RE-213J

PRODUCTION SPECTRUM OF HIGH ENERGY ELECTRONS FROM HIGH ENERGY COSMIC RAY COLLISIONS

June 1965

Grumman

RESEARCH DEPARTMENT

GRUMMAN AIRCRAFT ENGINEERING CORPORATION

BETHPAGE NEW YORK

Grumman ~~Research Department~~ Report RE-213J

PRODUCTION SPECTRUM OF HIGH ENERGY ELECTRONS FROM HIGH ENERGY COSMIC RAY COLLISIONS*

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June 1965

*Supported in part by the National Aeronautics and Space Administration

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Summary. - The distribution of charged pions from proton-proton collisions is combined with the cosmic ray flux and the distribution of electrons from π-μ-e decay to give the production spectrum of electrons from cosmic ray collisions. The proton-proton collisions are described by the Landau hydrodynamical model. Calculations are performed for electron energies between 20 and 10^8 GeV. The electron production spectrum is seen to exhibit approximately an $E_e^{-3.3}$ behavior, where E_e is the electron energy.

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1. - Introduction.

Among the possible sources for high energy electrons in the galaxy are collisions of cosmic ray protons with the interstellar gas nuclei. The electrons appear as secondaries which result from the decay of pions produced in the cosmic ray nuclear interactions. This mechanism for electron production has been investigated by several authors [1-5]. However, most of the calculations have been concerned with cosmic ray protons of energy less than 100 GeV; in addition, several of the calculations have been based upon simplified models of cosmic ray collisions. Consequently, it was felt useful to calculate the production spectrum of high energy secondary electrons using a more sophisticated model of proton-proton collisions. Because the Landau-Milekhin hydrodynamic model provides a good description of pion production at high energies [6], it has been adopted in the calculation.

Section 2 contains the general formulation of the problem. The pion production and the π-μ-e decay scheme are discussed in Sections 3 and 4, respectively. The derivation of the electron production spectrum is given in Section 5, and the results of the calculations are presented and discussed in Section 6.

2. - General formulation.

When cosmic ray protons collide with the interstellar hydrogen gas, and the collision energy is sufficiently above the threshold for production of additional particles, charged pi mesons are produced in abundance. The charged pions rapidly decay, chiefly into mu mesons and neutrinos. The muons likewise decay, into electrons and neutrinos. We calculate below the distribution of electrons produced via these processes.

In the collision of a cosmic ray proton, having energy E_p, with the interstellar hydrogen, let $f_{p\pi}(E_p, E_\pi)\,dE_\pi$ represent the distribution of pions produced that fall in the energy range E_π to $E_\pi + dE_\pi$. Let $f_{\pi e}(E_\pi, E_e)\,dE_e$ represent the distribution of electrons, with energy in the range E_e to $E_e + dE_e$, produced from the decay of a pion with energy E_π. Then the distribution of electrons produced per pion from the collision of a proton of energy E_p is given by

$$(1) \qquad f_{pe}(E_p, E_e)\,dE_e = \left[\int dE_\pi f_{p\pi}(E_p, E_\pi)\, f_{\pi e}(E_\pi, E_e) \right] dE_e \; .$$

The integration extends over all pion energies (for a given cosmic ray energy E_p) that could contribute an electron with energy E_e. The number of pions produced per unit volume and time by the collisions of cosmic rays in the energy range E_p to $E_p + dE_p$ with the interstellar gas will be given by

$$N_\pi(E_p)\,dE_p = j(E_p)\,n_H\sigma_p(E_p)\,\nu_\pi(E_p)\,dE_p\ , \tag{2}$$

where j is the differential cosmic ray proton flux; n_H is the interstellar gas density (protons/cm^3); σ_p is the inelastic cross section for proton-proton collisions, and ν_π is the number of pions produced per collision.

The number of electrons in the range E_e to $E_e + dE_e$ produced by cosmic ray protons in the energy range E_p to $E_p + dE_p$ is given by $N_\pi dE_p f_{pe} dE_e$. Integration over the incident cosmic ray energies gives the total number of electrons per unit volume and time in the energy range E_e to $E_e + dE_e$ resulting from cosmic ray proton-proton collisions in space:

$$\begin{aligned} q_e(E_e)\,dE_e &= \left[\int dE_p N_\pi(E_p)\,f_{pe}(E_p,\ E_e)\right]dE_e \\ &= \left[n_H\int dE_p\, j(E_p)\,\sigma_p(E_p)\,\nu_\pi(E_p)\int dE_\pi f_{p\pi}(E_p,E_\pi)\,f_{\pi e}(E_\pi,E_e)\right]dE_e\ . \end{aligned} \tag{3}$$

In Sections 3 and 4, respectively, the specific forms of the pion distribution $f_{p\pi}$ and the electron distribution $f_{\pi e}$ are discussed.

3. - Pion production from proton-proton collisions.

The Landau hydrodynamical model of high energy nucleon-nucleon collisions describes the nucleons as colliding relativistic fluids, and their subsequent decay as the expansion and decomposition of the composite nucleonic fluid into pions, nucleons, kaons, etc. This model is theoretically plausible and, in the light of present cosmic ray data, very acceptable for incident protons with greater than 50 GeV energy. This can be seen from a review of some predictions for the model[6-9]. The pion multiplicity is predicted to follow an $E_p^{\frac{1}{4}}$ law; this is in agreement with a large portion of available data[10]. Experimental data on the asymmetrical angular distribution of the produced pions can be reasonably fit by the hydrodynamical model[6,11]. Observations indicate that a large fraction of the available kinetic energy is carried off by a small fraction of the collision products. This is also found in the hydrodynamical model.

It has been shown[6] that the Landau model predicts the pion energy spectrum to be of the form

$$f_{p\pi}(E_p, E_\pi) = \frac{A_\pi}{E_\pi(2\pi L)^{\frac{1}{2}}} \exp\left(-\frac{1}{2L}\,\ell n^2 \frac{E_\pi}{2m_\pi \Gamma}\right), \tag{4}$$

where

$$L = \tfrac{1}{2}\,\ell n(E_p/m_p) + 1.6\,, \tag{5}$$

(6) $$\Gamma = \left[\frac{E_p/m_p + 1}{2}\right]^{\frac{1}{2}},$$

A_π is a constant equal to 1.1867, and m_π and m_p are the pion and proton rest energy, respectively (the speed of light is taken as unity throughout). The multiplicity of all charged pions created (+ and -) is expressed by

(7) $$\nu_\pi = K_\pi E_p^{\frac{1}{4}} .$$

From physical considerations, the range of energies possible for the pions produced in a proton-proton collision is restricted. The maximum pion energy in the Landau model depends on the incident proton energy in the manner [6]

(8) $$\varepsilon_\pi^{max} = 2m_\pi \Gamma \exp \left[(2L)^{\frac{1}{2}}\right] ,$$

and the minimum pion energy is given by

(9) $$\varepsilon_\pi^{min} = 2m_\pi \Gamma \exp \left[-(2L)^{\frac{1}{2}}\right] .$$

4. - Electron spectrum from pion decay.

The energy distribution of electrons from charged pi meson decay has been developed in detail elsewhere, and to date calculations have been made for pion energies up to 10^{11} eV [12]. Spectra are presented for several pion energies in Fig. 1 (solid curves). E_e^{max} is the maximum energy the electron can attain for a given pion energy. The normalization of $f_{\pi e}$ is such that $\int f_{\pi e} dE_e = 1$.

Because the electron distribution follows an essentially triangular form, it is convenient for high energy electrons to approximate the distribution by a linear function of E_e,

$$f_{\pi e} = a - bE_e \ ,$$

where both a and b will depend on the pion energy. The triangular approximation chosen is defined by the conditions that i) the distribution function vanishes at $E_e = E_e^{max}$, and ii) the normalization of $f_{\pi e}$ to unity is retained. The approximate electron distribution may then be written

$$f_{\pi e} = 2\left[\frac{1}{E_e^{max}} - \frac{E_e}{\left(E_e^{max}\right)^2}\right] . \tag{10}$$

The dashed lines in Fig. 1 represent $f_{\pi e}$ as given by eq. (10).

The maximum electron energy is given by [12]

(11) $$E_e^{max} = (E_\pi w + P_\pi v)/r \ ,$$

where P_π represents the pion momentum, and

(12) $$w = \tfrac{1}{2}(r + 1/r) \quad , \quad v = \tfrac{1}{2}(r - 1/r) \ ,$$

(13) $$r = m_\pi/m_\mu \ .$$

For large pion energies, $E_e^{max} \approx E_\pi$. The electron distribution function therefore becomes

(14) $$f_{\pi e} = 2\left[\frac{1}{E_\pi} - \frac{E_e}{E_\pi^2}\right] \ .$$

5. - Electron production spectrum.

In the construction of the electron production spectrum the usual power law form is assumed for the differential cosmic ray proton intensity, $j(E_p) = K_p E_p^{-\eta} (\mathrm{cm}^{-2}\ \mathrm{sr}^{-1}\ \mathrm{GeV}^{-1}\ \mathrm{sec}^{-1})$. At high energies the cross section for pion production σ_p is essentially energy independent. Substitution of these expressions into eq. (3), together with the pion spectrum of eq. (4), the electron spectrum of eq. (14), and the pion multiplicity of eq. (7), yields for the distribution of electrons from cosmic ray collisions

$$q_e(E_e)\,dE_e = \left[2 n_H \sigma_p A_\pi K_p K_\pi \int_{E_{p,1}}^{E_{p,2}} dE_p E_p^{-\eta+\frac{1}{4}} \times \int_{E_{\pi,1}}^{E_{\pi,2}} dE_\pi \frac{1}{E_\pi (2\pi L)^{\frac{1}{2}}} \exp\left(-\frac{1}{2L} \ell n^2 \frac{E_\pi}{2m_\pi \Gamma}\right)\left(\frac{1}{E_\pi} - \frac{E_e}{E_\pi{}^2}\right) \right] dE_e \,. \tag{15}$$

To complete the integrations of eq. (15), the regions of permitted E_π and E_p values must be determined. Energy momentum conservation restricts the pions that produce electrons of energy E_e to an energy range $E_\pi^L(E_e) \leq E_\pi \leq E_\pi^H(E_e)$. Similarly, in the proton-proton collisions, there are limitations on the energies of the protons $\left(E_p^A \leq E_p \leq E_p^D\right)$ that can give rise to pions within the above pion energy range.

It is possible to deduce expressions for the limiting energies of pions capable of yielding an electron of energy E_e from the behavior of E_e^{max} and E_e^{min} with pion energy, where E_e^{max} and E_e^{min} are the maximum and minimum energy that an electron can acquire from a pion of energy E_π. The maximum energy an electron can acquire from a pion of energy E_π was given in eq. (11). The minimum energy is given by [12]

$$E_e^{min} = \begin{cases} (E_\pi w - P_\pi v)/r & , \quad E_\pi > wm_\pi \\ m_e & , \quad E_\pi < wm_\pi \end{cases} \tag{16}$$

Because E_e^{max} increases monotonically with the energy E_π of the parent pion, the lowest energy pion contributing to an electron of energy E_e is the pion whose E_e^{max} value equals E_e. The lowest contributing pion energy is found, by inverting eq. (11), to be

$$E_\pi^{L} = \begin{cases} (E_e w - P_e v) r & , \quad E_e > wm_e \\ m_\pi & , \quad E_e < wm_e \end{cases} \tag{17}$$

In a similar manner, the highest energy pion that can decay into an electron of energy E_e is found from the expression for E_e^{min} to be

$$E_\pi^{H} = (E_e w + P_e v) r \; . \tag{18}$$

For the high energy electrons under consideration, these expressions reduce to

$$(19) \qquad E_\pi^{H} = r^2 E_e \quad , \quad E_\pi^{L} = E_e \qquad (E_e \gg wm_e) \ .$$

The continuous band of pion energies resulting from a proton-proton collision $(\varepsilon_\pi^{min} \leq E_\pi \leq \varepsilon_\pi^{max})$ was given as a function of the incident proton energy in eqs. (8) and (9), and is pictured in Figs. 2 and 3. However, eq. (19) shows that only pions of certain energies $(E_\pi^{L} \leq E_\pi \leq E_\pi^{H})$ could produce an electron of a given energy E_e. The cut off energies, $E_\pi^{L}(E_e)$ and $E_\pi^{H}(E_e)$, for an electron of energy E_e are superimposed in Figs. 2 and 3. This limitation on pion energy imposes, in turn, a corresponding restriction on the energies permitted the protons that produce the pions.

The pion energy limits required to complete the first integration of eq. (15) suggest a division of the area of $E_\pi - E_p$ integration into three regions (see Figs. 2 and 3). The proton energy boundaries of these regions are defined by

$$(20) \qquad \varepsilon_\pi^{max}(E_p^{A}) = E_\pi^{L} \quad , \quad \varepsilon_\pi^{min}(E_p^{D}) = E_\pi^{H} \ ,$$

$$(21) \qquad \varepsilon_\pi^{min}(E_p^{B}) = E_\pi^{L} \quad , \quad \varepsilon_\pi^{max}(E_p^{C}) = E_\pi^{H} \ .$$

It is found that for the high energy protons which we are considering $(E_p/m_p \gg 1)$

(22) $$E_p^A/m_p = \exp\left\{\left[\Delta_e^{\frac{1}{2}} - 1\right]^2 - 3.2\right\} ,$$

(23) $$E_p^B/m_p = \exp\left\{\left[\Delta_e^{\frac{1}{2}} + 1\right]^2 - 3.2\right\} ,$$

(24) $$E_p^C/m_p = \exp\left\{\left[(\Delta_e + 4 \ell n\ r)^{\frac{1}{2}} - 1\right]^2 - 3.2\right\} ,$$

(25) $$E_p^D/m_p = \exp\left\{\left[(\Delta_e + 4 \ell n\ r)^{\frac{1}{2}} + 1\right]^2 - 3.2\right\} ,$$

with

(26) $$\Delta_e = 2\ \ell n(E_e\ e^{2.1}/m_\pi \sqrt{2}) .$$

Figures 2 and 3 depict the two possible topologies of integration, the former corresponding to $E_p^B < E_p^C$ and the latter to $E_p^B > E_p^C$. The topologies coincide at an electron energy of

(27) $$E_e^{BC} = m_\pi \sqrt{2} \exp\left[\tfrac{1}{2}\ \ell n^2(r/e) - 2.1\right] ,$$

which is approximately 1000 GeV.

The evaluation of eq. (15) is described in Appendix A. It is found there that regions II and III make no significant contribution to q_e, and may be neglected. The pion and proton energy limits in eq. (15) are then

$$E_{\pi,1} = E_{\pi}^{L} = E_e \quad , \quad E_{\pi,2} = \varepsilon_{\pi}^{max}(E_p) \; , \tag{28}$$

$$E_{p,1} = E_p^{A}$$

$$E_{p,2} = \begin{cases} E_p^{B} \; , & E_e < E_e^{BC} \\ E_p^{C} \; , & E_e > E_e^{BC} \end{cases} \; . \tag{29}$$

The final expressions derived in Appendix A for the electron production spectrum q_e are rather lengthy and will not be repeated here.

6. - Results and discussion.

The astrophysical quantities and constants used in the expression for q_e are taken from the suggested values of GINZBURG and SYROVATSKII [(4)], specifically: $\eta = 2.6$, $K_\pi = 2.2$ (when E_p is given in GeV), and $K_p = 1.5$ (cm^{-2} sr^{-1} GeV^{-1} sec^{-1}). The expression for the electron distribution, q_e, was evaluated on an electronic computer (IBM 7094) for an electron energy range of 20 to 10^8 GeV. The results are shown in Fig. 4. It is seen that the form of the production spectrum resembles a power law

$$q_e(E_e)\,dE_e = A_e n_H \sigma_p E_e^{-z(E_e)}\,dE_e\,, \tag{30}$$

where $z(E_e)$ is a slowly varying function of the electron energy; to lowest order, $z \sim 3.3$ and $A_e \sim 0.28$ (cm^{-2} sr^{-1} GeV^{-1} sec^{-1}).

The spectrum of GINZBURG and SYROVATSKII [(4)], which follows a similar power law but with an exponent of 2.8, is plotted for comparison in Fig. 4. Both distributions are normalized to unit $n_H\sigma_p$. The present results are seen to diverge from the Ginzburg-Syrovatskii curve at high electron energies. In a later GINZBURG-SYROVATSKII paper [(5)], however, an asymptotic $E_e^{-2.64 \pm 0.5}$ dependence is found for electrons at energies greater than about 1 GeV, although there the calculations are limited to energies less than 5 GeV.

Although our interest in this paper is the effect on the electron production of using a more detailed model of proton-proton collisions, an estimate can also be made of the equilibrium density of secondary electrons, which allows for the various loss mechanisms for the electrons, e.g., magnetic bremsstrahlung and inverse Compton scattering. Several authors [1-4,13] have proposed models for the equilibrium electron density calculation. For high energy electrons in space, estimates of the energy losses by the various possible mechanisms indicate that magnetic bremsstrahlung is the predominant dissipative process [13,14]. If we incorporate our source function into a model which considers magnetic bremsstrahlung alone, we find that the equilibrium electron density $N_e(E_e)$ in a galactic magnetic field B will be [3,13]

$$N_e(E_e) = a_e \frac{E_e^{-(z+1)}}{z-1} \quad \left(\frac{\text{electrons}}{\text{cm}^3 \text{ sr GeV}}\right) \tag{31}$$

where

$$a_e = 2.64 \times 10^5 \frac{A_e n_H \sigma_p}{B_\perp^2 (\text{gauss})} . \tag{32}$$

If z is chosen as 3.3, the resultant high energy electron density is within the range of current astrophysical predictions.

* * *

We wish to thank Mr. Norman Greenspan for suggesting the solution of the integrals in Appendix B. It is also a pleasure to acknowledge fruitful discussions with Dr. S. N. Milford.

REFERENCES

(1) S. HAYAKAWA and H. OKUDA: Progr. Theor. Phys., 28, 517 (1962).

(2) F. C. JONES: J. Geophys. Res., 68, 4399 (1963).

(3) J. B. POLLACK and G. G. FAZIO: Nuclear Interactions of Cosmic Rays as the Source of the Synchrotron Radiating Particles of Our Galaxy, preprint (1964).

(4) V. L. GINZBURG and S. I. SYROVATSKII: Soviet Phys. - JETP, 18, 245 (1964).

(5) V. L. GINZBURG and S. I. SYROVATSKII: Soviet Astron. - AJ, 8, 342 (1964).

(6) M.S. SPERGEL, M. LIEBER, and S.N. MILFORD: Pion Production in High Energy Cosmic Ray Collisions, submitted to Nuovo Cimento for publication.

(7) G. A. MILEKHIN: Soviet Phys. - JETP, 8, 829 (1959).

(8) S. AMAI, H. FUKUDA, G. ISO, and M. SATO: Progr. Theor. Phys., 17, 241 (1957).

(9) S. Z. BELENKII and L. D. LANDAU: Usp. Fiz. Nauk, 56, 309 (1955).

(10) P. K. MALHOTRA: High Energy Interactions, Proceedings of the International Conference on Cosmic Rays, 5, 40 (1963).

(11) L. A. SANKO, Z. S. TAKIBAEV, and P. A. USIK: High Energy Interactions, Proceedings of the International Conference on Cosmic Rays, 5, 58 (1963).

(12) J. H. SCANLON and S. N. MILFORD: Astrophys. J., 141, 718 (1965).

(13) V. L. GINZBURG: Progress in Elementary Particle and Cosmic Ray Physics, IV, 337 (1958).

(14) R. J. GOULD and G. R. BURBIDGE: Ann. d'Astrophys., 28, 171 (1965).

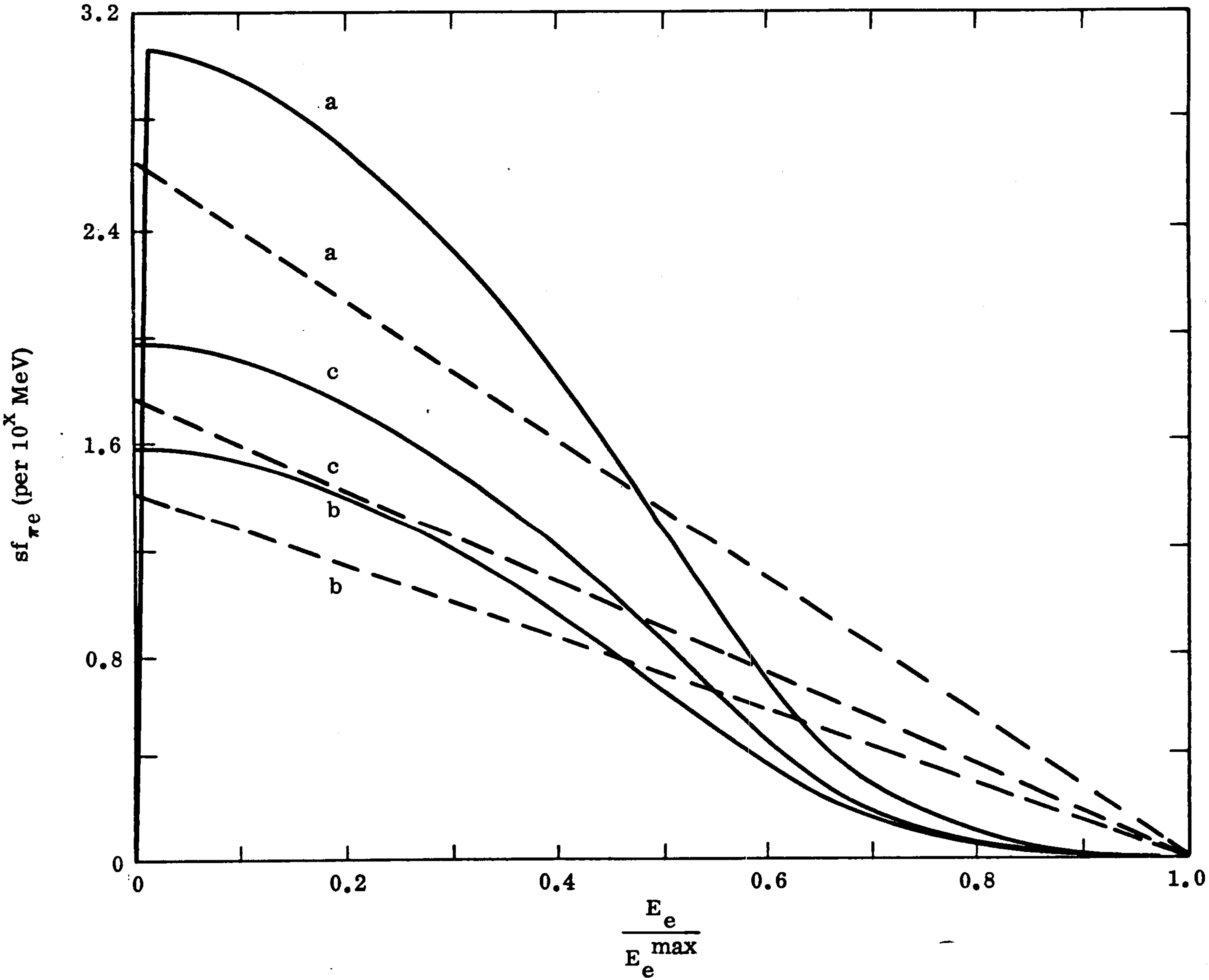

Fig. 1 Calculated (solid curves) and approximated (dashed lines) energy spectra of electrons from π-μ-e decay. (a) E_π = 740 MeV (s = 1, x = 3); (b) E_π = 5.6 GeV (s = 4), x = 3); (c) E_π = 112 GeV (s = 1, x = 5)

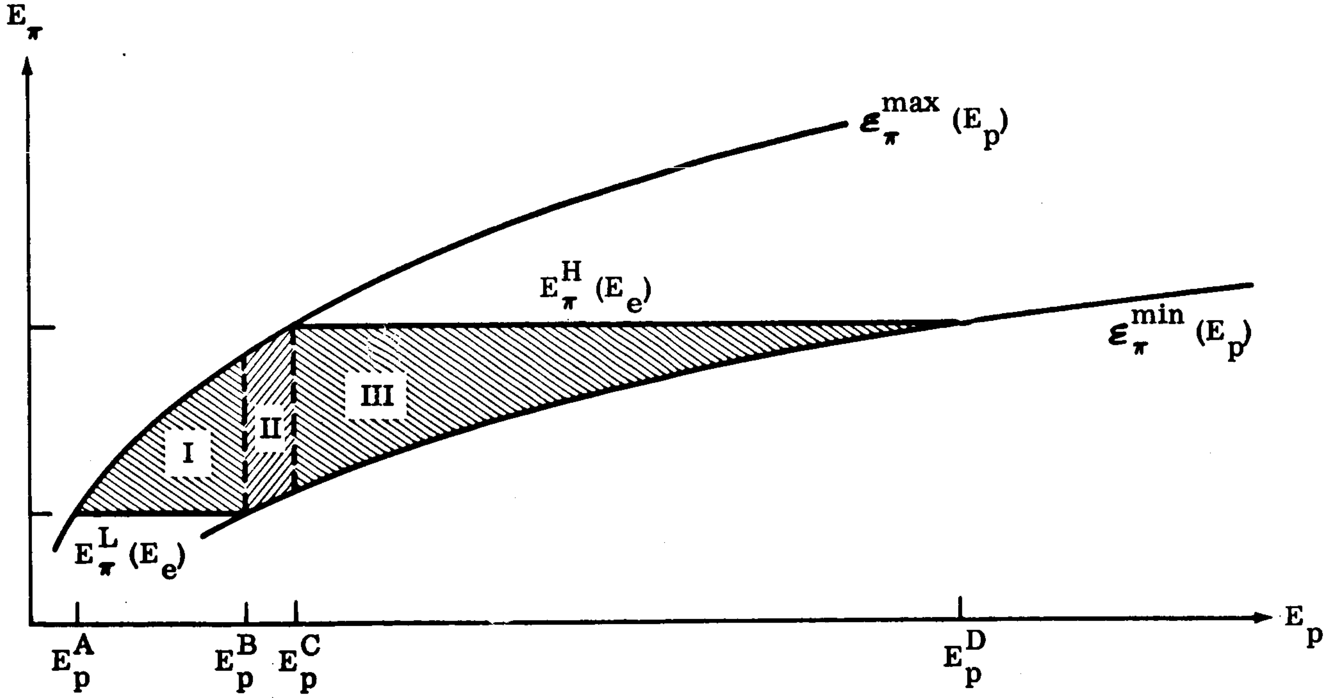

Fig. 2 Regions of $E_\pi - E_p$ integration for $E_p^B < E_p^C$

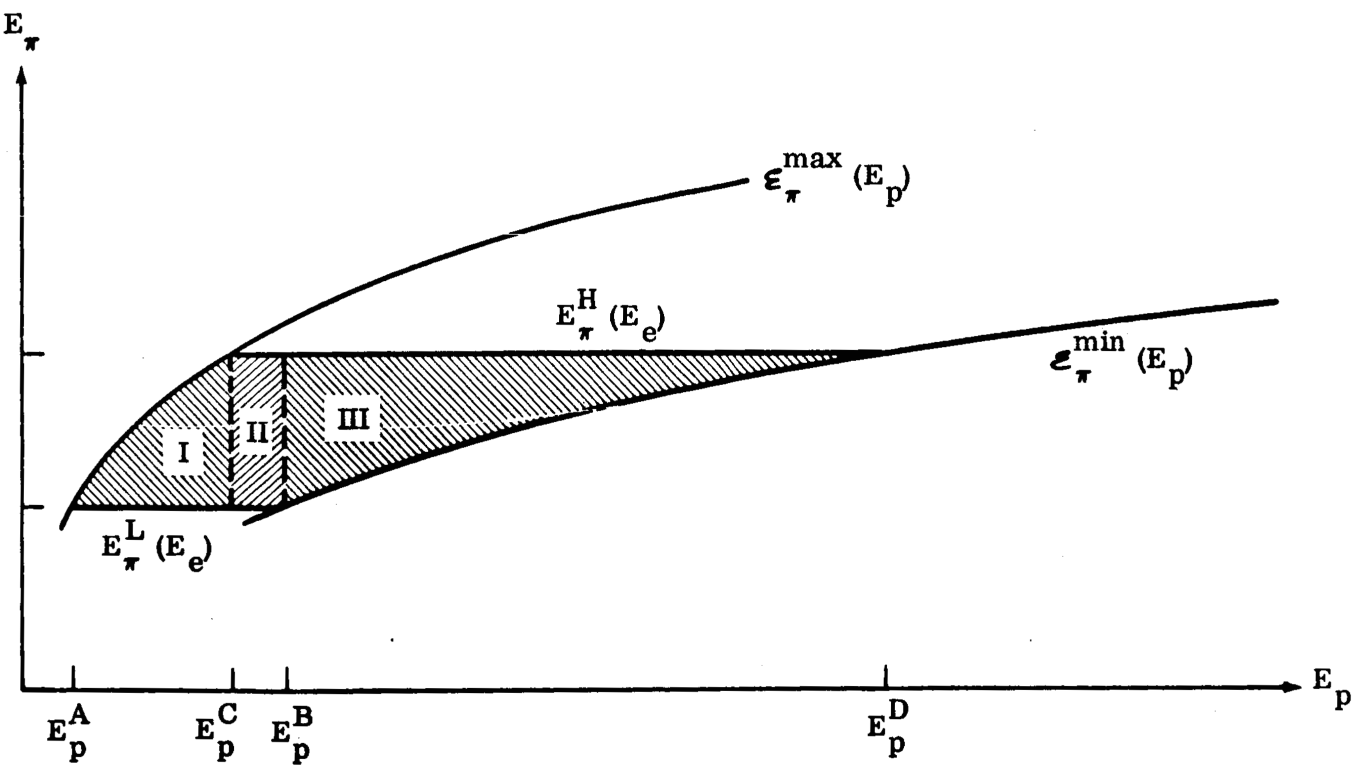

Fig. 3 Regions of $E_\pi - E_p$ integration for $E_p{}^B > E_p{}^C$

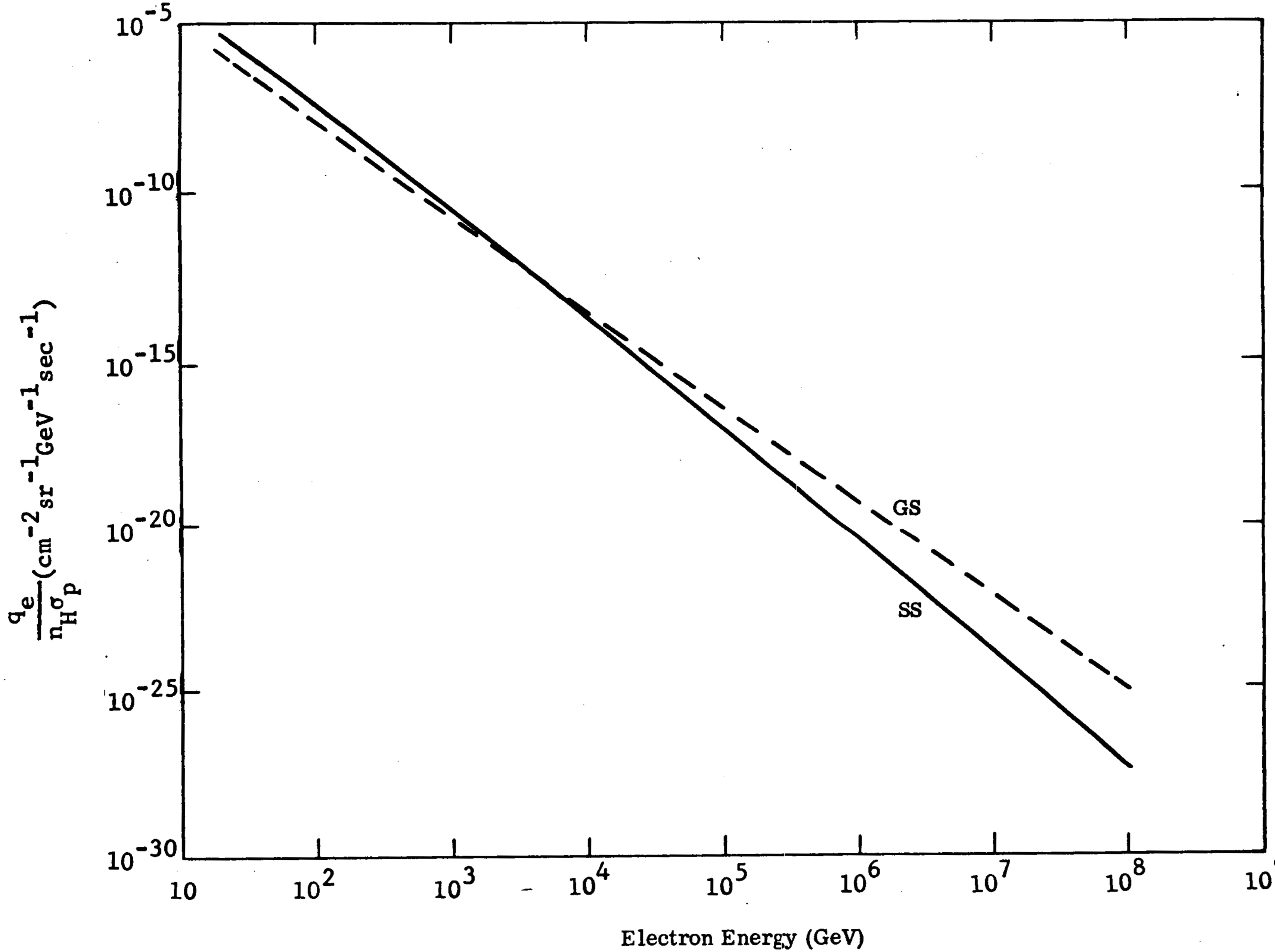

Fig. 4 Electron production spectrum. (SS) Present calculation, $q_e/n_H\sigma_p \sim 0.28\ E_e^{-3.3}$; (GS) GINZBURG and SYROVATSKII [(4)], $q_e/n_H\sigma_p \sim 0.09\ E_e^{-2.8}$.

APPENDIX A

Below we consider the explicit evaluation of the electron production spectrum represented by eq. (15). It is convenient to rewrite this expression for q_e in the form

$$\text{(A.1)} \qquad q_e(E_e)\,dE_e = 2n_H\sigma_p A_\pi K_p K_\pi \left[f_e^{(1)} - E_e f_e^{(2)} \right] dE_e \,,$$

where

$$\text{(A.2)} \quad f_e^{(\beta)} = \int_{E_{p,1}}^{E_{p,2}} dE_p \frac{E_p^{\frac{1}{4}-\eta}}{(2\pi L)^{\frac{1}{2}}} \int_{E_{\pi,1}}^{E_{\pi,2}} dE_\pi E_\pi^{-\beta-1} \exp\left(-\frac{1}{2L}\,\ell n^2 \frac{E_\pi}{2m_\pi\Gamma}\right)$$

and $\beta = 1, 2$.

The successive transformations

$$\text{(A.3)} \qquad z = \ell n\,(E_\pi/2m_\pi\Gamma)$$

and

$$\text{(A.4)} \qquad t = \frac{z}{(2L)^{\frac{1}{2}}} + \beta(\tfrac{1}{2}L)^{\frac{1}{2}}$$

convert $f_e^{(\beta)}$ to

$$\text{(A.5)} \quad f_e^{(\beta)} = \tfrac{1}{2}(2m_\pi)^{-\beta} \int_{E_{p,1}}^{E_{p,2}} dE_p E_p^{\frac{1}{4}-\eta}\Gamma^{-\beta} \exp(\tfrac{1}{2}\beta^2 L)\left[\Phi(t_2) - \Phi(t_1)\right] ,$$

where Φ represents the error function

(A.6) $$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt\, e^{-t^2} .$$

Numerical evaluation of the behavior of the integrand of $f_e^{(\beta)}$ in eq. (A.5) shows that it is a rapidly decreasing function of proton energy for the electron energies under consideration. The contributions from regions II and III to the integral $f_e^{(\beta)}$ were found to be down by many orders of magnitude from that of region I, permitting region I to be taken, with no error in precision, as the sole region of integration.

The limits on the pion energies in eq. (A.5) immediately follow from eq. (28) and are seen to be

(A.7) $$t_2 = \beta(\tfrac{1}{2}L)^{\frac{1}{2}} + 1$$

and

(A.8) $$t_1 = (\beta - 1)(\tfrac{1}{2}L)^{\frac{1}{2}} + \Lambda_e/(\tfrac{1}{2}L)^{\frac{1}{2}} ,$$

where

(A.9) $$\Lambda_e = \tfrac{1}{2} \ln \left(E_e\, e^{1.6}/m_\pi \sqrt{2}\right) .$$

In order to put eq. (A.5) into a homogeneous form, we recall the definition of L and introduce the variable

(A.10) $$u = (\tfrac{1}{2}L)^{\frac{1}{2}} .$$

With the definitions

(A.11) $$\zeta = 4\eta - 5 + 2\beta - \beta^2 \quad , \quad \alpha^2 = \zeta + \beta^2 ,$$

(A.12) $$K_e^{(\beta)} = 2m_p^{5/4-\eta}(m_\pi \sqrt{2})^{-\beta} \exp (0.8\alpha^2) ,$$

then $f_e^{(\beta)}$ in eq. (A.5) may be written

(A.13) $$f_e^{(\beta)} = K_e^{(\beta)} \left(I_1^{(\beta)} - I_2^{(\beta)} \right) ,$$

in which the I's are the integrals

(A.14) $$I_1^{(\beta)} = 2 \int_{u_1}^{u_2} du\, e^{-\zeta u^2}\, \Phi(\beta u + 1) ,$$

(A.15) $$I_2^{(\beta)} = 2 \int_{u_1}^{u_2} du\, u e^{-\zeta u^2}\, \Phi\left[(\beta - 1)u + \Lambda_e/u \right] .$$

The new integration limits are seen from eqs. (22), (24), (29), and (A.10) to be

(A.16)
$$u_1 = \tfrac{1}{2}\left[\Delta_e^{\frac{1}{2}} - 1\right]$$

$$u_2 = \begin{cases} \tfrac{1}{2}\left[\Delta_e^{\frac{1}{2}} + 1\right] & , \quad E_e < E_e^{BC} \\ \tfrac{1}{2}\left[(\Delta_e + 4 \ln r)^{\frac{1}{2}} - 1\right] & , \quad E_e > E_e^{BC} \end{cases} .$$

Remembering that

$$\frac{d\Phi(y)}{du} = \frac{2}{\sqrt{\pi}} e^{-y^2} \frac{dy}{du}$$

we can solve both of these integrals with the aid of integration by parts. The first yields directly

$$(A.17)\quad I_1^{(\beta)} = \frac{1}{\zeta}\left[e^{-\zeta u^2}\,\Phi(\beta u + 1)\right]_{u_2}^{u_1} + \frac{\beta}{\zeta\alpha}\,e^{-\zeta/\alpha^2}\left[\Phi\left(\alpha u + \frac{\beta}{\alpha}\right)\right]_{u_1}^{u_2} .$$

Integration by parts of $I_2^{(\beta)}$ gives

$$(A.18)\qquad I_2^{(\beta)} = \frac{1}{\zeta}\left[e^{-\zeta u^2}\,\Phi\left(\frac{\Lambda_e}{u} + \beta' u\right)\right]_{u_2}^{u_1} + T(u_1, u_2) ,$$

$$(A.19)\quad T(u_1, u_2) = \frac{2}{\zeta\sqrt{\pi}}\,e^{-2\Lambda_e\beta'}\int_{u_1}^{u_2} du\left(\beta' - \frac{\Lambda_e}{u^2}\right)\exp\left[-(\alpha' u)^2 - \left(\frac{\Lambda_e}{u}\right)^2\right] ,$$

in which

$$(A.20)\qquad \beta' = \beta - 1 \quad , \quad \alpha'^2 = \zeta + \beta'^2 .$$

Integrals of the form appearing in eq. (A.19) are evaluated in Appendix B. From the results given there $T(u_1, u_2)$ may be written as

$$(A.21) \qquad T(u_1, u_2) = \frac{\beta'}{2\zeta\alpha'} e^{-2\Lambda_e\beta'} \left\{ e^{2\Lambda_e\alpha'} \left[\Phi\left(\frac{\Lambda_e}{u} + \alpha' u\right) \right]_{u_1}^{u_2} \right.$$

$$\left. - e^{-2\Lambda_e\alpha'} \left[\Phi\left(\frac{\Lambda_e}{u} - \alpha' u\right) \right]_{u_1}^{u_2} \right\}$$

$$+ \frac{1}{2\zeta} e^{-2\Lambda_e\beta'} \left\{ e^{2\Lambda_e\alpha'} \left[\Phi\left(\frac{\Lambda_e}{u} + \alpha' u\right) \right]_{u_1}^{u_2} \right.$$

$$\left. + e^{-2\Lambda_e\alpha'} \left[\Phi\left(\frac{\Lambda_e}{u} - \alpha' u\right) \right]_{u_1}^{u_2} \right\} .$$

Combining eqs. (A.17), (A.18), and (A.21) with eq. (A.13) provides a complete expression for the electron production spectrum of eq. (A.1).

APPENDIX B

An integral of the form

$$\text{(B.1)} \qquad I = \int_a^b \frac{dx}{x^2} \exp\left[-(cx)^2 - \left(\frac{k}{x}\right)^2\right]$$

appears in the evaluation of the electron production spectrum of eq. (A.19). The substitution $x = 1/y$ recasts I as

$$\text{(B.2)} \qquad I = -\int_{1/a}^{1/b} dy \exp\left[-\left(\frac{c}{y}\right)^2 - (ky)^2\right] .$$

It will be seen that an integral of the form of eq. (B.2) also appears in eq. (A.19).

With the transformation $z = c/y + ky$, I may be rewritten

$$\text{(B.3)} \quad I = \frac{1}{k}\int_{z_b^+}^{z_a^+} dz \exp(-z^2 + 2ck) - \frac{c}{k}\int_{1/a}^{1/b} \frac{dy}{y^2} \exp\left[-\left(\frac{c}{y}\right)^2 - (ky)^2\right] ,$$

where

$$\text{(B.4)} \qquad z_b^+ = cb + k/b \quad , \quad z_a^+ = ca + k/a .$$

Similarly, letting $z = c/y - ky$ gives

$$\text{(B.5)}\quad I = -\frac{1}{k}\int_{z_b^-}^{z_a^-} dz\, \exp(-z^2 - 2ck) + \frac{c}{k}\int_{1/a}^{1/b} \frac{dy}{y^2} \exp\left[-\left(\frac{c}{y}\right)^2 - (ky)^2\right],$$

where

$$\text{(B.6)}\qquad z_b^- = cb - k/b \quad , \quad z_a^- = ca - k/a .$$

Adding eqs. (B.3) and (B.5), and recalling the definition of the error function Φ, we obtain

$$\text{(B.7)}\quad I = \frac{\sqrt{\pi}}{4k}\left\{e^{2ck}\left[\Phi\left(\frac{c}{y} + ky\right)\right]_{1/b}^{1/a} - e^{-2ck}\left[\Phi\left(\frac{c}{y} - ky\right)\right]_{1/b}^{1/a}\right\} .$$