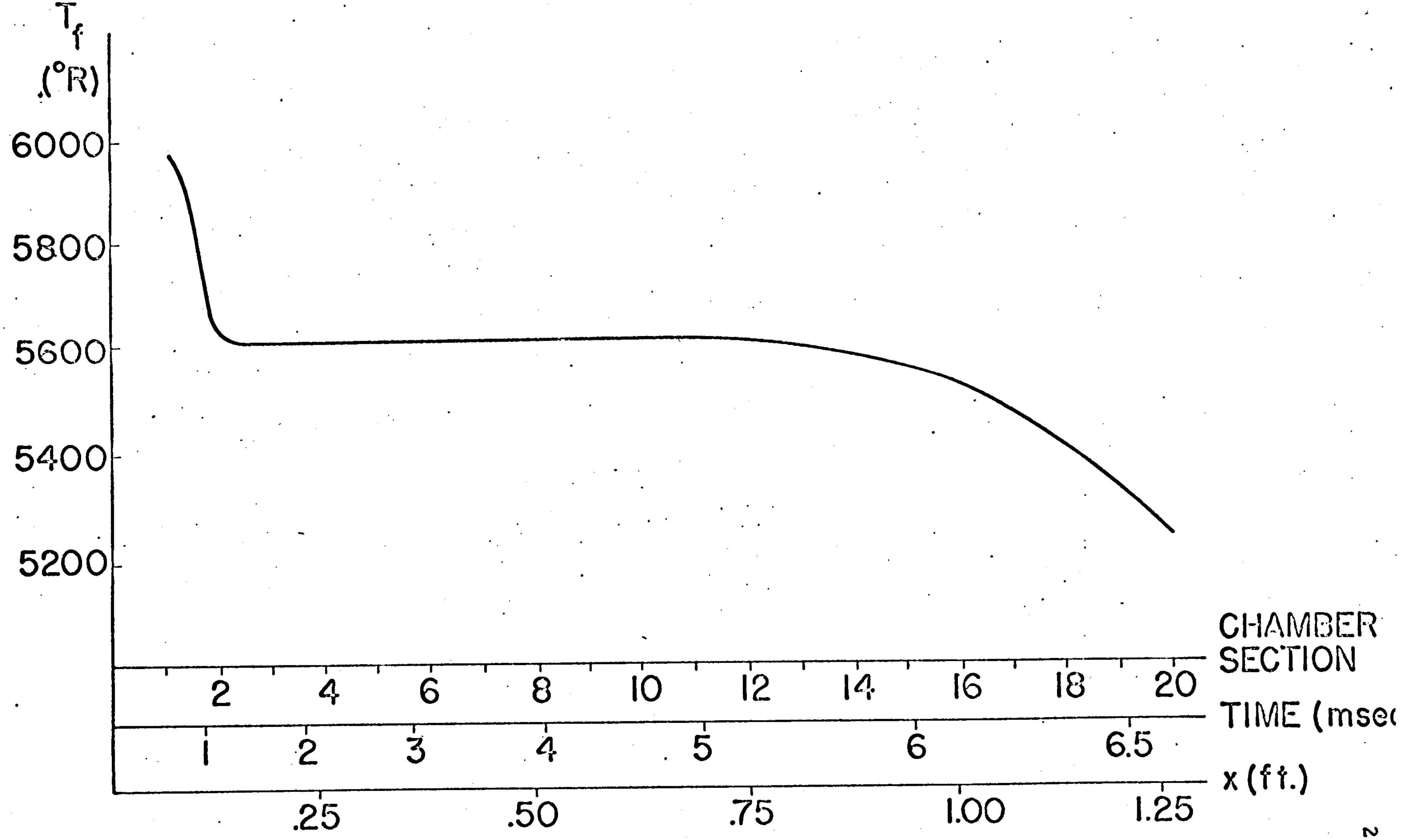


FIG. 6 FLAME TEMPERATURE GRADIENT 50/50 AEROZINE

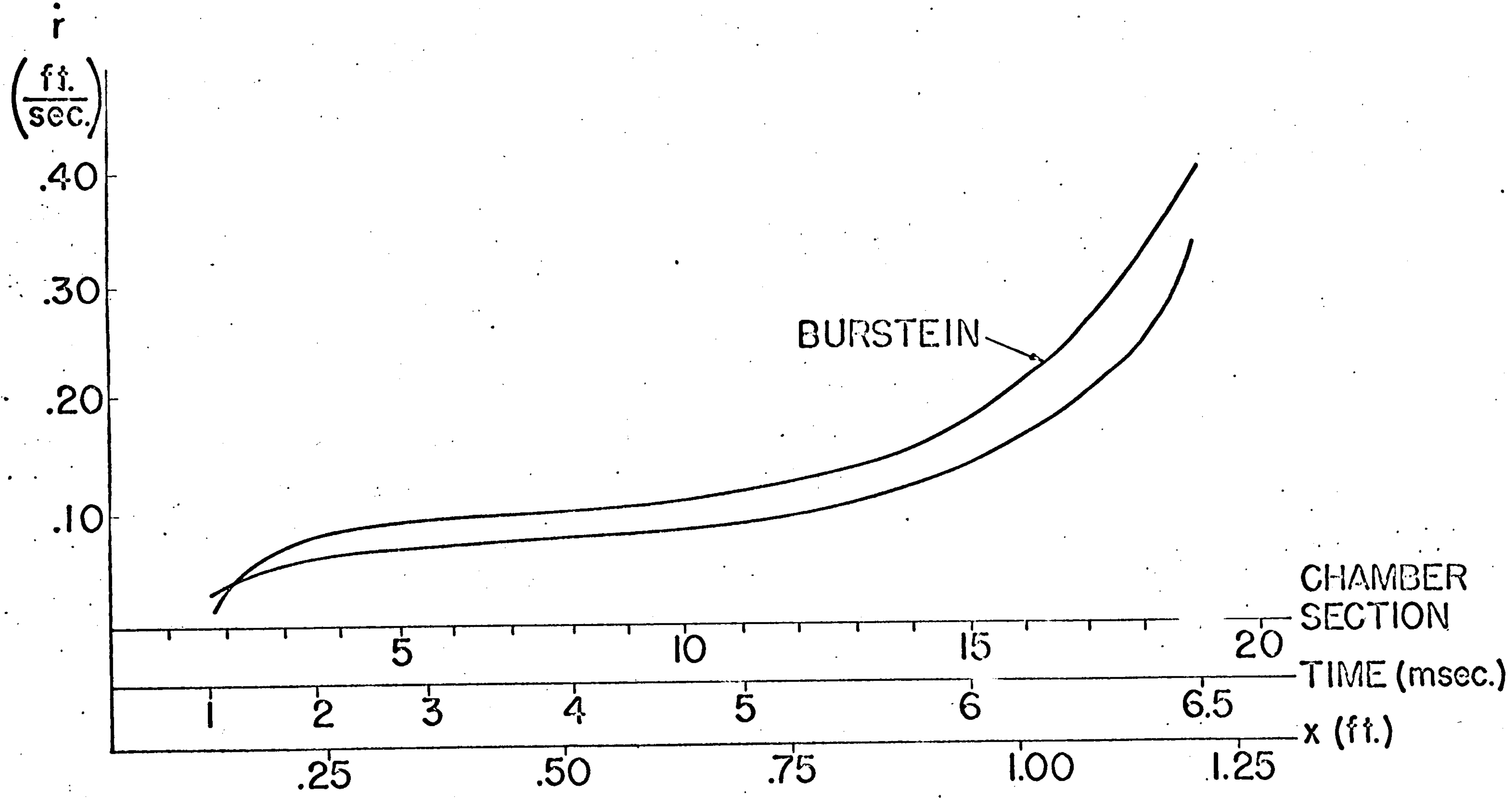


FIG. 7 VAPORIZATION RATE VS. CHAMBER LOCATION 50/50 AEROZINE

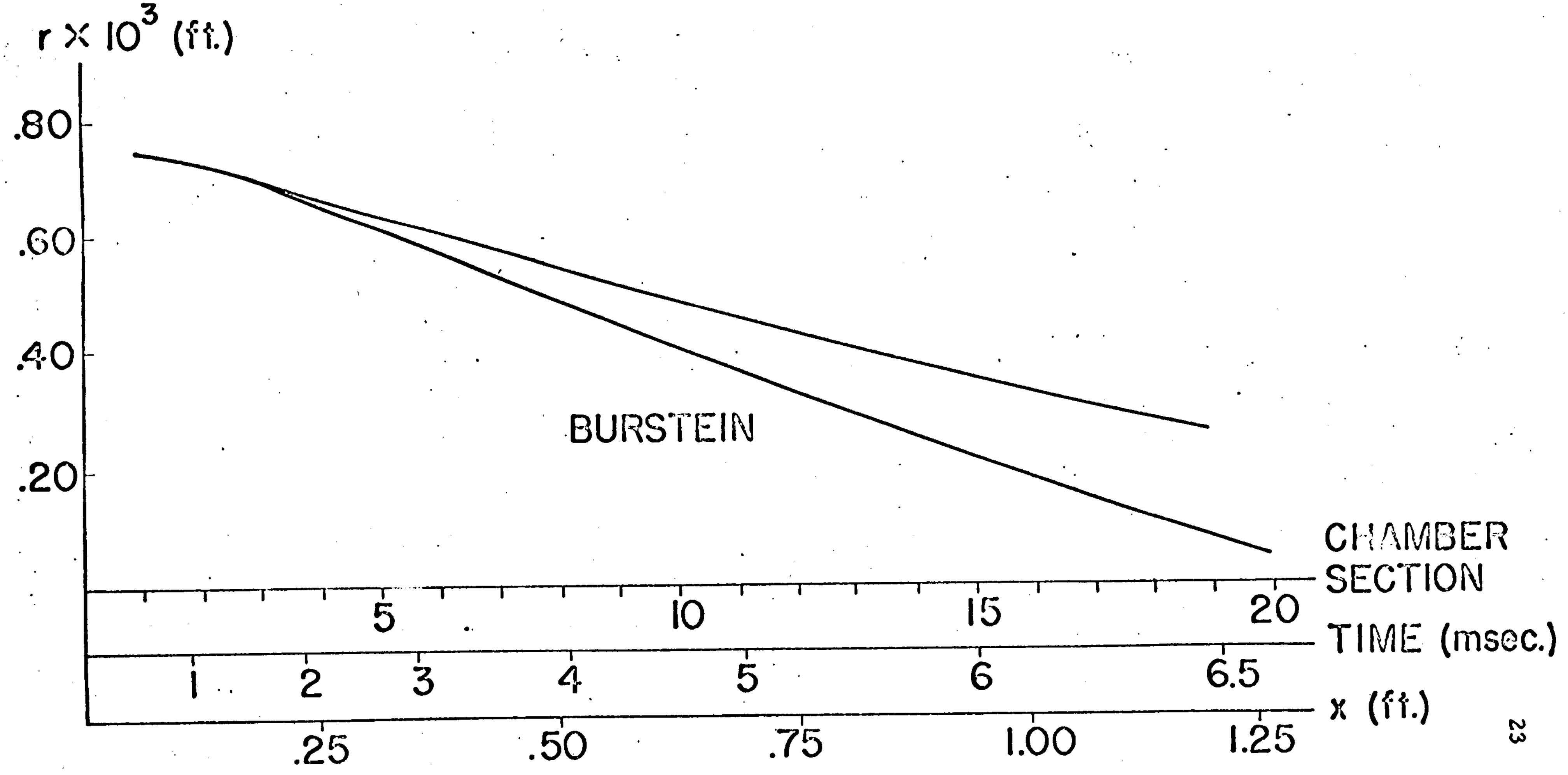


FIG. 8 DROPLET RADIUS VS. CHAMBER LOCATION 50/50 AEROZINE

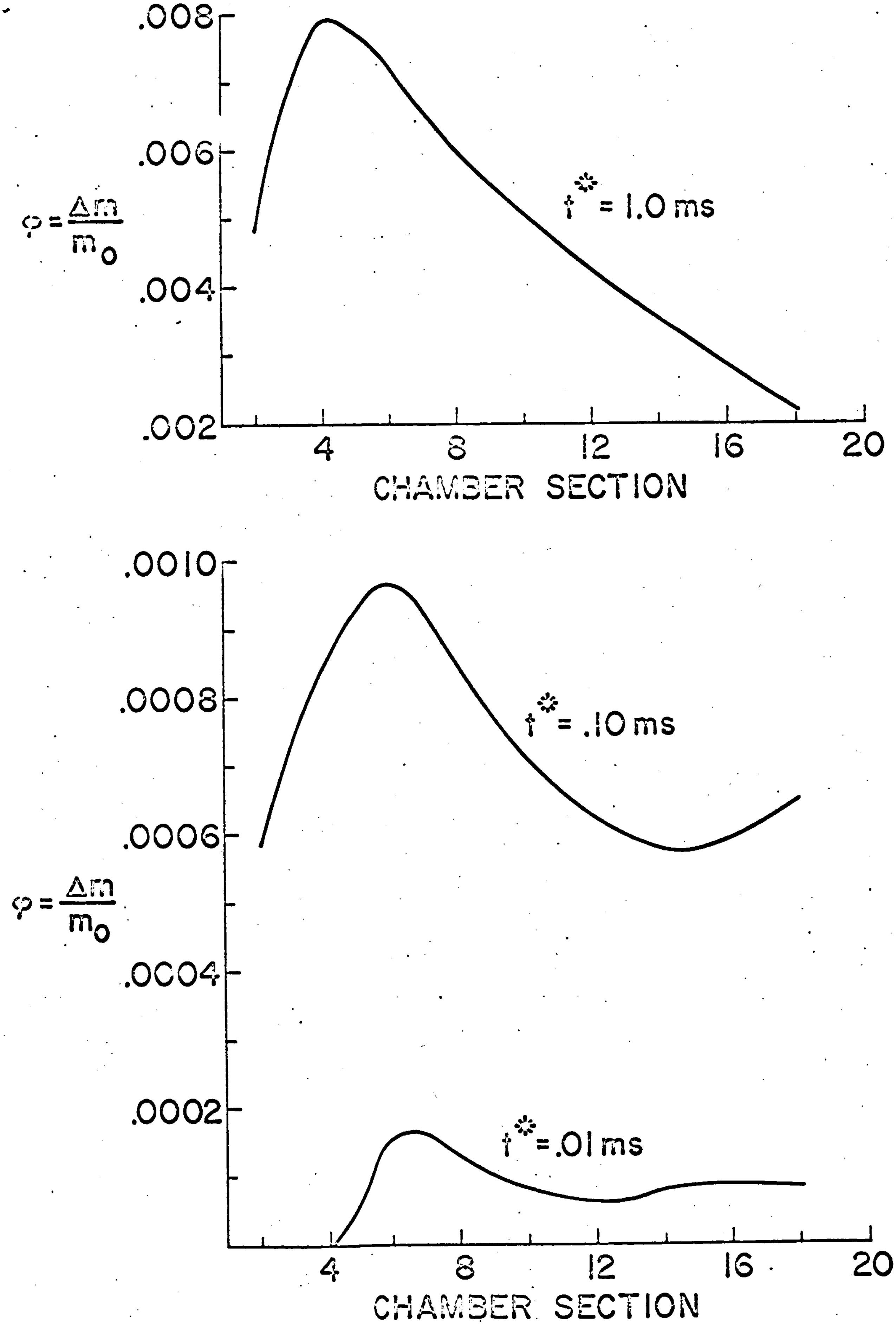


FIG. 9 ADDITIONAL MASS RELEASE DUE TO UNSTEADY HEAT INPUT

II. WAVE EQUATIONS

The conservation equations were derived employing sources and sinks and including viscous and heat conduction terms. These equations will be employed in deriving the wave and compatibility equations to be applied to the rocket motor. Although it is contemplated at this time not to include the viscous and heat conduction terms, they are considered in case their possible importance becomes apparent in the future.

The development of the wave equation was not pursued further at this time since it is felt that other work has priority. This includes debugging the non-oscillatory aerothermochemical computer program, developing the droplet evaporation program, and generating the functional form of the wave driving source term. These "wave" equations are being given at this time due to the suggestion of Dr. George Sotter (letter 6 April 1965). I trust that they will be helpful. Reference should be made to Report No. 64-3, 1 October 1964, for the gas dynamic equations which are used as the basis for the work given below.

The mass continuity equation remains unchanged,

$$\frac{\partial \rho_g}{\partial t} + \nabla \cdot \rho_g \vec{u} = \nabla \cdot \bar{\varphi} \tag{2.1}$$

where ρ_g is the gas density and subscript can be dropped,

t is the time,

$\vec{u}$ is the gas velocity,

∇ is the differential operator,

and $\nabla \cdot \bar{\varphi}$ is the mass source term and is obtained from

$$\int_{\nu_\ell} (\nabla \cdot \rho_\ell \vec{w}) \delta\nu_\ell \equiv \int_{\nu_g} \nabla \cdot \bar{\varphi} \ \delta\nu_g \tag{2.2}$$

where ν_ℓ is the liquid droplet volume,

ρ_ℓ is the droplet density,

ν_g is the gas-system volume,

and $\vec{w}$ is the liquid droplet surface velocity with respect to its center of mass. The source term $(\nabla \cdot \bar{\varphi})$ is a function of space and time and must be generated from the injector geometry and the results of the non-oscillatory aerothermochemical program to give (r,θ,z) distribution, and the nonsteady droplet evaporation program to give the time (t) variation.

In order to use Newton's Law, the body and surface forces acting on the mass in an elemental volume must be found and these are

$$\bar{\sum} \text{Force} = \int_{\nu_g} \rho_g \vec{F}_b \delta\nu_g + \int_S \bar{p}_n dS \tag{2.3}$$

where

$$\vec{p}_n = - \vec{n}p + 2\mu(\vec{n} \cdot \nabla)\vec{u} + \mu(\bar{n} \times \bar{\xi}) - \frac{2}{3} \mu \Delta \vec{n} , \tag{2.4}$$

$\Delta \equiv \nabla \cdot \vec{u}$,

$\vec{\xi} \equiv \nabla \times \vec{u}$,

$\vec{F}_b$ are body forces,

and μ is the viscosity.

The surface integrals are changed to volume integrals by Gauss's theorem so that we obtain

$$\int \rho_g \frac{d\vec{u}}{dt}\,\delta\nu_g + \int_{\nu_\ell} (\rho_\ell\,\delta\nu_\ell)\,\frac{d\bar{v}}{dt} + \int (\vec{u} - \vec{v})(\nabla\cdot\bar{\varphi})\,\delta\nu_g =$$

$$= \int \Big[\, \rho_g \vec{F}_{body} - \nabla p + 2(\nabla\cdot\mu\,\nabla)\,\vec{u} + \nabla\times(\mu\vec{\xi}) -$$

$$- \frac{2}{3}\,\nabla(\mu\Delta) \Big]\,\delta\nu_g \qquad (2.5)$$

The body forces on the liquid droplets are included in the drag term. In addition, any distortions and subsequent drag in the gas field due to the presence of the droplets are also included in the drag term. The effect of the pressure gradient across the droplet is neglected except that of work done by the expanding vapors against the gas, and this shows up in the energy equation below. Where the droplets are very large, then it can be envisaged that the pressure gradient can shear these droplets - but it is assumed that the droplets are not that large.

Since the volume ν_g is arbitrary, we can drop the integral signs and expand the viscous terms. If the variable viscosity and body force terms are neglected, then the equation (2.5) reduces to

$$\rho\,\frac{d\vec{u}}{dt} + \vec{F}_{drag} + (\vec{u} - \vec{v})(\nabla\cdot\bar{\varphi}) = -\,\nabla p -$$

$$-\,\mu\,(\nabla\times\vec{\xi}\,) + \frac{4}{3}\,\mu\,\nabla\,\Delta \qquad (2.6)$$

where $\vec{F}_{drag} \equiv \int_{\nu_\ell} (\rho_\ell\,\delta\nu_\ell)\,\frac{d\bar{v}}{dt}$,

∇p are "pressure" forces,

$\mu(\nabla\times\vec{\xi})$ are rotational flow terms,

and $\mu(\nabla\Delta)$ are compressibility terms.

The energy equation is more involved. It is decided to write the first law of thermodynamics for both the droplet and the gas subsystems and to add these. For the gas the first law can be written as

$$\int \frac{dQ_g}{dt}\,\delta\nu_g = \int \frac{dE^o_g}{dt} + \int \frac{dWk_g}{dt}\,\delta\nu_g \tag{2.7}$$

and for the droplet,

$$\int \frac{dQ_\ell}{dt}\,\delta\nu_\ell = \int \frac{dE^o_\ell}{dt} + \int \frac{dWk_\ell}{dt}\,\delta\nu_\ell \tag{2.8}$$

where Q is the thermodynamic heat transferred, E^o the stagnation internal energy, and Wk the thermodynamic work. The sum of these equations, i.e., equations (2.7) and (2.8) apply to the gas-droplet system. Let us now expand these terms and combine them. The internal energy of the gas is expressed as

$$\begin{aligned}\int \frac{dE^o_g}{dt} &= \int \frac{d}{dt}\left(\rho_g \delta\nu_g e^o_g\right)\\ &= \int \rho_g \delta\nu_g \frac{de^o_g}{dt} + \int e^o_g \frac{d}{dt}\left(\rho_g \delta\nu_g\right)\\ &= \int \rho_g \delta\nu_g \frac{de^o_g}{dt} + \int e^o_g \left(\nabla \cdot \vec{\varphi}\right)\delta\nu_g\end{aligned} \tag{2.9}$$

where the source term comes from the mass continuity equation. A similar expansion occurs for the droplet internal energy term,

$$\begin{aligned}\int \frac{dE^o_\ell}{dt} &= \int \rho_\ell \delta\nu_\ell \frac{de^o_\ell}{dt} + \int e^o_\ell \frac{d}{dt}\left(\rho_\ell \delta\nu_\ell\right)\\ &= \int \rho_\ell \delta\nu_\ell \frac{de^o_\ell}{dt} - \int \left(e^o_\ell + \lambda\right)\left(\nabla \cdot \vec{\varphi}\right)\delta\nu_g\end{aligned} \tag{2.10}$$

The term $\int \rho_\ell \delta\nu_\ell \frac{de^o_\ell}{dt}$ represents droplet heating, the latter term $\int (e^o_\ell + \lambda)(\nabla\cdot\vec{\varphi})\delta\nu$ represents the energy convected away by the evaporated portion of the droplet system. The symbol λ is the heat of evaporation, and this energy is shown to be abstracted from the internal energy of the droplet system; however, it is noted that this energy can be directly related to the heat transfer by equation (2.7) and its form becomes a matter of bookkeeping. The heat transfer term $\frac{dQ_g}{dt}$ can also be divided into that transferred: (1) to the rocket chamber walls, Q_w; (2) to the droplet, Q_d; (3) to a part of the droplet to evaporate it, $(\nabla\cdot\bar{\varphi})\lambda$; and (4) to the vapors in heating them from the droplet to ambient temperature, Q_v. It was decided to include part (3) in the internal energy term, and part (4) as a redistribution of energy within the gaseous subsystem. The negative sign occurs since it is assumed that a process of evaporation only can occur from the droplet.

Let us define

$$\int \frac{d\bar{Q}}{dt}\,\delta\nu \equiv \int \frac{dQ_g}{dt}\,\delta\nu_g + \int \frac{dQ_\ell}{dt}\,\delta\nu_\ell \tag{2.11}$$

where $\bar{Q}$ then becomes the thermodynamic heat for the droplet-gas system, i.e., the heat transfer to the rocket chamber walls and any other environment. If we combine equations (2.11), (2.10), (2.9), (2.8), and (2.7), we obtain

$$\begin{aligned}\int \frac{d\bar{Q}}{dt}\,\delta\nu = {} & \int \rho_g \delta\nu_g \frac{de^o_g}{dt} + \int e^o_g\,(\nabla\cdot\vec{\varphi})\delta\nu_g + \\ & + \int \rho_\ell \delta\nu_\ell - \int (e^o_\ell + \lambda)(\nabla\cdot\vec{\varphi})\delta\nu_g + \\ & + \int \frac{dWk_g}{dt}\,\delta\nu_g + \int \frac{dWk_\ell}{dt}\,\delta\nu_\ell\end{aligned} \tag{2.12}$$

Now let us consider the work terms. For the gaseous subsystems, the work is divided into four parts, namely, (1) the work required to accelerate the liquid droplets, (2) the work of the body forces, (3) the work of the pressure forces, and (4) the work done on the gas by the expanding vapors. Thus,

$$\begin{aligned}\int \frac{dWk}{dt}\,\delta\nu_g = &- \left\{ \iint_{\nu_g} \rho\bar{u} \cdot \bar{F}_b\,\delta\nu_g + \int_S \vec{u} \cdot \Big[\bar{n}p + \right.\\ &\left. + \frac{2}{3}\,\mu\Delta\bar{n} - 2\mu(\bar{n} \cdot \nabla)\,\vec{u} - \mu(\bar{n} \times \bar{\xi})\Big]\delta S \right\} - \\ &- \int_{S_\ell} \bar{v}_{vapors} \cdot \bar{n}\,p\,\delta S_\ell + \bar{F}_{drag} \cdot \bar{v} \qquad (2.13)\end{aligned}$$

where after the equal sign the first term is the work done by virtue of the body force, the second, the pressure, the third the expanding vapors, and the fourth the drag. In the third factor the symbol $\bar{v}_{vapors}$ appears, and it is the velocity of the envelope of the expanding vapors from the droplet. Its value is not known and varies from $\bar{v}$, the droplet velocity, to $\bar{u}$, the gas velocity. It is assumed that the expansion occurs at thermodynamic equilibrium such that the work done by these expanding vapors on the gas, as determined by the vapor thermodynamic properties, is equal to the work received by the gas,as determined by the gas thermodynamic properties. Thus, these works can be equated and this is done below.

The work done by the evaporating droplet is that due to drag and the evaporating vapors. The work done by the droplets on the gas by virtue of the gas viscosity and subsequent gas fluid field distortion is assumed to be included in the drag term. Thus,

$$\int \frac{dWk_{\ell}}{dt}\,\delta\nu_{\ell} = + \int_{S_{\ell}} \bar{v}_{vapors} \cdot \bar{n}\; p\; \delta S_{\ell} - F_{drag} \cdot \bar{v} \qquad (2.14)$$

The sum of the work done by the gas-droplet system is then

$$\int \frac{dWk_{g}}{dt}\,\delta\nu_{g} + \int \frac{dWk_{\ell}}{dt}\,\delta\nu_{\ell} = - \Big\{ -(\bar{u} \cdot \nabla)p - p\Delta + \mu\nabla^{2}(\bar{u} \cdot \bar{u}) +$$

$$+ \mu\nabla \cdot (\bar{\xi} \times \bar{u}) - \frac{2}{3}\,\mu\Delta^{2} - \frac{2}{3}\,\mu(\bar{u} \cdot \nabla)\Delta \Big\}\,\delta\nu_{g} \qquad (2.15)$$

where the body and variable viscosity terms are neglected. Again it is noted that the pressure gradient effects across the droplet are neglected. Now consider the term

$$\int p\Delta\;\delta\nu_{g} \equiv \int p(\nabla \cdot \bar{u})\delta\nu_{g}$$

From continuity

$$(\nabla \cdot \bar{u}) = \frac{1}{\rho_{g}}\left[(\nabla \cdot \bar{\varphi}) - \frac{d\rho_{g}}{dt}\right]$$

so that

$$\int p\Delta\;\delta\nu_{g} = \int \frac{p}{\rho_{g}}\left[(\nabla \cdot \bar{\varphi}) - \frac{d\rho_{g}}{dt}\right]\delta\nu_{g} \qquad (2.16)$$

Before we combine all terms, note that from equation (2.12) the droplet internal energy term can be expanded thusly,

$$\int \rho_{\ell}\delta\nu_{\ell}\,\frac{de^{o}_{\ell}}{dt} = \int \rho_{\ell}\delta\nu_{\ell}\,\frac{d}{dt}\left(e_{\ell} + \frac{v^{2}}{2}\right) =$$

$$= \int \rho_{\ell}\delta\nu_{\ell} + \int \rho_{\ell}\delta\nu_{\ell}\,\frac{d\bar{v}}{dt} \cdot v \qquad (2.17)$$

It is noted that the first term is related to the static temperature of the droplet and the second term is equivalent to the product of the drag force and the droplet velocity. Thus we have obtained from "paper and pencil operations" the inclusion of this drag work which accrues due to its equality to the change of kinetic energy of the droplets. This term is included in the form given in equation (2.17) since it offers the convenience of relating the heat transfer to the droplet which occurs at its static temperature, and the inclusion of the semi-empirical coefficient of drag from which the drag work and thus the subsequent droplet dynamics can be determined.

Combining terms we have

$$\begin{aligned} \int \frac{d\bar{Q}}{dt}\,\delta\nu_g - \int \rho_\ell \delta\nu_\ell \frac{de_\ell}{dt} = &- \int (\nabla\cdot\bar{\varphi})(\lambda+e^o_\ell)\delta\nu_g + \\ &+ \int \rho_g \delta\nu_g \frac{de^o_g}{dt} + \int e^o_g\,(\nabla\cdot\bar{\varphi})\delta\nu_g + \int \rho_\ell\delta\nu_\ell\,\bar{v}\cdot\frac{d\bar{v}}{dt} - \\ &- \int \Big\} -(\bar{u}\cdot\nabla)p - \frac{p}{\rho_g}\left[(\nabla\cdot\bar{\varphi}) - \frac{d\rho_g}{dt}\right] + \mu\nabla^2(\bar{u}\cdot\bar{u}) + \\ &+ \mu\nabla\cdot(\bar{\xi}\times\bar{u}) - \frac{2}{3}\mu\Delta^2 - \frac{2}{3}\mu(\bar{u}\cdot\nabla)\Delta + \Big\{\,\delta\nu_g \end{aligned} \tag{2.18}$$

Now let us combine and define terms. Let $\bar{Q}$ be the heat transferred to the chamber walls and conducted to the adjacent gaseous elements; then

$$\int \frac{d\bar{Q}}{dt}\,\delta\nu_g - \int \rho_\ell\delta\nu_\ell\frac{de_\ell}{dt} \equiv \int [q + (\nabla\cdot\kappa\nabla)T]\,\delta\nu_g \tag{2.19a}$$

$$\int [e^o_g + \frac{p}{\rho_g}]\,(\nabla\cdot\bar{\varphi})\delta\nu_g \equiv \int (\nabla\cdot\bar{\varphi})h^o_g\,\delta\nu_g \tag{2.19b}$$

$$\int (\rho_\ell \delta\nu_\ell)\, \bar{v} \cdot \frac{d\bar{v}}{dt} \equiv \int (\bar{F}_{drag} \cdot \bar{v})\delta\nu_g \qquad (2.19c)$$

$$\int (e^o_\ell + \lambda)(\nabla\cdot\bar{\varphi})\delta\nu_g \equiv \int e^o_{vapor}(\nabla\cdot\bar{\varphi})\delta\nu_g \qquad (2.19d)$$

where $\varkappa$ is the thermal conduction coefficient. Thus, since $\delta\nu_g$ is arbitrary and dropping the subscript g for gas, we have

$$\begin{aligned} q + (\nabla\cdot\varkappa\nabla)T = & - (\nabla\cdot\bar{\varphi})\; e^o_{vapor} + \rho\,\frac{de^o}{dt} + \\ & + (\nabla\cdot\bar{\varphi})h^o + \bar{F}\cdot\bar{v} - \frac{p}{\rho}\,\frac{d\rho}{dt} + (\bar{u}\cdot\nabla)p - \\ & - \mu\nabla^2\,(\bar{u}\cdot\bar{u}) - \mu\nabla\cdot(\bar{\xi}\times\bar{u}) + \frac{2}{3}\,\mu\Delta^2 + \\ & + \frac{2}{3}\,\mu\,(\bar{u}\cdot\nabla)\Delta \end{aligned} \qquad (2.20)$$

An alternate form of equation (2.20) and possibly more convenient is obtained by using the gas dynamic operator for source flow,

$$\frac{d}{dt} = -\,\Theta\,(\nabla\cdot\bar{u}) + (\nabla\cdot\bar{\varphi}) \qquad (2.21)$$

where Θ is any property. Thus we obtain

$$\begin{aligned} q + (\nabla\cdot\varkappa\nabla)T = & - (\nabla\cdot\bar{\varphi})\; e^o_{vapor} + \\ & + \frac{\partial}{\partial t}\,(\rho e^o) + (\nabla\cdot u\rho h^o) + \bar{F}_{drag}\cdot\bar{v} - \\ & - \mu\nabla^2(\bar{u}\cdot\bar{u}) - \mu\nabla(\bar{\xi}\times\bar{u}) + \frac{2}{3}\,\mu\,\Delta^2 + \\ & + \frac{2}{3}\,\mu(\bar{u}\cdot\nabla)\Delta \end{aligned} \qquad (2.22)$$

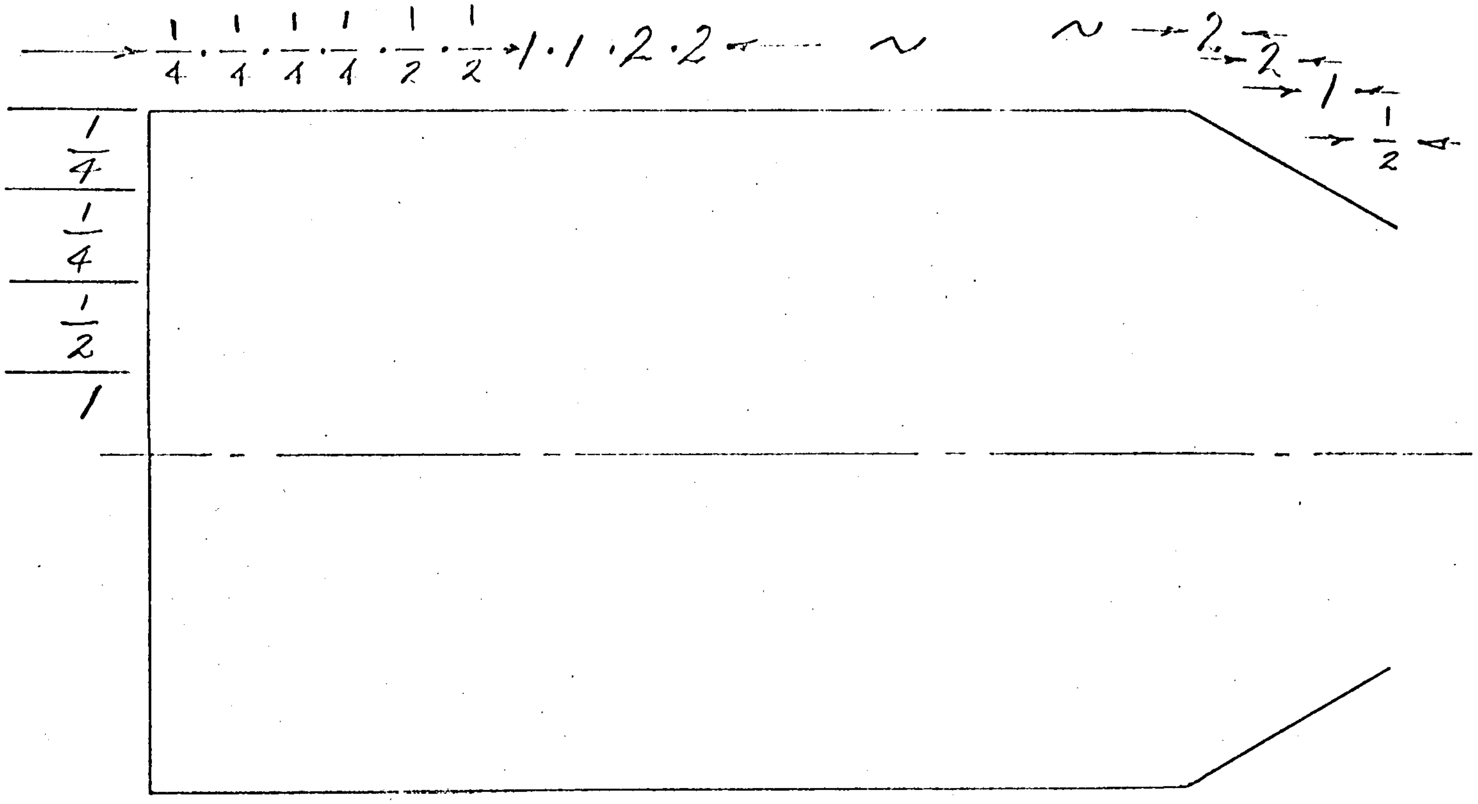
The wave dynamic equations are then equations (2.1), (2.6), and (2.22).

III. MOTOR INSTRUMENTATION

I should like to restrict myself at this time to instrumentation for nonoscillatory pressure data. This is described in detail below.

It is initially assumed that the gas dynamics occurring in a rocket motor chamber can be represented adequately by a one-dimensional approach. The theoretical results are then used to set up the undisturbed reacting fluid field in which the wave will propagate. The wave source term, $(\nabla\cdot\varphi)$, is multidimensional and is generated by the information obtained from the spray distribution of the injector, the one-dimensional droplet ballistics obtained from the results of the above-mentioned one-dimensional aerothermochemical program, and from the results of the nonsteady droplet evaporation analysis. Thus, from the one-dimensional aerothermochemical analysis, the initial coefficients of the independent variables of the wave equation are obtained. In addition, since the gases are moving out of the chamber through the nozzle, then a convective dissipation exists for the waves as they are swept out of the chamber or broadened by the longitudinal velocity gradients. These terms appear in the time dependent aerothermochemical program.

Some of the initial conditions to the droplet ballistics program of the aerothermochemical program are the spray characteristics, namely, the mean droplet size, and the standard normal deviation. Although this information is obtained principally from Priem's report TR-67, and from some of our experimental results, the difference in injector design and operating conditions preclude predicting the spray characteristics with certainty. Thus, from an experimental verification of the one-dimensional aerothermochemical analysis, a "quantitative" insight is obtained of the spray characteristics for that particular injector,



Pressure port location on rocket engine
(not to scale)

and the description of the fluid field in which the wave propagates would be more realistic.

The instrumentation for the non-oscillatory program is relatively simple. I would suggest that for the first half of the motor from the injector end, pressure ports be placed about one inch apart. Near and in the injector and up to two inches downstream, the pressure ports be closer spaced. A picture is worth a thousand words!!! A "scanivalve" arrangement can possibly be used; thus only one pressure transducer is necessary (in our case a Kistler #601). I do not know at this time whether a sufficiently fast scan motor is available. It would be preferable to obtain three consecutive non-oscillatory pressure readings prior to initiating any oscillations. From this data a longitudinal pressure gradient is obtained for the motor. The same can be calculated from the non-oscillatory theoretical program. Thus, the data (both experimental and theoretical) can be evaluated.

IV. NON-OSCILLATORY PROGRAM COMPUTATION

The next three sections deal with the computer program for the steady state aerothermodynamic analysis of the combustion chamber. We first present some details of the droplet and gas dynamic equations. In the following sections the input and output format of data is given in some detail. The program is operating giving reasonable results. It is intended to give you the present version along with the knowledge that even at the writing of this report some constants contained in the program will be changed in future versions. Final correction cards will be forthcoming when the program is to the satisfaction of the authors. (Note: A copy of the computer program will be given along with the FORTRAN deck).

In this section we briefly describe the basic equations of the droplet system and the gas dynamic system. First, it must be said that there are several options in the running of the steady state code. Normal operation is with all sense switches up. If sense switch 3 is down, extensive debugging printout will appear on tape 6. It is produced in both fuel and oxidizer evaporation routines as well as the gas dynamic routine. Its primary use was in the development of the program, and as such the meaning of the printed data will not be dwelt upon. If sense switch 4 is depressed, droplet breakup can be investigated as the Weber number criterion is used to determine if droplet breakup occurs. No time lag for breakup has been incorporated in this model, although it may be included.

The basic equations to be integrated in the fuel drop and oxidizer drop subroutines will not be described. They are basically the energy equation,

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momentum equation and continuity equation:

$$\frac{dT_\ell}{dt} = 3q_\ell/\rho_\ell c_p Ar$$

$$\frac{dv}{dt} = \frac{3}{8}\, C_D (u-v)^2\, \frac{\rho}{\rho_\ell}$$

$$\frac{dr}{dt} = -\,\frac{\dot{W}}{A} + \frac{r}{3}\,\frac{d\rho_\ell}{dt}\,\frac{1}{\rho_\ell}\;;\;\frac{d\rho_\ell}{dt} = \frac{dT_\ell}{dt}\cdot\frac{d\rho_\ell}{dT_\ell}$$

The main job of these two routines is to integrate these equations, although most of the numerical work is devoted to evaluating properties of the thermodynamic and transport properties.

Nomenclature for these equations and the units are

T_ℓ ; droplet temperature (^oR)

ρ_ℓ : droplet density $\rho_\ell = \rho_\ell(T_\ell)$ (lbm/ft^3)

c_p : droplet specific heat $\left(\frac{BTU}{lbm\,^oR}\right)$

A : droplet surface area $A = \pi r^2$ (ft^2)

r : droplet radius (ft)

v : droplet velocity (ft/sec)

u : combustion gas velocity (ft/sec)

ρ : combustion gas density (lbm/ft^3)

C_D : drag coefficient $C_D = 27/R_e^{.84}$; R_e is the Reynolds number

$$R_e = 2\cdot r\,|u-v|\,\rho/v$$

q_ℓ : heat transfer to liquid droplet (BTU/sec)

$\dot{W}$: evaporation rate (lbm/sec)

ν : viscosity (lbm/ft-sec)

The computation of the last two quantities require extensive knowledge of transport properties. For example, to compute $\dot{W}$ we have

$$\dot{W} = k_g A\alpha \quad ; \quad \alpha = p \ \mathrm{Ln}\,\frac{p}{p-p_\ell}$$

where k_g is the coefficient of mass transfer (lbm/sec-ft^2-atm),

A is the drop area (ft^2),

α is a correction factor to account for unidirectional diffusion (rather than just considering equimolar diffusion),

p, p_ℓ is the system pressure and vapor pressure (at T_ℓ)(atm).

K_g is somewhat complicated to compute but may be obtained from the empirical relation,

$$NU(k_g) = 2 + 0.6 S_c^{\frac{1}{3}} R_e^{\frac{1}{2}}$$

Here Nu is defined as $2k_g r/D_v(M/R\bar{T})$,

D_v is the diffusion coefficient (ft^2/sec),

$R = .729$ (atm ft^3/mole $^\circ$R),

$\bar{T}$: average film temperature ($^\circ$R) $= (T_\ell + T)/2$

T is the local flame temperature for the fuel drop ($^\circ$R) and for the oxidizer drop T is the equilibrium temperature ($^\circ$R) for the combustion gas,

$S_c = \mu/D_v\rho$ is the Schmidt number.

D_v is given by

$$D_v = \frac{(P_{ca}P_{cb})^{1/3}(T_{cb}T_{ca})^{5/12}}{\left(\frac{M_a M_b}{M_a + M_b}\right)^{\frac{1}{2}}} \, \frac{a}{p} \, T_R^b$$

If P the system pressure is in atm and $T_R = \overline{T}/(T_{ca}T_{cb})^{\frac{1}{2}}$ with $\overline{T}$ the average film temperature in °K, then D_v has the units of cm²/sec. It is understood that $P_{ca}, P_{cb}, T_{ca}, T_{cb}$ stand for the critical pressures (in atm) and critical temperature (in °K) of the droplet and average critical pressure of the main constituents of the combustion gas environment, respectively. This is converted to ft²/sec in the program.

The constants a and b have the following values:

Nonpolar gas pairs

$$a = 2.745 \times 10^{-4}$$

$$b = 1.823$$

H_2O and nonpolar gas

$$a = 3.640 \times 10^{-4}$$

$$b = 2.334$$

In the program it is to be observed that some early values were used. For most propellant combinations considered, however, H_2O is present in the combustion products. The continuity equation may now be integrated once dT_ℓ/dt is known. This computation will now be explained:

q_ℓ is defined by

$$q_\ell = q_T - \dot{W}\,\lambda(T_\ell)$$

for group of drops initially of equal size (p^{th} group).

Here

$$q_T = hA_\ell(T-T_\ell)Z \;, \quad Z = \frac{z}{e^z-1} \;, \quad z = \frac{\dot{W}c_{p_{vapor}}}{h\,A_\ell}$$

Note: $c_{p_{vapor}}$ includes the heats of formation so that the energy of dissociation is included.

h is the heat transfer coefficient and is obtained from the Nusselt correlation:

$$(Nu)_h = \frac{2hr}{k_m} = 2 + 0.6\, P_r^{1/3}\, Re^{\frac{1}{2}}$$

k_m is the mean film conductivity $\left(\frac{BTU}{sec\text{-}ft^2\text{-}{}^oR}\right)$

P_r is the Prandtl number $(c_p\mu/k_m) = 0.67$.

Because of the different types of fuels that are being used, some subroutines will be changed. The present program is operating and it is intended that you use to get acquainted with its many aspects if you desire. Continuing, we see that h may now be computed so that q_T may be evaluated. q is then calculated and dT_ℓ/dt is obtained from the energy balance.

The momentum equation is integrated, using the drag coefficient correlation given in the definition of terms of the basic droplet conservation equations.

The time dependent continuity equation, for the gas system, is

$$\frac{\partial \rho A}{\partial t} + \frac{\partial \rho u A}{\partial x} = \frac{\partial \varphi}{\partial x}\ ;$$

the momentum equation is

$$\frac{\partial \rho u A}{\partial t} + \frac{\partial \rho u^2 A}{\partial x} = v\,\frac{\partial \varphi}{\partial x} - f - A\,\frac{\partial p}{\partial x}$$

If time derivative terms are deleted and the above continuity equation is multiplied by u and subtracted from the momentum equation, the result is

$$\frac{\partial p}{\partial x} = -\frac{\rho}{g}\,\frac{\partial u^2/2}{\partial x} + \frac{u-v}{Ag}\,\frac{\partial \varphi}{\partial x} + \frac{f}{A}\quad ;\quad \varphi = \rho u A$$

p is the gas pressure (lbf/ft^2)

ρ is the gas density (lbm/ft^3)

u is the gas velocity (ft/sec)

v is the liquid velocity (ft/sec)

φ is the gas flow rate (lbm/sec)

f is the friction force (lbf/ft)

A is the chamber area (ft^2)

x is the distance (ft)

g is the gravitational constant (32.17 ft-lbm/lbf-sec^2).

Using forward finite difference approximations, the following difference equation is obtained:

$$p(x+\Delta x) = p(x) - \bar{\rho} g\left(\frac{u^2}{2}(x+\Delta x) - \frac{u^2}{2}(x)\right) + \frac{\overline{v-u}}{A}\cdot g\cdot(\varphi(x+\Delta x)-\varphi(x)) - \frac{f}{A}\Delta x$$

The frictional force is the total force which acts in accelerating the drops and is given by

$$f = \overline{\sum_p} N_p f_p = \overline{\sum_p} \left(\frac{m_\ell}{g}\frac{dv}{dt}\right)_p N_p$$

$(m_\ell)_p$ is the drop mass (lbm) of the pth group

$(\frac{dv}{dt})_p$ is the acceleration of that drop (ft/sec^2)

N_p is the number of drops in the pth group flowing per sec.

Here $$N_p = \varphi_o \cdot \mathcal{D}_p/(4/3\pi r_o{}^3 \rho_o)$$

r_o is the initial drop radius (ft)

ρ_o is the initial drop density (lbm/ft^3)

$_p$ is the distribution weight of the pth class of drops

φ_o is the total mass of injected propellant per sec.

This computation is carried out in the droplet subroutines, the result, f, being available for computation in the gas dynamic subroutine. The average value of f is computed in the droplet subroutines for the basic interval Δx(lbf/ft).

The time dependent energy equation is written as

$$q = \frac{\partial}{\partial t}[\rho A(e + \tfrac{1}{2}u^2)] + \frac{\partial}{\partial x}[\rho A u(e + p/\rho + \frac{u^2}{2})]$$

$$- e_v^o \frac{\partial \varphi}{\partial x} + fv$$

We delete the time dependent terms and use the steady state continuity expression $\varphi = \rho u A$ to obtain

$$(q + e_v^o \frac{d\varphi}{dx}) = \frac{d}{dx}[\varphi(h + \frac{u^2}{2})] + fv$$

letting $h = c_p T$ and writing the result in terms of forward finite differences,

$$[\varphi(c_p T + \frac{u^2}{2})]_{x+\Delta x} - [\varphi(c_p T + \frac{u^2}{2})]_x = [(q + e_v^o \frac{d\varphi}{dx}) - fv]\,\Delta x$$

or

$$T(x+\Delta x) = \frac{1}{c_p(x+\Delta x)}\left\{\frac{\varphi(x)}{\varphi(x+\Delta x)}(c_p T + \frac{u^2}{2})_x - \frac{u^2}{2}(x+\Delta x) + \right.$$

$$\left. + \frac{1}{\varphi(x+\Delta x)}[q + e_v^o \frac{d\varphi}{dx} - fv]\Delta x\right\}$$

Here, q is the negative of the time rate of change of internal energy of the drops $= - m_{\ell}\frac{d}{dt}(c_{v_\ell}T_\ell)(\frac{BTU}{sec}) = - q_\ell$;

e_v^o is the stagnation internal energy of the liquid droplets plus the latent heat of vaporization, $(e_\ell^o + \lambda)$;

λ is the latent heat;

φ is the total mass of gas flowing at a station in the chamber (lbm/sec);

T is the gas temperature (^oR);

u is the gas velocity (ft/sec);

f is the frictional force (lbf/ft) per unit distance;

v is the droplet velocity (ft/sec);

c_p is the gas specific heat $(\frac{BTU}{lbm-^oR})$; as thermodynamic data becomes better defined, then the appropriate specific heats are used. Where this is not possible, then the form of the energy equation will be changed so that the form of the thermodynamic data can be utilized.

The energy and momentum equations are solved for an approximate value of spatially advanced temperature and pressure. These variables are given the temporary names ω and π, and are weighted with first approximations to the advanced pressure and temperature

$$T^{(2)}(x+\Delta x) = \omega^{(1)}\cdot\theta + (1-\theta)\,T^{(1)}(x+\Delta x)$$

$$p^{(2)}(x+\Delta x) = \pi^{(1)}\cdot\theta + (1-\theta)\,p^{(1)}(x+\Delta x)$$

The superscripts refer to the approximation (2) second, (1) first, etc. θ is a weight factor or sometimes called a convergence parameter.

If
$$|\omega^{(n)} - T^{(n)}| < \epsilon \qquad (\epsilon \sim 1^oR) ,$$

we say we have convergence of the n^{th} iterate of the set of gas dynamic equations.

The gas velocity is computed from

$$u(x+\Delta x) = (\varphi(x+\Delta x)/A(x+\Delta x))\ R\omega^{(n-1)}/\pi^{(n-1)}$$

R is the gas constant (ft-lbf/lbm-°R).

The gas dynamic equations that are presented here are modified versions of the equations which have been programmed for the steady state model. It has become apparent that we will need several formulations of computation so that we will have a certain amount of flexibility of computation when the nonsteady calculation is performed. It is intended that the further improvements that will be made will become available to you as they occur.