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CONSERVATIVE TOLERANCE AND CONFIDENCE LIMITS

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ABSTRACT

This paper extends the validity of exponential tolerance and confidence limits, under certain restrictions, to the class of distributions with monotone failure rate. In particular, the usual exponential lower tolerance limit is shown to be conservative for the increasing failure rate class of distributions in the range of population coverages and confidence coefficients of practical interest. Conservative confidence limits are also obtained on tail probabilities and moments.

CONSERVATIVE TOLERANCE AND CONFIDENCE LIMITS

1. Introduction

A fundamental problem in statistical reliability theory and life testing is to obtain lower tolerance limits as a function of sample data, say $\underline{X} = (X_1, X_2, \ldots, X_n)$. That is, if X denotes the time to failure of an item with distribution F, then we seek a function $L(\underline{X})$ such that

$$P\{1 - F[L(\underline{X})] \geq 1 - q\} \geq 1 - \alpha .$$

We call $1 - q$ the population coverage for the interval $[L(\underline{X}), \infty]$ and $1 - \alpha$ the confidence coefficient. Another important problem is to obtain a function $M(\underline{X})$ such that

$$P\{1 - F(T) \geq M(\underline{X})\} \geq 1 - \alpha$$

for a specified time $T \geq 0$. Related problems are those of obtaining confidence limits on moments and percentiles.

Early papers in life testing (e.g. Epstein and Sobel (1953)) derived confidence limits assuming an exponential life distribution. Goodman and Madansky (1962) examine various criteria for goodness of tolerance intervals and certain optimum properties of the usual exponential tolerance limits are demonstrated. Recently, a great deal of effort has been devoted to obtaining various confidence limits for the Weibull distribution. Dubey (1962) obtains asymptotic confidence limits on $1 - F(T)$ and the failure rate for the class of Weibull distributions with non-decreasing failure rate. He also studies the properties of various estimators for Weibull parameters

(Dubey (1963)). Johns and Lieberman (1965) present a method for obtaining exact lower confidence limits for $1 - F(T)$ when F is the Weibull distribution with both scale and shape parameters unknown. Unlike Dubey, they do not require that the Weibull distribution in question have a non-decreasing failure rate. These confidence limits are obtained both for the censored and non-censored cases and are asymptotically efficient. Hanson and Koopmans (1964) obtain upper tolerance limits for the class of distributions with increasing hazard rate and lower tolerance limits for the class of distributions with PF_2 density, f (i.e. $\log f(x)$ is concave where finite). However, they do not assume non-negative random variables.

Assuming that the sample data arises from a distribution with monotone failure rate (either non-decreasing or non-increasing and $F(0^-) = 0$) we obtain conservative confidence limits for most reliability parameters of interest. These confidence limits are, in part, derived from the exponential distribution. Since in many cases these are optimum confidence limits when the failure distribution is exponential (Goodman and Madansky (1962)), they are, in this sense, best possible for the class of distributions with monotone failure rate. (See Barlow and Proschan (1965) Chapter 2 and Appendix 2 for a discussion of distributions with monotone failure rate and a test for its validity.) They also have the advantage that they are convenient to compute and are <u>not</u> based on a strong, non-verifiable, parametric assumption. Since these confidence limits are derived in part from the exponential distribution this paper, in a sense, represents a new justification for the use of exponential confidence limits in reliability theory.

2. Summary and Discussion of Results.

Let $X_1 \le X_2 \le \dots \le X_r \le \dots \le X_n$ denote an ordered sample from a life distribution F. We shall only allow the possibility of censorship on the right. Our methods will be used to obtain confidence bounds for more general types of censorship in another paper.

We say that a distribution F is IFR (DFR) if and only if $\ln [1 - F(x)]$ is concave where finite (convex on $[0,\infty]$). If F with density f is IFR (DFR) then the failure rate $\frac{f(t)}{1 - F(t)}$ is non-decreasing (non-increasing) in t. Barlow and Proschan (1964) obtain inequalities for expected values of statistics based on the exponential assumption when in fact the true distribution has a monotone hazard rate.

IFR Results

Let

$$\hat{\theta}_{r,n} = \frac{\sum_{1}^{r} X_i + (n-r) X_r}{r}$$

and

$$C_{1-\alpha,q}(r) = \begin{cases} \dfrac{-2r \ln(1-q)}{\chi^2_{1-\alpha}(2r)} & \text{if } \chi^2_{1-\alpha}(2r) > -2n \ln(1-q) \\ \dfrac{r}{n} & \text{if } \chi^2_{1-\alpha}(2r) \leq -2n \ln(1-q) \end{cases}$$

where $\chi^2_{1-\alpha}(2r)$ is the $(1-\alpha)$-th percentage point of a chi-square distribution with $2r$ degrees of freedom.

THEOREM 1. If F is IFR, $F(0^-) = 0$, $\zeta_q = \sup \{ x \mid F(x) \leq q \}$ then

(1) $P \{ 1 - F[C_{1-\alpha,q}(r) \hat{\theta}_{r,n}] \geq 1 - q \} \geq 1 - \alpha$

(2) $P \{ \zeta_q \geq C_{1-\alpha,q}(r) \hat{\theta}_{r,n} \} \geq 1 - \alpha$.

Mathematically (1) and (2) are equivalent statements. When $\chi^2_{1-\alpha}(2r) > -2n \ln(1-q)$, the lower tolerance limit provided by (1) is identical with the exponential tolerance limit. Amazingly enough, the exponential lower tolerance limits provide conservative tolerance limits for most cases of practical interest. For example, if $1 - \alpha > 1 - e^{-1} \sim .633$ and $1 - q > e^{-r/n}$, then the inequality $\chi^2_{1-\alpha}(2r) > -2n \ln(1-q)$ holds. In the sense of being "most stable" (see Goodman and Madansky)1962)) this is the best lower tolerance limit for the exponential distribution and hence a "sharp" conservative tolerance limit. If the full sample is known this is "best" for $r = n$.

Let

$$C^*_{\alpha,q}(r) = \begin{cases} \dfrac{-2r \ln(1-q)}{\chi^2_\alpha(2r)} & \text{if } \chi^2_\alpha(2r) < -2 \ln(1-q) \\ r & \text{if } \chi^2_\alpha(2r) \geq -2 \ln(1-q). \end{cases}$$

THEOREM 2. If F is IFR, $F(0^-) = 0$, $\zeta_q = \sup \{ x | F(x) \leq q \}$ then

(3) $P \{ F[C^*_{\alpha,q}(r)\ \hat{\theta}_{r,n}] \geq q \} \geq 1 - \alpha$

(4) $P \{ \zeta_q \leq C^*_{\alpha,q}(r)\ \hat{\theta}_{r,n} \} \geq 1 - \alpha$.

In this case the exponential upper tolerance limits are valid when $\chi^2_\alpha(2r) \leq -2 \ln(1-q)$. Unfortunately this inequality does not hold for all values of r $(1 \leq r \leq n)$ in the range of population coverage values, q, of greatest practical interest. A table follows which gives the largest values of r such that

$$\chi^2_\alpha(2r) \leq -2 \ln(1 - q) .$$

Table 1

Largest values of r such that the exponential upper tolerance limit is a conservative upper tolerance limit for the IFR class.

(i.e. $\chi^2_\alpha(2r) \le -2 \ln(1 - q)$)

q	$1 - \alpha = .90$	$1 - \alpha = .95$	$1 - \alpha = .99$
	r	r	r
.70	3	3	4
.75	3	3	5
.80	3	4	5
.85	4	4	6
.90	4	5	6
.95	5	6	8
.97	6	7	8
.98	6	7	9
.99	7	8	10
.999	10	11	14

The upper tolerance limit given in (3) is a significant improvement over the tolerance limit given by Hanson and Koopmans (1964) for the IFR class. However they do not restrict attention to non-negative random variables. Also they do not obtain a lower tolerance limit for the IFR class.

THEOREM 3. If F is IFR, $F(0^-) = 0$ and $T > 0$ is specified, then

(5) $$P\left\{1 - F(T) \geq \delta\left(\frac{r}{n}\hat{\theta}_{r,n} - T\right)\exp\left[-\frac{\chi^2_{1-\alpha}(2r)\,T}{2r\,\hat{\theta}_{r,n}}\right]\right\} \geq 1 - \alpha$$

where

$$\delta(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 . \end{cases}$$

Johns and Lieberman (1965) study the problem of obtaining lower confidence limits on $1 - F(T)$ for the Weibull distribution. (5) is more convenient than their result. However, if $\frac{r}{n}\hat{\theta}_{r,n} < T$ our result is trivial. If $\frac{r}{n}\hat{\theta}_{r,n} > T$, then it is identical with the exponential lower confidence limit. In reliability applications where it is desired to establish high reliability the mean, hopefully, will far exceed T and therefore it seems quite likely that $\hat{\theta}_{r,n}$ will also.

THEOREM 4. If F is IFR, $F(0^-) = 0$ and $\theta = {}_0\!\int^\infty x\,dF(x)$, then

(6) $$P\{\theta \leq k_{\alpha,r}\,\hat{\theta}_{r,n}\} \geq 1 - \alpha$$

$$(7) \qquad P\left\{\theta \geq \frac{\left[1 - \exp\left(-\frac{\chi^2_\alpha(2r)}{2n}\right)\right]}{\chi^2_\alpha(2r)}\, 2r\, \hat{\theta}_{r,n}\right\} \geq 1 - \alpha ,$$

where

$$k_{\alpha,r} = \begin{cases} r & \text{if } \chi^2_\alpha(2r) \geq 2 \\ \dfrac{2r}{\chi^2_\alpha(2r)} & \text{if } \chi^2_\alpha(2r) < 2 . \end{cases}$$

Similar confidence limits can be obtained for higher order moments.

The upper confidence limit on θ in (6) is the usual exponential confidence limit when $\chi^2_\alpha(2r) < 2$. Unfortunately this condition is not satisfied for values of r greater than 3 or 4 at the usual significance levels.

In acceptance sampling the following hypothesis testing problem is considered:

$$H_o : \theta = \theta_o$$

$$\text{versus} \qquad H_1 : \theta < \theta_o .$$

The rejection region for the exponential case is of the form:

$$\text{Reject } H_o \text{ if } \quad \hat{\theta}_{r,n} \leq \frac{\theta_o\, \chi^2_\alpha(2r)}{2r} .$$

If $\chi^2_\alpha(2r) < 2$, then by (6) this test is also a size α test for the IFR case; i.e.

$$P\left\{\hat{\theta}_{r,n} \leq \frac{\theta_o\, \chi^2_\alpha(2r)}{2r} \;\middle|\; F \text{ IFR} ;\ \theta > \theta_o\right\} \leq \alpha .$$

DFR Results

As we might expect, if a useful exponential confidence limit exists for a problem relative to IFR distributions, then no useful exponential confidence limit exists for the same problem relative to DFR distributions and conversely.

THEOREM 5. If F is DFR, $F(0^-) = 0$, $\zeta_q = \sup \{ x \mid F(x) \leq q \}$ and $\chi^2_{1-\alpha}(2r) < -2n \ln(1 - q)$, then

$$(8) \quad P \left\{ 1 - F\left[\frac{-2r \ln(1 - q)}{\chi^2_{1-\alpha}(2r)} \hat{\theta}_{r,n}\right] \geq 1 - q \right\} \geq 1 - \alpha$$

$$(9) \quad P \left\{ \zeta_q \geq \frac{-2r \ln(1 - q)}{\chi^2_{1-\alpha}(2r)} \hat{\theta}_{r,n} \right\} \geq 1 - \alpha .$$

If $\chi^2_{1-\alpha}(2r) \geq -2n \ln(1 - q)$ we can only make the trivial statement

$$P \{ \zeta_q \geq 0 \} \geq 1 - \alpha .$$

For most cases of practical interest -- high confidence and high population coverage -- (8) is not a useful result.

THEOREM 6. If F is DFR, $F(0^-) = 0$, $\zeta_q = \sup \{ x \mid F(x) \leq q \}$ and $\chi^2_{\alpha}(2r) > -2 \ln(1 - q)$, then

$$(10) \quad P \left\{ F\left[\frac{-2r \ln(1 - q)}{\chi^2_{\alpha}(2r)} \hat{\theta}_{r,n}\right] \geq q \right\} \geq 1 - \alpha$$

$$(11) \quad P\left\{ \xi_q \leq \frac{-2r \ln(1-q)}{\chi^2_\alpha(2r)} \hat{\theta}_{r,n} \right\} \geq 1 - \alpha \ .$$

The upper confidence limit is trivial when $\chi^2_\alpha(2r) \leq -2 \ln(1-q)$.

Table 2

Smallest values of r such that the exponential upper tolerance limit is a conservative upper tolerance limit for the DFR class.

(i.e. $\chi^2_\alpha(2r) > -2 \ln(1-q)$)

q	$1-\alpha = .90$	$1-\alpha = .95$	$1-\alpha = .99$
	r	r	r
.70	4	4	5
.75	4	5	6
.80	4	5	6
.85	5	5	7
.90	5	6	7
.95	6	7	9
.97	7	8	10
.98	8	9	10
.99	8	9	11
.999	11	12	15

THEOREM 7. If F is DFR, $F(0^-) = 0$ and $T > 0$ is specified, then

$$(12) \quad P\left\{1 - F(T) \geq \delta(T - r\hat{\theta}_{r,n}) \exp\left[\frac{-\chi^2_{1-\alpha}(2r)\ T}{2r\ \hat{\theta}_{r,n}}\right]\right\} \geq 1 - \alpha$$

where

$$\delta(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

as before.

THEOREM 8. If F is DFR, $F(0^-) = 0$, $\int_0^\infty x\, dF(x) = \theta$ and $\chi^2_\alpha(2r) < 2n$, then

$$(13) \quad P\left\{\theta \geq \frac{2r\ \hat{\theta}_{r,n}}{\chi^2_\alpha(2r)}\right\} \geq 1 - \alpha$$

while if $\chi^2_\alpha(2r) \geq 2n$, then

$$(14) \quad P\left\{\theta \geq \frac{r}{n}\ \hat{\theta}_{r,n} \exp\left[1 - \frac{\chi^2_\alpha(2r)}{2n}\right]\right\} \geq 1 - \alpha .$$

(13) holds for significance levels of practical interest when $r = n$.

3. Proofs of Theorems in Section 2.

Let Y denote a random variable with distribution G . If X has a continuous distribution F , note that $Y = G^{-1}F(X)$ has distribution G We will repeatedly use the following lemma.

Lemma. If $G^{-1}F(x)$ is convex non-decreasing for $x \geq 0$, $G^{-1}F(0) = 0$ and $Y_i = G^{-1}F(X_i)$, then

$$(15) \qquad F\left[\sum_{i=1}^{n} a_i X_i\right] \underset{(\geq)}{\leq} G\left[\sum_{i=1}^{n} a_i Y_i\right]$$

when $a_i \geq 0$ and $\sum_{i=1}^{n} a_i = 1$ ($a_i \geq 1$ or $a_i = 0$, $i = 1,2,\ldots,n$).

The proof is obvious.

In what follows it will be convenient to let

$$G(x) = \begin{cases} 1 - e^{-x} & x \geq 0 \\ 0 & x < 0 . \end{cases}$$

Proof of Theorem 1.

Since (1) and (2) are mathematically equivalent we need only prove (2). By the lemma we have

$$G\left[\sum_{i=1}^{r} \frac{Y_i + (n-r)Y_r}{n}\right] \geq F\left[\sum_{i=1}^{r} \frac{X_i + (n-r)X_r}{n}\right]$$

since $G^{-1}F(x)$ is convex when F is IFR. Now choose $k_{1-\alpha}$ so that

$$P\left\{ G\left[\sum_{i=1}^{r} \frac{Y_i + (n-r)Y_r}{n}\right] \leq k_{1-\alpha}\right\} = 1 - \alpha$$

i.e. $\ln(1 - k_{1-\alpha}) = \frac{-\chi^2_{1-\alpha}(2r)}{2n}$. Since F is IFR we know (Barlow and Proschan (1965), p. 27) that

$$F(t;\zeta_q) \geq b(t;\zeta_q) = \begin{cases} 0 & t < \zeta_q \\ 1 - (1-q)^{t/\zeta_q} & t \geq \zeta_q \end{cases}$$

where ζ_q is the (unknown) q-th quantile.

Hence

$$G\left[\frac{\sum_{1}^{r} Y_i + (n - r)Y_r}{n}\right] \geq F\left(\frac{r}{n}\hat{\theta}_{r,n};\ \zeta_q\right) \geq b\left(\frac{r}{n}\hat{\theta}_{r,n};\ \zeta_q\right)$$

and

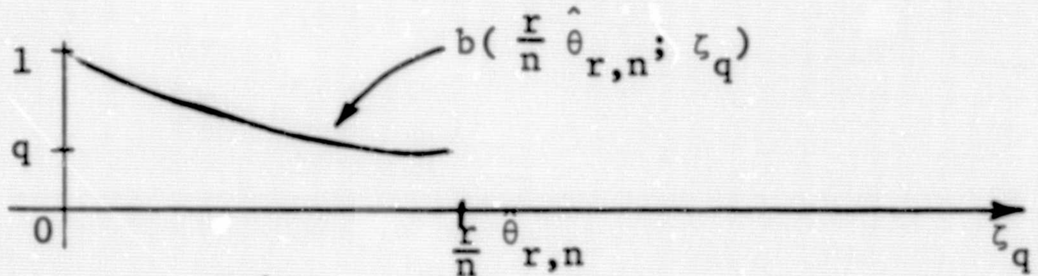
$$P\ \{\ b\ (\ \tfrac{r}{n}\hat{\theta}_{r,n};\ \zeta_q) \leq k_{1-\alpha}\ \} \geq 1 - \alpha.$$

Since $b(t;\ \zeta_q)$ is non-increasing in ζ_q we have

$$P\ \{\ \zeta_q \geq b^{-1}(\tfrac{r}{n}\hat{\theta}_{r,n};\ k_{1-\alpha})\ \} \geq 1 - \alpha$$

where the inverse is taken with respect to ζ_q .

Case 1. $k_{1-\alpha} \geq q$ (i.e. $\chi^2_{1-\alpha}(2r) \geq -2n\ \ln(1 - q)$). From the following figure



we see that $b(\frac{r}{n}\hat{\theta}_{r,n};\ \zeta_q) \leq k_{1-\alpha}$ if and only if

$$\zeta_q \geq \frac{-2r\ \ln(1 - q)\ \hat{\theta}_{r,n}}{\chi^2_{1-\alpha}(2r)} .$$

Case 2. $k_{1-\alpha} < q$ (i.e. $\chi^2_{1-\alpha}(2r) \leq -2n\ \ln(1 - q)$). In this case $b(\frac{r}{n}\hat{\theta}_{r,n};\ \zeta_q) \leq k_{1-\alpha}$ if and only if $\zeta_q > \frac{r}{n}\hat{\theta}_{r,n}$.

In either case we have

$$P\ \{\ \zeta_q \geq C_{1-\alpha,\ q}(r)\ \hat{\theta}_{r,n}\ \} \geq 1 - \alpha$$

where

$$C_{1-\alpha,q}(r) = \begin{cases} \dfrac{-2r \ln(1-q)}{\chi^2_{1-\alpha}(2r)} & \text{if } \chi^2_{1-\alpha}(2r) > -2n \ln(1-q) \\[2ex] \dfrac{r}{n} & \text{if } \chi^2_{1-\alpha}(2r) \le -2n \ln(1-q) . \end{cases}$$

Proof of Theorem 2. Again we need only prove statement (4). We use the following inequality which follows from the IFR assumption and the lemma:

$$G[\sum_1^r Y_i + (n-r)\, Y_r] \le F[\sum_1^r X_i + (n-r)\, X_r] .$$

We choose k_α so that

$$P\ \{ G\ [\sum_1^r Y_i + (n-r)\, Y_r] \ge k_\alpha \} = 1 - \alpha$$

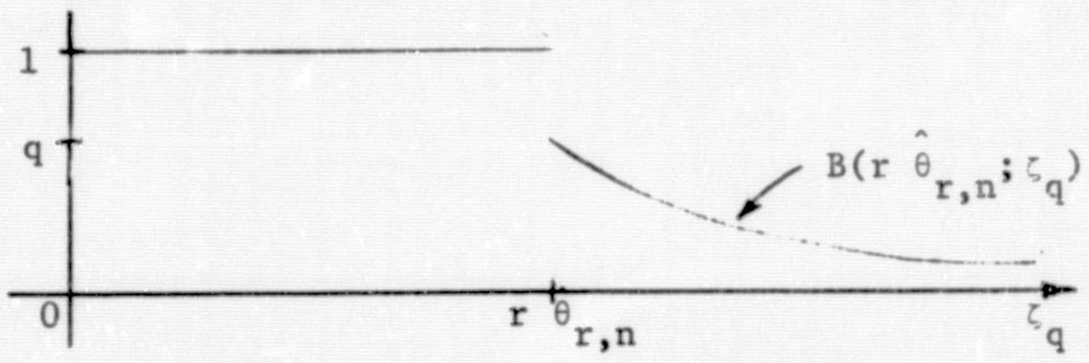
i.e. $\ln(1-k_\alpha) = \dfrac{-\chi^2_\alpha(2r)}{2}$. From Barlow and Proschan (1965) p. 27 we have the sharp bound

$$F(t;\ \zeta_q) \le B(t;\ \zeta_q) = \begin{cases} 1 - (1-q)^{t/\zeta_q} & t \le \zeta_q \\[2ex] 1 & t > \zeta_q . \end{cases}$$

Since $B(t;\ \zeta_q)$ is decreasing in ζ_q we have

$$P\left\{ B(r\ \hat{\theta}_{r,n};\ \zeta_q) \ge k_\alpha \right\} = P\left\{ \zeta_q \le B^{-1}(r\ \hat{\theta}_{r,n};\ k_\alpha) \right\} \ge 1 - \alpha .$$

Case 1. $k_\alpha > q$ (i.e. $\chi^2_\alpha(2r) \ge -2 \ln(1-q)$). From the following figure we see that $B(r\ \hat{\theta}_{r,n};\ \zeta_q) \ge k_\alpha$ if and only if $\zeta_q < r\ \hat{\theta}_{r,n}$.



Case 2. $k_\alpha \leq q$ (i.e. $\chi^2_\alpha(2r) < -2 \ln(1-q)$). In this case

$B(r\,\hat{\theta}_{r,n};\ \zeta_q) \geq k_\alpha$ if and only if $\zeta_q \leq \dfrac{-2r \ln(1-q)}{\chi^2_\alpha(2r)}$.

Cases 1 and 2 together establish statement (4). ||

Proof of Theorem 3. Again we use the inequality

$$G\left[\frac{\sum_1^r Y_i + (n-r)\,Y_r}{n}\right] \geq F\left[\frac{\sum_1^r X_i + (n-r)\,X_r}{n}\right]$$

and choose $k_{1-\alpha}$ so that

$$P\left\{ G\left[\frac{\sum_1^r Y_i + (n-r)\,Y_r}{n}\right] \leq k_{1-\alpha}\right\} = 1-\alpha$$

i.e. $\ln(1-k_{1-\alpha}) = \dfrac{\chi^2_{1-\alpha}(2r)}{2n}$. Let $p = F(T)$. We again use the

sharp bound

$$F(t;\,p) \geq b(t;\,p) = \begin{cases} 0 & t < T \\ 1-(1-p)^{t/T} & t \geq T\ . \end{cases}$$

Then

$$P\left\{ b\left(\frac{r}{n}\,\hat{\theta}_{r,n};\ p\right) \leq k_{1-\alpha}\right\} \geq 1-\alpha\ .$$

Since $b(t;\,p)$ is increasing in p , we have

$$P\left\{ 1-F(T) \geq 1-b^{-1}\left(\frac{r}{n}\,\hat{\theta}_{r,n};\ k_{1-\alpha}\right)\right\} \geq 1-\alpha$$

(5) follows when we recall that $b(\frac{r}{n}\hat{\theta}_{r,n};\, p) = 0$ when $T > \frac{r}{n}\hat{\theta}_{r,n}$. ||

Proof of Theorem 4. To show (6) use the sharp bound

$$F(t;\,\theta) \le B(t;\,\theta) = \begin{cases} 1 - e^{-t/\theta} & t \le \theta \\ 1 & t > \theta \end{cases}$$

(Barlow and Proschan (1965), p. 27)

together with

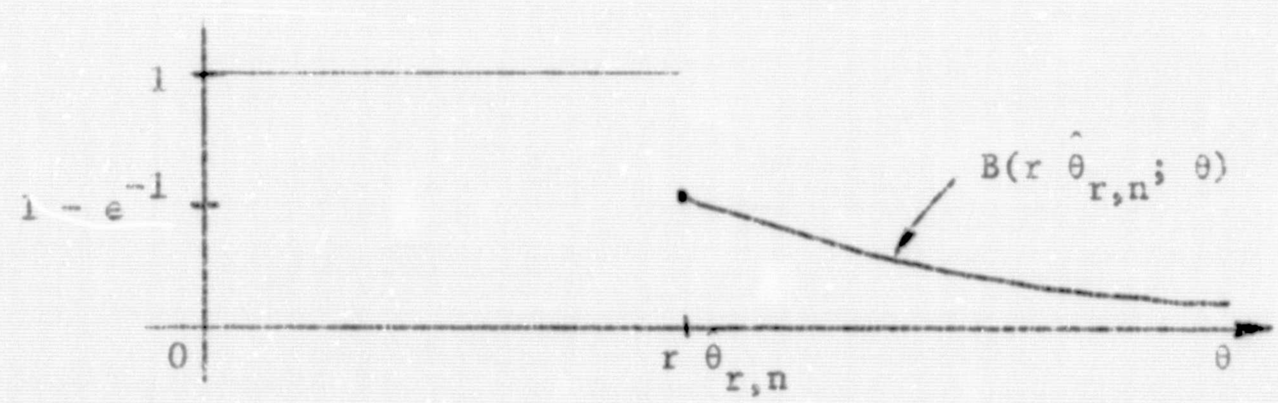
$$G\left[\sum_{1}^{r} Y_i + (n-r)\,Y_r\right] \le F(r\,\hat{\theta}_{r,n};\,\theta) \le B(r\,\hat{\theta}_{r,n};\,\theta)$$

to establish

$$P\ \{\, B(r\,\hat{\theta}_{r,n};\,\theta) \ge k_\alpha \,\} \ge 1 - \alpha$$

where $\ln(1 - k_\alpha) = \dfrac{-\chi_\alpha^2(2r)}{2}$ as before.

Case 1. $k_\alpha > 1 - e^{-1}$ (i.e. $\chi_\alpha^2(2r) > 2$). From the following figure we see that $B(r\,\hat{\theta}_{r,n};\,\theta) \ge k_\alpha$ if and only if $\theta < r\,\hat{\theta}_{r,n}$.



Case 2. $k_\alpha \leq 1 - e^{-1}$ (i.e. $\chi^2_\alpha(2r) \leq 2$). Also we see that

$B(r\,\hat{\theta}_{r,n};\,\theta) \geq k_\alpha$ if and only if $\theta \leq \dfrac{r\,\hat{\theta}_{r,n}}{\chi^2_\alpha(2r)}$. Hence the result.

To show (7). Use the sharp bound

$$F(t;\,\theta) \geq b(t;\,\theta) = \begin{cases} 0 & t < \theta \\ 1 - e^{-wt} & t \geq \theta \end{cases}$$

where w depends on t and satifies

(16) $\int_0^t e^{-wx}\,dx = \theta$,

(see Barlow and Proschan (1965) p. 28),

together with

$$G\left[\frac{\sum_1^r Y_i + (n-r)\,Y_r}{n}\right] \geq F\left(\frac{r}{n}\,\hat{\theta}_{r,n};\,\theta\right) \geq b\left(\frac{r}{n}\,\hat{\theta}_{r,n};\,\theta\right)$$

to assert

$$P\left\{ b\left(\frac{r}{n}\,\hat{\theta}_{r,n};\,\theta\right) \leq k_{1-\alpha} \right\} \geq 1 - \alpha$$

where $\ln(1 - k_{1-\alpha}) = \dfrac{-\chi^2_\alpha(2r)}{2n}$ as before. Notice that $w = w(\theta)$ is a function of θ and is decreasing in θ . Hence

$$P\left\{ 1 - \exp\left[-w(\theta)\,\frac{r}{n}\,\hat{\theta}_{r,n}\right] \leq k_{1-\alpha} \right\}$$

$$= P\left\{ w(\theta) \leq \frac{-\ln(1 - k_{1-\alpha})}{\frac{r}{n}\,\hat{\theta}_{r,n}} \right\} \geq 1 - \alpha$$

or

$$P\left\{\theta \geq w^{-1}\left[\frac{-\ln(1 - k_{1-\alpha})}{\frac{r}{n}\hat{\theta}_{r,n}}\right]\right\} \geq 1 - \alpha .$$

Since $\theta = \dfrac{1 - \exp\left[-w \frac{r}{n} \hat{\theta}_{r,n}\right]}{w}$ by (16) we have

$$w^{-1}\left[\frac{-\ln(1 - k_{1-\alpha})}{\frac{r}{n}\hat{\theta}_{r,n}}\right] = \frac{\left[1 - \exp\left(\frac{-\chi_{\alpha}^{2}(2r)}{2n}\right)\right] 2r\, \hat{\theta}_{r,n}}{\chi_{\alpha}^{2}(2r)}$$

which establishes (7).

We omit proofs of the DFR results since they are a straightforward application of the same techniques applied to bounds on DFR distributions.

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13. ABSTRACT

This paper extends the validity of exponential tolerance and confidence limits, under certain restrictions, to the class of distributions with monotone failure rate. In particular, the usual exponential lower tolerance limit is shown to be conservative for the increasing failure rate class of distributions in the range of population coverages and confidence coefficients of practical interest. Conservative confidence limits are also obtained on tail probabilities and moments.

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