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IMPROVED ANALYTIC LONGITUDINAL RESPONSE ANALYSIS FOR AXISYMMETRIC LAUNCH VEHICLES

VOLUME I - LINEAR ANALYTIC MODEL

by J. S. Archer and C. P. Rubin

Prepared under Contract No. NAS 1-4351 *by*
TRW SPACE TECHNOLOGY LABORATORIES
Redondo Beach, Calif.
for Langley Research Center

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1. INTRODUCTION

This report describes an improved linear analytical model and digital program developed for the calculation of axisymmetric launch vehicle steady-stage response to applied axisymmetric sinusoidal loads. The detailed computer programming manual for the digital program is contained in Volume II of this report.

In the evaluation of launch vehicle behavior, it is necessary to study the response of the entire vehicle to a wide variety of dynamic loadings to insure the structural integrity and stability of the system. Much effort has already gone into the development of techniques to calculate the vehicle response to lateral and longitudinal loadings using distributed and lumped spring-mass models and techniques for theoretical and empirical modeling of the vehicle behavior.[1,2] However, experimental data indicate that these procedures are unsatisfactory in several respects. For example, accurate representation of important structural shell characteristics and realistic coupling of the fluids with the detailed structural behavior of tank walls and bulkheads are omitted.

The approach described herein overcomes the above noted deficiencies. A finite element technique is utilized to construct the total launch vehicle stiffness matrix $\left[K\right]$ and mass matrix $\left[M\right]$ by subdividing the prototype structure into a set of (1) axisymmetric shell components, (2) fluid components, and (3) spring-mass components. In this way, it is possible to represent as separate shell units the fairing, interstage structure, bulkheads, tank walls and engine thrust structure, and to conveniently provide for the inertial and stiffness characteristics of equipment, engines and vehicle supporting structure.

The stiffness and mass matrices for the complete launch vehicle are obtained by superposition of the stiffness and mass matrices of the individual shell, fluid and spring-mass components which are computed using a Rayleigh-Ritz approach. Fluid motions are assumed to be consistent with the shell component distortions. The superposition technique assures displacement compatibility and force equilibrium at the joints between components. After the complete system stiffness and mass matrices have been formulated, displacement boundary conditions are introduced by removing appropriate rows and columns corresponding to points on the vehicle and its supports which are rigidly restrained from motion.

The coupled system natural frequencies and mode shapes are obtained from the eigenvalue equation constructed with the total stiffness and mass matrices

$$\left[K\right] \left\{a\right\} - p^2 \left[M\right] \left\{a\right\} = 0 \tag{1.1}$$

in which p is the circular frequency of the system and $\left\{a\right\}$ is the modal vector whose components are the longitudinal, radial and rotational displacements at discrete points on the vehicle. The steady-state response due to simple harmonic loads is determined using a standard modal response procedure which expresses the total displacement, velocity, acceleration and force responses as the linear superposition of the individual modal responses based on an assumed modal damping.

The procedure will handle shell components with a wide range of geometries. It includes shell effects in the tank and bulkhead structure, but avoids the need for including detailed local deformation, such as at shell discontinuities, which are unimportant in determining the total dynamic behavior of the vehicle. The approach has the capability of representing the tank or stage of most interest in great detail and those of least interest with minimum detail, as desired, thereby minimizing the computation time required and remaining within the maximum limitations of standard eigenvalue routines. The formulation of the problem is subdivided into well-defined portions, leading to efficient coding and easy modification for later incorporation of asymmetric shell behavior and even more detailed treatment of the fluid behavior.

The analytical procedure discussed herein is summarized in Figure 1. The launch vehicle analytical model is discussed in Section 2. The equations for the shell and fluid component stiffness and mass matrices are developed in Sections 3 and 4, respectively. The coordinate representation which forms the framework for the vehicle model, and the construction of stiffness and mass matrices for the complete launch vehicle, are discussed in Sections 5 and 6, respectively. The method for computing the dynamic response is discussed in Section 7. The computer program arrangement is described in Section 8 and detailed input data requirements for the computer program are itemized in Section 9. A complete list of symbols is provided in Section 10.

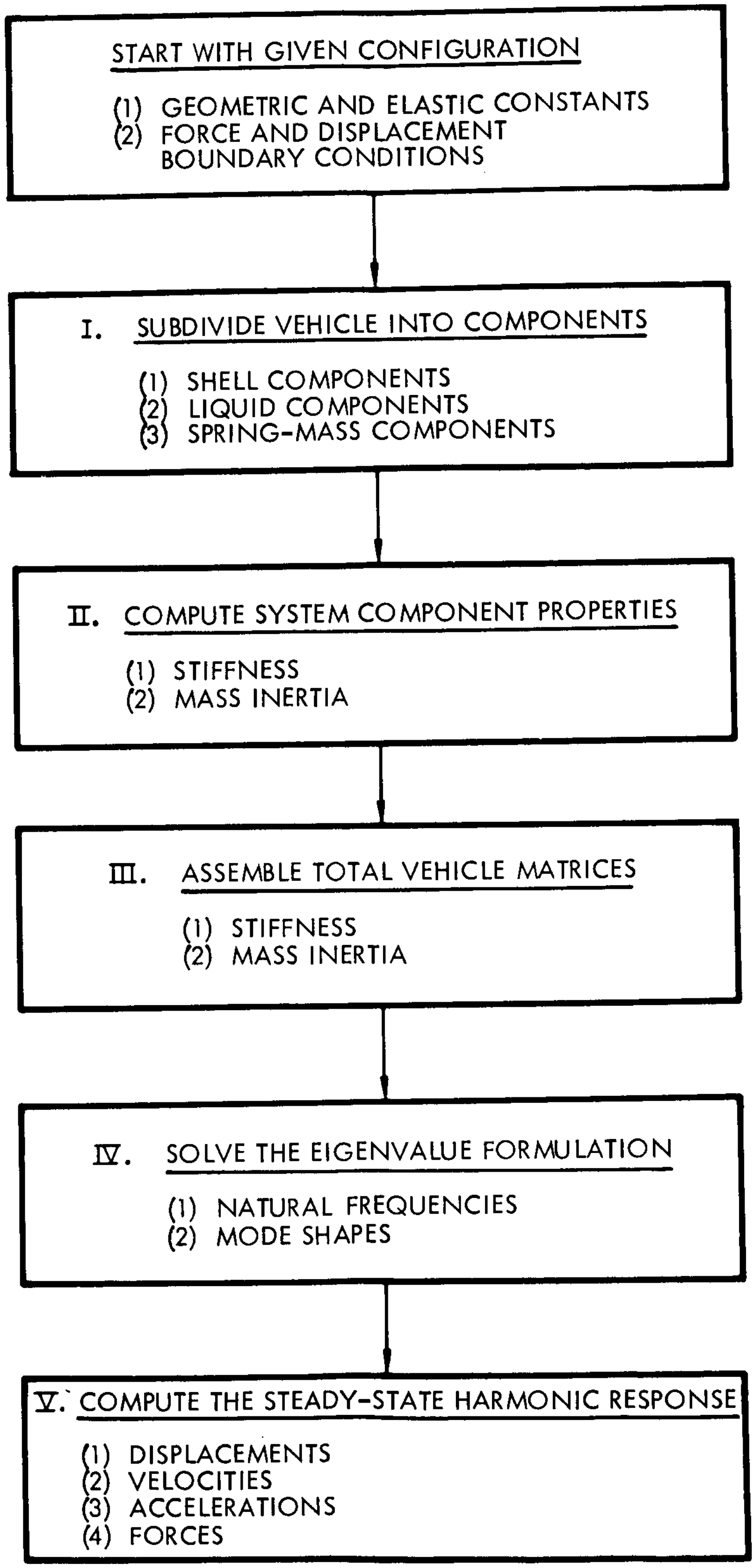


Figure 1. Summary of Technical Approach

2. ANALYTICAL MODEL

As illustrated in Figure 2, the vehicle structure is subdivided into a consistent set of shell components, a, fluid components, b, and multi-coordinate spring-mass components, c. The total vehicle which may be represented is limited to one with not more than six (6) fluid components. The total number of shell components which may be represented shall not exceed forty (40). The characteristics of the spring-mass components representing such equipment as engines and mass-elastic supports are provided directly by low order ($\leq$ 10) stiffness and mass matrices. The total number of spring-mass components may not exceed thirty (30). The vehicle behavior is described in terms of motions of discrete points on the vehicle located at intersections of shell components, at lumped masses, and at intermediate points on the shell elements. The number of nonfixed degrees-of-freedom by which the behavior of the system is described may not exceed eighty (80).

The specific shell components to be used are conical frustums (which include cylindrical shells as a special case) and ellipsoidal bulkheads (which include hemispherical shells as a special case). Within the domain of thin shell theory, the shell components may have orthotropic properties and a linear thickness variation in the meridional direction. Local thickening of a shell at a bulkhead or wall joint may be handled by using an equivalent local hoop stiffener which is provided as input in the form of an additional spring-mass component. Initial static stresses based on membrane theory are accounted for in determining the stiffness matrix for the shell components.

The most general fluid component may be in contact with an ellipsoidal upper bulkhead, a conical tank wall, and a conical or ellipsoidal lower bulkhead. The bulkhead shell elements may be convex down or up with fluid at any desired depth on either side, both sides or neither side. The tank configurations which are considered are illustrated in Figure 3.

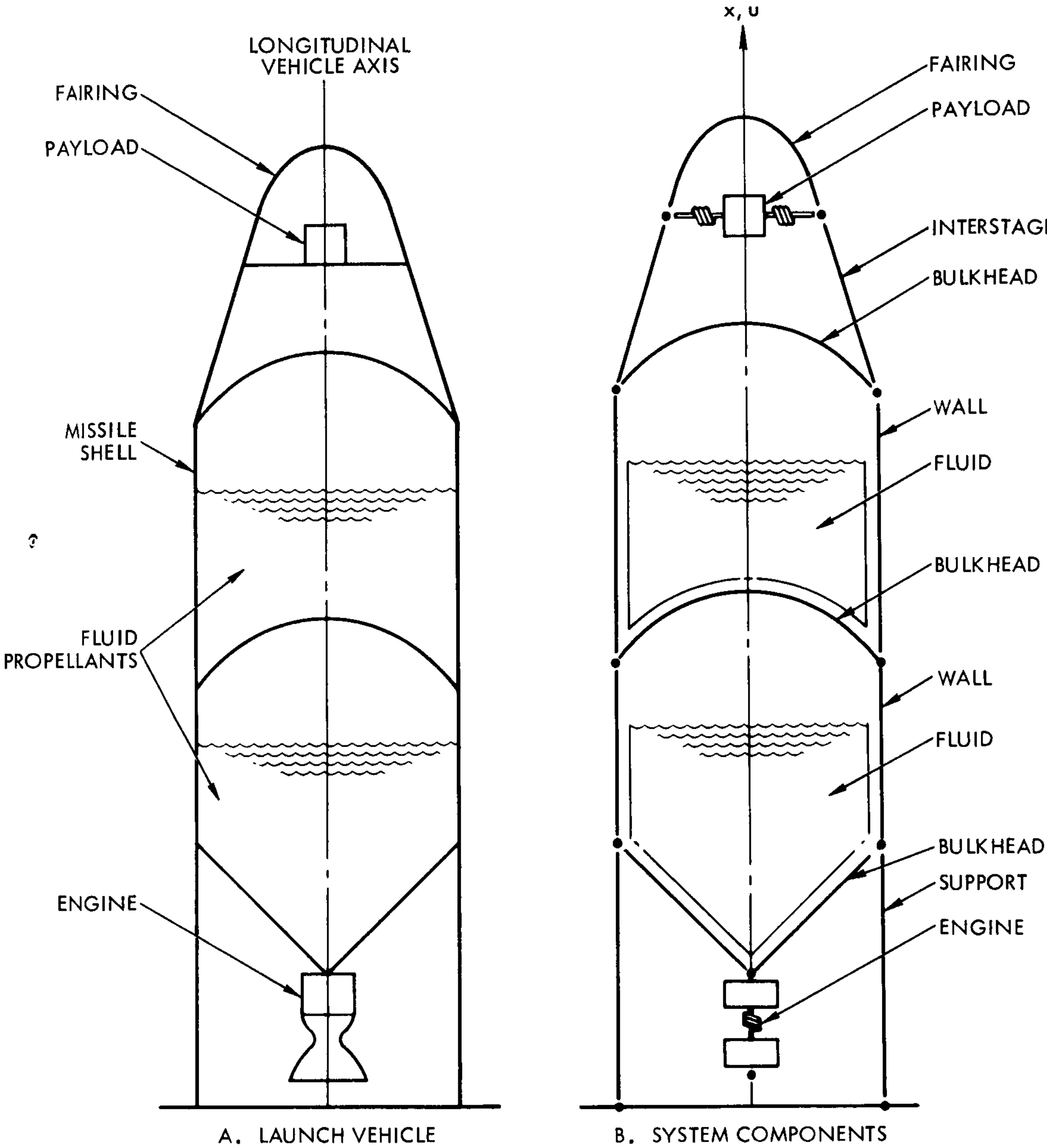


Figure 2. Vehicle Components

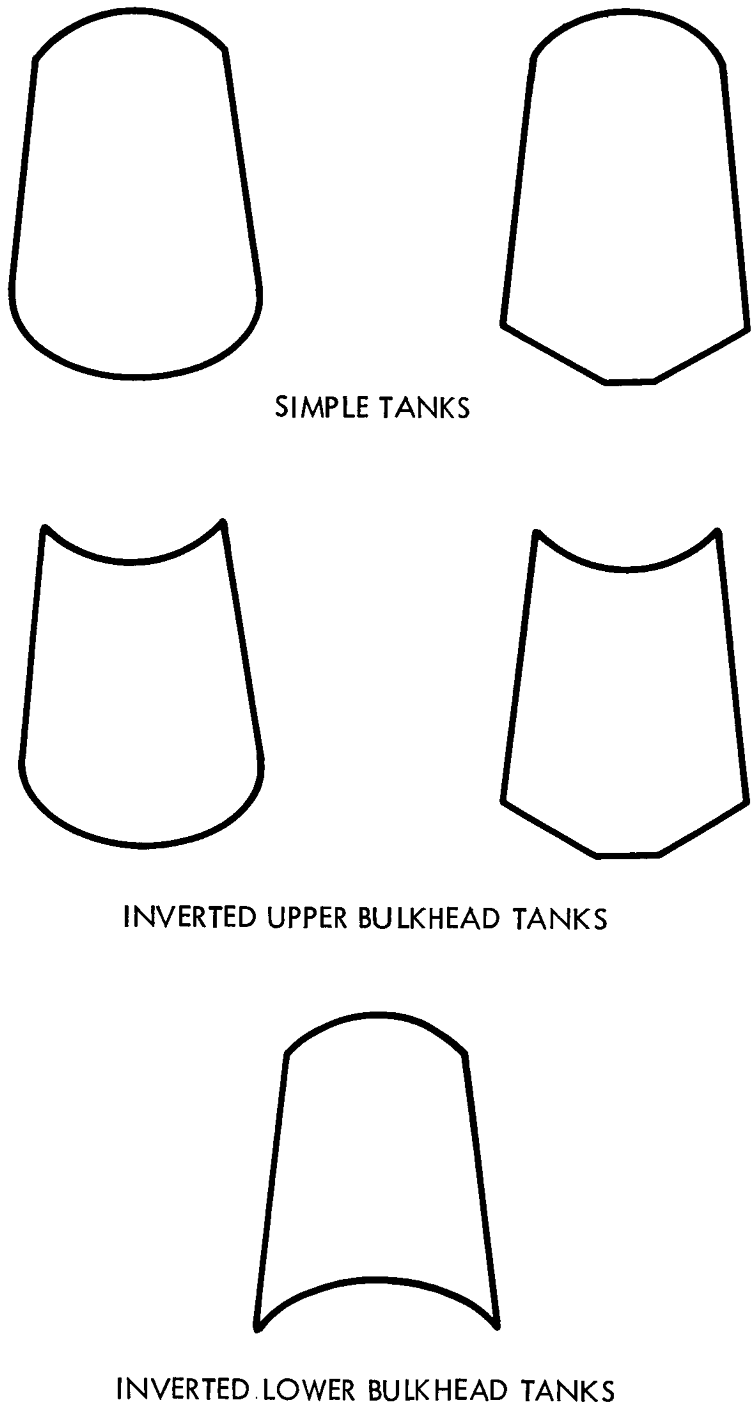


Figure 3. Tank Configurations

3. SHELL COMPONENT STIFFNESS AND MASS MATRICES

The displacements of an individual shell component (Figure 4) are approximated in the Rayleigh-Ritz manner by a finite series of functions having the form

$$
\begin{aligned}
u(\xi) &= \sum_{k=1}^{\overline{U}} \bar{a}_k u_k(\xi) \;, \qquad \overline{U} \leq 11 \\
v(\xi) &= \sum_{\ell=1}^{\overline{V}} \bar{\beta}_\ell v_\ell(\xi) \;, \qquad \overline{V} \leq 11
\end{aligned}
\tag{3.1}
$$

in which ξ is a dimensionless variable and $0 \leq \xi \leq 1$. For conical frustums, Figure 5, $\xi = s/\ell \sin \phi_o$. For convex upward ellipsoidal shells, Figure 6, $\xi = \phi/\phi_o$, and for convex downward ellipsoidal shells, $\xi = (\pi - \phi)/(\pi - \phi_o)$.

The assumed mode shapes $u_k(\xi)$ and $v_k(\xi)$ consist of polynomial terms sufficient to represent all modes of shell distortion, including longitudinal stretching, radial dilatation and rigid body displacements. The specific shape of the assumed modes is determined within the limitations of a tenth order polynomial, as follows:

$$
\begin{aligned}
u_k(\xi) &= \sum_{n=0}^{10} a_{kn} \xi^n \\
v_\ell(\xi) &= \sum_{n=0}^{10} b_{\ell n} \xi^n
\end{aligned}
\tag{3.2}
$$

in which $\left[a_{kn}\right] = \left[A\right]_{(\overline{U} \times 11)}$ and $\left[b_{\ell n}\right] = \left[B\right]_{(\overline{V} \times 11)}$ define the polynomial functions associated with the local coordinates $(\bar{a}_k)$ and $(\bar{\beta}_\ell)$, respectively. $\left[A\right]$ and $\left[B\right]$ are furnished as input to provide maximum flexibility in the selection of the assumed coordinate functions. It should be understood that the $\bar{a}_k$ and the $\bar{\beta}_\ell$ are the unknown generalized coordinates associated with the shell component, whereas the a_{kn} and $b_{\ell n}$ are definite arbitrarily specified coefficients which determine the functions $u_k(\xi)$ and $v_\ell(\xi)$.

3.1 SHELL STIFFNESS MATRIX

The shell stiffness matrix is constructed from the potential energy function V derived in Appendix A. For the generalized displacement (Equation 3.1), the stiffness matrix is defined by

$$
\left[\overline{K}_a\right]_{\left((\overline{U}+\overline{V})x(\overline{U}+\overline{V})\right)} = \left[\begin{array}{ccc|ccc}
\frac{\partial^2 V}{\partial\overline{a}_1\partial\overline{a}_1} & \cdots & \frac{\partial^2 V}{\partial\overline{a}_1\partial\overline{a}_{\overline{U}}} & \frac{\partial^2 V}{\partial\overline{a}_1\partial\overline{\beta}_1} & \cdots & \frac{\partial^2 V}{\partial\overline{a}_1\partial\overline{\beta}_{\overline{V}}} \\
\vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial^2 V}{\partial\overline{a}_{\overline{U}}\partial\overline{a}_1} & \cdots & \frac{\partial^2 V}{\partial\overline{a}_{\overline{U}}\partial\overline{a}_{\overline{U}}} & \frac{\partial^2 V}{\partial\overline{a}_{\overline{U}}\partial\overline{\beta}_1} & \cdots & \frac{\partial^2 V}{\partial\overline{a}_{\overline{U}}\partial\overline{\beta}_{\overline{V}}} \\
\hline
\frac{\partial^2 V}{\partial\overline{\beta}_1\partial\overline{a}_1} & \cdots & \frac{\partial^2 V}{\partial\overline{\beta}_1\partial\overline{a}_{\overline{U}}} & \frac{\partial^2 V}{\partial\overline{\beta}_1\partial\overline{\beta}_1} & \cdots & \frac{\partial^2 V}{\partial\overline{\beta}_1\partial\overline{\beta}_{\overline{V}}} \\
\vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial^2 V}{\partial\overline{\beta}_{\overline{V}}\partial\overline{a}_1} & \cdots & \frac{\partial^2 V}{\partial\overline{\beta}_{\overline{V}}\partial\overline{a}_{\overline{U}}} & \frac{\partial^2 V}{\partial\overline{\beta}_{\overline{V}}\partial\overline{\beta}_1} & \cdots & \frac{\partial^2 V}{\partial\overline{\beta}_{\overline{V}}\partial\overline{\beta}_{\overline{V}}}
\end{array}\right] \qquad (3.3)
$$

which is consistent with the Rayleigh-Ritz procedure and in which, for example

$$
\begin{aligned}
\frac{\partial^2 V}{\partial\overline{a}_k\partial\overline{\beta}_\ell} = 2\pi\int_s \gamma a \Bigg\{ & \left[C_{11}\frac{\partial\epsilon_\phi}{\partial\overline{a}_k} + C_{12}\frac{\partial\epsilon_\theta}{\partial\overline{a}_k}\right]\frac{\partial\epsilon_\phi}{\partial\overline{\beta}_\ell} + \left[C_{12}\frac{\partial\epsilon_\phi}{\partial\overline{a}_k} + C_{22}\frac{\partial\epsilon_\theta}{\partial\overline{a}_k}\right]\frac{\partial\epsilon_\theta}{\partial\overline{\beta}_\ell} \\
& + \left[C_{33}\frac{\partial K_\phi}{\partial\overline{a}_k} + C_{34}\frac{\partial K_\theta}{\partial\overline{a}_k}\right]\frac{\partial K_\phi}{\partial\overline{\beta}_\ell} + \left[C_{34}\frac{\partial K_\phi}{\partial\overline{a}_k} + C_{44}\frac{\partial K_\theta}{\partial\overline{a}_k}\right]\frac{\partial K_\theta}{\partial\overline{\beta}_\ell} \\
& + N_\phi^o\frac{\partial\rho}{\partial\overline{a}_k}\frac{\partial\rho}{\partial\overline{\beta}_\ell}\Bigg\} ds
\end{aligned} \qquad (3.4)
$$

and

$$k = 1 \ldots \overline{U}$$

$$\ell = 1 \ldots \overline{V}$$

The strains (ϵ_ϕ, ϵ_θ), curvatures (K_ϕ, K_θ) and meridional rotation ρ are defined in terms of the displacements in Appendix A by Equations (A.13) through (A.17). The quantities C_{11}, C_{12}, C_{22}, C_{33}, C_{34} and C_{44} are the orthotropic stress-strain coefficients which are functions of the dimensionless parameter ξ. For the present analytical model, these coefficients are approximated by the following polynomial equations:

$$\left\{\overline{C}(\xi)\right\} = \begin{Bmatrix} C_{11}(\xi) \\ C_{12}(\xi) \\ C_{22}(\xi) \end{Bmatrix} = \left\{\overline{C}(0)\right\} + \xi \left(\left\{\overline{C}(1)\right\} - \left\{\overline{C}(0)\right\}\right) \tag{3.5}$$

$$\left\{\overline{\overline{C}}(\xi)\right\} = \begin{Bmatrix} C_{33}(\xi) \\ C_{34}(\xi) \\ C_{44}(\xi) \end{Bmatrix} = -\frac{9}{2}\left(\xi - \frac{1}{3}\right)\left(\xi - \frac{2}{3}\right)(\xi - 1)\left\{\overline{\overline{C}}(0)\right\}$$

$$+ \frac{27}{2}\xi\left(\xi - \frac{2}{3}\right)(\xi - 1)\left\{\overline{\overline{C}}\left(\frac{1}{3}\right)\right\}$$

$$- \frac{27}{2}\xi\left(\xi - \frac{1}{3}\right)(\xi - 1)\left\{\overline{\overline{C}}\left(\frac{2}{3}\right)\right\}$$

$$+ \frac{9}{2}\xi\left(\xi - \frac{1}{3}\right)\left(\xi - \frac{2}{3}\right)\left\{\overline{\overline{C}}(1)\right\}$$

The quantities $\left\{\overline{C}(0)\right\}$, $\left\{\overline{C}(1)\right\}$, $\left\{\overline{\overline{C}}(0)\right\}$, $\left\{\overline{\overline{C}}\left(\frac{1}{3}\right)\right\}$, $\left\{\overline{\overline{C}}\left(\frac{2}{3}\right)\right\}$ and $\left\{\overline{\overline{C}}(1)\right\}$ are provided as input.

N_ϕ^o is the initial meridional stress caused by tank pressures and longitudinal vehicle accelerations (prestress and static longitudinal acceleration stress). These initial stresses are derived in Appendix B using membrane theory, which is a reasonable first order approximation, except in very localized areas where the bending stresses predominate.

This simplifies the calculation of the initial stresses which are readily obtained from the membrane equations of equilibrium. Expressions for the initial stresses for conical shell components are given in Equations (B. 1) through (B. 6), for the upright ellipsoidal bulkheads in Equations (B. 7) through (B. 11), and for the inverted ellipsoidal bulkheads in Equations (B. 12) through (B. 15). The computation of these stresses is included as an integral part of the digital program.

After substitution of the displacements (3. 1) into (3. 3), the shell component stiffness matrix becomes

$$
\begin{aligned}
\left[\overline{K}_a\right] = 2\pi \int r \Big(& K1 \cdot \{\dot{u}\} \cdot \{\dot{u}\}^T + K2 \{\ddot{u}\} \cdot \{\ddot{u}\}^T + K3 \left[\{\dot{u}\} \cdot \{\ddot{u}\}^T + \{\ddot{u}\} \cdot \{\dot{u}\}^T \right] \\
& + K4 \left[\{v\} \cdot \{\dot{v}\}^T + \{\dot{v}\} \cdot \{v\}^T \right] + K5 \{v\} \cdot \{v\}^T + K6 \{\dot{v}\} \cdot \{\dot{v}\}^T \\
& + K7 \{\ddot{v}\} \cdot \{\ddot{v}\}^T + K8 \left[\{\dot{v}\} \cdot \{\ddot{v}\}^T + \{\ddot{v}\} \cdot \{\dot{v}\}^T \right] \\
& + K9 \left[\{v\} \cdot \{\dot{u}\}^T + \{\dot{u}\} \cdot \{v\}^T \right] + K10 \left[\{\dot{v}\} \cdot \{\dot{u}\}^T + \{\dot{u}\} \cdot \{\dot{v}\}^T \right] \\
& + K11 \left[\{\ddot{v}\} \cdot \{\dot{u}\}^T + \{\dot{u}\} \cdot \{\ddot{v}\}^T \right] + K12 \left[\{\dot{v}\} \cdot \{\ddot{u}\}^T + \{\ddot{u}\} \cdot \{\dot{v}\}^T \right] \\
& + K13 \left[\{\ddot{v}\} \cdot \{\ddot{u}\}^T + \{\ddot{u}\} \cdot \{\ddot{v}\}^T \right] \Big) r_1 d\phi
\end{aligned}
\tag{3.6}
$$

in which

$$
\{u\} = \begin{Bmatrix} u_1(\xi) \\ \vdots \\ u_{\overline{U}}(\xi) \\ --- \\ 0_1 \\ \vdots \\ 0_{\overline{V}} \end{Bmatrix}
\qquad\qquad
\{v\} = \begin{Bmatrix} 0_1 \\ \vdots \\ 0_{\overline{U}} \\ ---- \\ v_1(\xi) \\ \vdots \\ v_{\overline{V}}(\xi) \end{Bmatrix}
$$

$$\frac{1}{r_1}\frac{d}{d\phi}\left\{u\right\} = \left\{\dot{u}\right\} = \begin{bmatrix} A \\ \hdashline 0 \end{bmatrix}\begin{bmatrix}\dot{D}\end{bmatrix}\left\{\xi^n\right\}$$

$$\frac{1}{r_1}\frac{1}{d\phi}\left\{\dot{u}_k\right\} = \left\{\ddot{u}_k\right\} = \begin{bmatrix} A \\ \hdashline 0 \end{bmatrix}\begin{bmatrix}\dot{D}\end{bmatrix}\begin{bmatrix}\dot{D}\end{bmatrix}\left\{\xi^n\right\}$$

$$\frac{1}{r_1}\frac{d}{d\phi}\left\{v_\ell\right\} = \left\{\dot{v}_\ell\right\} = \begin{bmatrix} 0 \\ \hdashline B \end{bmatrix}\begin{bmatrix}\dot{D}\end{bmatrix}\left\{\xi^n\right\}$$

$$\frac{1}{r_1}\frac{d}{d\phi}\left\{\dot{v}_\ell\right\} = \left\{\ddot{v}_\ell\right\} = \begin{bmatrix} 0 \\ \hdashline B \end{bmatrix}\begin{bmatrix}\dot{D}\end{bmatrix}\begin{bmatrix}\dot{D}\end{bmatrix}\left\{\xi^n\right\}$$

$$\begin{bmatrix}\dot{D}\end{bmatrix} = D\left\{\begin{matrix} 0 & 0 & 0 & . & . & . & 0 \\ 1 & 0 & 0 & . & . & . & 0 \\ 0 & 2 & 0 & . & . & . & 0 \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ 0 & 0 & 0 & . & . & 10 & 0 \end{matrix}\right\}_{(11\mathrm{x}11)}$$

and

$$\begin{aligned} D &= \frac{1}{r_1\phi_o} && \text{for convex upward ellipsoid,} \\ &= \frac{1}{r_1(\phi_o - \pi)} && \text{for convex downward ellipsoid,} \qquad (3.7) \\ &= \frac{\sin\phi_o}{/L/} && \text{for a cone.} \end{aligned}$$

The constants K1 - K13 are defined as follows:

$$K1 = \left[C_{11} \sin^2 \phi + C_{33} \frac{1}{r_1^2} \left(\sin \phi + \frac{\dot{r}_1}{r_1} \cos \phi\right)^2 \right.$$

$$\left. - C_{34} \frac{2 \cos^2 \phi}{r r_1} \left(\sin \phi + \frac{\dot{r}_1}{r_1} \cos \phi\right) + C_{44} \frac{\cos^4 \phi}{r^2}\right] + N_\phi^o \cos^2 \phi$$

$$K2 = C_{33} \cos^2 \phi$$

$$K3 = -C_{33} \frac{\cos \phi}{r_1} \left(\sin \phi + \frac{\dot{r}_1}{r_1} \cos \phi\right) + C_{34} \frac{\cos^3 \phi}{r}$$

$$K4 = C_{12} \frac{\cos \phi}{r}$$

$$K5 = C_{22} \frac{1}{r^2}$$

$$K6 = C_{11} \cos^2 \phi + C_{33} \frac{1}{r_1^2} \left(\cos \phi - \frac{\dot{r}_1}{r_1} \sin \phi\right)^2$$

$$+ C_{34} \frac{2 \sin \phi \cos \phi}{r r_1} \left(\cos \phi - \frac{\dot{r}_1}{r_1} \sin \phi\right) + C_{44} \frac{\sin^2 \phi \cos^2 \phi}{r^2} + N_\phi^o \sin^2 \phi$$

$$K7 = C_{33} \sin^2 \phi$$

$$K8 = C_{33} \frac{\sin \phi}{r_1} \left(\cos \phi - \frac{\dot{r}_1}{r_1} \sin \phi\right) + C_{34} \frac{\sin^2 \phi \cos \phi}{r}$$

$$K9 = -C_{12} \frac{\sin \phi}{r}$$

$$K10 = -C_{11} \sin\phi\cos\phi - C_{33}\frac{1}{r_1^2}\left(\cos\phi - \frac{\dot{r}_1}{r_1}\sin\phi\right)\left(\sin\phi + \frac{\dot{r}_1}{r_1}\cos\phi\right)$$

$$- C_{34}\frac{\sin\phi\cos\phi}{rr_1}\left(\sin\phi + \frac{\dot{r}_1}{r_1}\cos\phi\right) + C_{34}\frac{\cos^2\phi}{rr_1}\left(\cos\phi - \frac{\dot{r}_1}{r_1}\sin\phi\right)$$

$$+ C_{44}\frac{\sin\phi\cos^3\phi}{r^2} + N_\phi^o \sin\phi\cos\phi$$

$$K11 = -C_{33}\frac{1}{r_1}\left(\sin\phi + \frac{\dot{r}_1}{r_1}\cos\phi\right)\sin\phi + C_{34}\frac{\sin\phi\cos^2\phi}{r}$$

$$K12 = C_{33}\left(\cos\phi - \frac{\dot{r}_1}{r}\sin\phi\right)\frac{\cos\phi}{r_1} + C_{34}\frac{\sin\phi\cos^2\phi}{r}$$

$$K\ 13 = C_{33}\sin\phi\cos\phi \qquad (3.8)$$

3.2 SHELL MASS MATRIX

The shell mass matrix associated with the generalized coordinates $\bar{\alpha}_k$, $\bar{\beta}_\ell$ is derived by operating on the expression for the kinetic energy T. For each shell component, the mass matrix is defined by

$$
p^2\left[\overline{M}_a\right]_{\left((\overline{U}+\overline{V})x(\overline{U}+\overline{V})\right)} = \left[\begin{array}{ccc:ccc}
\frac{\partial^2 T}{\partial\overline{\alpha}_1\partial\overline{\alpha}_1} & \cdots & \frac{\partial^2 T}{\partial\overline{\alpha}_1\partial\overline{\alpha}_{\overline{U}}} & & & \\
\vdots & & \vdots & & (\text{Zero}) & \\
\frac{\partial^2 T}{\partial\overline{\alpha}_{\overline{U}}\partial\overline{\alpha}_1} & \cdots & \frac{\partial^2 T}{\partial\overline{\alpha}_{\overline{U}}\partial\overline{\alpha}_{\overline{U}}} & & & \\
\hdashline
 & & & \frac{\partial^2 T}{\partial\overline{\beta}_1\partial\overline{\beta}_1} & \cdots & \frac{\partial^2 T}{\partial\overline{\beta}_1\partial\overline{\beta}_{\overline{V}}} \\
 & (\text{Zero}) & & \vdots & & \vdots \\
 & & & \frac{\partial^2 T}{\partial\overline{\beta}_{\overline{V}}\partial\overline{\beta}_1} & \cdots & \frac{\partial^2 T}{\partial\overline{\beta}_{\overline{V}}\partial\overline{\beta}_{\overline{V}}}
\end{array}\right] \tag{3.9}
$$

After substitution of Equations (3.1) into the kinetic energy expression in Equation (3.9), the mass matrix assumes the form[3]

$$
\left[\overline{M}_a\right] = 2\pi\gamma_a \int tr\left(\left\{u\right\}\cdot\left\{u\right\}^T + \left\{v\right\}\cdot\left\{v\right\}^T\right) ds \tag{3.10}
$$

where γ_a is the shell mass density.

The detailed equations for the calculation of the stiffness and mass matrices of the shell components are presented in Appendix D. The equations are written in matrix notation to accommodate the Gaussian weighted matrix[5] integration scheme.

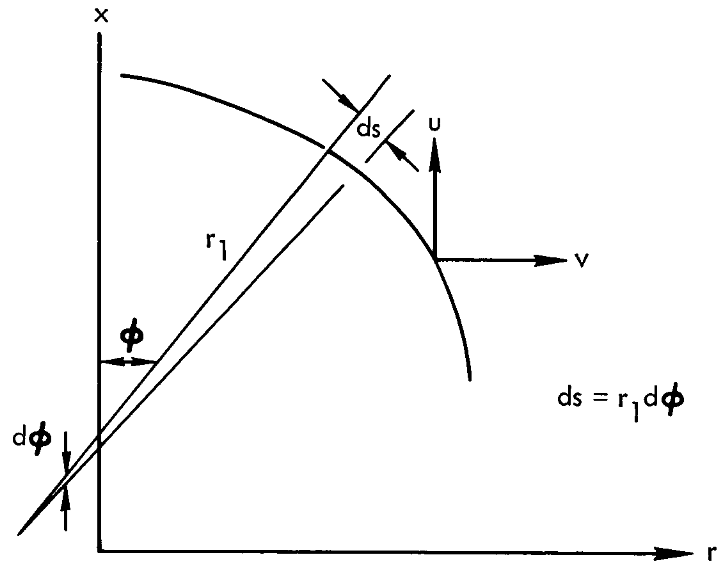


Figure 4. Displacements of Shell of Revolution

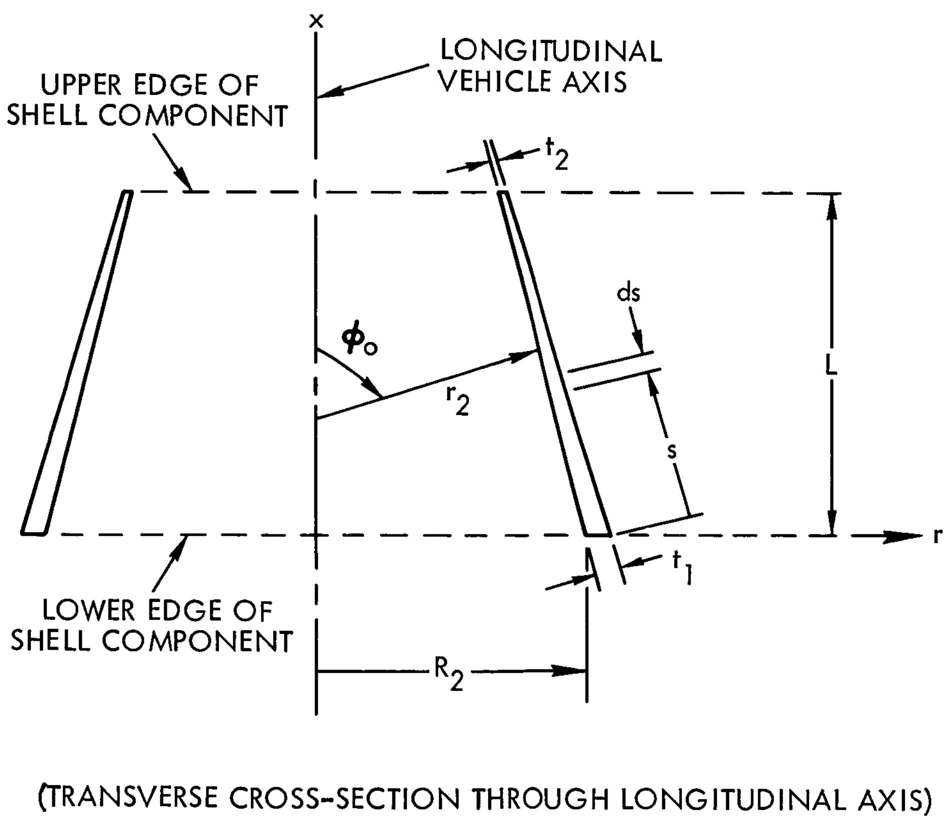
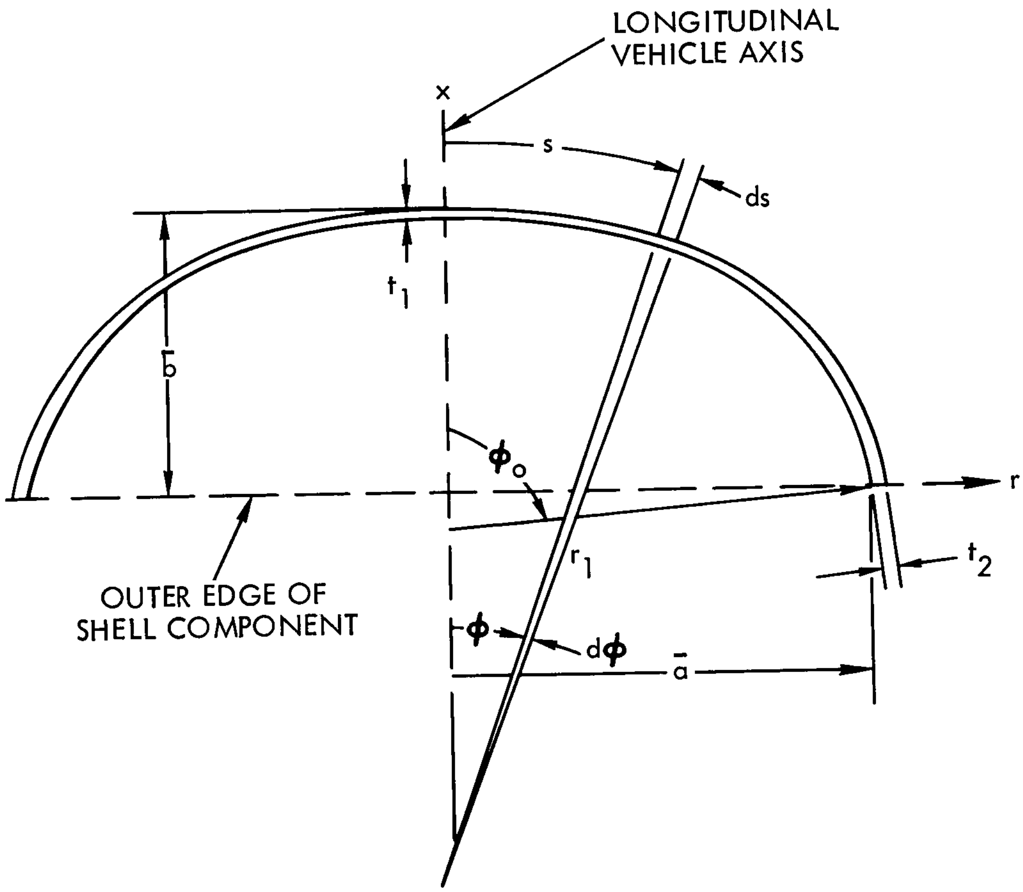


Figure 5. Conical Shell Component



(TRANSVERSE CROSS-SECTION THROUGH LONGITUDINAL AXIS)

Figure 6. Ellipsoidal Bulkhead Component

4. FLUID COMPONENT MASS MATRIX

In addition to the mass and stiffness matrices for the shell components, the inertial effects due to the presence of liquid propellants in the vehicle fuel tanks must be considered. The linear analytical model does not include, however, the effective fluid stiffness caused by changes in the fluid head during shell distortions as this is a higher order nonlinear effect.

The general fluid component b is enclosed by three shell components, consisting of conical and ellipsoidal shells of revolution. The specific tank configurations which are included in the present model have been illustrated in Figure 3. For a typical fluid component, as shown in Figure 7, the tank is divided into three shells in which the upper bulkhead is referred to as shell a1, the tank wall as shell a2, and the lower bulkhead as shell a3.

The fluid motions are a function of the generalized displacements for shell components, a1, a2 and a3. For the $u_k(\xi)$ or $v_\ell(\xi)$ of each shell element defined by Equation (3.2), there is an associated fluid motion $\hat{u}_m(x)$ and $\hat{v}_m(x, \hat{r})$, where $1 \leq m \leq \overline{W}$ and $\overline{W} = (\overline{U} + \overline{V})_{a1} + (\overline{U} + \overline{V})_{a2} + (\overline{U} + \overline{V})_{a3}$. $\hat{u}_m(x)$ is the fluid displacement parallel to the x-axis (longitudinal axis of the vehicle) associated with the tank shell generalized displacement m. $\hat{v}_m(x, \hat{r})$ is the fluid displacement parallel to the r-axis (radial axis of the vehicle) associated with the tank shell generalized displacement m.

The general form of the fluid mass matrix consistent with the fluid displacement may be expressed as[3]

$$\left[\overline{M}_b\right]_{(\overline{W}x\overline{W})} = 2\pi\gamma_b \int_{-\overline{H}_3}^{H} \int_0^{r} \hat{r} \left(\left\{\hat{u}(x)\right\}\left\{\hat{u}(x)\right\}^T + \left\{\hat{v}(x, \hat{r})\right\}\left\{\hat{v}(x, \hat{r})\right\}^T\right) d\hat{r}\, dx \tag{4.1}$$

where

$$\left\{\hat{u}(x)\right\} \equiv \begin{Bmatrix} \hat{u}_1(x) \\ \vdots \\ \hat{u}_{\overline{W}}(x) \end{Bmatrix}, \quad \left\{\hat{v}(x, \hat{r})\right\} \equiv \begin{Bmatrix} \hat{v}_1(x, \hat{r}) \\ \vdots \\ \hat{v}_{\overline{W}}(x, \hat{r}) \end{Bmatrix} \tag{4.2}$$

γ_b is the fluid mass density and matrix $\left[\overline{M}_b\right]$ is of order $\overline{W}$.

The fluid motion $\hat{u}_m(x)$ is assumed independent of location $\hat{r}$ and is obtained by treating the fluid as incompressible and inviscid. From these assumptions, $\hat{u}_m(x)$ is equal to the change in volume below a given location x divided by the corresponding tank cross sectional area. Thus (see Figure 7),

$$\left\{\hat{u}(x)\right\} = -\frac{2}{r^2}\int_{-\bar{H}_3}^{x} r\left(\cot\phi\left\{\overline{u(\xi)}\right\}_b + \left\{\overline{v(\xi)}\right\}_b\right)dx \tag{4.3}$$

where

$$\left\{\overline{u(\xi)}\right\}_b = \left\{\begin{matrix} u_1(\xi) \\ \vdots \\ u_{\bar{U}_1}(\xi) \\ ---- \\ 0_1 \\ \vdots \\ 0_{\bar{V}_1} \\ ---- \\ u_1(\xi) \\ \vdots \\ u_{\bar{U}_2}(\xi) \\ ---- \\ 0_1 \\ \vdots \\ 0_{\bar{V}_2} \\ ---- \\ u_1(\xi) \\ \vdots \\ u_{\bar{U}_3}(\xi) \\ ---- \\ 0_1 \\ \vdots \\ 0_{\bar{V}_3} \end{matrix}\right\}_{(\bar{W}\times 1)} \quad \text{and} \quad \left\{\overline{v(\xi)}\right\}_b = \left\{\begin{matrix} 0_1 \\ \vdots \\ 0_{\bar{U}_1} \\ ---- \\ v_1(\xi) \\ \vdots \\ v_{\bar{V}_1}(\xi) \\ ---- \\ 0_1 \\ \vdots \\ 0_{\bar{U}_2} \\ ---- \\ v_1(\xi) \\ \vdots \\ v_{\bar{V}_2}(\xi) \\ ---- \\ 0_1 \\ \vdots \\ 0_{\bar{U}_3} \\ ---- \\ v_1(\xi) \\ \vdots \\ v_{\bar{V}_3}(\xi) \end{matrix}\right\}_{(\bar{W}\times 1)} \tag{4.4}$$

Fluid sloshing motions which disturb the planar character of the assumed longitudinal motion are beyond the scope of this treatment, but may be superimposed as generalized sloshing modes independent of the shell distortions. Consistent with the assumed longitudinal fluid motion $\hat{u}_k(x)$ and the axisymmetric nature of the linear model, the radial fluid motion $\hat{v}_k(x, \hat{r})$ varies linearly with space coordinate $\hat{r}$. At a particular longitudinal location x, the radial fluid motion is a function of the radial motion of the adjacent tank shell boundary and of the longitudinal fluid motion. Thus

$$\left\{\hat{v}(x, \hat{r})\right\} = \frac{\hat{r}}{r}\left(\cot\phi\left\{u(\xi)\right\}_b + \left\{v(\xi)\right\}_b - \cot\phi\left\{\hat{u}(x)\right\}\right) \qquad (4.5)$$

Upon substituting Equations (4.3) and (4.5) into (4.1) and integrating with respect to r, one obtains

$$\left[\bar{M}_b\right] = \pi\gamma_b \int_{-\bar{H}_3}^{H} \left(\frac{4}{r^2}\left\{\frac{r^2}{2}\hat{u}(x)\right\}\left\{\frac{r^2}{2}\hat{u}(x)\right\}^T + \frac{1}{2}\left\{r\hat{v}(x, r)\right\}\left\{r\hat{v}(x, r)\right\}^T\right)dx \qquad (4.6)$$

where

$$\left\{\frac{r^2}{2}\hat{u}(x)\right\} = -\int_{-\bar{H}_3}^{x} r\left(\cot\phi\left\{\overline{u(\xi)}\right\}_b + \left\{\overline{v(\xi)}\right\}_b\right)dx$$

and

$$\left\{r\hat{v}(x, r)\right\} = r\cot\phi\left\{\overline{u(\xi)}\right\}_b + r\left\{\overline{v(\xi)}\right\}_b - r\cot\phi\left\{\hat{u}(x)\right\} \qquad (4.7)$$

The detailed expressions developed for evaluating Equation (4.6) for various cases are summarized in Appendix C.

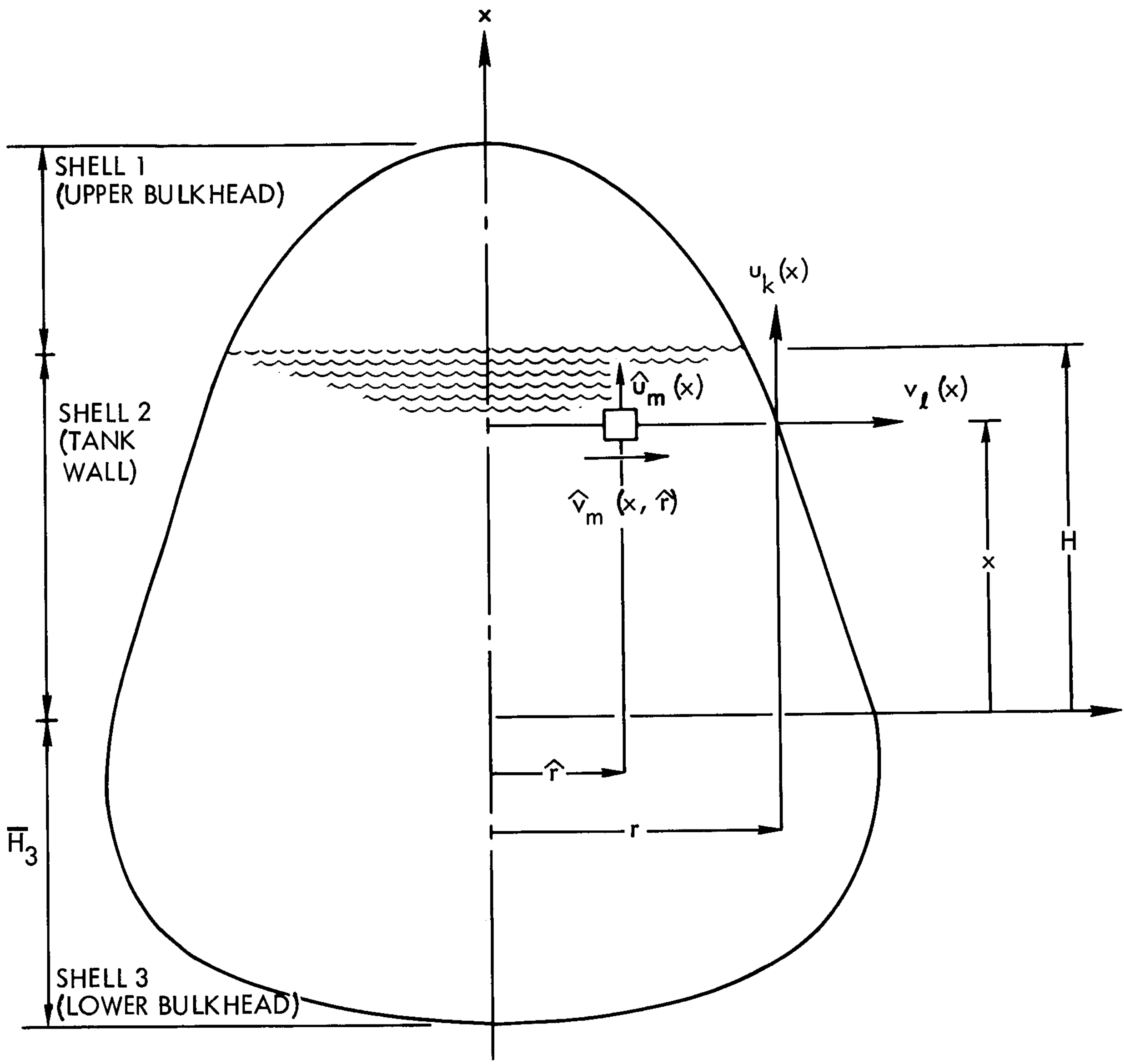


Figure 7. Definition of Fluid Motions

5. LAUNCH VEHICLE COORDINATE SYSTEMS

5.1 LOCAL COORDINATE DISTORTIONS

The matrices for the shell and fluid components are derived in Sections 3 and 4, respectively, using a system of generalized coordinate displacements (local coordinates), as given by Equations (3.1) and (3.2) to describe the shell distortion. These equations may be written in matrix notation in the following form:

$$u(\xi) = \lfloor \bar{a}_k \rfloor [A] \{\xi\} = \lfloor \xi \rfloor [A]^T \{\bar{a}_k\} \tag{5.1}$$

and

$$v(\xi) = \lfloor \bar{\beta}_\ell \rfloor [B] \{\xi\} = \lfloor \xi \rfloor [B]^T \{\bar{\beta}_\ell\} \tag{5.2}$$

where

$$\begin{aligned} \lfloor \bar{a}_k \rfloor &= \lfloor \bar{a}_1 \; . \; . \; . \; \bar{a}_{\bar{U}} \rfloor \\ \lfloor \bar{\beta}_\ell \rfloor &= \lfloor \bar{\beta}_1 \; . \; . \; . \; \bar{\beta}_{\bar{V}} \rfloor \\ \lfloor \xi \rfloor &= \lfloor 1 (\xi)(\xi)^2 \; . \; . \; . \; (\xi)^{10} \rfloor \end{aligned} \tag{5.3}$$

$\bar{U}$ and $\bar{V}$ are constants provided as input which define the number of local coordinates selected for representing shell distortions in the longitudinal and radial directions, respectively, for a particular component.

5.2 SYSTEM COORDINATE POINT DISPLACEMENTS

In order to work with reference to a space frame, however, it is necessary to transform the generalized coordinates to space coordinates designated as system coordinates. The system coordinates represent displacement and rotations at specific points on each shell component, connections of spring-mass components, and applications of force inputs at arbitrary stations along the vehicle.

For each shell component, system coordinates are provided to represent displacements in the longitudinal and radial directions at equally spaced intervals along the shell meridian, and tangential rotations at the edges of the shell (see Figures 8, 9 and 10). Longitudinal displacements $\{u(\xi_i)\}$

at locations $\xi_i = (U - i)/(U - 1)$, where i = 1, 2, ..., U are expressed [using Equation (5.1)] as

$$\left\{u(\xi_i)\right\}_{(U\mathrm{x}1)} = \left[U\right]\left[A\right]^T \left\{\overline{a_k}\right\} \tag{5.4}$$

where

$$\left[U\right] = \begin{bmatrix} 1_1 & \xi_1 & \cdots & (\xi_1)^{10} \\ \vdots & \vdots & & \vdots \\ 1_U & \xi_U & \cdots & (\xi_U)^{10} \end{bmatrix} \tag{5.5}$$

Longitudinal displacements $\left\{v(\xi_j)\right\}$ at locations $\xi_j = (V - j)/(V - 1)$, where j = 1, 2, ..., V, are similarly expressed (using Equation (5.2) as

$$\left\{v(\xi_j)\right\}_{(V\mathrm{x}1)} = \left[V\right]\left[B\right]^T \left\{\overline{\beta}_\ell\right\} \tag{5.6}$$

where

$$\left[V\right] = \begin{bmatrix} 1_1 & \xi_1 & \cdots & (\xi_1)^{10} \\ \vdots & \vdots & & \vdots \\ 1_V & \xi_V & \cdots & (\xi_V)^{10} \end{bmatrix} \tag{5.7}$$

The scalar quantities U and V are constants provided as input which define the number of system coordinate point displacements to be provided in the longitudinal and radial directions, respectively, for a given shell component. These are related to $\overline{U}$ and $\overline{V}$, as discussed in Section 5.3. The shell rotations, $\rho 1$ and $\rho\xi$, are evaluated at $\xi = 1$ and 0, respectively, and by Equation (A.17) can be shown to have the following form:

$$\left\{\begin{matrix} \rho 1 \\ \rho 2 \end{matrix}\right\} = \left[D_1\right]\left[A\right]^T \left\{\overline{a}\right\} + \left[D_2\right]\left[B\right]^T \left\{\overline{\beta}\right\} \tag{5.8}$$

where

$$\left[D_1\right] = D_1 \begin{bmatrix} 0 & 1 & 2 & \cdots & 9 & 10 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(2x11)}$$

and

$$\left[D_2\right] = D_2 \begin{bmatrix} 0 & 1 & 2 & \cdots & 9 & 10 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(2x11)} \tag{5.9}$$

$$D_1 = \frac{\cos\phi_o \sin\phi_o}{L} \quad \text{for conical frustums}$$

$$\frac{\cos\phi_o}{\left(r_1\right)_o \phi_o} \quad \text{for convex upward ellipsoidal bulkheads, and}$$

$$\frac{\cos\phi_o}{\left(r_1\right)_o (\phi_o - \pi)} \quad \text{for convex downward ellipsoidal bulkheads}$$

$$D_2 = \frac{\sin^2\phi_o}{L} \quad \text{for conical frustums,}$$

$$\frac{\sin\phi_o}{\left(r_1\right)_o \phi_o} \quad \text{for convex upward elipsoidal bulkheads, and}$$

$$\frac{\sin\phi_o}{\left(r_1\right)_o (\phi_o - \pi)} \quad \text{for convex downward ellipsoidal bulkheads.}$$

where $\left(r_1\right)_o$ is r_1 evaluated at $\phi = \phi_o$

The total vector of system coordinates displacements for shell component "a" is defined from Equations (5.4), 5.6) and 5.8) as

$$\{\alpha_a\}_{(\bar{U}+\bar{V})\times 1} = \begin{Bmatrix} u(\xi_i) \\ v(\xi_j) \\ \rho 1 \\ \rho 2 \end{Bmatrix} \tag{5.10}$$

As indicated in Figures 8, 9 and 10, each displacement and rotation coordinate is identified by a number C where $1 \leq C \leq N_C$ and N_C is defined as the total number of system coordinates used in the vehicle model. The identification numbers C may be arbitrarily specified on the vehicle independent of location and, hence, must be identified with the vector $\{\alpha_a\}$ components in the input data. For this purpose, a vector $\{C_a\}$ is provided for each shell component a with the identification numbers illustrated in Figures 8, 9 and 10 arranged consistent with Equations (5.4), (5.6) and (5.10). Thus, in general,

$$\{C_a\} = \begin{Bmatrix} C_1 \\ \vdots \\ C_U \\ C_{U+1} \\ \vdots \\ C_{U+V} \\ C_{\rho 1} \\ C_{\rho 2} \end{Bmatrix} \tag{5.11}$$

For ellipsoidal bulkhead components, vector elements C_{U+V} and $C_{\rho 2}$ are omitted.

5.3 LOCAL TO SYSTEM COORDINATE TRANSFORMATION FOR SHELL COMPONENT

The local coordinate displacements for shell component "a" are related to the system coordinate displacements by the transformation $[T_a]$ as follows:

Consolidating the arrangement of local coordinate displacements, as defined in Equations (5.1), (5.2) and (5.3), let

$$\left\{\bar{\alpha}_a\right\}_{(\bar{U}+\bar{V}) \times 1} = \begin{Bmatrix} \bar{\alpha}_k \\ \bar{\beta}_\ell \end{Bmatrix} \tag{5.12}$$

Consistent with the system coordinate displacement vector $\left\{\alpha_a\right\}$ given by Equations (5.4), (5.6) and (5.8), the transformation matrix $\left[T_a\right]$ which relates local to system coordinates, is defined by the equation

$$\left\{\alpha_a\right\} = \left[T_a\right]^{-1}\left\{\bar{\alpha}_a\right\} \tag{5.13}$$

where, in general,

$$\left[T_a\right]^{-1} = \left[\begin{array}{c|c} [U] & 0 \\ \hline 0 & [V] \\ \hline [D_1] & [D_2] \end{array}\right] \left[\begin{array}{c|c} [A]^T & 0 \\ \hline 0 & [B]^T \end{array}\right] \tag{5.14}$$

Thus

$$\left\{\bar{\alpha}_a\right\} = \left[T_a\right]\left\{\alpha_a\right\} \tag{5.15}$$

It is apparent that computation of the matrix $\left[T_a\right]$ requires inversion of the matrix $\left[T_a\right]^{-1}$, which must therefore be nonsingular, square, and of order $(\bar{U} + \bar{V})$. This may require modification of the general Equation (5.14), as discussed below. Special attention must be given the scalar quantities U, V, $\bar{U}$, $\bar{V}$ and the matrices $[A]$, $[B]$, $[D_1]$, $[D_2]$ to satisfy the above conditions consistent with the shell component boundary conditions.

For ellipsoidal bulkheads, the $[A]$ and $[B]$ matrices are selected so that the displacement v(0) and the rotation $\rho 2$ are equal to zero to satisfy the conditions imposed by axial symmetry. This implies that for ellipsoidal bulkheads $U = \bar{U}$, $V = \bar{V}$, which requires that the last row of $[V]$, $[D_1]$ and $[D_2]$ in Equation (5.14) be removed. For a conical bulkhead, $[B]$ is selected so that v(0) is zero. This case requires that the last row of $[V]$ be removed, and $U + 1 = \bar{U}$, $V = \bar{V}$. For cylindrical shells, $U = \bar{U}$, $V = \bar{V} - 2$. For other cases, $U + V + 2 = U + V$ is a necessary condition. The following example matrices fulfill the above requirements.

For nonbulkhead components:

$$[A]_{(\overline{U}\times 11)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & . & . & . & . & . & . & . \\ 0 & 0 & 1 & 0 & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 1 & . & . & . & . & . & . & 0 \end{bmatrix} \tag{5.16}$$

(Illustrated for $\overline{U} = 4$)

$$[B]_{(\overline{V}\times 11)} = \begin{bmatrix} 1 & 0 & 0 & . & . & . & . & . & . & . & 0 \\ 0 & 1 & 0 & 0 & . & . & . & . & . & . & . \\ 0 & 0 & 1 & 0 & 0 & . & . & . & . & . & . \\ 0 & 0 & 0 & 1 & 0 & 0 & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 1 & 0 & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & . & . & . & . \end{bmatrix} \tag{5.17}$$

(Illustrated for $\overline{V} = 6$)

For ellipsoidal bulkhead components:

$$[A]_{(\overline{U}\times 11)} = \begin{bmatrix} 1 & 0 & 0 & . & . & . & . & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & . & . & . & 0 \\ 0 & 0 & 0 & 1 & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 1 & . & . & . & . & . & 0 \end{bmatrix} \tag{5.18}$$

(Illustrated for $\overline{U} = 4$)

$$[B]_{(\overline{V}\times 11)} = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 1 & 0 & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 1 & . & . & . & . & . & 0 \end{bmatrix} \tag{5.19}$$

(Illustrated for $\overline{V} = 4$)

For the simple unit diagonal $\left[A\right]$ and $\left[B\right]$ matrices illustrated, the matrix $\left[T_a\right]^{-1}$ may be poorly conditioned and difficult to invert accurately if $\overline{U}$ and/or $\overline{V}$ are equal to or greater than six (6). For these cases it is recommended that Shifted Chebyshev Polynomial Coefficients[10] (see Table 1) be used for the $\left[A\right]$ and $\left[B\right]$ matrices to improve the accuracy of the matrix $\left[T_a\right]$ calculation. Using Table 1, Equations (5.16), (5.17), (5.18) and (5.19) become, respectively,

For nonbulkhead components:

$$\left[A\right]_{(\overline{U}x11)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & . & . & . & . & . & . & . \\ 1 & -8 & 8 & 0 & . & . & . & . & . & . & . \\ -1 & 18 & -48 & 32 & . & . & . & . & . & . & 0 \end{bmatrix} \quad (5.20)$$

(Illustrated for $\overline{U} = 4$)

$$\left[B\right]_{(\overline{V}x11)} = \begin{bmatrix} 1 & 0 & 0 & . & . & . & . & . & . & . & 0 \\ -1 & 2 & 0 & 0 & . & . & . & . & . & . & . \\ 1 & -8 & 8 & 0 & 0 & . & . & . & . & . & . \\ -1 & 18 & -48 & 32 & 0 & 0 & . & . & . & . & . \\ 1 & -32 & 160 & -256 & 128 & 0 & . & . & . & . & . \\ -1 & 50 & -400 & 1120 & -1280 & 512 & 0 & . & . & . & . \end{bmatrix} \quad (5.21)$$

(Illustrated for $\overline{V} = 6$)

For ellipsoidal bulkhead components:

$$\left[A\right]_{(\overline{U}x11)} = \begin{bmatrix} 1 & 0 & 0 & . & . & . & . & . & . & . & 0 \\ 1 & 0 & 8 & 0 & . & . & . & . & . & . & 0 \\ -1 & 0 & -48 & 32 & 0 & . & . & . & . & . & . \\ 1 & 0 & 160 & -256 & 128 & . & . & . & . & . & 0 \end{bmatrix} \quad (5.22)$$

(Illustrated for $\overline{U} = 4$)

$$\left[B\right]_{(\overline{V}x11)} = \begin{bmatrix} 0 & 2 & 0 & 0 & . & . & . & . & . & . & 0 \\ 0 & -8 & 8 & 0 & . & . & . & . & . & . & . \\ 0 & 18 & -48 & 32 & 0 & . & . & . & . & . & . \\ 0 & -32 & 160 & -256 & 128 & . & . & . & . & . & 0 \end{bmatrix} \tag{5.23}$$

(Illustrated for $\overline{V} = 4$)

5.4 LOCAL TO SYSTEM COORDINATE TRANSFORMATION FOR FLUID COMPONENT

As illustrated in Figure 7, and discussed in Section 4, the distortion of each fluid component "b" is a function of the distortion of the three enclosing shell components a1, a2 and a3. The local to system coordinate transformation matrix $\left[T_b\right]$ for the fluid component is thus obtained as a combination of the shell component transformations $\left[T_a\right]$ for a = a1, a2 and a3, as follows:

Consolidating the arrangement of local coordinate displacements, as defined by Equations (4.4) and (5.12), let

$$\left\{\bar{\alpha}_b\right\} = \begin{Bmatrix} \left\{\bar{\alpha}_{a1}\right\} \\ \left\{\bar{\alpha}_{a2}\right\} \\ \left\{\bar{\alpha}_{a3}\right\} \end{Bmatrix} \tag{5.24}$$

Consistent with Equation (5.10), let the consolidated vector of system coordinate displacements be

$$\left\{\alpha_b\right\} = \begin{Bmatrix} \left\{\alpha_{a1}\right\} \\ \left\{\alpha_{a2}\right\} \\ \left\{\alpha_{a3}\right\} \end{Bmatrix} \tag{5.25}$$

Corresponding to Equations (5.15), (5.24) and (5.25), the transformation matrix $\left[T_b\right]$ is defined by

$$\left\{\bar{a}_b\right\} = \left[T_b\right]\left\{a_b\right\} \tag{5.26}$$

where

$$\left[T_b\right] = \left\{\begin{array}{c:c:c} \left[T_{a1}\right] & 0 & 0 \\ \hdashline 0 & \left[T_{a2}\right] & 0 \\ \hdashline 0 & 0 & \left[T_{a3}\right] \end{array}\right\}_{(\overline{W}\times\overline{W})} \tag{5.27}$$

From Equations (5.10) and (5.25), it is noted that the coordinate identification vector for fluid component b is obtained as a combination of the shell coordinate identification vectors $\left\{C_a\right\}$, defined in Equation (5.11), for a = a1, a2 and a3, as follows:

$$\left\{C_b\right\} = \left\{\begin{array}{c} \left\{C_{a1}\right\} \\ \left\{C_{a2}\right\} \\ \left\{C_{a3}\right\} \end{array}\right\} \tag{5.28}$$

Table 1. Shifted Chebyshev Polynomial Coefficients for Use in Constructing Matrices [A] and [B]

1	0	0	0	0	0	0	0	0	0	0
-1	2	0	0	0	0	0	0	0	0	0
1	-8	8	0	0	0	0	0	0	0	0
-1	18	-48	32	0	0	0	0	0	0	0
1	-32	160	-256	128	0	0	0	0	0	0
-1	50	-400	1120	-1280	512	0	0	0	0	0
1	-72	840	-3584	6192	-6144	2048	0	0	0	0
-1	98	-1568	9408	-26880	39424	-28672	8192	0	0	0
1	-128	2688	-21504	84480	-180224	212992	-131072	32768	0	0
-1	162	-4320	44352	-228096	658944	-1118208	1105920	-589824	131072	0
1	-200	6600	-84880	549120	-2050048	4659200	-6553600	5570560	-2621440	524288

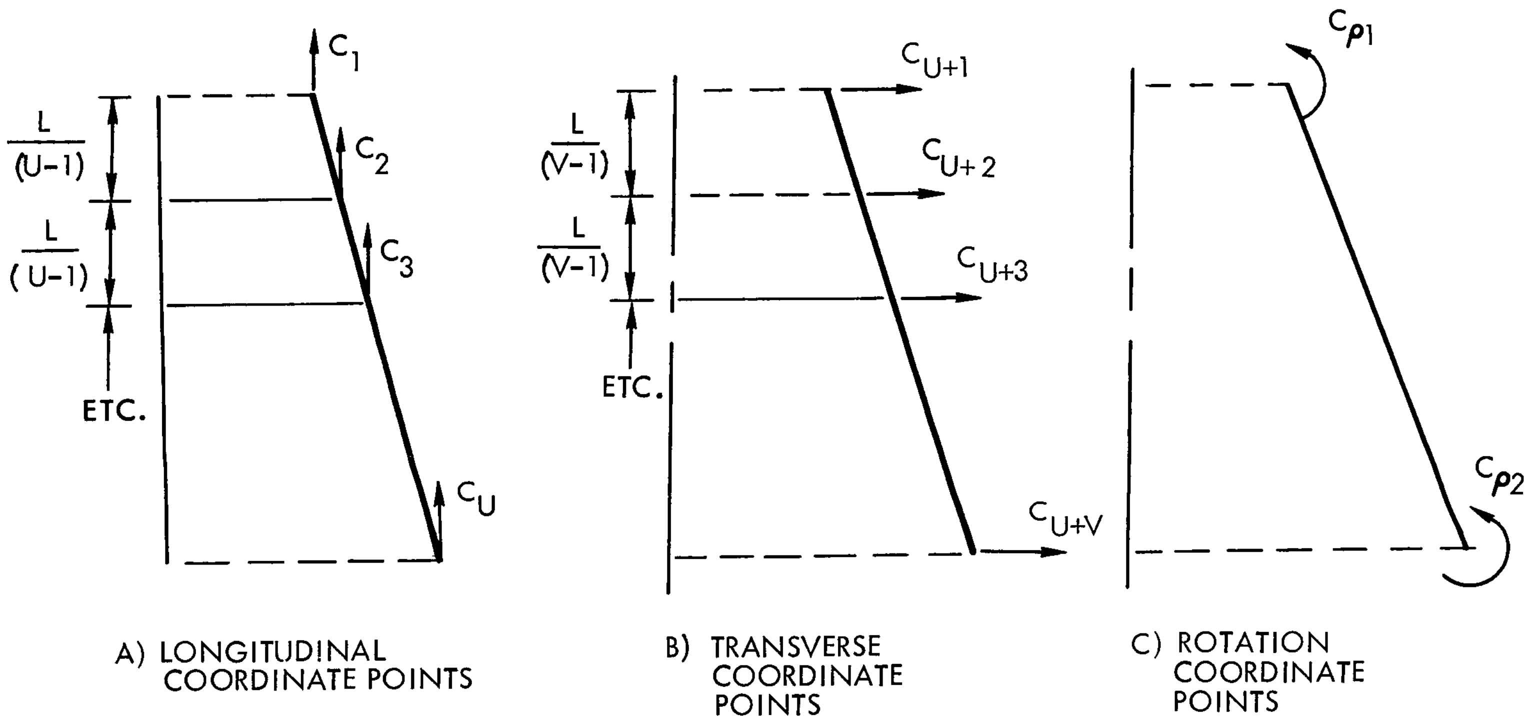


Figure 8. Conical Shell Component System Coordinate Arrangement

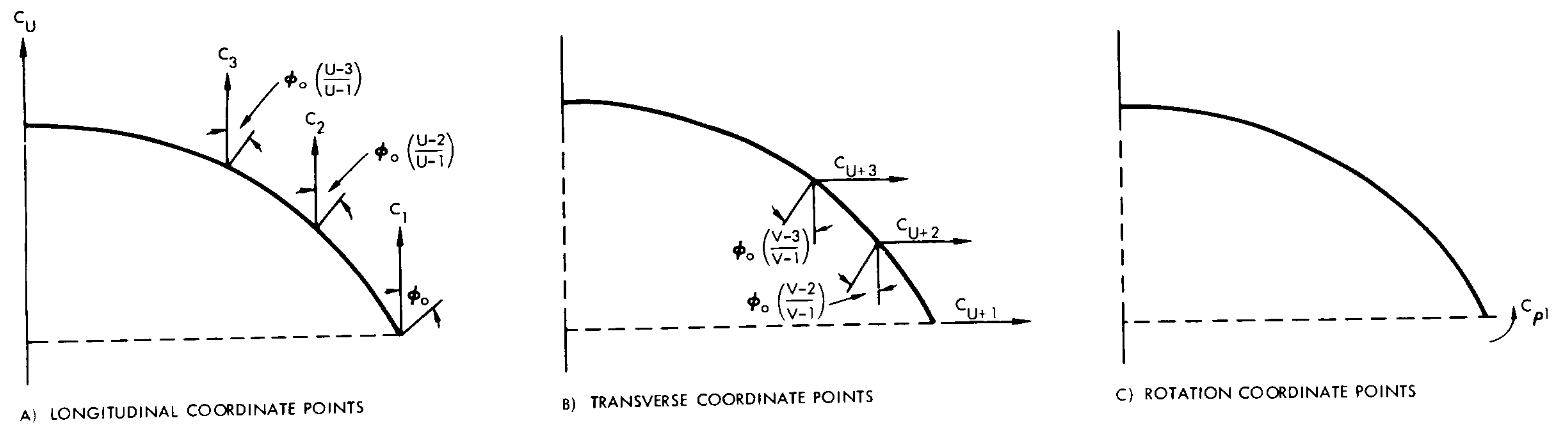


Figure 9. Convex Upward Ellipsoidal Shell, Component System Coordinate Arrangement

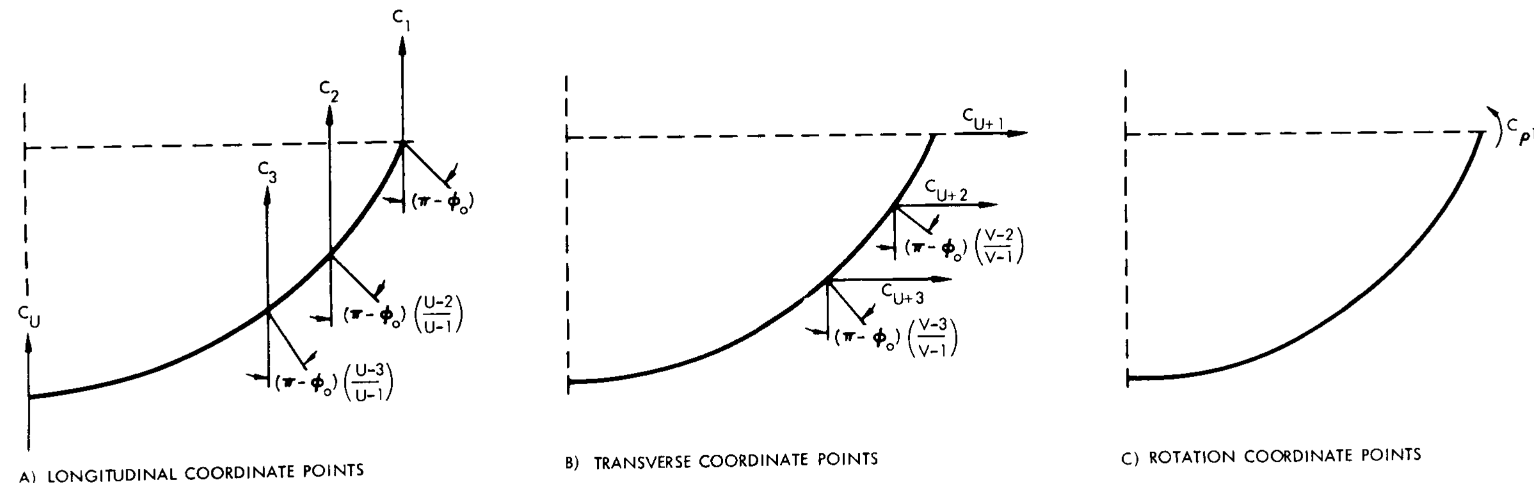


Figure 10. Convex Downward Ellipsoidal Shell, Component System Coordinate Arrangement

6. LAUNCH VEHICLE STIFFNESS AND MASS MATRIX SYNTHESIS

In order to construct the total vehicle stiffness and mass matrices referenced to a common coordinate system, the individual component stiffness and mass matrices are expressed in terms of the system coordinates developed in Section 5.2. For each shell and fluid component e, the matrices, transformed[4] to system coordinates, become

$$\left[K_e\right] = \left[T_e\right]^T \left[\overline{K}_e\right] \left[T_e\right]$$

and (6.1)

$$\left[M_e\right] = \left[T_e\right]^T \left[\overline{M}_e\right] \left[T_e\right]$$

where $\left[T_e\right]$ is the transformation matrix defined in Equation (5.13) for shell components and in Equation (5.26) for fluid components. $\left[\overline{K}_e\right]$ and $\left[\overline{M}_e\right]$ are the stiffness and mass matrices related to the local coordinate system. The stiffness and mass matrices for the spring mass components do not undergo the transformation Equation (6.1), since they already exist as input data in terms of the system coordinate displacements.

The total system stiffness and mass matrices are synthesized by expanding each of the component matrices Equation (6.1) into an enlarged matrix which is of the same order as the total system matrix and which is related to the set of system coordinates for the total vehicle, as shown in Figure 11. This is accomplished by superimposing each of the component matrices after an additional transformation, as described below.

As indicated in Figures 8, 9, 10 and 11 and discussed in Section 5.2, each system displacement and rotation coordinate is defined by an identification number $C(C = 1, 2, \ldots, N_c)$ arbitrarily specified to suit the convenience of the analyst. It is desirable, however, to reserve the identification numbers (N_c), $(N_c - 1)$, $\ldots$, $(N_c - N_o + 1)$ for the N_o system coordinates to be used for rigid supports to preserve a one-to-one correspondence in the digital program output between the identification numbers and the coordinate row-column location in the total system matrices.

The arrangement of the coordinates in the total system displacement vector $\{a\}$ of order N_c is made to coincide with the coordinate identification numbers C. Thus, the transformation matrix $[\Delta_e]$ relating the total system displacements $\{a\}$ to the component e system displacements $\{a_e\}$ is defined as

$$\{a_e\} = [\Delta_e]\{a\} . \tag{6.2}$$

The elements δ_{rs} of $[\Delta_e]$ are determined to be zero or unity from the relations

$$\begin{aligned} \delta_{rs} &= 1 \text{ when } s = C_r \\ &= 0 \text{ when } s \neq C_r \end{aligned} \tag{6.3}$$

where C_r is the r-th element of the coordinate identification vector $\{C_e\}$ for component e [see Equations (5.11) and 5.28)].

The total stiffness and mass matrices are obtained using transformation matrices $[\Delta_e]$ in the following summation equations

$$[K]_{(N_C \times N_C)} = \sum_{a=1}^{N_S} [\Delta_a]^T [K_a][\Delta_a] + \sum_{c=1}^{N_M} [\Delta_c]^T [K_c][\Delta_c] \tag{6.4}$$

and

$$[M]_{(N_C \times N_C)} = \sum_{a=1}^{N_S} [\Delta_a]^T [M_a][\Delta_a] + \sum_{b=1}^{N_F} [\Delta_b]^T [M_b][\Delta_b] + \sum_{c=1}^{N_M} [\Delta_c]^T [M_c][\Delta_c] \tag{6.5}$$

in which N_C, N_S, N_F and N_M are the total number of system coordinates, shell components, fluid components and spring-mass components, respectively. The matrices $[K_a]$, $[M_a]$ and $[M_b]$ are, respectively, the stiffness and mass matrices for the shell components and the mass matrix for the fluid components, as defined by Equation (6.1).

The superposition technique assures displacement compatibility and force equilibrium at the joints between components. Displacement boundary conditions are imposed on the total stiffness and mass matrix by removing appropriate rows and columns of coefficients corresponding to points on the vehicle and its support which are rigidly restrained from motion. Due to storage limitations in the digital program, the order of the resulting total stiffness and mass matrix must not exceed 80.

Additional limitations are placed on the size of the component matrices utilized in Equations (6.4) and (6.5). The shell component stiffness and mass matrices, $\left[K_a\right]$ and $\left[M_a\right]$, must have an order no larger than 22. For the fluid mass matrix, $\left[M_b\right]$, the order must not exceed 35, and for the spring-mass component stiffness and mass matrices, $\left[K_c\right]$ and $\left[M_c\right]$, the order must not exceed 10.

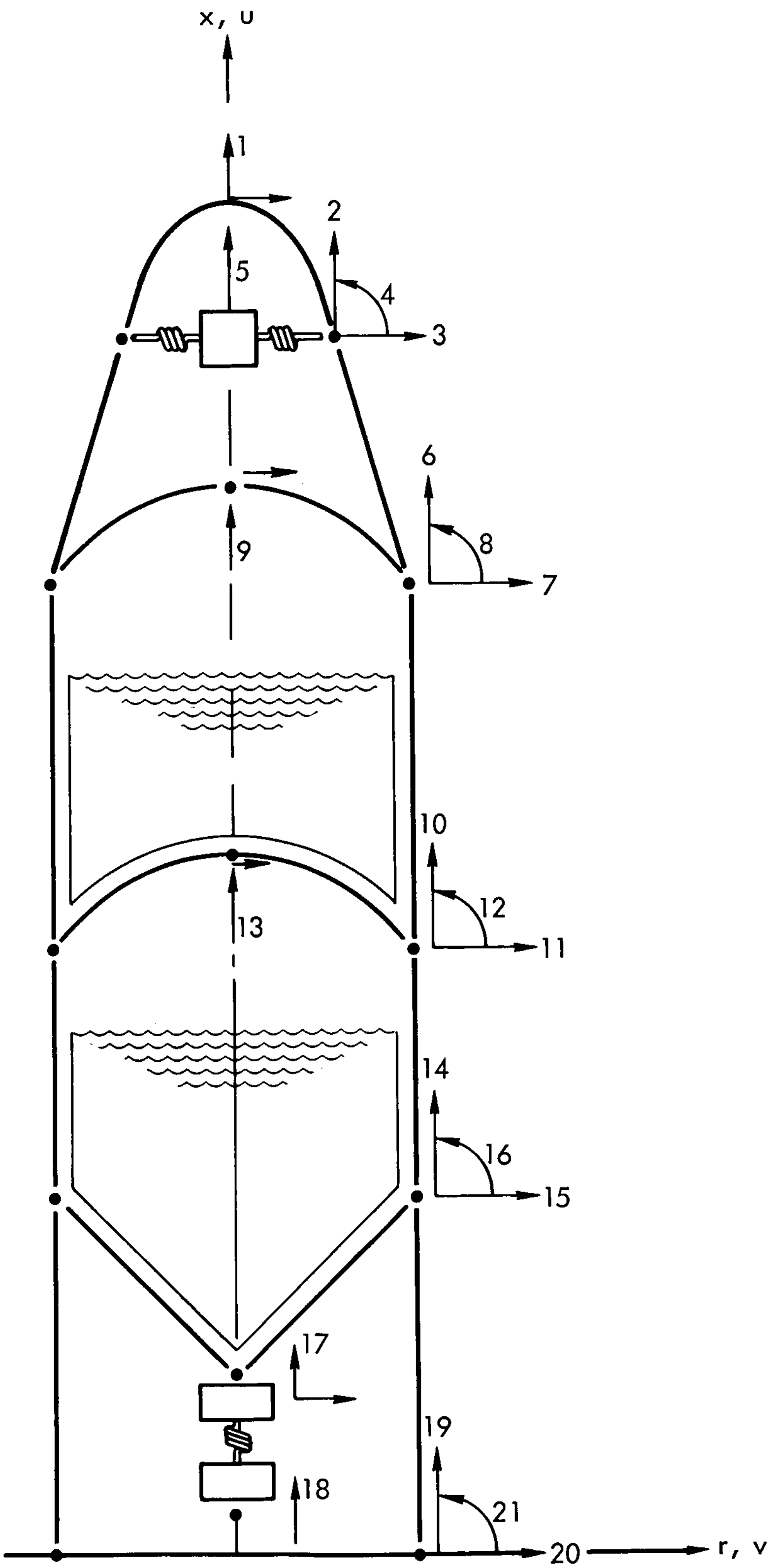


Figure 11. Vehicle System Coordinates

7. DYNAMIC RESPONSE EQUATIONS

7.1 NATURAL FREQUENCY EQUATIONS

The total stiffness and mass matrices which are derived in Section 6 are used for computation of the natural frequencies and mode shapes from the eigenvalue equation

$$\left[K\right]\left\{a\right\} - p^2 \left[M\right]\left\{a\right\} = 0 \tag{7.1}$$

in which p is the circular frequency of the launch vehicle and $\left\{a\right\}$ is the modal vector whose elements are the longitudinal, radial and rotational system coordinate displacements defined in Section 5 and illustration in Figure 11. This equation is solved to obtain the natural frequencies p_t and the mode shapes for all modes, t, which are arranged in a square modal matrix $\left\{a\right\}$ of order $N_c - N_o$. Each column, $\left\{a_t\right\}$, of $\left[a\right]$, is the mode t displacement vector with system coordinate elements, whereas each row, $\left\lfloor a_s \right\rfloor$, of $\left[a\right]$, is the system coordinate s displacement vector with natural mode elements.

7.2 STEADY-STATE RESPONSE EQUATIONS

The steady-state response due to simple harmonic loads of frequency ω is determined using a standard modal technique. The elements of the load vector $\left\{P\right\}$ of order $(N_c - N_o)$ represent axisymmetric forces (longitudinal and radial) or moments, depending on whether the associated coordinate is a displacement or a rotation. The displacement response $\left\{R\right\}$ at coordinates $s\left(s = 1, 2, \ldots, (N_c - N_o)\right)$ on the launch vehicle is expressed as the linear superposition of the individual modal responses based on an assumed modal damping factor η_k which is the ratio of the actual damping to the critical damping for each mode and has the form[4]

$$\left\{R\right\} = \left\{\overline{R} \sin \left\{\omega t - \overline{\delta}\right\}\right\}$$

where

$$\left\{\overline{R}^2\right\} = \left(\left[a\right]\left\{\frac{Q_t \sin \delta_t}{m_t\left(p_t^2 z_t\right)}\right\}\right)^2 + \left(\left[a\right]\left\{\frac{Q_t \cos \delta_t}{m_t\left(p_t^2 z_t\right)}\right\}\right)^2 \tag{7.2}$$

$$\left\{\bar{\delta}\right\} = \left\{\tan^{-1}\left(\frac{\lfloor a_s \rfloor \left\{\dfrac{Q_t \sin \delta_t}{m_t\left(p_t^2 z_t\right)}\right\}}{\lfloor a_s \rfloor \left\{\dfrac{Q_t \cos \delta_t}{m_t\left(p_t^2 z_t\right)}\right\}}\right)\right\} \qquad (7.3)$$

$$Q_t = \{a_t\}^T \{P\}$$

$$m_t = \{a_t\}^T [M] \{a_t\}$$

$$p_t^2 z_t = \left[\left(p_t^2 - \omega^2\right)^2 + 4p_t^2 \eta_t^2 \omega^2\right]^{1/2} \qquad (7.4)$$

and

$$\delta_t = \tan^{-1} \frac{2\eta_t \left(\dfrac{p_t}{\omega}\right)}{\left(\dfrac{p_t}{\omega}\right)^2 - 1}$$

In these equations, $\delta_t = \pi$ when $p_t = 0$, and $\delta_t = 0$ when $p_t \neq 0$, $\omega = 0$. $\{\bar{R}\}$ and $\{\bar{\delta}\}$ represent, respectively, the vectors of steady-state displacement amplitude and the phase angle by which the forcing function leads (+) or lags (-) the response. The velocity $\{\dot{R}\}$ and acceleration $\{\ddot{R}\}$ responses are obtained from the relations

$$\{\dot{R}\} = \omega \left\{\bar{R} \sin\left(\omega t - \left(\frac{\pi}{2} + \bar{\delta}\right)\right)\right\}$$

$$\{\ddot{R}\} = -\omega^2 \left\{\bar{R} \sin\left(\omega t - \bar{\delta}\right)\right\} \qquad (7.5)$$

The internal forces (or moments) $\{S_a\}$ acting at each point along the vehicle on each shell component a are obtained from the equation

$$\{S_a\}_{\left((\bar{U}+\bar{V}) \times 1\right)} = \left\{\bar{S} \sin\left(\omega t - \delta\right)\right\} \qquad (7.6)$$

where

$$\left\{\bar{S}^2\right\}_{\left((\bar{U}+\bar{V}) \times 1\right)} = \left(\left[K_a\right]\left[\Delta_a\right]\left\{\bar{R} \sin \bar{\bar{\delta}}\right\}\right)^2 + \left(\left[K_a\right]\left[\Delta_a\right]\left\{\bar{R} \cos \bar{\bar{\delta}}\right\}\right)^2$$

$$\left\{\bar{\bar{\delta}}\right\}_{\left((\bar{U}+\bar{V}) \times 1\right)} = \left\{\tan^{-1}\left(\frac{\lfloor K_s \rfloor \left[\Delta_a\right]\left\{\bar{R} \sin \bar{\delta}\right\}}{\lfloor K_s \rfloor \left[\Delta_a\right]\left\{\bar{R} \cos \bar{\delta}\right\}}\right)\right\} \tag{7.7}$$

$\left\{\bar{S}\right\}$ and $\left\{\bar{\bar{\delta}}\right\}$ represent, respectively, the amplitude and phase angle of the internal forces. $\lfloor K_s \rfloor$ is row s $\left(s = 1, 2, \cdots, (\bar{U} + \bar{V})_a\right)$ of the shell element stiffness matrix $\left[K_a\right]$. $\left\{\bar{R}\right\}$ and $\left\{\bar{\delta}\right\}$ are obtained from Equation (7.3).

8. DIGITAL PROGRAM ARRANGEMENT

The analytic model developed in the previous sections is used as the basis for a digital computer program to determine the vibration characteristics and steady-state response of a launch vehicle subjected to axisymmetric sinusoidal loads. The program is written in Fortran IV language for use on an IBM 7094 computer having 32K magnetic core storage locations. The functional operations and the overall program arrangement are illustrated in Figure 12. Each of the operations enclosed by a block represents an independent link in the computer program. This feature facilitates the task of making future modifications or expansions of the program.

The input data necessary to initiate the program sequence is discussed in detail in Section 9. After the data has been processed, the program proceeds to develop the stiffness and mass matrices for each of the shell components. The functional operations required to perform these computations and the form of the equations used in the digital program are provided in Appendix D. Consistent with the matrix formulation of the basic equations, the numerical integration is performed using a sixteen point Gaussian[5] weighted matrix integration scheme.

In a similar fashion, the fluid mass matrix for each fluid component is constructed. The functional flow diagram and the basic equations required to describe the three fluid tank configurations considered in the present analytical model are presented in detail in Appendix E. Unlike the formulation for the shell stiffness and mass matrices, the equations for the fluid mass matrix involve a double integration. For this computation, a double Lagrangian[6] weighted matrix integration scheme was found most suitable. This technique employs two 11-point Lagrangian weighting matrices in sequence to provide a 22 point approximation.

The component matrix construction is concluded with the setting up of the stiffness and mass matrices, provided as input data, of the spring-mass components.

The shell, fluid and spring-mass component stiffness and mass matrices are then synthesized into a total vehicle system stiffness and mass matrix, according to the steps presented in Section 6. Utilizing these matrices, the natural frequency equation, Equation (7. 1), is formulated and subsequently solved using a standard digital eigenvalue routine which solves matrices up to order 80.

The program user now has the option of 1) continue the analysis and move directly to the computation of the steady-state response, or 2) to temporarily stop the solution after the free vibrations stage for the purpose of examining the output before proceeding with the computation of the steady-state response. This option enables the user to examine the results of the computation for the natural frequencies and mode shapes before determining the input for the modal damping and for the frequencies at which the launch vehicle is forced.

Checks on the accuracy and consistency of the various computations are performed throughout the program. The shell component stiffness matrices are subjected to longitudinal rigid body displacements to establish automatically that equilibrium is satisfied. For each shell component, the "equilibrium check" appears as output in the following form

$$\text{Equilibrium Check} = \frac{\sum \text{Longitudinal Rigid Body Displacement Forces}}{\sum_{i=1}^{U} k_{ii}} \tag{8.1}$$

where k_{ii} are the diagonal components of the shell component stiffness matrix $\left[K_a\right]$ associated with the system coordinates and U is defined in Section 5.2 as the total number of longitudinal system coordinates associated with component a. If equilibrium is satisfied, Equation (8. 1) will be equal to zero.

In a similar manner, the mass matrices for the shell and fluid components are subjected to unit accelerations to verify that the rigid body mass characteristics have been accurately represented. The "mass check" also appears as output and has the form

$$\text{Mass Check} = \frac{M_{input}}{M_{computed}} \tag{8.2}$$

where M_{input} is the total mass of the element, provided as input, and $M_{computed}$ is the rigid body mass computed in the program. The elements of the mass matrix $\left[M_e\right]$ are corrected by the "mass check" factor to provide the correct total mass representation for the structural component e, since $M_{computed}$, in general, will not agree with M_{input} due to the physical nature of the problem.

An additional check on the accuracy of the program is provided in the form of a $\left[T_a\right]$ "inverse check." In developing the $\left[T_a\right]$ transformation, Equation (5.15), it is necessary to take the inverse of $\left[T_a\right]^{-1}$. An indication of the conditioning of this matrix is furnished by the product

$$\text{Inverse Check} = \left[T_a\right]\left[T_a\right]^{-1} \tag{8.3}$$

which appears as output. The deviation of this product from a unit diagonal matrix provides an estimate of the accuracy of the computation for $\left[T_a\right]$. The incorporation of the three checks discussed above is a useful aid in assuring the reliability of the digital solution.

An additional feature which is incorporated in the program is the capability to stack cases, i.e., to solve numerous related problems in sequence by simply stacking the data input for each of the cases. The stacking capability can be utilized for

1) the solution of the complete problem, which may include the steady-state response or

2) the solution of the steady-state response using natural mode data previously stored on tape.

This feature provides for an efficient use of machine time by eliminating the need for reloading the program deck for successive cases.

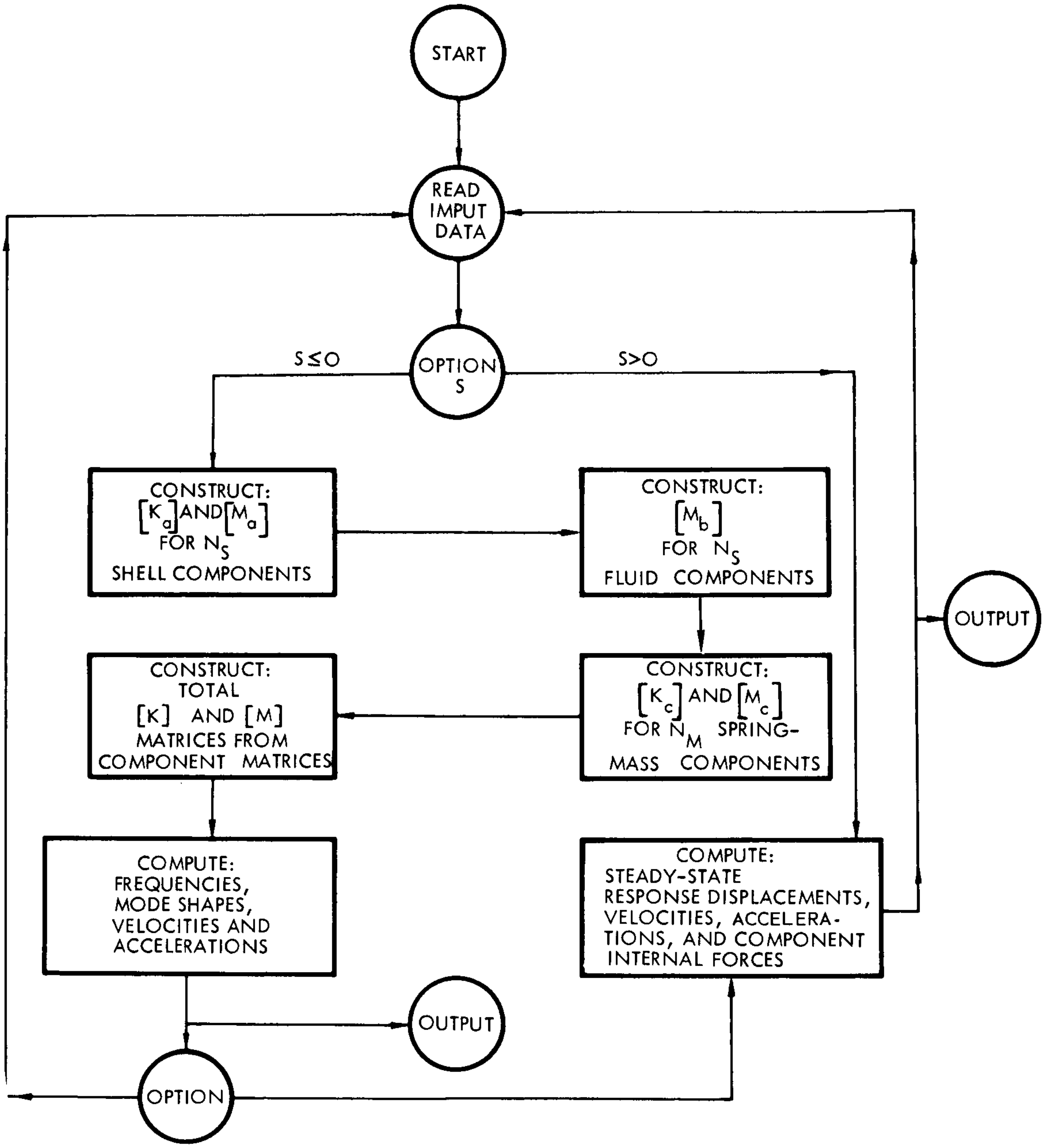


Figure 12. Digital Program Arrangement

9. DATA SETUP

9.1 VEHICLE SUBDIVISION

In order to prepare the input data, it is first necessary to subdivide the launch vehicle into a consistent set of axisymmetric shell components a, fluid components b, and spring-mass components c. Size limitations of the program require that

1) The total number of shell components shall not exceed forty (40),

2) The total number of fluid components shall not exceed six (6),

3) The total number of spring-mass components shall not exceed thirty (30), and

4) The order of the spring-mass stiffness and mass matrices must not exceed ten (10).

Subsequent to the vehicle subdivision, the location of the system coordinate displacements must be determined. The internal operations of the digital program require that

1) Longitudinal, radial and rotational coordinate displacements must be placed at each junction of two or more shell elements

2) Longitudinal and radial coordinate displacements are placed at each point lying on the vehicle longitudinal axes.

These locations are defined as the terminal points of each shell element. Additional longitudinal and radial coordinate displacements may be placed, as desired, at intermediate points uniformly spaced between the terminal points. Longitudinal, radial and rotational coordinates may also be used to describe the motions of the spring-mass elements.

The coordinate displacements are then identified by a consecutively numbered sequence. The arrangement of these numbers on the vehicle is arbitrary and is left up to the discretion of the analyst. However, the radial coordinate points lying on the vehicle axis are not to be identified with a number. This is necessary because the program must assume these displacements equal to zero, as is required by the axisymmetry of the vehicle, and thus they do not contribute to the degrees-of-freedom of the system. These are defined as "unnumbered" coordinates and are utilized only in the preparation of the data input sheets for determining the value of U and V (see Figure 11).

In general, for a supported structure, some of the "numbered" coordinates may be assumed fixed or restrained from motion. The total number of "numbered" coordinates which are not fixed must not exceed eighty (80). In addition, the numbering sequence should be arranged so that the fixed coordinates are numbered last to insure consistency in the program output identification with the coordinate identification numbers.

The total number of coordinates for a shell component may not exceed 22, of which in general the number of longitudinal displacement coordinates may not exceed 11, and the number of lateral displacement coordinates may not exceed 10 (see Section 5.3). For the case of three shell components surrounding a fluid component, the total number of coordinates for the composite tank structure may not exceed 29.

9.2 INPUT DESCRIPTION

System Input Data

1) Heading

HHEAD	is one line of BCD characters which will be printed as the title of the printed output. The number of BCD words (6 characters per word) must not exceed 11.

2) Input Parameters

N_C	is the total number of system coordinates which include the fixed coordinates N_O. $N_C - N_O$ must not exceed 80.
N_S	is the total number of shell components. N_S cannot be zero and must not exceed 40.
N_F	is the total number of fluid components. N_F must not exceed 6.
N_M	is the total number of spring-mass components. N_M must not exceed 30.

N_O	is the total number of fixed coordinates. $N_C - N_O$ must not exceed 80.

3) Applied Loads and Forcing Frequencies (Section 7.2)

N_L	is the total number of discrete applied loads. N_L must not exceed 80.
C_i	is the applied load coordinates C_1, C_2, C_3, $\cdots$, C_{NL}.
P_i	is the discrete applied loads P_1, P_2, P_3, $\cdots$, P_{NL}.
N_W	is the number of sets of forcing function frequencies.
f_i, Δf_i, m_i	f is the frequency of the forcing function in cycles per second ($\omega = 2\pi f$). This program will compute the steady-state response for the frequencies f_i, $f_i + \Delta f_i$, $f_i + 2\Delta f_i$, $\cdots$, $f_i + (m_i - 1)\Delta f_i$, $i = 1, 2, \cdots, N_W$.

4) Modal Damping Factor

N_{ET}	is the number of input η. Program will generate a complete table of η by setting $\eta_{(N_{ET}+1)}$, $\eta_{(N_{ET}+2)}$, $\cdots$, $\eta_{(N_C-N_O)}$ equal to η_{NET}
η_k	is the ratio of the assumed damping to the critical damping in mode k

$$\eta_1, \eta_2, \eta_3, \cdots, \eta_{NET}$$

5) Ratio of Acceleration

g	is the ratio of the vehicle acceleration to the acceleration of gravity.

6) Steady-State Response Option

S — is a fixed point word which controls the option of computing steady-state response.

$S < 0$ — indicates that the computation of the steady-state response is not included. The necessary data for the steady-state computation is saved on Tape 1 and 2.

$S = 0$ — indicates that the computation of the steady-state response is included.

$S > 0$ — is the option to compute the steady-state response only. The necessary data should be available on Tape 1 and 2. In addition the following System Input Data must be provided:

$S = 1$, Heading [Item 1)], Input Parameters [Item 2)], and Modal Damping Factors [Item 4)].

$S = 2$, Heading [Item 1)], Input Parameters [Item 2)], Applied Loads and Forcing Frequencies [Item 3)], Modal Damping Factors [Item 4)], and the option word opt_4.

7) Print Options

opt_1 — is an option word which controls the output of stiffness matrix and mass matrix of the shell and the fluid components.

$opt_1 = 1$, print the component matrices

$opt_1 = 0$, suppress the printing of component matrices.

opt_2 — is an option word which controls the printing of total stiffness matrix and total mass matrix.

$opt_2 = 1$, print the total stiffness and mass matrices.

opt_2 = 0, suppress the printing of total stiffness and mass matrices.

opt_3 is an option word which sets the rigid body frequency to zero for computing the response.

opt_3 = 1, set the first frequency to zero.

opt_3 = 0, do not set the first frequency to zero.

opt_4 is an option word which controls the computation and printing of forces for the steady-state response.

opt_4 = 1, compute and print the forces.

opt_4 = 0, do not compute and print the forces.

N_{EI} is the number of frequencies, mode shapes, velocities and accelerations that will be printed as the final output.

8) Polynomial Matrices (Section 5.3)

N_P is the total number of polynomial matrices.

$\bar{U}_k$ is the number of rows of polynomial matrix $[A]_{\bar{U}_k x 11}$. ($\bar{U}_k \leq 11$)

$\bar{V}_k$ is the number of rows of polynomial matrix $[B]_{\bar{V}_k x 11}$. ($\bar{V}_k \leq 11$)

$[A]_k$ is $\bar{U}_k$ x 11 polynomial matrix.

$[B]_k$ is $\bar{V}_k$ x 11 polynomial matrix.

$k = 1, 2, 3, \cdots, N_P$

The input sequence of the polynomial matrices establishes the identification number k which is referred by the shell components.

The subscript k is used as the polynomial matrix identification number by the shell components.

Shell Component Input Data

1) I.D. Number

a	is the identification number for shell component a where $0 < a \leq N_S$
	+ a indicates a conical shell component - a indicates an ellispoidal shell component

2) Coordinates

U, V	are the total number of system coordinates.
$\bar{U}$, $\bar{V}$	are the total number of local coordinates $\bar{U}$, $\bar{V}$ must not exceed 11.

3) Coordinate I.D. Vector (Figures 8, 9, and 10)

$(ID)_i$	is the identification vector which is used to position the elements for building total stiffness and mass matrices. The length of the vector must be equal to $\bar{U} + \bar{V}$ and the number must not be greater than N_C.

4) Polynomial Matrix Identification Number

k	is the polynomial matrix identification number which refers to polynomial matrices $[A]_k$ and $[B]_k$ in the system input data.

5) Shell Geometric Data (Figures 4 and 5)

ϕ_o	is the meridional angle for conical shell and is the edge meridional angle for ellipsoidal shell. ϕ_o is input in degrees.
L	is the height of conical shell + L indicates converging upward - L indicates converging downward L = 0 for ellipsoidal shell input

R_2 — is the lower radius of conical shell
$R_2 = 0$ for ellipsoidal shell input

$\bar{b}$ — is the height of ellipsoidal shell
$+\bar{b}$ indicates convex upward
$-\bar{b}$ indicates convex downward
$\bar{b} = 0$ for conical shell input

$\bar{a}$ — is the radius of the base of ellipsoidal shell, $\bar{a} = 0$ for conical shell input

6) Orthotropic Shell Constants and Thickness (Equation 3.5)

$(C_{11})_p$, $(C_{12})_p$, $(C_{22})_p$ — are orthotropic shell constants at two points $\xi = 0, 1$ which are represented by $p = 1, 2$, respectively.

$(C_{33})_p$, $(C_{34})_p$, $(C_{44})_p$ — are orthotropic shell constants at four points $\xi = 0, 1/3, 2/3, 1$ which are represented by $p = 1, 2, 3, 4$, respectively.

$(t)_p$ — are shell thickness at two points $\xi = 0, 1$ which are represented by $p = 1, 2$, respectively.

7) Mass Density and Total Mass

γ_a — is the mass density of the shell component.

M_a — is the total mass of the shell component.

$M_a \neq 0$, the ratio of the total mass M_a to the computed mass $\bar{M}_a$ will be used as the scaling factor for the mass matrix. When $opt_1 = 1$, the scaling factor will be printed as the mass check of the mass matrix.

$M_a = 0$, no scaling factor will be used for the mass matrix. When $opt_1 = 1$, the computed mass will be printed.

8) Initial Stress Data (Figures (B.1), (B.2) and)B.3))

H_i	is the depth of interior fluid.
w_i	is the weight density of interior fluid.
p_i	is the uniform interior pressure.
H_e	is the depth of exterior fluid.
w_e	is the weight density of exterior fluid.
p_e	is the uniform exterior pressure
W	is the reactive force at upper edge of conical shell.

+ W produces tensile stresses.
- W produces compressive stresses.
W = 0 for ellipsoidal shells.

Fluid Component Input Data

1) I.D. Number

b	is the identification number for fluid component b where $0 < b \leq N_F$.

2) Associated Shell Components (Figure 7)

a_1, a_2, and a_3	are the identification numbers of the associated shell components

3) Fluid Data (Figure C.1)

H	is the depth of fluid component.
γ	is the mass density of fluid component.
M	is the mass of fluid component. $M \neq 0$, the ratio of the total mass M to the computed mass $\bar{M}$ will be used as the scaling factor for the mass matrix. When $opt_1 = 1$, the scaling factor will be printed as the mass check of the mass matrix.

M = 0, no scaling factor will be used for the mass matrix. When $opt_1 = 1$, the computed mass will be printed.

Spring-Mass Component Input Data

1) I.D. Number

c — is the identification number of spring-mass component c where $0 < c \leq N_M$.

2) Stiffness and Mass Matrices

n — is the order of the spring-mass component n must not exceed 10.

$[K]_c$ — is n x n stiffness matrix of spring-mass component.

$[M]_c$ — is n x n mass matrix of spring-mass component.

3) Coordinate I.D. Vector

IDC_i — is the identification vector which is used to position the elements for building the total stiffness and mass matrices. The length of the vector must be equal to n and the number must not be greater than N_C.

9.3 SAMPLE INPUT DATA SHEETS

Sample input data sheets are included to illustrate the input format. The data must be arranged in the order shown, that is,

1) System input data
2) Shell component input data
3) Fluid component input data
4) Spring-mass component input data

THE SYSTEM INPUT DATA

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DATE ______ PAGE ____ OF ____

NAME ______ PRIORITY ______

PROBLEM NO. ______ KEYPUNCHED BY ______

NO. OF CARDS ______ VERIFIED BY ______

1	7	73
HHEAD		
		X3

SYMBOL	PRE (1, 19, 37, 55)	LOC. (2, 20, 38, 56)	VALUE (7, 25, 43, 61)	EXP. (17, 35, 53, 71)	73
N_C	I	NC			
N_S	I	NS			
N_F	I	NF			
N_M	I	NM			
N_O	I	NØ			
N_L	I	NL			
C_1	I	LDCRD			
C_2	I				
.	.				
.	.				
.	.				
.	.				
C_{N_L}	I				
P_1		ALØAD			
P_2					
.					
.					
.					
P_{N_L}					
N_W	I	NW			
ω_1		ØMEGA			
$\Delta\omega_1$					
m_1	I				
.					
.					
ω_{N_W}					
$\Delta\omega_{N_W}$					
m_{N_W}	I				
N_{ET}	I	IETAK			
η_1		ETAK			
η_2					
.					

X3

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SYMBOL	PRE	LOC.	VALUE	EXP.
.				
.				
T_{NET}				
g		G		
S	I	INS		
opt_1	I	IØPT1		
opt_2	I	IØPT2		
opt_3	I	IØPT3		
opt_4	I	IØPT4		
N_{EI}	I	NEI		
N_P	I	NP		
	E	ND		
$\bar{U}_k$	I	UBAR		
$\bar{V}_k$	I	VBAR		
$[A]_k$	M	APØLY	11,11	
		OI,OJ		
$[B]_k$	M	BPØLY	11,11	
		OI,OJ		

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SYMBOL	PRE	LOC.	VALUE	EXP.
	E	ND		
$k=1,2,3 \ldots N_P$				

X3

THE SHELL COMPONENT INPUT DATA

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SYMBOL	P/E (1, 19, 37, 55)	LOC. (2, 20, 38, 56)	VALUE (7, 25, 43, 61)	EXP. (17, 35, 53, 71)
a	I	INA		
U	I	IU		
V	I	IV		
$\bar{U}$	I	IUB		
$\bar{V}$	I	IVB		
$(ID)_1$	I	IDVT		
$(ID)_2$	I			
.	.			
.	.			
.	.			
$(ID)_{\bar{U}+\bar{V}}$	I			
k	I	INK		
φ_o		PHIN		
L		XL		
R_2		R2IN		
$\bar{b}$		BBAR		
$\bar{a}$		ABAR		
$(C_{11})_1$		C11I		
$(C_{11})_2$				
$(C_{12})_1$		C12I		
$(C_{12})_2$				
$(C_{22})_1$		C22I		
$(C_{22})_2$				
$(C_{33})_1$		C33I		
$(C_{33})_2$				
$(C_{33})_3$				
$(C_{33})_4$				
$(C_{34})_1$		C34I		
$(C_{34})_2$				
$(C_{34})_3$				
$(C_{34})_4$				
$(C_{44})_1$		C44I		
$(C_{44})_2$				

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PROBLEM NO. ______ KEYPUNCHED BY ______

NO. OF CARDS ______ VERIFIED BY ______

X3

SYMBOL	PRE	LOC.	VALUE	EXP.
$(C_{44})_3$				
$(C_{44})_4$				
$(t)_1$		THI		
$(t)_2$				
Y_a		DESTA		
M_a		AM		
H_i		HI		
w_i		DESTI		
P_i		PRI		
H_e		HE		
w_e		DESTE		
P_e		PRE		
W		WFØRC		
	E	ND		

X3

THE FLUID COMPONENT INPUT DATA

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DATE ____________ PAGE ____ OF ____

NAME ____________ PRIORITY ____________

PROBLEM NO. ____________ KEYPUNCHED BY ____________

NO. OF CARDS ____________ VERIFIED BY ____________

X3

SYMBOL	P R E	LOC.	VALUE	EXP.
b	I	INB		
a_1	I	IDA1		
a_2	I	IDA2		
a_3	I	IDA3		
H		HF		
γ		DESTF		
M		FM		
	E	ND		

X3

THE SPRING-MASS COMPONENT INPUT DATA IV-12

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PROBLEM NO.

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SYMBOL	PRE (1, 19, 37, 55)	LOC. (2, 20, 38, 56)	VALUE (7, 25, 43, 61)	EXP. (17, 35, 53, 71)
c	I	INC		
n	I	IN		
$[K]_c$	M	CK	10, 10	
		OI, OJ		
$[M]_c$	M	CM	10, 10	
		OI, OJ		
$(IDC)_1$	I	IDVTC		
$(IDC)_2$	I			
.	.			
.	.			
.	.			
$(IDC)_n$	I			
	E	ND		

X3

10. NOTATION

$\lfloor\ \rfloor_{(1xj)}$	Row matrix of order j
$\{\ \}_{(ix1)}$	Column matrix of order i
$[\ ]_{(ixj)}$	Rectangular matrix with i rows and j columns
a, b, c	Identification number for the shell, fluid and spring-mass components respectively
a, b	Semimajor and semiminor axes, respectively, of an ellipsoidal bulkhead
$\bar{a}$, $\bar{b}$	Edge radius and height, respectively, of an ellipsoidal bulkhead
a1, a2, a3	Identification numbers for the shell components which enclose a fluid component
$[A]_{(\bar{U}x11)} \equiv [a_{kn}]$	Polynomial matrix associated with $u_k(\xi)$
$[B]_{(\bar{V}x11)}$ $[b_{\ell n}]$	Polynomial matrix associated with $v_\ell(\xi)$
$\{C_e\}$	Coordinate identification vector for component e
C_r	r^{th} component of $\{C_e\}$
$C_{11}(\xi)$, $C_{12}(\xi)$, $C_{22}(\xi)$	Orthotropic stress-strain coefficients
$C_{33}(\xi)$, $C_{34}(\xi)$, $C_{44}(\xi)$	Orthotropic moment-curvature coefficients
ds	Differential meridional distance along shell
$(\ )^{\cdot} \equiv \frac{d(\ )}{ds}$	Derivative with respect to meridional distance s
D_1, D_2	Constants used to determine the rotation vector $\{\rho\}$
$[D_1]_{(2x11)}$, $[D_2]_{(2x11)}$	Matrices used in the definition of the rotation vector $\{\rho\}$

e	General identification number for vehicle components which may stand for a or b
g	Ratio of vehicle acceleration to acceleration of gravity
H	Total fluid level measured positive upward from the base of a2
$\bar{H}_1, \bar{H}_2, \bar{H}_3$	Fluid levels associated with tank shell sections a1, a2, a3
$\left[K\right]_{(N_c x N_c)}$	Total launch vehicle stiffness matrix
$\left[K_a\right]_{\left((\bar{U}+\bar{V})x(\bar{U}+\bar{V})\right)}$	Stiffness matrix for shell component a associated with the system coordinates
$\left[\bar{K}_a\right]_{\left((\bar{U}+\bar{V})x(\bar{U}+\bar{V})\right)}$	Stiffness matrix for shell component a associated with the local coordinates
$\left[K_c\right]$	Stiffness matrix of spring-mass component c associated with the system coordinates
$\lfloor K_s \rfloor$	Row s $\left(s = 1, 2, \cdots (\bar{U} + \bar{V})_a\right)$ of $\left[K_a\right]$
K_ϕ	Shell meridional curvature
K_θ	Shell hoop curvature
L	Length of conical shell
m_t	Generalized mass for mode t
$\left[M\right]$	Total mass matrix for the launch vehicle
$\left[M_a\right]$	Mass matrix for shell component a associated with the system coordinates
$\left[\bar{M}_a\right]$	Mass matrix for shell component a associated with the local coordinates
$\left[M_b\right]$	Mass matrix for fluid component b associated with the system coordinates

$\left[\bar{M}_b\right]$	Mass matrix for fluid component b associated with the local coordinates
$\left[M_c\right]$	Mass matrix for spring-mass component c associated with the system coordinates
N_C	Total number of system coordinates used in the vehicle model
N_F	Total number of fluid components used in the vehicle model
N_M	Total number of spring-mass components used in the vehicle model
N_O	Total number of fixed coordinates
N_S	Total number of shell components used in the vehicle model
N_ϕ^o	Initial meridional stress
p	Circular frequency of the vehicle
p_t	Circular frequency of the vehicle for mode t
$\{P\}\left((N_C-N_O)x1\right)$	Applied load vector
Q_t	Generalized force acting on mode t
r	Radial distance from the vehicle longitudinal axis to each point on the shell
$\hat{r}$	Radial distance from the vehicle longitudinal axis to each point in the fluid
r_1	Meridional radius of curvature of the shell
r_2	Hoop radius of curvature of the shell
$\{R\}\left(N_C x1\right)$	Displacement response vector
$\{\bar{R}\}$	Displacement response amplitude vector

$\{S_a\}_{((\bar{U}+\bar{V})\times 1)}$	Vector of internal forces or moments acting at each point of shell component a
$\{\bar{S}\}$	Amplitude vector of internal forces $\{S_a\}$
t	Identification number for a particular mode of vibration
t	Time
$[T_a]_{((\bar{U}+\bar{V})\times(\bar{U}+\bar{V}))}$	Transformation matrix which relates local to system coordinates in shell component a
$[T_b]_{(\bar{W}\times\bar{W})}$	Transformation matrix which relates local to system coordinates in fluid component b
$\{U\}_{((\bar{U}+\bar{V})\times 1)}$, $\{V\}_{((\bar{U}+\bar{V})\times 1)}$	Longitudinal and radial displacement vectors, respectively, for shell components
$u_k(\xi)$, $v(\xi)$	Generalized longitudinal and radial displacements, respectively, for shell components
$\{\hat{u}(x)\}_{(\bar{W}\times 1)} = \{\hat{u}_m(x)\}$ and $\{\hat{v}(x, \hat{r})\}_{(\bar{W}\times 1)} = \{\hat{v}_m(x, \hat{r})\}$	Longitudinal and radial fluid displacements, respectively
U, V	Total number of longitudinal and radial system coordinates, respectively, associated with each shell component
$\bar{U}$, $\bar{V}$	Total number of longitudinal and radial local coordinates, respectively, associated with each shell component
$[U]_{(U\times 11)}$, $[V]_{(V\times 11)}$	Constant matrices used in the definition of the system coordinates
w_i, w_e	Fluid weight densities interior and exterior, respectively, to each shell component
$\bar{w}_i$, $\bar{w}_e$	Effective fluid weight densities equal to, respectively, gw_i and gw_e
$\bar{W}$	The sum of $\bar{U} + \bar{V}$ for the three shells surrounding a fluid component

x	Longitudinal axis of the launch vehicle
z_t	Structural impedance for mode t
$\{\alpha_a\}_{((\bar{U}+\bar{V})x1)}$	Modal vector whose components are the longitudinal, radial and rotational system coordinate displacements for shell component a
$\{\bar{\alpha}_a\}_{((\bar{U}+\bar{V})x1)}$	Consolidated vector of local coordinates
$\{\bar{\alpha}_k\}_{(\bar{U}x1)}$, $\{\bar{\beta}_\ell\}_{(\bar{V}x1)}$	Generalized coordinates in the longitudinal and radial directions, respectively
$[\alpha]_{(N_C-N_O)x(N_C-N_O)}$	Modal matrix with each column representing a vehicle mode shape
$\lfloor \alpha_s \rfloor_{(1x(N_C-N_O))}$	Each row of $[\alpha]$ which is a system coordinate "s" displacement vector with natural mode elements
$\{\alpha_t\}_{(N_C-N_O)x1}$	Each column of $[\alpha]$ which is a mode "t" displacement vector with system coordinate elements
γ_a, γ_b	Mass densities for shell and fluid components respectively
$\{\bar{\delta}\}_{((N_C-N_O)x1)}$	Vector of steady-state displacement phase angles
$\{\bar{\bar{\delta}}\}_{((\bar{U}+\bar{V})x1)}$	Vector of phase angles for the internal forces of each shell component
$[\Delta e]_{((\bar{U}+\bar{V})xN_C)} = [\delta_{rs}]$	Transformation matrix relating total system displacements to component 3 system displacements
ϵ_ϕ, ϵ_θ	Meridional and hoop strains, respectively, for the shell components
η_t	Ratio of actual damping to the critical damping for each mode t
$\{\xi\}$	Nondimensional variable describing location on each shell component

ρ Meridional rotation of the shell components

ϕ Meridional angle

ω Frequency of the forcing function in radians per second

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APPENDIX A

SHELL GEOMETRY AND ENERGY EXPRESSIONS

The additional potential energy of a shell of revolution due to axisymmetric deformations is given in the form

$$V = \frac{1}{2} \int_s 2\pi r \left(N_\phi \epsilon_\phi + N_\theta \epsilon_\theta + M_\phi K_\phi + M_\theta K_\theta + N_\phi^{\,o} \rho^2 \right) ds \qquad \text{(A. 1)}$$

in which the last term represents the work done by the initial meridional stress, $N_\phi^{\,o}$.[7] The initial hoop stress does not make a similar contribution to the potential energy since there is zero rotation in the hoop direction. In the notation of Flugge,[8] the strains (ϵ_ϕ, ϵ_θ), curvatures (K_ϕ, K_θ) and the meridional rotation ρ are expressed in terms of the displacements $\overline{v}$ and $\overline{w}$ (see Figure A. 1) as follows:

$$\epsilon_\phi = \frac{1}{r_1} \left(\frac{d\overline{v}}{d\phi} + \overline{w} \right) \qquad \text{(A. 2)}$$

$$\epsilon_\theta = \frac{1}{r_2} \left(\overline{v} \cot \phi + \overline{w} \right) \qquad \text{(A. 3)}$$

$$K_\phi = \frac{1}{r_1} \frac{d}{d\phi} \left[\frac{1}{r_1} \left(\frac{d\overline{w}}{d\phi} - \overline{v} \right) \right] \qquad \text{(A. 4)}$$

$$K_\theta = \frac{\cot \phi}{r_2} \left[\frac{1}{r_1} \cdot \left(\frac{d\overline{w}}{d\phi} - \overline{v} \right) \right] \qquad \text{(A. 5)}$$

$$\rho = \frac{1}{r_1} \left(\frac{d\overline{w}}{d\phi} - \overline{v} \right) \qquad \text{(A. 6)}$$

where r_1 and r_2 are the radii of curvature of the shell in the meridional and hoop directions, respectively. Hookes law for an orthotropic shell with the principal directions in the hoop and meridional directions takes the form:[9]

$$\begin{Bmatrix} N_\phi \\ N_\theta \\ M_\phi \\ M_\theta \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 \\ 0 & 0 & C_{33} & C_{34} \\ 0 & 0 & C_{34} & C_{44} \end{bmatrix} \begin{Bmatrix} \epsilon_\phi \\ \epsilon_\theta \\ K_\phi \\ K_\theta \end{Bmatrix} \tag{A.7}$$

The shell configurations to be considered for the bulkheads and the tank walls are listed below with their corresponding geometric parameters defined:

1. Conical Shell (Figure A. 2)

   a) General Case

$$\begin{aligned} \phi &= \phi_o \\ r_1 &= \infty \\ r_2 &= (R_2/\sin\phi_o) - S\cot\phi_o \end{aligned} \tag{A.8}$$

   b) Cylinder (radius = R)

$$\begin{aligned} \phi &= \frac{\pi}{2} \\ r_1 &= \infty \\ r_2 &= R \end{aligned} \tag{A.9}$$

2. Ellipsoid (Figure A. 3)

   a) General Case (a = semimajor axis, b = semiminor axis)

$$\begin{aligned} r_1 &= \frac{a^2 b^2}{\left(a^2 \sin^2\phi + b^2 \cos^2\phi\right)^{3/2}} \\ r_2 &= \frac{a^2}{\left(a^2 \sin^2\phi + b^2 \cos^2\phi\right)^{1/2}} \end{aligned} \tag{A.10}$$

b) Hemisphere (radius = R)

$$r_1 = R$$
$$r_2 = R \tag{A.11}$$

For the present analysis, the longitudinal displacement u and the radial displacement v will be more convenient. They are related to the displacements $\overline{v}$ and $\overline{w}$ by the transformation

$$\overline{v} = -u \sin \phi + v \cos \phi$$
$$\overline{w} = u \cos \phi + v \sin \phi \tag{A.12}$$

Substitution of this transformation into the strains, curvature and rotation yields

$$\epsilon_\phi = \frac{1}{r_1} \left(-\frac{du}{d\phi} \sin \phi + \frac{dv}{d\phi} \cos \phi\right) \tag{A.13}$$

$$\epsilon_\theta = \frac{v}{r} \tag{A.14}$$

$$K_\phi = \frac{1}{r_1} \frac{d}{d\phi} \left[\frac{1}{r_1} \left(\frac{du}{d\phi} \cos \phi + \frac{dv}{d\phi} \sin \phi\right)\right] \tag{A.15}$$

$$K_\theta = \frac{\cos \phi}{r} \left[\frac{1}{r_1} \left(\frac{du}{d\phi} \cos \phi + \frac{dv}{d\phi} \sin \phi\right)\right] \tag{A.16}$$

$$\rho = \frac{1}{r_1^2} \left(\frac{du}{d\phi} \cos \phi + \frac{dv}{d\phi} \sin \phi\right) \tag{A.17}$$

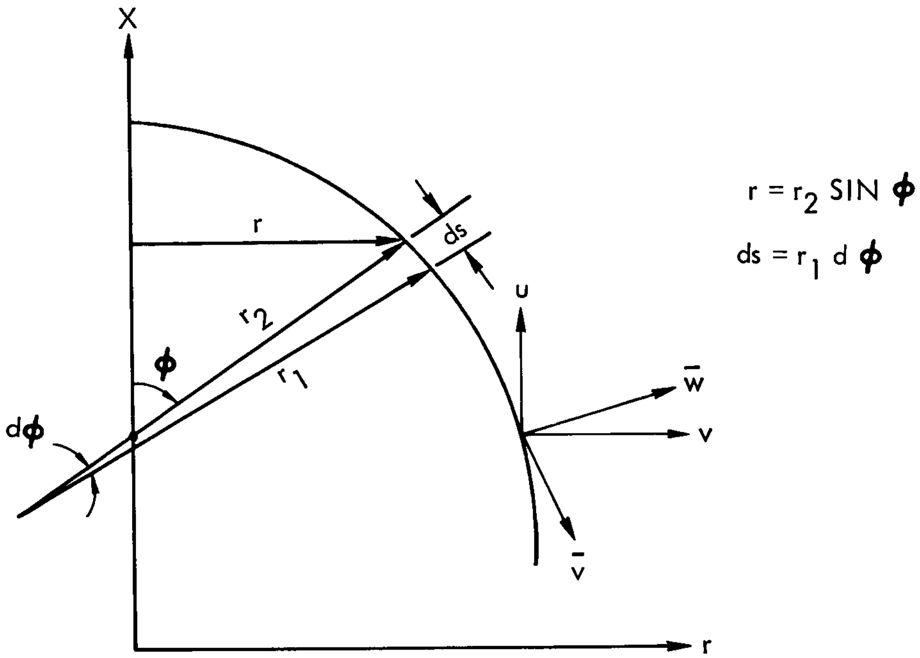


Figure A.1. Meridian of Shell of Revolution

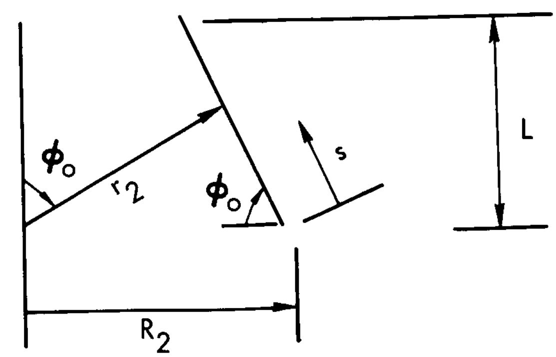


Figure A.2. Conical Shell

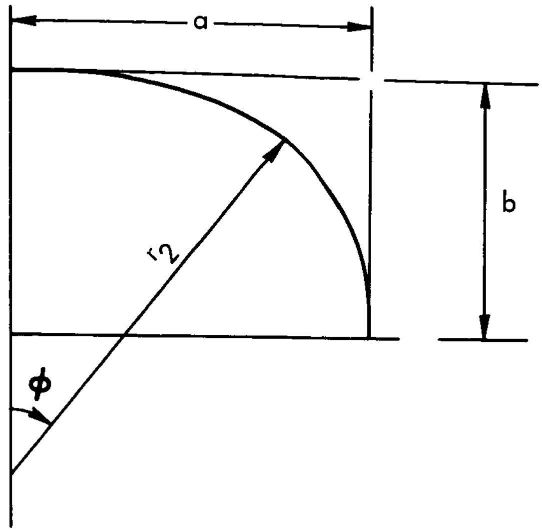


Figure A. 3. Ellipsoidal Shell

APPENDIX B

INITIAL STRESSES IN SHELL ELEMENTS

In the formulation of the shell stiffness matrix, the effect of the initial meridional stresses is included. Expressions for these stresses are developed below for each of the shell elements to be considered. These derivations are based on membrane theory which is a good first order approximation except in very localized areas where bending predominates.

Conical Element

The derivation of the initial stresses in a conical element is divided into three steps. The notation is listed below and shown in Figure B. 1.

R_2	lower radius of the conic
ϕ_o	meridional angle
L	length of element
$\overline{w}_i$	effective density of interior fluid
h_i	location of interior fluid surface
$\overline{w}_e$	effective density of exterior fluid
h_e	location of exterior fluid surface
p_i	uniform internal pressure
p_e	uniform external pressure
$\pm W$	reactive force at top of conic $\left\{ \begin{array}{l} + \text{ produces tensile stress} \\ - \text{ produces compressive stress} \end{array} \right\}$

1. Stress due to uniform pressures and reactive force

$$N_{\phi_1} = \frac{1}{2\pi r \sin\phi_o}\left[W + (p_i - p_e)\,\pi\left(r^2 - R_1^{\,2}\right)\right] \qquad 0 \leq x \leq L \qquad \text{(B. 1)}$$

2. Stress due to interior fluid

Case 1: $h_i \leq L$

$$N_{\phi_2} = \frac{-1}{r \sin \phi_o} \int_x^{h_i} pr\, dr$$

$$= \frac{-1}{r \sin \phi_o} \int_x^{h_i} \bar{w}_i (h_i - x)(R_2 - x \cot \phi_o)(-d\,x \cot \phi_o)$$

$$= + \frac{\bar{w}_i \cot \phi_o}{r \sin \phi_o} \left[h_i (h_i - x) R_2 - \frac{1}{2}\left(h_i^2 - x^2\right)(R_2 + h_i \cot \phi_o) + \frac{1}{3}\left(h_i^3 - x^3\right) \cot \phi_o \right] \qquad x \leq h_i \leq L \qquad \text{(B. 2)}$$

Case 2: $h_i > L$

$$N_{\phi_2} = \frac{-1}{r \sin \phi_o} \int_x^{L} pr\, dr$$

$$= + \frac{\bar{w}_i \cot \phi_o}{r \sin \phi_o} \left[h_i (L - x) R_2 - \frac{1}{2}(L^2 - x^2)(R_2 + h_i \cot \phi_o) + \frac{1}{3}(L^3 - x^3) \cot \phi_o \right] \qquad 0 \leq x \leq L \qquad \text{(B. 3)}$$

3. Stress due to exterior fluid

Case 1: $h_e \leq L$

$$N_{\phi_3} = \frac{-1}{r \sin \phi_o} \int_x^{h_e} pr\, dr$$

$$- \frac{\bar{w}_e \cot \phi_o}{r \sin \phi_o} \left[h_e (h_e - x) R_2 - \frac{1}{2}\left(h_e^2 - x^2\right)(R_2 + h_e \cot \phi_o) + \frac{1}{3}\left(h_e^3 - x^3\right) \cot \phi_o \right] \qquad 0 \leq x \leq h_e \leq L \qquad \text{(B. 4)}$$

Case 2: $h_e > L$

$$N_{\phi_3} = \frac{-1}{r \sin \phi_o} \int_x^L pr\, dr$$

$$= \frac{-\overline{w}_e \cot \phi_o}{r \sin \phi_o} \Big[h_e (L - x) R_2 - \frac{1}{2} (L^2 - x^2) (R_2 + h_e \cot \phi_o)$$

$$+ \frac{1}{3} (L^3 - x^3) \cot \phi_o \Big] \qquad 0 \leq x \leq L \qquad (B.5)$$

The total meridional initial stress on the conic will then be the sum of the above stresses

$$N_\phi = N_{\phi_1} + N_{\phi_2} + N_{\phi_3} \qquad (B.6)$$

Ellipsoidal Element

The expressions for the initial stress for an ellipsoidal element are developed separately for the upright and inverted bulkheads. The notation for both cases is listed below and shown in Figures B.2 and B.3.

$\bar{a}$	radius of base
$\bar{b}$	height of element
ϕ_o	slope of meridian (Note: The semimajor and semiminor axes, a and b, can be computed from $\bar{a}$, $\bar{b}$, and ϕ_o.)
$\overline{w}_i$	effective density of interior fluid
h_i	location of surface of interior fluid
$\overline{w}_e$	effective density of exterior fluid
h_e	location of surface of exterior fluid
p_i	uniform internal pressure
p_e	uniform external pressure

<u>Upright Bulkhead</u> (Figure B. 2)

1. Stress for exterior fluid

Case 1: $h_e \leq \bar{b}$

$$N_{\phi_1} = -\frac{p_e r_2}{2} + \frac{1}{2\pi r \sin\phi} \int_{\bar{r}}^{r} 2\pi r p \, dr$$

where

$$\bar{r} = a\sqrt{1 - \frac{\left(h_e + b - \bar{b}\right)^2}{b^2}} = a\sqrt{1 - \frac{H_e^{\,2}}{b^2}}$$

$$p = -\bar{w}_e \,(h_e - x) \qquad \text{(B. 7)}$$

$$x + (b - \bar{b}) = b\sqrt{1 - \frac{r^2}{a^2}}$$

This finally leads to

$$N_{\phi_1} = -\frac{p_e r_2}{2} - \frac{\bar{w}_e}{r \sin\phi}\left[\frac{H_e}{2}\,(r^2 - \bar{r}^2) - \frac{a^2}{3b^2}\,H_e^{\,3} + \frac{a^2 b}{3}\left(1 - \frac{r^2}{a^2}\right)^{3/2}\right] \quad \text{for } \bar{r} \leq r \leq \bar{a}$$

$$N_{\phi_1} = -\frac{p_e r_2}{2} \quad \text{for } 0 \leq r \leq \bar{r} \qquad \text{(B. 8)}$$

Case 2: $h_e > \bar{b}$

$$N_{\phi_1} = -\frac{p_e r_2}{2} + \frac{1}{2\pi r \sin\phi} \int_{o}^{r} 2\pi r p \, dr$$

$$= -\frac{p_e r_2}{2} - \frac{\bar{w}_e}{r \sin\phi}\left[\frac{H_e r^2}{2} + \frac{a^2 b}{3}\left(1 - \frac{r^2}{a^2}\right)^{3/2} - \frac{a^2 b}{3}\right] \quad \text{for } 0 \leq r \leq \bar{a} \quad \text{(B. 9)}$$

2. Stress for interior fluid

$$N_{\phi_2} = \frac{p_i r_2}{2} + \frac{1}{2\pi r \sin\phi} \int_{\bar{\bar{r}}}^{r} 2\pi r p \, dr \qquad \text{(B. 10)}$$

where

$$\bar{\bar{r}} = a \sqrt{1 - \frac{(h_i + b - b)^2}{b^2}} = a \sqrt{1 - \frac{H_i^2}{b^2}}$$

$$p = \bar{w}_i \, (h_i - x)$$

or, after integration,

$$N_{\phi_2} = \frac{p_i r_2}{2} + \frac{\bar{w}_i}{r \sin\phi} \left[\frac{H_i}{2} (r^2 - \bar{\bar{r}}^2) - \frac{a^2}{3b^2} H_i^3 + \frac{a^2 b}{3} \left(1 - \frac{r^2}{a^2}\right)^{3/2} \right]$$

for $\bar{\bar{r}} < r \leq \bar{a}$

(B. 11)

$$N_{\phi_2} = \frac{p_i r_2}{2}$$

for $0 \leq r \leq \bar{\bar{r}}$

<u>Inverted Bulkhead</u> (Figure B. 3)

1. Stress for interior fluid

$$N_{\phi_1} = \frac{p_i r_2}{2} + \frac{1}{2\pi r \sin\phi} \int_{o}^{r} 2\pi r p \, dr \qquad \text{(B. 12)}$$

for

$$r \leq \bar{r} = a \sqrt{1 - \frac{(h_i + b - \bar{b})^2}{b^2}}$$

$$= a \sqrt{1 - \frac{H_i^2}{b^2}} \qquad h_i \geq 0$$

or

$$= \bar{a} \qquad h_i < 0$$

where

$$p = w_i \, (x - h_i) \qquad x \geq h_i$$

This leads to the following expressions for the stress:

$$N_{\phi_1} = \frac{p_i r_2}{2} - \frac{\overline{w}_i}{r \sin\phi}\left[\frac{H_i}{2} r^2 + \frac{a^2 b}{3}\left(1 - \frac{r^2}{a^2}\right)^{3/2} - \frac{a^2 b}{3}\right] \quad 0 \leq r \leq \bar{r}$$

$$N_{\phi_1} = \frac{p_i r_2}{2} - \frac{\overline{w}_i}{r \sin\phi}\left[\frac{H_i}{2} \bar{r}^2 + \frac{a^2 b}{3}\left(1 - \frac{\bar{r}^2}{a^2}\right)^{3/2} - \frac{a^2 b}{3}\right] \quad \bar{r} \leq r \leq \bar{a} \tag{B. 13}$$

2. Stress for lower fluid

$$N_{\phi_2} = -\frac{p_e r_2}{2} + \frac{1}{2\pi r \sin\phi}\int_o^r 2\pi r p \, dr$$

for

$$r \leq \bar{\bar{r}} = a\sqrt{1 - \frac{\left(h_e + b - \bar{b}\right)^2}{b^2}} = a\sqrt{1 - \frac{H_e^2}{b^2}} \tag{B. 14}$$

where

$$p = -\overline{w}_e \,(x - h_e) \qquad x \geq h_e$$

or, after integration

$$N_{\phi_2} = -\frac{p_e r_2}{2} + \frac{\overline{w}_e}{r \sin\phi}\left[\frac{H_e}{2} r^2 + \frac{a^2 b}{3}\left(1 - \frac{r^2}{a^2}\right)^{3/2} - \frac{a^2 b}{3}\right] \text{for } 0 \leq r \leq \bar{\bar{r}}$$

$$N_{\phi_2} = \frac{-p_e r_2}{2} + \frac{\overline{w}_e}{r \sin\phi}\left[\frac{H_e}{2} \bar{\bar{r}}^2 + \frac{a^2 b}{3}\left(1 - \frac{\bar{\bar{r}}^2}{a^2}\right)^{3/2} - \frac{a^2 b}{3}\right] \quad \bar{\bar{r}} \leq r \leq \bar{a} \tag{B. 15}$$

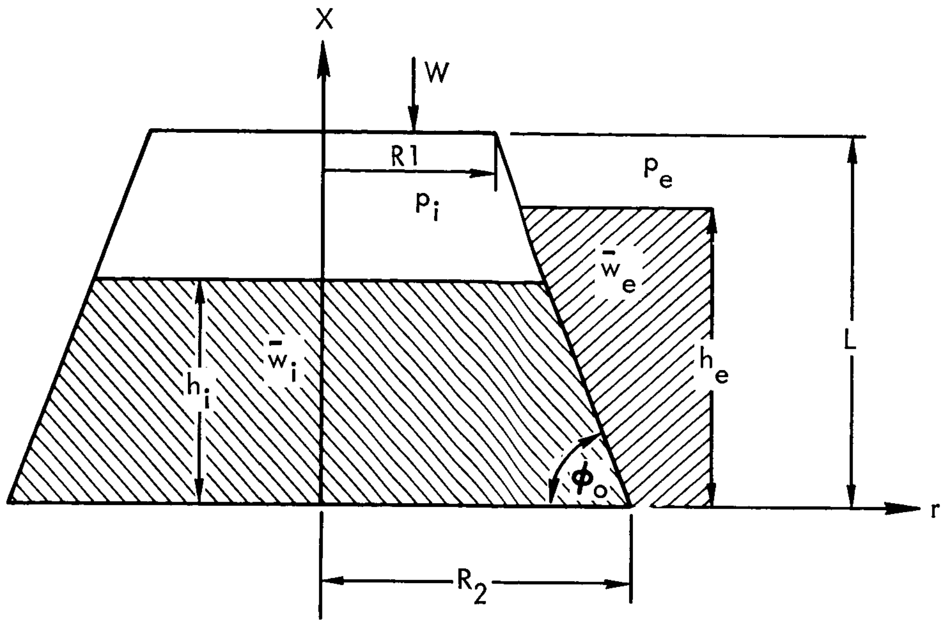


Figure B. 1. Conic

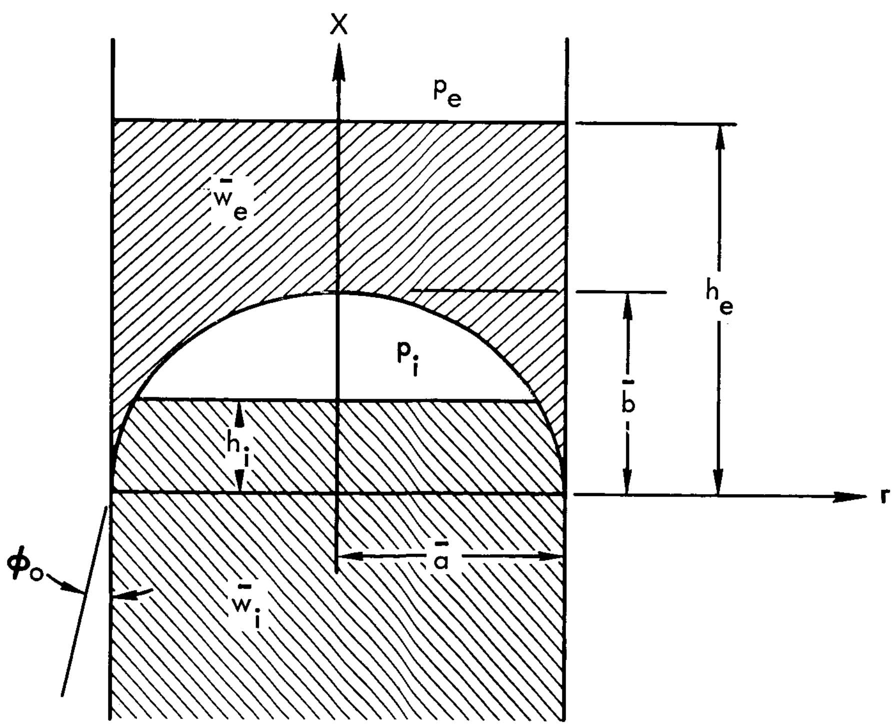


Figure B. 2. Upright Ellipsoidal Bulkhead

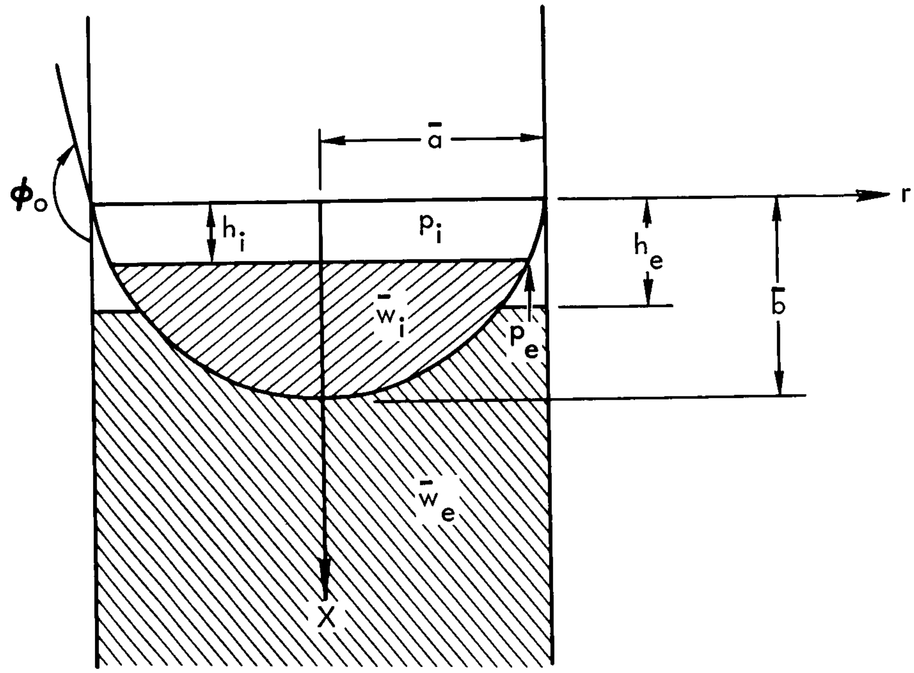


Figure B.3. Inverted Ellipsoidal Bulkhead

APPENDIX C

FLUID MASS MATRIX EXPRESSIONS

Detailed expressions are given below for evaluating Equation (4.4) to obtain the fluid component mass matrix. Three cases are involved depending upon whether the upper and lower tank bulkheads are convex upward or convex downward (see Section 2.0). A general description of the program operations utilized to construct the mass matrix for the fluid tanks is described in Appendix E. A brief flow chart followed by the relevant equations in a form suitable for a weighting matrix integration scheme is included to provide the user with a more basic understanding of the digital program.

C.1 CASE I. UPPER BULKHEAD CONVEX UPWARD, LOWER BULKHEAD CONVEX DOWNWARD

Equation (4.6) for the general tank shell can be expressed as the sum of contributions to $\overline{M}_b$ from fluid in three sections of the tank, as illustrated in Figure C.1 for Case I. One of these sections is that located above the base of the upper bulkhead (shell 1). A second lies below the top of lower bulkhead (shell 2). The third lies between the two bulkheads. In general, the motion of the fluid in any section may be affected by the generalized coordinate distortions associated with all three tank shell components. Thus, Equation (4.6) is expressed as

$$\frac{1}{\pi\gamma_b}\left[\overline{M}_b\right] = \sum_{m=m_o}^{m=3} \int_o^{\overline{H}(m)} \left(\frac{4}{r_{(m)}^2} \left\{ \frac{r_{(m)}^2}{2} \hat{u}_{(m)}(x) \right\} \left\{ \frac{r_{(m)}^2}{2} \hat{u}_{(m)}(x) \right\}^T + \frac{1}{2} \left\{ r_{(m)} \hat{v}_{(m)}(x, r) \right\} \left\{ r_{(m)} \hat{v}_{(m)}(x, r) \right\}^T \right) dx \tag{C.1}$$

where m_o = 1, 2 or 3, depending upon whether the fluid surface lies within the range of tank section (1), (2) or (3), as illustrated in Figure C.1. $\overline{H}_{(m)}$ is the depth of fluid within tank section (m), $r_{(m)}$ is the radius of shell (m) and, similarly, $\hat{u}(x)$ and $\hat{v}_{(m)}(x, r)$ represent the fluid motions within tank section m. The specific form of the column matrix $\left\{\hat{u}_{(m)}(x)\right\}$ in terms of the component shell generalized coordinates is

$$\left\{\frac{r_{(1)}^2}{2}\hat{u}_{(1)}(x)\right\} = \left\{\begin{array}{c} -\int_o^x r_{(1)} \cot\phi_{(1)}\left\{u_{(1)}(\xi)\right\} dx \\ -\int_o^x r_{(1)}\left\{v_{(1)}(\xi)\right\} dx \\ \hdashline -\int_o^{\overline{H}_{(2)}} r_{(2)} \cot\phi_{(2)}\left\{u_{(2)}(\xi)\right\} dx \\ -\int_o^{\overline{H}_{(2)}} r_{(2)}\left\{v_{(2)}(\xi)\right\} dx \\ \hdashline -\int_o^{\overline{H}_{(3)}} r_{(3)} \cot\phi_{(3)}\left\{u_{(3)}(\xi)\right\} dx \\ -\int_o^{\overline{H}_{(3)}} r_{(3)}\left\{v_{(3)}(\xi)\right\} dx \end{array}\right\} \tag{C. 2}$$

$$\left\{\frac{r_{(2)}^2}{2}\hat{u}_{(2)}(x)\right\} = \left\{\begin{array}{c} 0 \\ \hdashline -\int_o^x r_{(2)} \cot\phi_{(2)}\left\{u_{(2)}(\xi)\right\} dx \\ -\int_o^x r_{(2)}\left\{v_{(2)}(\xi)\right\} dx \\ \hdashline -\int_o^{\overline{H}_{(3)}} r_{(3)} \cot\phi_{(3)}\left\{u_{(3)}(\xi)\right\} dx \\ -\int_o^{\overline{H}_{(3)}} r_{(3)}\left\{v_{(3)}(\xi)\right\} dx \end{array}\right\} \tag{C. 3}$$

and

$$\left\{\frac{r_{(3)}^2}{2}\hat{u}_{(3)}(x)\right\} = \left\{\begin{array}{c} 0 \\ \hdashline 0 \\ \hdashline -\int_o^x r_{(3)} \cot \phi_{(3)} \left\{u_{(3)}(\xi)\right\} dx \\ -\int_o^x r_{(3)} \left\{v_{(3)}(\xi)\right\} dx \end{array}\right\} \qquad (C.4)$$

where

$$\left\{u_{(m)}(\xi)\right\} = \left\{\begin{array}{c} u_1(\xi) \\ \cdot \\ \cdot \\ \cdot \\ u_{\overline{U}}(\xi) \end{array}\right\}$$

and

$$\left\{v_{(m)}(\xi)\right\} = \left\{\begin{array}{c} v_1(\xi) \\ \cdot \\ \cdot \\ \cdot \\ v_{\overline{V}}(\xi) \end{array}\right\}$$

are the generalized coordinate displacement vectors for tank shell component m, and $\phi_{(m)}$ is the meridional angle for tank shell component m, and $\overline{U}$ and $\overline{V}$ are the number of longitudinal and radial generalized coordinates, respectively, in shell component m.

The column matrix $\left\{r_{(m)}\hat{v}_{(m)}(x, r)\right\}$ in terms of the component shell generalized coordinates is

$$\left\{r_{(1)}\hat{v}_{(1)}(x, r)\right\} = \left\{\begin{array}{c} r_{(1)} \cot \phi_{(1)} \left\{u_{(1)}(\xi)\right\} \\ r_{(1)} \left\{v_{(1)}(\xi)\right\} \\ \hdashline 0 \\ \hdashline 0 \end{array}\right\} - \frac{2 \cot \phi_{(1)}}{r_{(1)}} \left\{\frac{r_{(1)}^2}{2}\hat{u}_{(1)}(x)\right\} \qquad (C.5)$$

$$\left\{ r_{(2)} \hat{v}_{(2)}(x, r) \right\} = \left\{ \begin{array}{c} 0 \\ \hline r_{(2)} \cot \phi_{(2)} \left\{ u_{(2)}(\xi) \right\} \\ r_{(2)} \left\{ v_{(2)}(\xi) \right\} \\ \hline 0 \end{array} \right\} - \frac{2 \cot \phi_{(2)}}{r_{(2)}} \left\{ \frac{r_{(2)}^2}{2} \hat{u}_{(2)}(x) \right\} \quad (C.6)$$

$$\left\{ r_{(3)} \hat{v}_{(3)}(x, r) \right\} = \left\{ \begin{array}{c} 0 \\ \hline 0 \\ \hline r_{(3)} \cot \phi_{(3)} \left\{ u_{(3)}(\xi) \right\} \\ r_{(3)} \left\{ v_{(3)}(\xi) \right\} \end{array} \right\} - \frac{2 \cot \phi_{(3)}}{r_{(3)}} \left\{ \frac{r_{(3)}^2}{2} \hat{u}_{(3)}(x) \right\} \quad (C.7)$$

C.2 CASE II. UPPER BULKHEAD CONVEX DOWNWARD, LOWER BULKHEAD CONVEX DOWNWARD

The Case II configuration is illustrated in Figure C.2 which also defines the tank sections m for this case. Equation (C.1) for the fluid in tank section (1) is modified as follows. The fluid motion $\left\{\hat{u}(x)\right\}$ in tank section (1) is given by

$$
\left\{\hat{u}_{(1)}(x)\right\} = \frac{2}{\left(r_{(2)}^{\,2} - r_{(1)}^{\,2}\right)} \left\{ \begin{array}{l}
\displaystyle\int_o^x r_{(1)} \cot\phi_{(1)} \left\{u_{(1)}(\xi)\right\} dx \\
\displaystyle\int_o^x r_{(1)} \left\{v_{(1)}(\xi)\right\} dx \\
\hdashline
\displaystyle -\int_o^x r_{(2)} \cot\phi_{(2)} \left\{u_{(2)}(\xi)\right\} dx - \int_o^{\overline{H}_2} r_{(2)} \cot\phi_{(2)} \left\{u_{(2)}(\xi)\right\} dx \\
\displaystyle -\int_o^x r_{(2)} \left\{v_{(2)}(\xi)\right\} dx - \int_o^{\overline{H}_2} r_{(2)} \left\{v_{(2)}(\xi)\right\} dx \\
\hdashline
\displaystyle -\int_o^{\overline{H}_3} r_{(3)} \cot\phi_{(3)} \left\{u_{(3)}(\xi)\right\} dx \\
\displaystyle -\int_o^{\overline{H}_3} r_{(3)} \left\{v_{(3)}(\xi)\right\} dx
\end{array} \right\} \tag{C.8}
$$

The fluid motion $\left\{\hat{v}(x, \hat{r})\right\}$ in tank section (1) is defined by

$$
\left\{\hat{v}_{(1)}(x, \hat{r})\right\} = \frac{\left(\hat{r} - r_{(2)}\right)}{\left(r_{(1)} - r_{(2)}\right)} \left(\left\{ \begin{array}{c}
\cot\phi_{(1)} \left\{u_{(1)}(\xi)\right\} \\
\left\{v_{(1)}(\xi)\right\} \\
\hdashline
0 \\
\hdashline
0
\end{array} \right\} - \cot\phi_{(1)} \left\{\hat{u}_{(1)}(x)\right\} \right)
$$

$$
- \frac{\left(\hat{r} - r_{(1)}\right)}{\left(r_{(1)} - r_{(2)}\right)} \left(\left\{ \begin{array}{c}
0 \\
\hdashline
\cot\phi_{(2)} \left\{u_{(2)}(\xi)\right\} \\
\left\{v_{(2)}(\xi)\right\} \\
\hdashline
0
\end{array} \right\} - \cot\phi_{(2)} \left\{\hat{u}_{(2)}(x)\right\} \right) \tag{C.9}
$$

Upon substitution of Equations (C. 1) and (C. 2) into Equation (4. 1) and integration with respect to $\hat{r}$ between the limits of $r_{(1)}$ and $r_{(2)}$, one obtains the following expression for the m = 1 portion of Equation (C. 1).

$$\frac{1}{\pi \gamma_b}\left[\overline{M}_b\right]_{(1)}$$

$$= \int_o^{\overline{H}_{(1)}} \frac{4}{\left(r_{(2)}^2 - r_{(1)}^2\right)} \left\{\frac{\left(r_{(2)}^2 - r_{(1)}^2\right)}{2}\hat{u}_{(1)}(x)\right\}\left\{\frac{\left(r_{(2)}^2 - r_{(1)}^2\right)}{2}\hat{u}_{(1)}(x)\right\}^T dx$$

$$+ \int_o^{\overline{H}_{(1)}} \frac{\left(r_{(2)}^4 + 8r_{(2)}r_{(1)}^3 - 6r_{(2)}^2 r_{(1)}^2 - 3r_{(1)}^4\right)}{6\left(r_{(1)} - r_{(2)}\right)^2}\left\{\hat{v}_{(1,1)}(x,r)\right\}\left\{\hat{v}_{(1,1)}(x,r)\right\}^T dx$$

$$+ \int_o^{\overline{H}_{(1)}} \frac{\left(r_{(2)}^4 + 2r_{(2)}r_{(1)}^3 - 2r_{(1)}r_{(2)}^3 - r_{(1)}^4\right)}{6\left(r_{(1)} - r_{(2)}\right)^2}\left(\left\{\hat{v}_{(1,1)}(x,r)\right\}\left\{\hat{v}_{(1,2)}(x,r)\right\}^T + \left\{\hat{v}_{(1,2)}(x,r)\right\}\left\{\hat{v}_{(1,1)}(x,r)\right\}^T\right) dx$$

$$+ \int_o^{\overline{H}_{(1)}} \frac{\left(3r_{(2)}^4 - 8r_{(1)}r_{(2)}^3 + 6r_{(1)}^2 r_{(2)}^2 - r_{(1)}^4\right)}{6\left(r_{(1)} - r_{(2)}\right)^2}\left\{\hat{v}_{(1,2)}(x,r)\right\}\left\{\hat{v}_{(1,2)}(x,r)\right\}^T dx \tag{C. 10}$$

where

$$\left\{ \frac{\left(r_{(2)}^{2} - r_{(1)}^{2}\right)}{2}\, \hat{u}_{(1)}(x) \right\} = \left\{ \begin{array}{l} + \int_{o}^{x} r_{(1)} \cot \phi_{(1)} \left\{ u_{(1)}(\xi) \right\} dx \\ + \int_{o}^{x} r_{(1)} \left\{ v_{(1)}(\xi) \right\} dx \\ \hline - \int_{o}^{x+\overline{H}_{(2)}} r_{(2)} \cot \phi_{(2)} \left\{ u_{(2)}(\xi) \right\} dx \\ - \int_{o}^{x+\overline{H}_{(2)}} r_{(2)} \left\{ v_{(2)}(\xi) \right\} dx \\ \hline - \int_{o}^{\overline{H}_{(3)}} r_{(3)} \cot \phi_{(3)} \left\{ u_{(3)}(\xi) \right\} dx \\ - \int_{o}^{\overline{H}_{(3)}} r_{(3)} \left\{ v_{(3)}(\xi) \right\} dx \end{array} \right\} \tag{C. 11}$$

$$\left\{ \hat{v}_{(1,1)}(x, r) \right\} = \left\{ \begin{array}{c} \cot \phi_{(1)} \left\{ u_{(1)}(\xi) \right\} \\ \left\{ v_{(1)}(\xi) \right\} \\ \hline 0 \\ \hline 0 \end{array} \right\} - \frac{2 \cot \phi_{(1)}}{\left(r_{(2)}^{2} - r_{(1)}^{2}\right)} \left\{ \frac{\left(r_{(2)}^{2} - r_{(1)}^{2}\right)}{2}\, \hat{u}_{(1)}(x) \right\} \tag{C. 12}$$

$$\left\{ \hat{v}_{(1,2)}(x, r) \right\} = \left\{ \begin{array}{c} 0 \\ \hline \cot \phi_{(2)} \left\{ u_{(2)}(\xi) \right\} \\ \left\{ v_{(2)}(\xi) \right\} \\ \hline 0 \end{array} \right\} - \frac{2 \cot \phi_{(2)}}{\left(r_{(2)}^{2} - r_{(1)}^{2}\right)} \left\{ \frac{\left(r_{(2)}^{2} - r_{(1)}^{2}\right)}{2}\, \hat{u}_{(1)}(x) \right\} \tag{C. 13}$$

The expressions for the m = 2 and m = 3 portions of Equation (C. 1) remain unchanged. In summary for evaluating the equivalent of Equation (C. 1) for Case II tank configuration, use Equations (C. 10), (C. 11), (C. 12) and (C. 13) for m = 1, and Equations (C. 1), (C. 3), (C. 4), (C 6) and (C. 7) for m = 2 and m = 3.

C. 3 CASE III. UPPER BULKHEAD CONVEX UPWARD, LOWER BULKHEAD CONVEX UPWARD

The Case III configuration is illustrated in Figure C. 3 which also defines the tank sections m for this case. Equation (C. 1) for the fluid in tank section (3) is modified as follows. The fluid motion $\left\{\hat{u}(x)\right\}$ in tank section (3) is given by

$$\left\{\hat{u}_{(3)}(x)\right\} = \frac{\left(r_{(2)}^{2} - r_{(3)}^{2}\right)}{2} \left\{\begin{array}{c} 0 \\ \hline -\int_{o}^{x} r_{(2)} \cot \phi_{(2)} \left\{u_{(2)}(\xi)\right\} \\ -\int_{o}^{x} r_{(2)} \left\{v_{(2)}(\xi)\right\} \\ \hline \int_{o}^{x} r_{(3)} \cot \phi_{(3)} \left\{u_{(3)}(\xi)\right\} \\ \int_{o}^{x} r_{(3)} \left\{v_{(3)}(\xi)\right\} \end{array}\right\} \qquad \text{(C. 14)}$$

The fluid motion $\left\{\hat{v}(x, \hat{r})\right\}$ in tank section (3) is given by

$$
\begin{aligned}
\left\{\hat{v}_{(3)}(x, \hat{r})\right\} = \frac{\left(\hat{r} - r_{(2)}\right)}{\left(r_{(3)} - r_{(2)}\right)} & \left(\left\{ \begin{array}{c} 0 \\ \hline 0 \\ \hline \cot \phi_{(3)} \left\{u_{(3)}(\xi)\right\} \\ \left\{v_{(3)}(\xi)\right\} \end{array} \right\} - \cot \phi_{(3)} \left\{\hat{u}_{(3)}(x)\right\} \right) \\
- \frac{\left(\hat{r} - r_{(3)}\right)}{\left(r_{(3)} - r_{(2)}\right)} & \left(\left\{ \begin{array}{c} 0 \\ \hline \cot \phi_{(2)} \left\{u_{(2)}(\xi)\right\} \\ \left\{v_{(2)}(\xi)\right\} \\ \hline 0 \end{array} \right\} - \cot \phi_{(2)} \left\{\hat{u}_{(3)}(x)\right\} \right)
\end{aligned}
\tag{C. 15}
$$

Upon substitution of Equations (C. 14) and (C. 15) into Equation (4. 1) and integration with respect to $\hat{r}$ between the limits of $r_{(3)}$ and $r_{(2)}$, one obtains the following expression for the $m = 3$ portion of Equation (C. 1).

$$\frac{1}{\pi\gamma_b}\left[\overline{M}_b\right]_{(3)}$$

$$= \int_o^{\overline{H}_{(3)}} \frac{4}{\left(r_{(2)}^{\ 2} - r_{(3)}^{\ 2}\right)} \left\{\frac{\left(r_{(2)}^{\ 2} - r_{(3)}^{\ 2}\right)}{2} \hat{u}_{(3)}(x)\right\} \left\{\frac{\left(r_{(2)}^{\ 2} - r_{(3)}^{\ 2}\right)}{2} \hat{u}_{(3)}(x)\right\}^T dx$$

$$+ \int_o^{\overline{H}_{(3)}} \frac{\left(r_{(2)}^{\ 4} + 8r_{(2)}r_{(3)}^{\ 3} - 6r_{(2)}^{\ 2}r_{(3)}^{\ 2} - 3r_{(3)}^{\ 4}\right)}{6\left(r_{(3)} - r_{(2)}\right)^2} \left\{\hat{v}_{(3,3)}(x,r)\right\} \left\{\hat{v}_{(3,3)}(x,r)\right\}^T dx$$

$$+ \int_o^{\overline{H}_{(3)}} \frac{\left(r_{(2)}^{\ 4} + 2r_{(2)}r_{(3)}^{\ 3} - 2r_{(2)}^{\ 3}r_{(3)} - r_{(3)}^{\ 4}\right)}{6\left(r_{(3)} - r_{(2)}\right)^2} \left(\left\{\hat{v}_{(3,3)}(x,r)\right\} \left\{\hat{v}_{(3,2)}(x,r)\right\}^T + \left\{\hat{v}_{(3,2)}(x,r)\right\} \left\{\hat{v}_{(3,3)}(x,r)\right\}^T\right) dx$$

$$+ \int_o^{\overline{H}_{(3)}} \frac{\left(3r_{(2)}^{\ 4} - 8r_{(2)}r_{(3)}^{\ 3} + 6r_{(2)}^{\ 2}r_{(3)}^{\ 2} - r_{(3)}^{\ 4}\right)}{6\left(r_{(3)} - r_{(2)}\right)^2} \left\{\hat{v}_{(3,2)}(x,r)\right\} \left\{\hat{v}_{(3,2)}(x,r)\right\}^T dx$$

(C. 16)

where

$$\left\{\frac{\left(r_{(2)}^{\ 2} - r_{(3)}^{\ 2}\right)}{2} \hat{u}_{(3)}(x)\right\} = \left\{\begin{array}{c} 0 \\ \hline -\int_o^x r_{(2)} \cot\phi_{(2)} \left\{u_{(2)}(\xi)\right\} dx \\ -\int_o^x r_{(2)} \left\{v_{(2)}(\xi)\right\} dx \\ \hline +\int_o^x r_{(3)} \cot\phi_{(3)} \left\{u_{(3)}(\xi)\right\} dx \\ +\int_o^x r_{(3)} \left\{v_{(3)}(\xi)\right\} dx \end{array}\right\} \tag{C. 17}$$

$$\left\{\hat{v}_{(3,3)}(x,r)\right\} = \left\{\begin{array}{c} 0 \\ \hdashline 0 \\ \hdashline \cot\phi_{(3)}\left\{u_{(3)}(\xi)\right\} \\ \left\{v_{(3)}(\xi)\right\} \end{array}\right\} - \frac{2\cot\phi_{(3)}}{\left(r_{(2)}^{\;2} - r_{(3)}^{\;2}\right)}\left\{\frac{\left(r_{(2)}^{\;2} - r_{(3)}^{\;2}\right)}{2}\hat{u}_{(3)}(x)\right\} \tag{C.18}$$

$$\left\{\hat{v}_{(3,2)}(x,r)\right\} = \left\{\begin{array}{c} 0 \\ \hdashline \cot\phi_{(2)}\left\{u_{(2)}(\xi)\right\} \\ \left\{v_{(2)}(\xi)\right\} \\ \hdashline 0 \end{array}\right\} - \frac{2\cot\phi_{(2)}}{r_{(2)}^{\;2} - r_{(3)}^{\;2}}\left\{\frac{\left(r_{(2)}^{\;2} - r_{(3)}^{\;2}\right)}{2}\hat{u}_{(3)}(x)\right\} \tag{C.19}$$

Equation (C. 1) remains valid for m = 1 and m = 2, however, Equations (C.2) and (C. 3), respectively, become

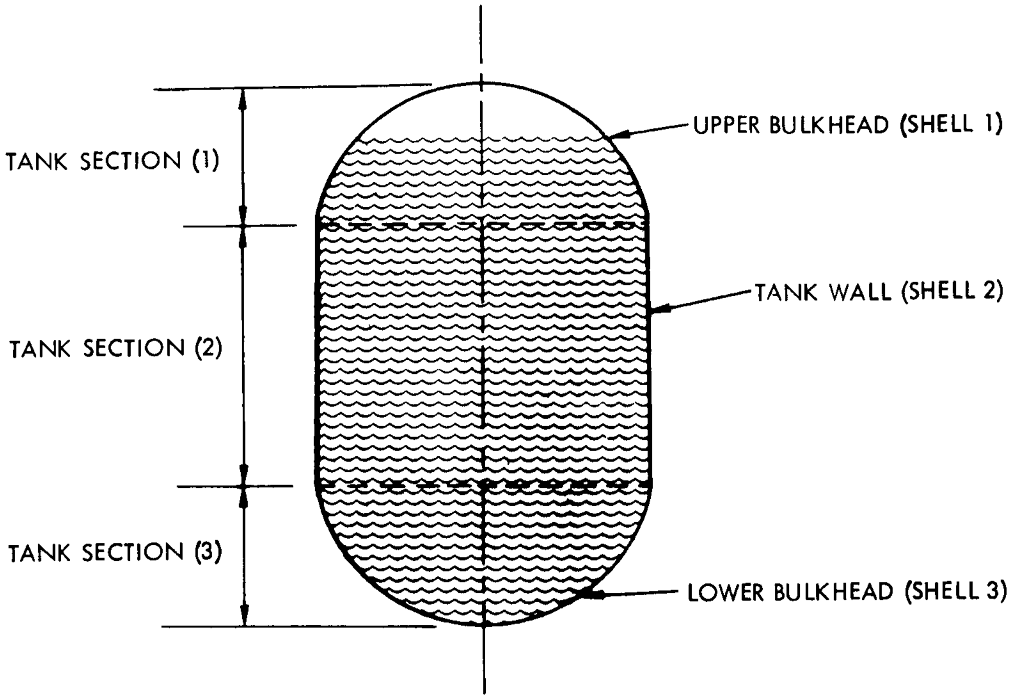
$$\left\{ \frac{r_{(1)}^{2}}{2} \hat{u}_{(1)}(x) \right\} = \left\{ \begin{array}{l} -\int_{o}^{x} r_{(1)} \cot \phi_{(1)} \left\{ u_{(1)}(\xi) \right\} dx \\ -\int_{o}^{x} r_{(1)} \left\{ v_{(1)}(\xi) \right\} dx \\ \hline -\int_{\overline{H}_{(3)}}^{\overline{H}_{(2)}} r_{(2)} \cot \phi_{(2)} \left\{ u_{(2)}(\xi) \right\} dx \\ -\int_{\overline{H}_{(3)}}^{\overline{H}_{(2)}} r_{(2)} \left\{ v_{(2)}(\xi) \right\} dx \\ \hline +\int_{o}^{\overline{H}_{(3)}} r_{(3)} \cot \phi_{(3)} \left\{ u_{(3)}(\xi) \right\} dx \\ +\int_{o}^{\overline{H}_{(3)}} r_{(3)} \left\{ v_{(3)}(\xi) \right\} dx \end{array} \right\} \qquad \text{(C. 20)}$$

$$
\left\{ \frac{r_{(2)}^{2}}{2} \hat{u}_{(2)}(x) \right\} = \left\{ \begin{array}{l} 0 \\ \hdashline - \int_{\overline{H}_{(3)}}^{x} r_{(2)} \cot \phi_{(2)} \left\{ u_{(2)}(\xi) \right\} dx \\ \\ - \int_{\overline{H}_{(3)}}^{x} r_{(2)} \left\{ v_{(2)}(\xi) \right\} dx \\ \hdashline + \int_{o}^{\overline{H}_{(3)}} r_{(3)} \cot \phi_{(3)} \left\{ u_{(3)}(\xi) \right\} dx \\ \\ + \int_{o}^{\overline{H}_{(3)}} r_{(3)} \left\{ v_{(3)}(\xi) \right\} dx \end{array} \right\} \tag{C. 21}
$$

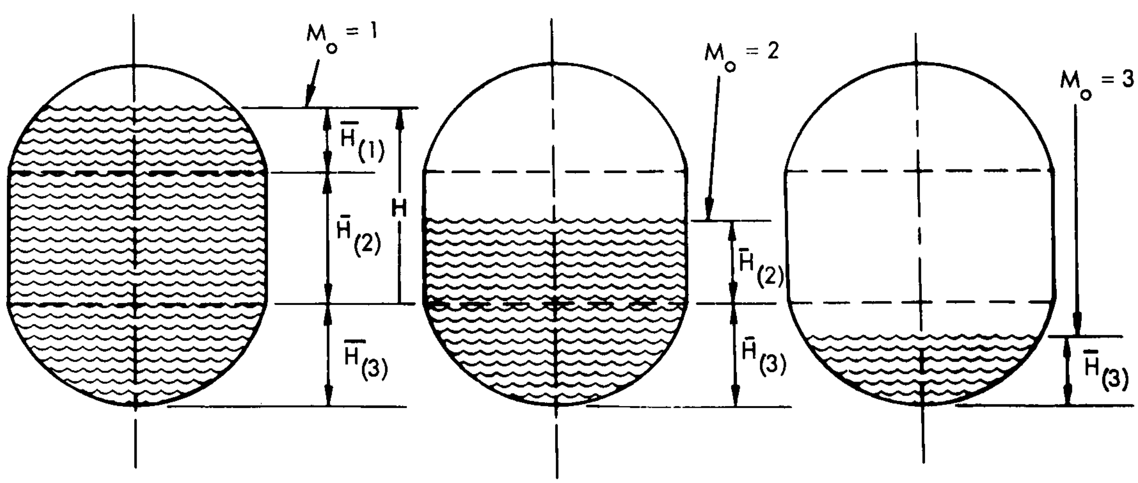
In summary for evaluating the equivalent of Equation (C. 1) for Case III tank configuration, use Equations (C. 1), (C. 20) and (C. 5) for $m = 1$, use Equations (C. 1) with limits of integration

$$
\int_{\overline{H}_{(3)}}^{\overline{H}_{(2)}}
$$

Equations (C. 21) and (C. 6) for $m = 2$, and use Equations (C. 16), (C. 14) and (C. 15) for $m = 3$.



A. DEFINITION OF TANK SECTIONS



B. TANK SECTION FLUID LEVELS

Figure C.1. Case I Tank Configuration

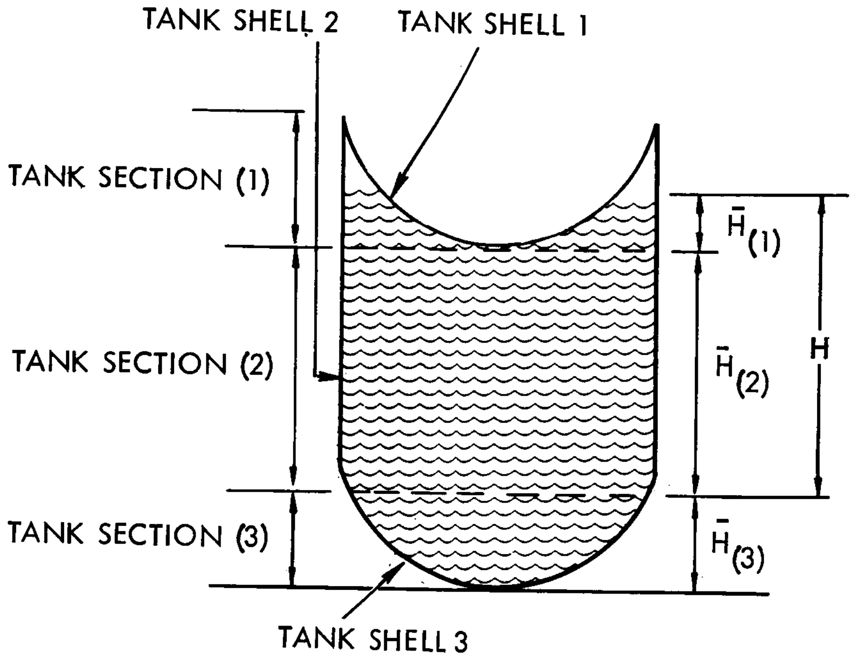


Figure C. 2. Case II Tank Configuration

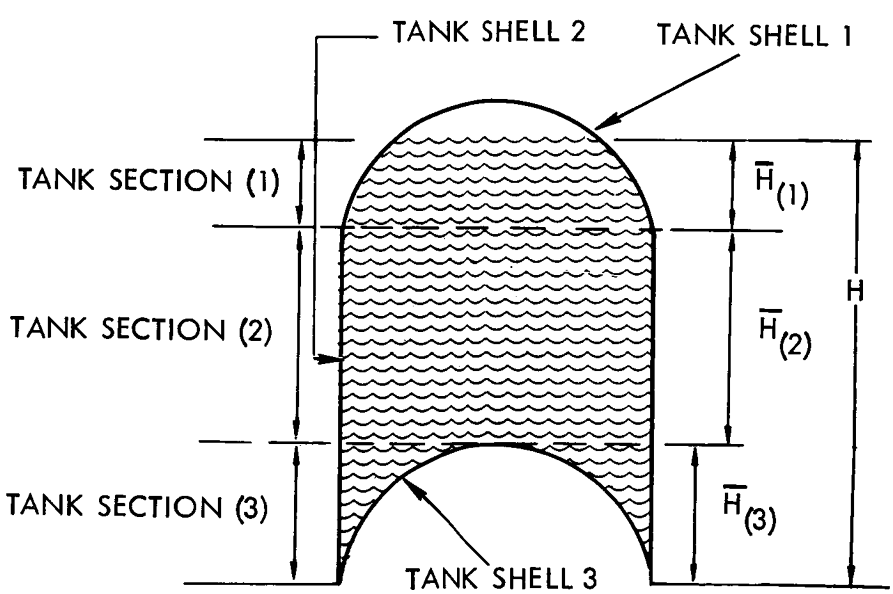


Figure C. 3. Case III Tank Configuration

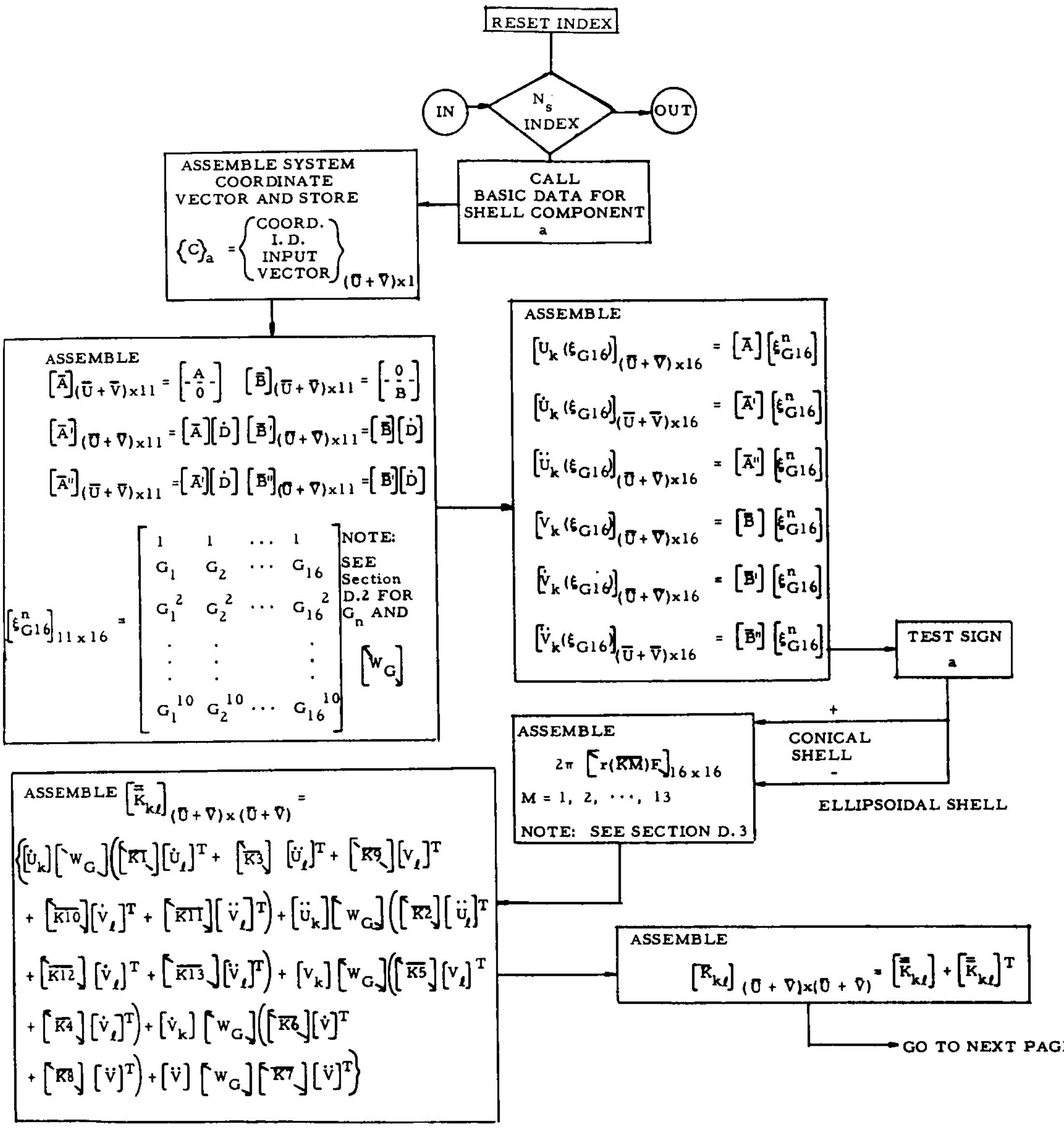
APPENDIX D

CALCULATION OF THE SHELL STIFFNESS AND MASS MATRICES

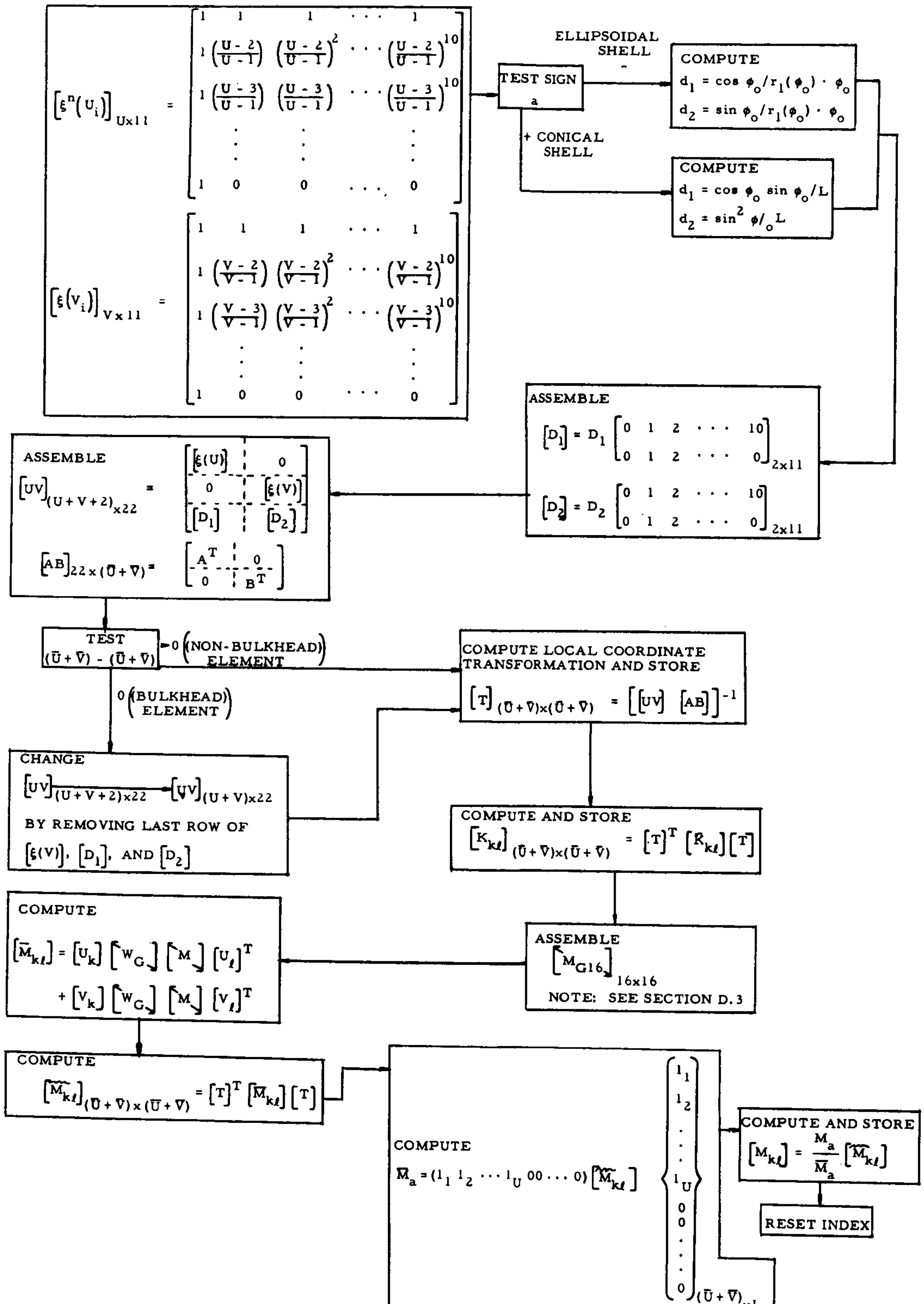
In this section the equations for the shell stiffness and mass matrices presented in Section 3.0 are rewritten in a form more suitable for digital programming. As previously mentioned, integration of these equations is accomplished using a sixteen (16) point Gaussian weighting matrix integration scheme.[5]

These operations may be performed in a straightforward fashion using matrix operations. The equations in this section are, therefore, written in matrix notation and are identical to the equations which are coded into the digital program. For a more complete understanding of the steps required in the construction of the shell stiffness and mass matrices, a functional flow chart depicting the program operations is presented in Section D.1. The weighting matrix coefficients are given in Section D.2 and other detailed expressions are provided in Sections D.3, D.4 and D.5 for elements $\overline{KM}$, M and for the initial stresses N_{ϕ}.

D.1 FUNCTIONAL FLOW CHART FOR CALCULATION OF SHELL STIFFNESS AND MASS MATRICES



D.1 FUNCTIONAL FLOW CHART FOR CALCULATION OF SHELL STIFFNESS AND MASS MATRICES (Continued)



D.2 GAUSSIAN WEIGHTING MATRIX TABLES

DATA FOR $\left[\xi^n_{G16}\right]$			DATA $\left\lceil W_G \right\rfloor$		
G1	0.00529	95325	W1	0.01357	62297
G2	0.02771	24885	W2	0.03112	67620
G3	0.06718	43988	W3	0.04757	92558
G4	0.12229	77958	W4	0.06231	44857
G5	0.19106	18778	W5	0.07479	79944
G6	0.27099	16112	W6	0.08457	82597
G7	0.35919	82246	W7	0.09130	17075
G8	0.45249	37451	W8	0.09472	53052
G9	0.54750	62549	W9	0.09472	53052
G10	0.64080	17754	W10	0.09130	17075
G11	0.72900	83888	W11	0.08457	82597
G12	0.80893	81222	W12	0.07479	79944
G13	0.87770	22042	W13	0.06231	44857
G14	0.93281	56012	W14	0.04757	92558
G15	0.97228	75115	W15	0.03112	67620
G16	0.99470	04675	W16	0.01357	62297

D.3 ASSEMBLY OF $\left[r\overline{KMF}\right]$ FOR M = 1, 2, $\cdots$, 13:

The equations for $\overline{KM}$ and F are presented below and are evaluated at the points $\xi = G_1, G_2, \cdots, G_{16}$.

$$\overline{K1} = \frac{1}{2} K1 = \frac{1}{2}\left[C_{11} \sin^2 \phi + C_{33} \frac{1}{r_1^2} \left(\sin \phi + \frac{\dot{r}_1}{r_1} \cos \phi \right)^2 \right.$$

$$\left. - C_{34} \frac{2 \cos^2 \phi}{r \cdot r_1} \left(\sin \phi + \frac{\dot{r}_1}{r} \cos \phi \right) + C_{44} \frac{\cos^4 \phi}{r^2} + N_\phi^\circ \cos^2 \phi \right]$$

$$\overline{K2} = \frac{1}{2} K2 = \frac{1}{2} \left(C_{33} \cos^2 \phi \right)$$

$$\overline{K3} = K3 = -C_{33} \frac{\cos \phi}{r_1} \left(\sin \phi + \frac{\dot{r}_1}{r_1} \cos \phi \right) + C_{34} \frac{\cos^3 \phi}{r}$$

$$\overline{K4} = K4 = C_{12} \frac{\cos \phi}{r}$$

$$\overline{K5} = \frac{1}{2} K5 = \frac{1}{2} \left(C_{22} \frac{1}{r^2} \right)$$

$$\overline{K6} = \frac{1}{2} K6 = \frac{1}{2} \left[C_{11} \cos^2 \phi + C_{33} \frac{1}{r_1^2} \left(\cos \phi - \frac{\dot{r}_1}{r_1} \sin \phi \right)^2 \right.$$

$$+ C_{34} \frac{2 \sin \phi \cos \phi}{r \cdot r_1} \left(\cos \phi - \frac{\dot{r}_1}{r_1} \sin \phi \right)$$

$$\left. + C_{44} \frac{\sin^2 \phi \cos^2 \phi}{r^2} + N_\phi^\circ \sin^2 \phi \right]$$

$$\overline{K7} = \frac{1}{2} K7 = \frac{1}{2} \left(C_{33} \sin^2 \phi \right)$$

$$\overline{K8} = K8 = C_{33} \frac{\sin \phi}{r_1} \left(\cos \phi - \frac{\dot{r}_1}{r_1} \sin \phi \right) + C_{34} \frac{\sin^2 \phi \cos \phi}{r}$$

$$\overline{K9} = K9 = -C_{12} \frac{\sin \phi}{r}$$

$$\overline{K10} = K10 = \left[-C_{11} \sin\phi \cos\phi - C_{33} \frac{1}{r_1^2} \left(\cos\phi - \frac{\dot{r}_1}{r_1} \sin\phi \right) \left(\sin\phi + \frac{\dot{r}_1}{r_1} \cos\phi \right) \right.$$

$$- C_{34} \frac{\sin\phi \cos\phi}{r \cdot r_1} \left(\sin\phi + \frac{\dot{r}_1}{r_1} \cos\phi \right)$$

$$+ C_{34} \frac{\cos^2\phi}{r \cdot r_1} \left(\cos\phi - \frac{\dot{r}_1}{r_1} \sin\phi \right)$$

$$\left. + C_{44} \frac{\sin\phi \cos^3\phi}{r^2} + N_\phi^\circ \sin\phi \cos\phi \right]$$

$$\overline{K11} = K11 = \left[-C_{33} \frac{1}{r_1} \left(\sin\phi + \frac{\dot{r}_1}{r_1} \cos\phi \right) \sin\phi + C_{34} \frac{\sin\phi \cos^2\phi}{r} \right]$$

$$\overline{K12} = K12 = \left[C_{33} \left(\cos\phi - \frac{\dot{r}_1}{r} \sin\phi \right) \frac{\cos\phi}{r_1} + C_{34} \frac{\sin\phi \cos^2\phi}{r} \right]$$

$$\overline{K13} = K13 = C_{33} \sin\phi \cos\phi$$

$$\frac{\dot{r}_1}{r} = \frac{-3(k^2 - 1) \sin\phi \cos\phi}{\left[(k^2 - 1) \sin^2\phi + 1 \right]}$$

$M = 2\pi\gamma_a rtF$

and

$$F = \frac{|L|}{\sin\theta_o} \text{ for conic}$$

$= r_1\phi_o$ for convex upward ellipsoidal bulkhead

$= r_1(\phi_o - \pi)$ for convex downward ellipsoidal bulkhead

D.4 INITIAL STRESS FOR CONIC

CONIC

INPUT

$\phi_o,\ R_2,\ L,\ W$

$w_i,\ h_i,\ p_i,\ p_e$

DEFINE

$$N_{\phi_1} = \frac{1}{2\pi r \sin\phi_o}\left[gW + (p_i - p_e)\pi\left(r^2 - R_1'^2\right)\right]$$

$$N_{\phi_2} = \frac{g w_i L^3 \cot\phi_o}{r \sin\phi_o}\left[\frac{R_2 h_i}{L^2}\left(\frac{h_i}{L} - \xi\right) - \frac{1}{2}\left(\frac{R_2}{L} + \frac{h_i}{L}\cot\phi_o\right)\left(\frac{h_i^2}{L^2} - \xi^2\right) + \frac{1}{3}\left(\frac{h_i^3}{L^3} - \xi^3\right)\cot\phi_o\right]$$

$$N_{\phi_3} = \frac{g w_i L^3 \cot\phi_o}{r \sin\phi_o}\left[\frac{R_2 h_i}{L^2}(1 - \xi) - \frac{1}{2}\left(\frac{R_2}{L} + \frac{h_i}{L}\cot\phi_o\right)(1 - \xi^2) + \frac{1}{3}(1 - \xi^3)\cot\phi_o\right]$$

where

$$r = L\left(\frac{R_2}{L} - \xi\cot\phi_o\right)$$

$\frac{h_i}{L} \leq 1$

$$N_\phi = N_{\phi_1} + N_{\phi_2} \qquad 0 < \xi < \frac{h_i}{L}$$

$$= N_{\phi_1} \qquad \frac{h_i}{L} \leq \xi < 1$$

$\frac{h_i}{L} > 1$

$$N_\phi = N_{\phi_1} + N_{\phi_3} \qquad 0 < \xi < 1$$

D.5 INITIAL STRESS FOR ELLIPSOIDAL BULKHEAD

ELLIPSOIDAL BULKHEAD

INPUT

$\bar{a}, \ \pm\bar{b}, \ \phi_o$

$h_i, \ w_i, \ h_e, \ w_e, \ p_i, \ p_e$

$$b = \bar{b}\,\frac{1 - \bar{b}\alpha}{1 - 2\bar{b}\alpha}\,, \quad \alpha = \frac{\cot\phi_o}{\bar{a}}$$

$$a = \frac{\bar{a}}{\sqrt{2}}\left[1 + \left(1 + 4b^2\alpha^2\right)^{1/2}\right]^{1/2}$$

$$H_e = h_e + (b - \bar{b})\,, \quad H_i = h_i + (b - \bar{b})$$

$$N_{\phi_1} = (p_i - p_e)\,\frac{r_2}{2}$$

$$N_{\phi_2}(u,\ v,\ r, H) = \begin{cases} gw(H - b)\,\dfrac{r_2}{2} & r = 0 \\ \dfrac{gw}{r\sin\phi}\left[\dfrac{H}{2}\left(u^2 - v^2\right) + \dfrac{a^2 b}{3}\left(1 - \dfrac{u^2}{a^2}\right)^{3/2} - \dfrac{a^2 b}{3}\left(1 - \dfrac{v^2}{a^2}\right)^{3/2}\right] & r \neq 0 \end{cases}$$

TEST SIGN b

$-$: GO TO NEXT PAGE

$+$:

$h_e \geq \bar{b}$:

$$N_\phi = N_{\phi_1} + N_{\phi_2}(r,\ 0,\ -w_e,\ H_e) \qquad 0 \leq r < \bar{\bar{r}}$$

$$= N_{\phi_1} + N_{\phi_2}(r,\ \bar{\bar{r}},\ w_i,\ H_i) + N_{\phi_2}(r,\ 0,\ -w_e,\ H_e) \qquad \bar{\bar{r}} \leq r \leq \bar{a}$$

$h_e < \bar{b}$:

$h_e \leq h_i$:

$$N_\phi = N_{\phi_1} \qquad 0 \leq r < \bar{\bar{r}}$$

$$= N_{\phi_1} + N_{\phi_2}(r,\ \bar{\bar{r}},\ w_i,\ H_i) \qquad \bar{\bar{r}} \leq r \leq \bar{r}$$

$$= N_{\phi_1} + N_{\phi_2}(r,\ \bar{\bar{r}},\ w_i,\ H_i) + N_{\phi_2}(r,\ \bar{r},\ -w_e,\ H_e) \qquad \bar{r} < r \leq \bar{a}$$

$h_e > h_i$:

$$N_\phi = N_{\phi_1} \qquad 0 \leq r \leq \bar{r}$$

$$= N_{\phi_1} + N_{\phi_2}(r,\ \bar{r},\ -w_e,\ H_e) \qquad \bar{r} < r \leq \bar{\bar{r}}$$

$$= N_{\phi_1} + N_{\phi_2}(r,\ \bar{r},\ -w_e,\ H_e) + N_{\phi_2}(r,\ \bar{\bar{r}},\ w_i,\ H_i) \qquad \bar{\bar{r}} < r \leq \bar{a}$$

GO TO NEXT PAGE

$h_i \geq h_a$

$$
\begin{aligned}
N_\phi &= N_{\phi_1} + N_{\phi_2}(r, 0, -w_i, H_i) + N_{\phi_2}(r, 0, w_e, H_e) && 0 \leq r \leq \bar{r} \\
&= N_{\phi_1} + N_{\phi_2}(\bar{r}, 0, -w_i, H_i) + N_{\phi_2}(r, 0, w_e, H_e) && \bar{r} < r \leq \bar{\bar{r}} \\
&= N_{\phi_1} + N_{\phi_2}(\bar{r}, 0, -w_i, H_i) + N_{\phi_2}(\bar{\bar{r}}, 0, w_e, H_e) && \bar{\bar{r}} < r \leq \bar{a}
\end{aligned}
$$

$h_i \leq h_e \leq \bar{b}$

$$
\begin{aligned}
N_\phi &= N_{\phi_1} + N_{\phi_2}(r, 0, -w_i, H_i) + N_{\phi_2}(r, 0, w_e, H_e) && 0 \leq r \leq \bar{\bar{r}} \\
&= N_{\phi_1} + N_{\phi_2}(r, 0, -w_i, H_i) + N_{\phi_2}(\bar{\bar{r}}, 0, w_e, H_e) && \bar{\bar{r}} < r \leq \bar{r} \\
&= N_{\phi_1} + N_{\phi_2}(\bar{r}, 0, -w_i, H_i) + N_{\phi_2}(\bar{\bar{r}}, 0, w_e, H_e) && \bar{r} < r \leq \bar{a}
\end{aligned}
$$

Note: when

$$b > 0, \quad \bar{r} = a\sqrt{1 - \frac{\left(h_e + b - |\bar{b}|\right)^2}{b^2}}$$

$$\bar{\bar{r}} = a\sqrt{1 - \frac{\left(h_i + b - |\bar{b}|\right)^2}{b^2}}$$

$$b < 0, \quad \bar{r} = a\sqrt{1 - \frac{\left(h_i + b - |\bar{b}|\right)^2}{b^2}}$$

$$\bar{\bar{r}} = a\sqrt{1 - \frac{\left(h_e + b - |\bar{b}|\right)^2}{b^2}}$$

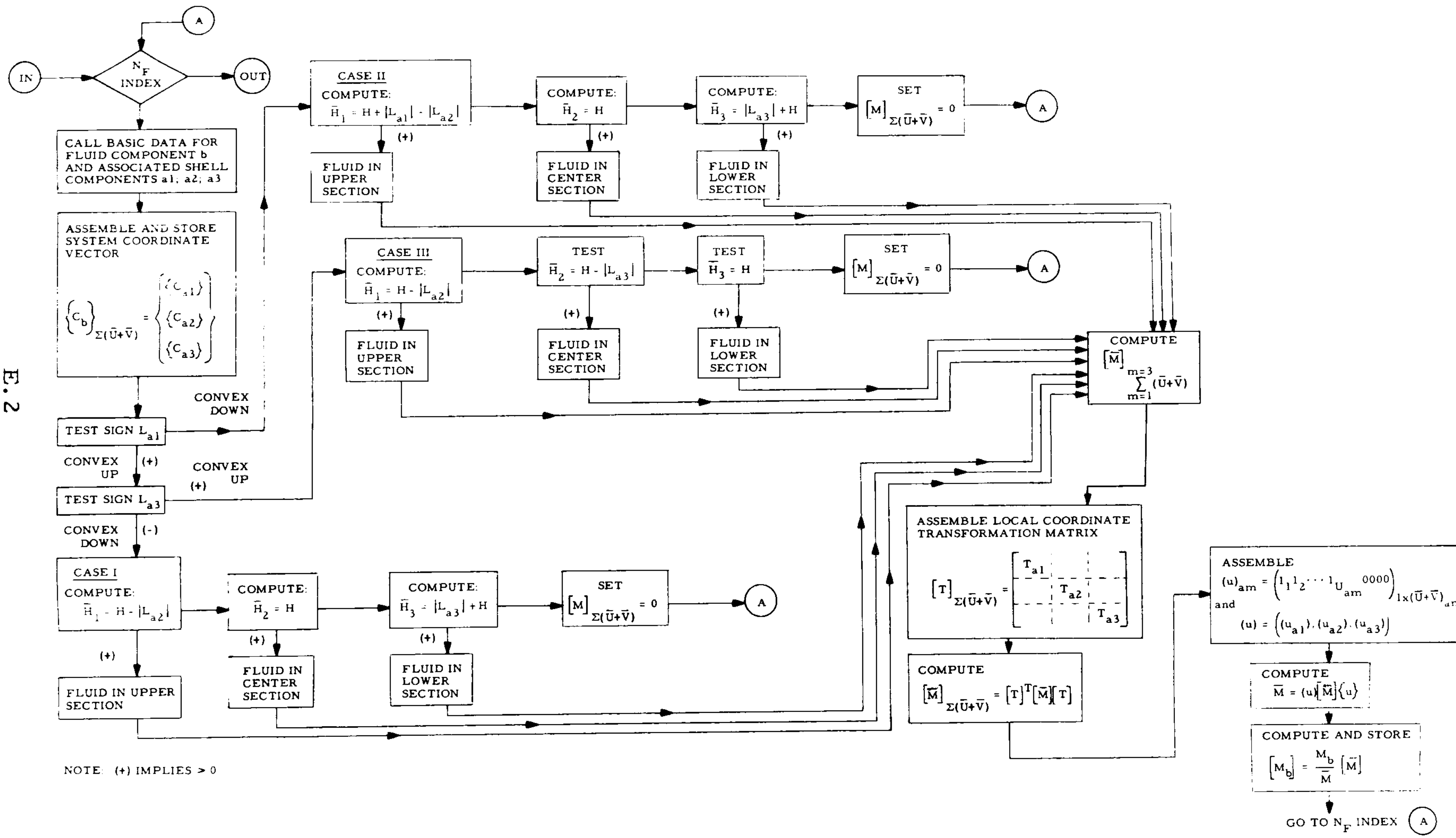
APPENDIX E

CALCULATION OF FLUID MASS MATRIX

The construction of the fluid mass matrix presented in Section 4.0 for three tank configurations differs from the construction of the shell inertial and stiffness characteristics insofar as the fluid equations involve double integrations. As mentioned in Section 4.0, these integrations are performed using a Lagrangian integration scheme. Although this technique is similar to the Gaussian method of integration used in the computation of the shell matrices described in Appendix D, only the Lagrangian technique is adaptable to double integration. The digital program utilizes a 21-point Lagrangian weighting matrix which is constructed by combining two 11-point matrices tabulated in the above reference. The increased number of points provides additional numerical accuracy. In the following pages, the fluid equations are rewritten to accommodate the integrations using the weighting matrix technique.* A functional flow chart outlining the complete mass matrix calculation is provided.

*The notation of this appendix corresponds to the notation utilized in the digital program.

E. 1 FUNCTIONAL FLOW CHART FOR CALCULATION OF FLUID MASS MATRIX



NOTE: (+) IMPLIES > 0

E. 2 GENERAL MATRIX EQUATION FOR MASS MATRIX $[\overline{M}]$

$$\frac{[\overline{M}]}{\sum_{m=1}^{m=3} (\overline{U}+\overline{V})_{a(m)}} = [\overline{M}_1] + [\overline{M}_2]$$

$$\frac{[\overline{M}_1]}{\sum_{m=1}^{m=3} (\overline{U}+\overline{V})_{am}} = \begin{bmatrix} [\overline{U}_k^{\,1}] \lceil K1 \rfloor \lceil W \rfloor [\overline{U}_k^{\,1}]^T \\ + [\overline{U}_k^{\,2}] \lceil K2 \rfloor \lceil W \rfloor [\overline{U}_k^{\,2}]^T \\ + [\overline{U}_k^{\,3}] \lceil K3 \rfloor \lceil W \rfloor [\overline{U}_k^{\,3}]^T \end{bmatrix}$$

$$\frac{[\overline{M}_2]}{\sum_{m=1}^{m=3} (\overline{U}+\overline{V})_{am}} = \begin{bmatrix} [\overline{V}_k^{\,1}] \lceil C1 \rfloor \lceil W \rfloor [\overline{V}_k^{\,1}]^T + [\overline{V}_k^{\,1}] \lceil C2 \rfloor \lceil W \rfloor [\overline{\overline{V}}_k^{\,1}]^T \\ [\overline{\overline{V}}_k^{\,1}] \lceil C2 \rfloor \lceil W \rfloor [\overline{V}_k^{\,1}]^T + [\overline{\overline{V}}_k^{\,1}] \lceil C3 \rfloor \lceil W \rfloor [\overline{\overline{V}}_k^{\,1}]^T \\ [\overline{V}_k^{\,2}] \lceil C4 \rfloor \lceil W \rfloor [\overline{V}_k^{\,2}]^T \\ [\overline{V}_k^{\,3}] \lceil C5 \rfloor \lceil W \rfloor [\overline{V}_k^{\,3}]^T + [\overline{V}_k^{\,3}] \lceil C6 \rfloor \lceil W \rfloor [\overline{\overline{V}}_k^{\,3}]^T \\ [\overline{\overline{V}}_k^{\,3}] \lceil C6 \rfloor \lceil W \rfloor [\overline{V}_k^{\,3}]^T + [\overline{\overline{V}}_k^{\,3}] \lceil C7 \rfloor \lceil W \rfloor [\overline{\overline{V}}_k^{\,3}]^T \end{bmatrix}$$

$$
\left[\bar{U}_k^{\,1}\right]_{\sum(U+V)\times 21} = \left[\begin{array}{l}
[U_{k1}]\,[KU_1]\,[\bar{W}] \\[2ex]
[V_{k1}]\,[KV_1]\,[\bar{W}] \\[1ex]
\hline
[U_{k2}]\,[KU_2]\,[\bar{\bar{W}}] + [\bar{U}_{k2}]\,[\overline{KU}_2]\,[\bar{W}] + [\bar{\bar{U}}_{k2}]\,[\overline{\overline{KU}}_2]\,[\bar{\bar{W}}] \\[2ex]
[V_{k2}]\,[KV_2]\,[\bar{\bar{W}}] + [\bar{V}_{k2}]\,[\overline{KV}_2]\,[\bar{W}] + [\bar{\bar{V}}_{k2}]\,[\overline{\overline{KV}}_2]\,[\bar{\bar{W}}] \\[1ex]
\hline
[U_{k3}]\,[KU_3]\,[\bar{\bar{W}}] \\[2ex]
[V_{k3}]\,[KV_3]\,[\bar{\bar{W}}]
\end{array}\right]
$$

$$
\left[\bar{U}_k^{\,2}\right]_{\sum(\bar{U}+\bar{V})\times 21} = \left[\begin{array}{c}
0 \\
\hline
[U_{k2}]\,[KU_2]\,[\bar{W}] + [\bar{\bar{U}}_{k2}]\,[\overline{\overline{KU}}_2]\,[\bar{\bar{W}}] \\
[V_{k2}]\,[KV_2]\,[\bar{W}] + [\bar{\bar{V}}_{k2}]\,[\overline{\overline{KV}}_2]\,[\bar{\bar{W}}] \\
\hline
[U_{k3}]\,[KU_3]\,[\bar{\bar{W}}] \\
[V_{k3}]\,[KV_3]\,[\bar{\bar{W}}]
\end{array}\right]
$$

$$
\left[\bar{U}_k^{\,3}\right]_{\sum(\bar{U}+\bar{V})\times 21} = \left[\begin{array}{c}
0 \\
\hline
[\bar{\bar{U}}_{k2}]\,[\overline{\overline{KU}}_2]\,[\bar{W}] \\
[\bar{\bar{V}}_{k2}]\,[\overline{\overline{KV}}_2]\,[\bar{W}] \\
\hline
[U_{k3}]\,[KU_3]\,[\bar{W}] \\
[V_{k3}]\,[KV_3]\,[\bar{W}]
\end{array}\right]
$$

$$
\left[\overline{V}_k^{\,1}\right]_{\sum(\overline{U}+\overline{V})\times 21} = \left[\begin{array}{cc} \left[U_{k1}\right] & \left[KU_4\right] \\ \left[V_{k1}\right] & \left[KV_4\right] \\ \hline \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c}{0} \end{array}\right] + \left[\overline{U}_k^{\,1}\right]\left[K4\right]
$$

$$
\left[\overline{\overline{V}}_k^{\,1}\right]_{\sum(\overline{U}+\overline{V})\times 21} = \left[\begin{array}{cc} \multicolumn{2}{c}{0} \\ \hline \left[\overline{U}_{k2}\right] & \left[\overline{KU}_5\right] \\ \left[\overline{V}_{k2}\right] & \left[\overline{KV}_5\right] \\ \hline \multicolumn{2}{c}{0} \end{array}\right] + \left[\overline{U}_k^{\,1}\right]\left[\overline{K5}\right]
$$

$$
\left[\overline{V}_k^{\,2}\right]_{\sum(\overline{U}+\overline{V})\times 21} = \left[\begin{array}{cc} \multicolumn{2}{c}{0} \\ \hline \left[U_{k2}\right] & \left[KU_5\right] \\ \left[V_{k2}\right] & \left[KV_5\right] \\ \hline \multicolumn{2}{c}{0} \end{array}\right] + \left[\overline{U}_k^{\,2}\right]\left[K5\right]
$$

$$\left[\overline{V}_k^{\,3}\right]_{\sum(\overline{U}+\overline{V})\times 21} = \begin{bmatrix} 0 \\ \hdashline 0 \\ \hdashline \left[U_{k3}\right] \left[KU_6\right] \\ \left[V_{k3}\right] \left[KV_6\right] \end{bmatrix} + \left[\overline{U}_k^{\,3}\right] \left[K6\right]$$

$$\left[\overline{\overline{V}}_k^{\,3}\right]_{\sum(\overline{U}+\overline{V})\times 21} = \begin{bmatrix} 0 \\ \hdashline \left[\overline{\overline{U}}_{k2}\right] \left[\overline{\overline{KU}}_5\right] \\ \left[\overline{\overline{V}}_{k2}\right] \left[\overline{\overline{KV}}_5\right] \\ \hdashline 0 \end{bmatrix} + \left[\overline{U}_k^{\,3}\right] \left[\overline{\overline{K5}}\right]$$

$$\left[U_{k1}\right]_{\overline{U}_{a1}\times 21} = \left[A_{a1}\right]\left[\xi_{a1}^{n}\right]$$

$$\left[V_{k1}\right]_{\overline{V}_{a1}\times 21} = \left[B_{a1}\right]\left[\xi_{a1}^{n}\right]$$

$$\left[U_{k2}\right]_{\overline{U}_{a2}\times 21} = \left[A_{a2}\right]\left[\xi_{a2}^{n}\right] \qquad \left[\overline{U}_{k2}\right]_{\overline{U}_{a2}\times 21} = \left[A_{a2}\right]\left[\overline{\xi}_{a2}^{n}\right]$$

$$\left[V_{k2}\right]_{\overline{V}_{a2}\times 21} = \left[B_{a2}\right]\left[\xi_{a2}^{n}\right] \;; \quad \left[\overline{V}_{k2}\right]_{\overline{V}_{a2}\times 21} = \left[B_{a2}\right]\left[\overline{\xi}_{a2}^{n}\right]$$

$$\left[U_{k3}\right]_{\overline{U}_{a3}\times 21} = \left[A_{a3}\right]\left[\xi_{a3}^{n}\right]$$

$$\left[V_{k3}\right]_{\overline{V}_{a3}\times 21} = \left[B_{a3}\right]\left[\xi_{a3}^{n}\right]$$

$$\left[\overline{\overline{U}}_{k2}\right]_{\overline{U}_{a2}\times 21} = \left[A_{a2}\right]\left[\overline{\overline{\xi}}_{a2}^{n}\right]$$

$$\left[\overline{\overline{V}}_{k2}\right]_{V_{a2}\times 21} = \left[B_{a2}\right]\left[\overline{\overline{\xi}}_{a2}^{n}\right]$$

Note: $\left[A_{am}\right]$ and $\left[B_{am}\right]$ are provided as initial input

$$\left[\xi^{n}_{am}\right]_{11\times 21} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \xi_{m0} & (0.95\,\xi_{m0} + 0.05\,\xi_{m1}) & (0.90\,\xi_{m0} + 0.10\,\xi_{m1}) & \cdots & \xi_{m1} \\ \xi^{2}_{m0} & (0.95\,\xi_{m0} + 0.05\,\xi_{m1})^{2} & (0.90\,\xi_{m0} + 0.10\,\xi_{m1})^{2} & \cdots & \xi^{2}_{m1} \\ \vdots & \vdots & \vdots & & \vdots \\ \xi^{10}_{m0} & (0.95\,\xi_{m0} + 0.05\,\xi_{m1})^{10} & (0.90\,\xi_{m0} + 0.10\,\xi_{m1}) & \cdots & \xi^{10}_{m1} \end{bmatrix}$$

for $m = 1, 2, 3$

$$\left[\bar{\xi}^{n}_{a2}\right]_{11\times 21} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \bar{\xi}_{2,1}(\xi_{10}) & \bar{\xi}_{2,2}(0.95\,\xi_{10} + 0.05\,\xi_{11}) & \bar{\xi}_{2,3}(0.90\,\xi_{10} + 0.10\,\xi_{11}) & \cdots\ \bar{\xi}_{2,21}(\xi_{11}) \\ \bar{\xi}^{2}_{2,1} & \bar{\xi}^{2}_{2,2} & \bar{\xi}^{2}_{2,3} & \cdots\ \bar{\xi}^{2}_{2,21} \\ \vdots & \vdots & \vdots & \vdots \\ \bar{\xi}^{10}_{2,1} & \bar{\xi}^{10}_{2,2} & \bar{\xi}^{10}_{2,3} & \cdots\ \bar{\xi}^{10}_{2,21} \end{bmatrix}$$

Note:
See Functional Relationship $\bar{\xi}_{a2}(\xi_{a1})$ defined below.

$$\left[\bar{\bar{\xi}}^{n}_{a2}\right]_{11\times 21} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \bar{\bar{\xi}}_{2,1}(\xi_{30}) & \bar{\bar{\xi}}_{2,2}(0.95\,\xi_{30} + 0.05\,\xi_{31}) & \bar{\bar{\xi}}_{2,3}(0.90\,\xi_{30} + 0.10\,\xi_{31}) & \cdots\ \bar{\bar{\xi}}_{2,21}(\xi_{31}) \\ \bar{\bar{\xi}}^{2}_{2,1} & \bar{\bar{\xi}}^{2}_{2,2} & \bar{\bar{\xi}}^{2}_{2,3} & \cdots\ \bar{\bar{\xi}}^{2}_{2,21} \\ \vdots & \vdots & \vdots & \vdots \\ \bar{\bar{\xi}}^{10}_{2,1} & \bar{\bar{\xi}}^{10}_{2,2} & \bar{\bar{\xi}}^{10}_{2,3} & \cdots\ \bar{\bar{\xi}}^{10}_{2,21} \end{bmatrix}$$

Note:
See functional relationship $\bar{\bar{\xi}}_{a2}\,(\xi_{a3})$ defined below.

$$F_1(r_{a1}, r_{a2}) = \frac{r_{a2}^4 + 8r_{a2}r_{a1}^3 - 6r_{a2}^2 r_{a1}^2 - 3r_{a1}^4}{6\left(r_{a1} - r_{a2}\right)^2}$$

$$F_2(r_{a1}, r_{a2}) = \frac{r_{a2}^4 + 2r_{a2}r_{a1}^3 - 2r_{a1}r_{a2}^3 - r_{a1}^4}{6\left(r_{a1} - r_{a2}\right)^2}$$

<u>Functional Relationship $\bar{\xi}_{a2}(\xi_{a1})$</u>

$$\bar{\xi}_{a2} = \frac{x}{|L|} = \frac{b}{|L|}\left[1 - \sqrt{1 - \frac{r^2}{a^2}}\right]$$

where

$$r^2 = f(\xi_{a1})$$

$$= \frac{a^4 \sin^2 \phi}{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)}$$

and

$$\xi_{a1} = \frac{\pi - \phi}{\pi - \phi_o}$$

<u>Functional Relationship $\bar{\bar{\xi}}_{a2}\ (\xi_{a3})$</u>

$$\bar{\bar{\xi}}_{a2} = \frac{1}{|L|}\left\{\bar{b} - b\left[1 - \sqrt{1 - \frac{r^2}{a2}}\right]\right\}$$

where

$$r^2 = f(\xi_{a3})$$

$$= \frac{a^4 \sin^2 \phi}{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)}$$

and

$$\xi_{a3} = \frac{\phi}{\phi_o}$$

E. 3 CASE I $\left(\begin{array}{l} \text{a1 - Convex Up Ellipsoid; a2 - Conic;} \\ \text{a3 - }\left\{\begin{array}{ll}\text{IA:} & \text{Convex Down Ellipsoid} \\ \text{IB:} & \text{Conic}\end{array}\right\}\end{array}\right)$

E. 3. 1 Integral Equation Relationships

$$\left[\overline{M}\right] = -4\pi\overline{\gamma}_1\left(\phi_o\right)_{a1}\int_{\xi_{10}}^{\xi_{11}}\left(\frac{r_1 \sin\phi}{r^2}\right)_{a1}\left\{\frac{r^2}{2}\hat{u}(x)\right\}_1\left\{\frac{r^2}{2}\hat{u}(x)\right\}_1^T d\xi_{a1}$$

$$+ 4\pi\overline{\gamma}_2\left(|L|\right)_{a2}\int_{\xi_{20}}^{\xi_{21}}\left(\frac{1}{r^2}\right)_{a2}\left\{\frac{r^2}{2}\hat{u}(x)\right\}_2\left\{\frac{r^2}{2}\hat{u}(x)\right\}_2^T d\xi_{a2}$$

$$\text{IA:}\left\{-4\pi\overline{\gamma}_3\left(\phi_o - \pi\right)_{a3}\int_{\xi_{30}}^{\xi_{31}}\left(\frac{r_1 \sin\phi}{r^2}\right)_{a3}\left\{\frac{r^2}{2}\hat{u}(x)\right\}_3\left\{\frac{r^2}{2}\hat{u}(x)\right\}_3^T d\xi_{a3}\right.$$

$$\text{IB:}\left\{+4\pi\overline{\gamma}_3\left(|L|\right)_{a3}\int_{\xi_{30}}^{\xi_{31}}\left(\frac{1}{r^2}\right)_{a3}\left\{\frac{r^2}{2}\hat{u}(x)\right\}_3\left\{\frac{r^2}{2}\hat{u}(x)\right\}_3^T d\xi_{a3}\right.$$

$$- \frac{\pi\overline{\gamma}_1\left(\phi_o\right)_{a1}}{2}\int_{\xi_{10}}^{\xi_{11}}\left(r_1 \sin\phi\right)_{a1}\left\{r\hat{v}(x, r)\right\}_1\left\{r\hat{v}(x, r)\right\}_1^T d\xi_{a1}$$

$$+ \frac{\pi\overline{\gamma}_2\left(|L|\right)_{a2}}{2}\int_{\xi_{20}}^{\xi_{21}}\left\{r\hat{v}(x, r)\right\}_2\left\{r\hat{v}(x, r)\right\}_2 d\xi_{a2}$$

$$\text{IA:}\left\{-\frac{\pi\overline{\gamma}_3\left(\phi_o - \pi\right)_{a3}}{2}\int_{\xi_{30}}^{\xi_{31}}\left(r_1 \sin\phi\right)_{a3}\left\{r\hat{v}(x, r)\right\}_3\left\{r\hat{v}(x, r)\right\}_3^T d\xi_{a3}\right.$$

$$\text{IB:}\left\{+\frac{\pi\overline{\gamma}_3\left(|L|\right)_{a3}}{2}\int_{\xi_{30}}^{\xi_{31}}\left\{r\hat{v}(x, r)\right\}_3\left\{r\hat{v}(x, r)\right\}_3^T d\xi_{a3}\right.$$

where

$$\left\{\frac{r^2}{2}\hat{u}(x)\right\}_1 = \left\{ \begin{array}{ll} & + \int_{\xi_{10}}^{\xi} \left(r \cdot r_1 \, \phi_o \cos\phi \left\{u(\xi)\right\}\right)_{a1} d\xi_{a1} \\ & + \int_{\xi_{10}}^{\xi} \left(r \cdot r_1 \, \phi_o \sin\phi \left\{v(\xi)\right\}\right)_{a1} d\xi_{a1} \\ \hline & - \int_{\xi_{20}}^{\xi_{21}} \left(r \, |L| \cot\phi_o \left\{u(\xi)\right\}\right)_{a2} d\xi_{a2} \\ & - \int_{\xi_{20}}^{\xi_{21}} \left(r \, |L| \left\{v(\xi)\right\}\right)_{a2} d\xi_{a2} \\ \hline \text{IA:} & \left\{ \begin{array}{l} \int_{\xi_{30}}^{\xi_{31}} \left(r \cdot r_1 \, (\phi_o - \pi) \cos\phi \left\{u(\xi)\right\}\right)_{a3} d\xi_{a3} \\ \int_{\xi_{30}}^{\xi_{31}} \left(r \cdot r_1 \, (\phi_o - \pi) \sin\phi \left\{v(\xi)\right\}\right)_{a3} d\xi_{a3} \end{array} \right. \end{array} \right\}$$

$$\text{IB:} \quad \left\{ \begin{array}{l} - \int_{\xi_{30}}^{\xi_{31}} \left(r|L| \cot\phi_o \left\{u(\xi)\right\}\right)_{a3} d\xi_{a3} \\ - \int_{\xi_{30}}^{\xi_{31}} \left(r|L| \left\{v(\xi)\right\}\right)_{a3} d\xi_{a3} \end{array} \right.$$

$$\left\{\frac{r^2}{2}\,\hat{u}(x)\right\}_2 = \left\{\begin{array}{l} 0 \\ \hline -\int_{\xi_{20}}^{\xi}\left(r|L|\cot\phi_o\left\{u(\xi)\right\}\right)_{a2} d\xi_{a2} \\ -\int_{\xi_{20}}^{\xi}\left(r|L|\left\{v(\xi)\right\}\right)_{a2} d\xi_{a2} \\ \hline \int_{\xi_{30}}^{\xi_{31}}\left(r\cdot r_1\,(\phi_o-\pi)\cos\phi\left\{u(\xi)\right\}\right)_{a3} d\xi_{a3} \\ \int_{\xi_{30}}^{\xi_{31}}\left(r\cdot r_1\,(\phi_o-\pi)\sin\phi\left\{v(\xi)\right\}\right)_{a3} d\xi_{a3} \end{array}\right\}$$

IA: (the last two rows above)

IB:

$$\left\{\begin{array}{l} -\int_{\xi_{30}}^{\xi_{31}}\left(r|L|\cot\phi_o\left\{u(\xi)\right\}\right)_{a3} d\xi_{a3} \\ -\int_{\xi_{30}}^{\xi_{31}}\left(r|L|\left\{v(\xi)\right\}\right)_{a3} d\xi_{a3} \end{array}\right.$$

$$\left\{\frac{r^2}{2}\hat{u}(x)\right\}_3 = \left\{\begin{array}{c} 0 \\ \hdashline 0 \\ \hdashline \end{array}\right.$$

IA:

$$\left\{\begin{array}{l} \int_{\xi_{30}}^{\xi}\left(r \cdot r_1\,(\phi_o - \pi)\cos\phi\left\{u(\xi)\right\}\right)_{a3} d\xi_{a3} \\ \\ \int_{\xi_{30}}^{\xi}\left(r \cdot r_1\,(\phi_o - \pi)\sin\phi\left\{v(\xi)\right\}\right)_{a3} d\xi_{a3} \end{array}\right.$$

IB:

$$\left\{\begin{array}{l} -\int_{\xi_{30}}^{\xi}\left(r\,|L|\cot\phi_o\left\{u(\xi)\right\}\right)_{a3} d\xi_{a3} \\ \\ -\int_{\xi_{30}}^{\xi}\left(r|L|\left\{v(\xi)\right\}\right)_{a3} d\xi_{a3} \end{array}\right.$$

$$\left\{r\hat{v}(x,r)\right\}_1 = \left\{\begin{array}{c} \left(r \cdot \cot\phi\left\{u(\xi)\right\}\right)_{a1} \\ \left(r\left\{v(\xi)\right\}\right)_{a1} \\ \hdashline 0 \\ \hdashline 0 \end{array}\right\} - \left(\frac{2\cot\phi}{r}\right)_{a1} \bullet \left\{\frac{r^2}{2}\hat{u}(x)\right\}_{a1}$$

$$\left\{r\hat{v}(x, r)\right\}_2 = \left\{\begin{array}{c} 0 \\ \hdashline \left(r \cdot \cot\phi_o \left\{u(\xi)\right\}\right)_{a2} \\ \left(r \cdot \left\{v(\xi)\right\}\right)_{a2} \\ \hdashline 0 \end{array}\right\} - \left(\frac{2\cot\phi_o}{r}\right)_{a2} \cdot \left\{\frac{r^2}{2}\hat{u}(x)\right\}_{a2}$$

$$\left\{r\hat{v}(x, r)\right\}_3 = \left\{\begin{array}{c} 0 \\ \hdashline 0 \\ \hdashline \left(r \cdot \cot\phi \left\{u(\xi)\right\}\right)_{a3} \\ \left(r \left\{v(\xi)\right\}\right)_{a3} \end{array}\right\} - \left(\frac{2\cot\phi}{r}\right)_{a3} \cdot \left\{\frac{r^2}{2}\hat{u}(x)\right\}_{a3}$$

E.3.2 Integration Limit Data

Fluid Surface in Section 1

$$\xi_{11} = \left(\bar{\phi}/\phi_o\right)_{a1}$$

$$\xi_{20} = \xi_{30} = 0$$

$$\xi_{10} = \xi_{21} = \xi_{31} = 1$$

$$\bar{\gamma}_1 = \bar{\gamma}_2 = \bar{\gamma}_3 = \gamma$$

Note:

$$\overline{\phi} = \sin^{-1}\left\{\frac{\overline{r}}{\left[a^2k^2 - \overline{r}^2(k^2 - 1)\right]^{1/2}}\right\}$$

$$k = a/b$$

$$\overline{r} = a\sqrt{1 - \frac{\left(\overline{H}_1 + b - \overline{b}\right)^2}{b^2}}$$

if $(\overline{H}_1 + b - \overline{b}) > 0$, $0 < \overline{\phi} < 90^o$, if < 0, $90^o < \overline{\phi} < 180^o$

<u>Fluid Surface in Section 2</u>

$$\xi_{21} = \left(\overline{H}_2/|L|\right)_{a2}$$

$$\xi_{10} = \xi_{11} = \xi_{31} = 1$$

$$\xi_{20} = \xi_{30} = 0$$

$$\overline{\gamma}_1 = 0$$

$$\overline{\gamma}_2 = \overline{\gamma}_3 = \gamma$$

<u>Fluid Surface in Section 3</u>

$$\text{IA: } \left\{\xi_{31} = \left(\frac{\pi - \overline{\overline{\phi}}}{\pi - \phi_o}\right)_{a3}\right.$$

$$\text{IB: } \left\{\xi_{31} = \left(\overline{H}_3/|L|\right)_{a3}\right.$$

$$\xi_{10} = \xi_{11} = 1$$

$$\xi_{20} = \xi_{21} = \xi_{30} = 0$$

$$\overline{\gamma}_1 = \overline{\gamma}_2 = 0$$

$$\overline{\gamma}_3 = \gamma$$

Note:

$$\bar{\bar{\phi}} = \sin^{-1}\left\{\frac{\bar{\bar{r}}}{-\left[a^2k^2 - \bar{\bar{r}}^2(k^2 - 1)\right]^{1/2}}\right\}$$

$$k = a/b$$

$$\bar{\bar{r}} = a\sqrt{1 - \frac{\left(b - \overline{H}_3\right)^2}{b^2}}$$

E. 3. 3 Specific Matrix Data

Note: Quantities $\left\{(\ \ )_{am}\right\}_{am,\,i}$ are evaluated at

points $\xi_{am,\,i}$ $(i = 1, 2, 3, \ldots 21)$ where

$\xi_{am,\,1} = \xi_{m0}$; $\xi_{am,\,2} = (0.95\ \xi_{m0} + 0.05\ \xi_{m1})$;

$\xi_{am,\,3} = (0.90\ \xi_{m0} + 0.10\ \xi_{m1})$; $\xi_{am,\,4} = (0.85\ \xi_{m0} + 0.15\ \xi_{m1})$;

$\ldots,\ \xi_{am,\,20} = (0.05\ \xi_{m0} + 0.95\ \xi_{m1})$; $\xi_{am,\,21} = \xi_{m1}$

$$K1_i = -4\pi\bar{\gamma}_1\left(\phi_o\right)_{a1}\left\{\left(\frac{r_1 \sin\phi}{r^2}\right)_{a1}\right\}_{a1,\,i} \cdot (\xi_{11} - \xi_{10})$$

$$K2_i = +4\pi\bar{\gamma}_2\ (|L|)_{a2}\left\{\left(\frac{1}{r^2}\right)_{a2}\right\}_{a2,\,i} \cdot (\xi_{21} - \xi_{20})$$

$$\text{IA:}\ \left\{K3_i = -4\pi\bar{\gamma}_3\left(\phi_o - \pi\right)_{a3}\left\{\left(\frac{r_1 \sin\phi}{r^2}\right)_{a3}\right\}_{a3,\,i}\right. \cdot (\xi_{31} - \xi_{30})$$

$$\text{IB:}\ \left\{K3_i = +4\pi\bar{\gamma}_3\ (|L|)_{a3}\left\{\left(\frac{1}{r^2}\right)_{a3}\right\}_{a3,\,i}\right. \cdot (\xi_{31} - \xi_{30})$$

$$K4_i = -\left\{\left(\frac{2\cot\phi}{r}\right)_{a1}\right\}_{a1,i}$$

$$K5_i = -2(\cot\phi_o)_{a2}\left\{\left(\frac{1}{r}\right)_{a2}\right\}_{a2,i} \quad ; \quad \overline{K}_{5i} = \overline{\overline{K}}_{5i} = 0$$

$$\text{IA:} \left\{ K6_i = -\left\{\left(\frac{2\cot\phi}{r}\right)_{a3}\right\}_{a3,i}\right.$$

$$\text{IB:} \left\{ K6_i = -2(\cot\alpha)_{a3}\left\{\left(\frac{1}{r}\right)_{a3}\right\}_{a3,i}\right.$$

$$C1_i = -\frac{\pi\overline{\gamma}_1}{2}\left(\phi_o\right)_{21}\left\{\left(r_1 \sin\phi\right)_{a1}\right\}_{a1,i} \cdot (\xi_{11} - \xi_{10})$$

$$C2_i = C3_i = 0$$

$$C4_i = +\frac{\pi\overline{\gamma}_2}{2}\,(|L|)_{a2}\,\{1\}_{a2,i} \cdot (\xi_{21} - \xi_{20})$$

$$\text{IA:} \left\{ C5_i = -\frac{\pi\overline{\gamma}_3}{2}\left(\phi_o - \pi\right)_{a3}\left\{\left(r_1 \sin\phi\right)_{a3}\right\}_{a3,i} \cdot (\xi_{31} - \xi_{30})\right.$$

$$\text{IB:} \left\{ C5_i = +\frac{\pi\overline{\gamma}_3}{2}\,(|L|)_{a3}\,\{1\}_{a3,i} \cdot (\xi_{31} - \xi_{30})\right.$$

$$C6_i = C7_i = 0$$

$$\left(KU_1\right)_i = +\left(\phi_o\right)_{a1}\left\{\left(r \cdot r_1 \cos\phi\right)_{a1}\right\}_{a1,i} \cdot (\xi_{11} - \xi_{10})$$

$$\left(KV_1\right)_i = +\left(\phi_o\right)_{a1}\left\{\left(r \cdot r_1 \sin\phi\right)_{a1}\right\}_{a1,i} \cdot (\xi_{11} - \xi_{10})$$

$$\left(KU_2\right)_i = -(|L|\cot\phi_o)_{a2}\left\{(r)_{a2}\right\}_{a2,\,i} \cdot (\xi_{21} - \xi_{20})\ ;$$

$$\overline{KU}_2 = \overline{\overline{KU}}_2 = \overline{KV}_2 = \overline{\overline{KV}}_2 = 0$$

$$\left(KV_2\right)_i = -(|L|)_{a2}\left\{(r)_{a2}\right\}_{a2,\,i} \cdot (\xi_{21} - \xi_{20})$$

$$\text{IA:}\ \begin{cases} \left(KU_3\right)_i = +\left(\phi_o - \pi\right)_{a3}\left\{(r \cdot r_1\ \cos\phi)_{a3}\right\}_{a3,\,i} \cdot (\xi_{31} - \xi_{30}) \\[2ex] \left(KV_3\right)_i = +\left(\phi_o - \pi\right)_{a3}\left\{(r \cdot r_1\ \sin\phi)_{a3}\right\}_{a3,\,i} \cdot (\xi_{31} - \xi_{30}) \end{cases}$$

$$\text{IB:}\ \begin{cases} \left(KU_3\right)_i = -(|L|\cot\phi_o)_{a3}\left\{(r)_{a3}\right\}_{a3,\,i} \cdot (\xi_{31} - \xi_{30}) \\[2ex] \left(KV_3\right)_i = -(|L|)_{a3}\left\{(r)_{a3}\right\}_{a3,\,i} \cdot (\xi_{31} - \xi_{30}) \end{cases}$$

$$\left(KU_4\right)_i = +\left\{(r\ \cot\phi)_{a1}\right\}_{a1,\,i}$$

$$\left(KV_4\right)_i = +\left\{(r)_{a1}\right\}_{a1,\,i}$$

$$\left(KU_5\right)_i = +(\cot\phi_o)_{a2}\left\{(r)_{a2}\right\}_{a2,\,i}$$

$$\left(KV_5\right)_i = +\left\{(r)_{a2}\right\}_{a2,\,i}\ ;\quad \overline{KU}_5 = \overline{\overline{KU}}_5 = \overline{KV}_5 = \overline{\overline{KV}}_5 = 0$$

$$\text{IA:}\quad \begin{cases} \left(KU_6\right)_i = +\left\{(r \cot \phi)_{a3}\right\}_{a3,\,i} \\[2ex] \left(KV_6\right)_i = +\left\{(r)_{a3}\right\}_{a3,\,i} \end{cases}$$

$$\text{IB:}\quad \begin{cases} \left(KU_6\right)_i = +(\cot \phi_o)_{a3}\left\{(r)_{a3}\right\}_{a3,\,i} \\[2ex] \left(KV_6\right)_i = +\left\{(r)_{a3}\right\}_{a3,\,i} \end{cases}$$

E. 4 CASE II $\left(\begin{array}{l} a1 \text{ - Convex Down Ellipsoid; } a2 \text{ - Conic;} \\ a3 \text{ - } \left\{\begin{array}{l}\text{IIA: Convex Down Ellipsoid} \\ \text{IIB: Conic}\end{array}\right\}\end{array}\right)$

E. 4. 1 Integral Equation Relationships

$$[\overline{M}] = -4\pi\overline{\gamma}_1 \left(\phi_o - \pi\right)_{a1} \int_{\xi_{10}}^{\xi_{11}} \frac{\left(r_1 \sin\phi\right)_{a1}}{\left(r_{a2}^2 - r_{a1}^2\right)} \left\{\frac{\left(r_{a2}^2 - r_{a1}^2\right)}{2} \hat{u}(x)\right\}_1 \left\{\frac{\left(r_{a2}^2 - r_{a1}^2\right)}{2} \hat{u}(x)\right\}_1^T d\xi_{a1}$$

$$+4\pi\overline{\gamma}_2 (|L|)_{a2} \int_{\xi_{20}}^{\xi_{21}} \left(\frac{1}{r^2}\right)_{a2} \left\{\frac{r^2}{2} \hat{u}(x)\right\}_2 \left\{\frac{r^2}{2} \hat{u}(x)\right\}_2^T d\xi_{a2}{}^*$$

$$\text{IIA: } \left\{ -4\pi\overline{\gamma}_3 \left(\phi_o - \pi\right)_{a3} \int_{\xi_{30}}^{\xi_{31}} \left(\frac{r_1 \sin\phi}{r^2}\right)_{a3} \left\{\frac{r^2}{2} \hat{u}(x)\right\}_3 \left\{\frac{r^2}{2} \hat{u}(x)\right\}_3^T d\xi_{a3}{}^* \right.$$

$$\text{IIB: } \left\{ +4\pi\overline{\gamma}_3 (|L|)_{a3} \int_{\xi_{30}}^{\xi_{31}} \left(\frac{1}{r^2}\right)_{a3} \left\{\frac{r^2}{2} \hat{u}(x)\right\}_3 \left\{\frac{r^2}{2} \hat{u}(x)\right\}_3^T d\xi_{a3}{}^* \right.$$

$$-\pi\overline{\gamma}_1 \left(\phi_o - \pi\right)_{a1} \int_{\xi_{10}}^{\xi_{11}} F_1 (r_{a1}, r_{a2}) \cdot \left(r_1 \sin\phi\right)_{a1} \left\{\hat{v}(x,r)\right\}_{1,1} \left\{\hat{v}(x,r)\right\}_{1,1}^T d\xi_{a1}$$

$$-\pi\overline{\gamma}_1 \left(\phi_o - \pi\right)_{a1} \int_{\xi_{10}}^{\xi_{11}} F_2 (r_{a1}, r_{a2}) \cdot \left(r_1 \sin\phi\right)_{a1} \left[\left\{\hat{v}(x,r)\right\}_{1,1} \left\{\hat{v}(x,r)\right\}_{1,2}^T + \left\{\hat{v}(x,r)\right\}_{1,2} \left\{\hat{v}(x,r)\right\}_{1,1}^T\right] d\xi_{a1}$$

$$+\pi\overline{\gamma}_1 \left(\phi_o - \pi\right)_{a1} \int_{\xi_{10}}^{\xi_{11}} F_1 (r_{a2}, r_{a1}) \cdot \left(r_1 \sin\phi\right)_{a1} \left\{\hat{v}(x,r)\right\}_{1,2} \left\{\hat{v}(x,r)\right\}_{1,2}^T d\xi_{a1}$$

$$+\frac{\pi\overline{\gamma}_2 (|L|)_{a2}}{2} \int_{\xi_{20}}^{\xi_{21}} \left\{r\hat{v}(x,r)\right\}_2 \left\{r\hat{v}(x,r)\right\}_2^T d\xi_{a2}{}^*$$

$$\text{IIA: } \left\{ -\frac{\pi\overline{\gamma}_3 \left(\phi_o - \pi\right)_{a3}}{2} \int_{\xi_{30}}^{\xi_{31}} \left(r_1 \sin\phi\right)_{a3} \left\{r\hat{v}(x,r)\right\}_3 \left\{r\hat{v}(x,r)\right\}_3^T d\xi_{a3}{}^* \right.$$

$$\text{IIB: } \left\{ +\frac{\pi\overline{\gamma}_3 (|L|)_{a3}}{2} \int_{\xi_{30}}^{\xi_{31}} \left\{r\hat{v}(x,r)\right\}_3 \left\{r\hat{v}(x,r)\right\}_3^T d\xi_{a3}{}^* \right.$$

---

*Identical with Case I.

where

$$\left\{\frac{\left(r_{a2}^2 - r_{a1}^2\right)}{2}\hat{u}(x)\right\}_1 = \left\{\begin{array}{ll}
 & -\int_{\xi_{10}}^{\xi}\left(r \cdot r_1\,(\phi_o - \pi)\cos\phi\,\{u(\xi)\}\right)_{a1} d\xi_{a1} \\
 & -\int_{\xi_{10}}^{\xi}\left(r \cdot r_1\,(\phi_o - \pi)\sin\phi\{v(\xi)\}\right)_{a1} d\xi_{a1} \\
\hline
 & -\int_{\xi_{20}}^{\xi_{21}}\left(r|L|\cot\phi_o\,\{u(\xi)\}\right)_{a2} d\xi_{a2} \\
 & \quad +\int_{\xi_{10}}^{\xi}\left(r\cot\phi_o\{u(\xi)\}\right)_{a2} \cdot \left(r_2(\phi_o - \pi)\sin\phi\right)_{a1} d\xi_{a1} \\
 & -\int_{\xi_{20}}^{\xi_{21}}\left(r|L|\{v(\xi)\}\right)_{a2} d\xi_{a2} \\
 & \quad +\int_{\xi_{10}}^{\xi}\left(r\{v(\xi)\}\right)_{a2} \cdot \left(r_2(\phi_o - \pi)\sin\phi\right)_{a1} d\xi_{a1} \\
\hline
\text{IIA:} & \left\{\begin{array}{l}
\int_{\xi_{30}}^{\xi_{31}}\left(r \cdot r_1(\phi_o - \pi)\cos\phi\,\{u(\xi)\}\right)_{a3} d\xi_{a3} \\
\int_{\xi_{30}}^{\xi_{31}}\left(r \cdot r_1(\phi_o - \pi)\sin\phi\,\{v(\xi)\}\right)_{a3} d\xi_{a3}
\end{array}\right. \\
\text{IIB:} & \left\{\begin{array}{l}
-\int_{\xi_{30}}^{\xi_{31}}\left(r|L|\cot\phi_o\{u(\xi)\}\right)_{a3} d\xi_{a3} \\
-\int_{\xi_{30}}^{\xi_{31}}\left(r|L|\,\{v(\xi)\}\right)_{a3} d\xi_{a3}
\end{array}\right.
\end{array}\right\}$$

Note: $\left\{\frac{r^2}{2}\hat{u}(x)\right\}_2$ and $\left\{\frac{r^2}{2}\hat{u}(x)\right\}_3$ are the same as for Case I.

$$\left\{\hat{v}(x,r)\right\}_{1,1} = \left\{\begin{array}{c} \left(\cot\phi\left\{u(\xi)\right\}\right)_{a1} \\ \left\{v(\xi)\right\}_{a1} \\ \hdashline 0 \\ \hdashline 0 \end{array}\right\} - \frac{2(\cot\phi)_{a1}}{\left(r_{a2}^2 - r_{a1}^2\right)} \cdot \left\{\frac{r_{a2}^2 - r_{a1}^2}{2}\,\hat{u}(x)\right\}_1$$

$$\left\{\hat{v}(x,r)\right\}_{1,2} = \left\{\begin{array}{c} 0 \\ \hdashline \left(\cot\phi_o\left\{u(\xi)\right\}\right)_{a2} \\ \left\{v(\xi)\right\}_{a2} \\ \hdashline 0 \end{array}\right\} - \frac{2(\cot\phi_o)_{a2}}{\left(r_{a2}^2 - r_{a1}^2\right)} \cdot \left\{\frac{r_{a2}^2 - r_{a1}^2}{2}\,\hat{u}(x)\right\}_1$$

Note: $\left\{\hat{v}(x,r)\right\}_2$ and $\left\{\hat{v}(x,r)\right\}_3$ same as for Case I.

E.4.2 Integration Limit Data

Fluid Surface in Section 1

$$\xi_{10} = \xi_{20} = \xi_{30} = 0 \;; \quad \xi_{31} = 1$$

$$\xi_{11} = \left(\frac{\pi - \overline{\overline{\phi}}}{\pi - \phi_o}\right)_{a1}$$

$$\xi_{21} = 1 - (|\overline{b}|)_{a1} / (|L|)_{a2}$$

$$\overline{\gamma}_1 = \overline{\gamma}_2 = \overline{\gamma}_3 = \gamma$$

Note:

$$\overline{\overline{\phi}} = \sin^{-1}\left\{\frac{\overline{\overline{r}}}{-\left[a^2k^2 - \overline{\overline{r}}^2(k^2 - 1)\right]^{1/2}}\right\}$$

$$k = a/b$$

$$\overline{\overline{r}} = a\sqrt{1 - \frac{\left(b - \overline{H}_1\right)^2}{b^2}}$$

Fluid Surface in Section 2

$$\xi_{10} = \xi_{11} = 0.5$$

$$\xi_{20} = \xi_{30} = 0$$

$$\xi_{31} = 1$$

$$\xi_{21} = (\overline{H}_2 / |L|)_{a2}$$

$$\overline{\gamma}_1 = 0 \;; \quad \overline{\gamma}_2 = \overline{\gamma}_3 = \gamma$$

Fluid Surface in Section 3

$$\xi_{10} = \xi_{11} = 0.5 \quad , \qquad \xi_{20} = \xi_{21} = \xi_{30} = 0$$

$$\text{IIA:} \left\{ \xi_{31} = \left(\overline{\overline{\phi}} / \phi_o\right)_{a3} \right.$$

$$\text{IIB:} \left\{ \xi_{31} = \left(\overline{H}_3 / |L|\right)_{a3} \right.$$

E.4.3 Specific Matrix Data

$$(K1)_i = -4\pi\overline{\gamma}_1\left(\phi_o - \pi\right)_{a1}\left\{\frac{\left(r_1 \sin \phi\right)_{a1}}{\left(r^2_{a2} - r^2_{a1}\right)}\right\}_{a1,i} \cdot (\xi_{11} - \xi_{10})$$

$$(K2)_i = +4\pi\overline{\gamma}_2\left(|L|\right)_{a2}\left\{\left(\frac{1}{r^2}\right)_{a2}\right\}_{a2,i} \cdot (\xi_{21} - \xi_{20})^*$$

$$\text{IIA:} \quad \left\{(K3)_i = -4\pi\overline{\gamma}_3\left(\phi_o - \pi\right)_{a3}\left\{\left(\frac{r_1 \sin \phi}{r^2}\right)_{a3}\right\}_{a3,i} \quad (\xi_{31} - \xi_{30})^*\right.$$

$$\text{IIB:} \quad \left\{(K3)_i = +4\pi\overline{\gamma}_3\left(|L|\right)_{a3}\left\{\left(\frac{1}{r^2}\right)_{a3}\right\}_{a3,i} \cdot (\xi_{31} - \xi_{30})^*\right.$$

$$(C1)_i = -\pi\overline{\gamma}_1\left(\phi_o - \pi\right)_{a1}\left\{F_1(r_{a1}, r_{a2}) \cdot \left(r_1 \sin \phi\right)_{a1}\right\}_{a1,i} \cdot (\xi_{11} - \xi_{10})$$

$$(C2)_i = -\pi\overline{\gamma}_1\left(\phi_o - \pi\right)_{a1}\left\{F_2(r_{a1}, r_{a2}) \cdot \left(r_1 \sin \phi\right)_{a1}\right\}_{a1,i} \cdot (\xi_{11} - \xi_{10})$$

$$(C3)_i = +\pi\overline{\gamma}_1\left(\phi_o - \pi\right)_{a1}\left\{F_1(r_{a2}, r_{a1}) \cdot \left(r_1 \sin \phi\right)_{a1}\right\}_{a1,i} \cdot (\xi_{11} - \xi_{10})$$

$$(C4)_i = +\frac{\pi\overline{\gamma}_2}{2}\left(|L|\right)_{a2}\left\{1\right\}_{a2,i} \cdot (\xi_{21} - \xi_{20})^*$$

$$\text{IIA:} \quad \left\{(C5)_i = -\frac{\pi\overline{\gamma}_3}{2}\left(\phi_o - \pi\right)_{a3}\left\{\left(r_1 \sin \phi\right)_{a3}\right\}_{a3,i} \cdot (\xi_{31} - \xi_{30})^*\right.$$

$$\text{IIB:} \quad \left\{(C5)_i = +\frac{\pi\overline{\gamma}_3}{2}\left(|L|\right)_{a3}\left\{1\right\}_{a3,i} \cdot (\xi_{31} - \xi_{30})^*\right.$$

$$(C6)_i = (C7)_i = 0^*$$

*Same as for Case I

$$\left(KU_1\right)_i = -\left(\phi_o - \pi\right)_{a1}\left\{\left(r \cdot r_1 \cos\phi\right)_{a1}\right\}_{a1,i}(\xi_{11} - \xi_{10})$$

$$\left(KV_1\right)_i = -\left(\phi_o - \pi\right)_{a1}\left\{\left(r \cdot r_1 \sin\phi\right)_{a1}\right\}_{a1,i}(\xi_{11} - \xi_{10})$$

$$\left(KU_2\right)_i = -\left(|L|\cot\phi_o\right)_{a2}\left\{(r)_{a2}\right\}_{a2,i}(\xi_{21} - \xi_{20})^*$$

$$\left(KV_2\right)_i = -\left(|L|\right)_{a2}\left\{(r)_{a2}\right\}_{a2,i}(\xi_{21} - \xi_{20})^*$$

$$\text{IIA:}\begin{cases}\left(KU_3\right)_i = \left(\phi_o - \pi\right)_{a3}\left\{\left(r \cdot r_1 \cos\phi\right)_{a3}\right\}_{a3,i}(\xi_{31} - \xi_{30})^* \\ \left(KV_3\right)_i = \left(\phi_o - \pi\right)_{a3}\left\{\left(r \cdot r_1 \sin\phi\right)_{a3}\right\}_{a3,i}(\xi_{31} - \xi_{30})^*\end{cases}$$

$$\text{IIB:}\begin{cases}\left(KU_3\right)_i = -\left(|L|\cot\phi_o\right)_{a3}\left\{(r)_{a3}\right\}_{a3,i}(\xi_{31} - \xi_{30})^* \\ \left(KV_3\right)_i = -\left(|L|\right)_{a3}\left\{(r)_{a3}\right\}_{a3,i}(\xi_{31} - \xi_{30})^*\end{cases}$$

$$\left(\overline{KU}_2\right)_i = \left(\phi_o - \pi\right)_{a1}(\cot\phi_o)_{a2}\left\{\left(r_1 \sin\phi\right)_{a1}(r)_{a2}\right\}_{a1,i}(\xi_{11} - \xi_{10})$$

$$\left(\overline{KV}_2\right)_i = \left(\phi_1 - \pi\right)_{a1}\left\{\left(r_1 \sin\phi\right)_{a1}(r)_{a2}\right\}_{a1,i}(\xi_{11} - \xi_{10})$$

$$\left(\overline{\overline{KU}}_2\right)_i = 0^*$$

$$\left(\overline{\overline{KV}}_2\right)_i = 0^*$$

$$\left(KU_4\right)_i = \left\{(\cot\phi)_{a1}\right\}_{a1,i}$$

$$\left(KV_4\right)_i = (1)$$

---

*Same as for Case I

$$\left(\overline{KU}_5\right)_i = (\cot\phi_o)_{a2}$$

$$\left(\overline{KV}_5\right)_i = (1)$$

$$\left(KU_5\right)_i = +(\cot\phi_o)_{a2}\left\{(r)_{a2}\right\}^*_{a2,i}$$

$$\left(KV_5\right)_i = +\left\{(r)_{a2}\right\}^*_{a2,i}$$

$$\text{IIA:}\begin{cases}\left(KU_6\right)_i = +\left\{(r\cot\phi)_{a3}\right\}^*_{a3,i}\\[2ex] \left(KV_6\right)_i = +\left\{(r)_{a3}\right\}^*_{a3,i}\end{cases}$$

$$\text{IIB:}\begin{cases}\left(KU_6\right)_i = +(\cot\phi_o)_{a3}\left\{(r)_{a3}\right\}^*_{a3,i}\\[2ex] \left(KV_6\right)_i = +\left\{(r)_{a3}\right\}^*_{a3,i}\end{cases}$$

$$\left(\overline{\overline{KU}}_5\right)_i = 0^*$$

$$\left(\overline{\overline{KV}}_5\right)_i = 0^*$$

$$(K4)_i = -2\left\{\frac{(\cot\phi)_{a1}}{\left(r^2\right)_{a2} - \left(r^2\right)_{a1}}\right\}_{a1,i}$$

$$(\overline{K5})_i = -2\left\{\frac{(\cot\phi)_{a2}}{\left(r^2\right)_{a2} - \left(r^2\right)_{a1}}\right\}_{a1,i}$$

---

*Same as for Case I

$$(K5)_i = -2\,(\cot\phi_o)_{a2}\left\{\left(\frac{1}{r}\right)_{a2}\right\}^*_{a2,\,i}$$

$$(\overline{\overline{K5}})_i = 0^*$$

$$\text{IIA: } \Big\{ (K6)_i = -\left\{\left(\frac{2\cot\phi}{r}\right)_{a3}\right\}^*_{a3,\,i}$$

$$\text{IIB: } \Big\{ (K6)_i = -2\,(\cot\phi_o)_{a3}\left\{\left(\frac{1}{r}\right)_{a3}\right\}^*_{a3,\,i}$$

*Same as for Case I

E.5 CASE III (a1 - Convex Up Ellipsoid; a2 - Conic; a3 - Convex Up Ellipsoid)

E.5.1 Integral Equation Relationships

$$[\overline{M}] = -4\pi\overline{\gamma}_1(\phi_o)_{a1}\int_{\xi_{10}}^{\xi_{11}}\left(\frac{r_1 \sin\phi}{r^2}\right)_{a1}\left\{\frac{r^2}{2}\hat{u}(x)\right\}_1\left\{\frac{r^2}{2}\hat{u}(x)\right\}_1^T d\,\xi_{a1}$$

$$+ 4\pi\overline{\gamma}_2(|L|)_{a2}\int_{\xi_{20}}^{\xi_{21}}\left(\frac{1}{r^2}\right)_{a2}\left\{\frac{r^2}{2}\hat{u}(x)\right\}_2\left\{\frac{r^2}{2}\hat{u}(x)\right\}_2^T d\,\xi_{a2}$$

$$- 4\pi\overline{\gamma}_3(\phi_o)_{a3}\int_{\xi_{30}}^{\xi_{31}}\frac{(r_1 \sin\phi)_{a3}}{\left(r_{a2}^2 - r_{a3}^2\right)}\left\{\frac{\left(r_{a2}^2 - r_{a3}^2\right)}{2}\hat{u}(x)\right\}_3\left\{\frac{\left(r_{a2}^2 - r_{a3}^2\right)}{2}\hat{u}(x)\right\}^T d\,\xi_{a3}$$

$$- \frac{\pi\overline{\gamma}_1(\phi_o)_{a1}}{2}\int_{\xi_{10}}^{\xi_{11}}(r_1 \sin\phi)_{a1}\left\{r\hat{v}(x,r)\right\}_1\left\{r\hat{v}(x,r)\right\}_1^T d\,\xi_{a1}$$

$$+ \frac{\pi\overline{\gamma}_2(|L|)_{a2}}{2}\int_{\xi_{20}}^{\xi_{21}}\left\{r\hat{v}(x,r)\right\}_2\left\{r\hat{v}(x,r)\right\}_2^T d\,\xi_{a2}$$

$$- \pi\overline{\gamma}_3(\phi_o)_{a3}\int_{\xi_{30}}^{\xi_{31}} F_1(r_{a3}; r_{a2})\cdot(r_1 \sin\phi)_{a3}\left\{\hat{v}(x,r)\right\}_{3,3}\left\{\hat{v}(x,r)\right\}_{3,3}^T d\,\xi_{a3}$$

$$- \pi\overline{\gamma}_3(\phi_o)_{a3}\int_{\xi_{30}}^{\xi_{31}} F_2(r_{a3}; r_{a2})\cdot(r_1 \sin\phi)_{a3}\left[\left\{\hat{v}(x,r)\right\}_{3,3}\left\{\hat{v}(x,r)\right\}_{3,2}^T + \left\{\hat{v}(x,r)\right\}_{3,2}\left\{\hat{v}(x,r)\right\}_{3,3}^T\right] d\,\xi_{a3}$$

$$+ \pi\overline{\gamma}_3(\phi_o)_{a3}\int_{\xi_{30}}^{\xi_{31}} F_1(r_{a2}; r_{a3})\cdot(r_1 \sin\phi)_{a3}\left\{\hat{v}(x,r)\right\}_{3,2}\left\{\hat{v}(x,r)\right\}_{3,2}^T d\,\xi_{a3}$$

where

$$\left\{\frac{r^2}{2}\hat{u}(x)\right\}_1 = \left\{\begin{array}{l}
+\displaystyle\int_{\xi_{10}}^{\xi}\left(r\cdot r_1\phi_o \cos\phi\left\{u(\xi)\right\}\right)_{a1} d\xi_{a1} \\
+\displaystyle\int_{\xi_{10}}^{\xi}\left(r\cdot r_1\phi_o \sin\phi\left\{v(\xi)\right\}\right)_{a1} d\xi_{a1} \\
\hdashline
-\displaystyle\int_{\xi_{20}}^{\xi_{21}}\left(r|L|\cot\phi_o\left\{u(\xi)\right\}\right)_{a2} d\xi_{a2} \\
\qquad +\displaystyle\int_{\xi_{30}}^{\xi_{31}}\left(r\cot\phi_o\left\{u(\xi)\right\}\right)_{a2}\cdot\left(r_1\phi_o\sin\phi\right)_{a3} d\xi_{a3} \\
-\displaystyle\int_{\xi_{20}}^{\xi_{21}}\left(r|L|\left\{v(\xi)\right\}\right)_{a2} d\xi_{a2} \\
\qquad +\displaystyle\int_{\xi_{30}}^{\xi_{31}}\left(r\left\{v(\xi)\right\}\right)_{a2}\cdot\left(r_1\phi_o\sin\phi\right)_{a3} d\xi_{a3} \\
\hdashline
-\displaystyle\int_{\xi_{30}}^{\xi_{31}}\left(r\cdot r_1\phi_o\cos\phi\left\{u(\xi)\right\}\right)_{a3} d\xi_{a3} \\
-\displaystyle\int_{\xi_{30}}^{\xi_{31}}\left(r\cdot r_1\phi_o\sin\phi\left\{v(\xi)\right\}\right)_{a3} d\xi_{a3}
\end{array}\right\}$$

$$
\left\{\frac{r^2}{2}\ \hat{u}(x)\right\}_2 = \left\{\begin{array}{l}
\qquad\qquad\qquad 0 \\
\hline
-\int_{\xi_{20}}^{\xi}\left(r\,|L|\cot\phi_o\{u(\xi)\}\right)_{a2} d\xi_{a2} \\
\qquad +\int_{\xi_{30}}^{\xi_{31}}\left(r\cot\phi_o\{u(\xi)\}\right)_{a2}\left(r_1\phi_o\ \sin\ \phi\right)_{a3} d\xi_{a3} \\
-\int_{\xi_{20}}^{\xi}\left(r\,|L|\,\{v(\xi)\}\right)_{a2} d\xi_{a2} \\
\qquad +\int_{\xi_{30}}^{\xi_{31}} r\{v(\xi)\}_{a2}\left(r_1\phi_o\ \sin\ \phi\right)_{a3} d\xi_{a3} \\
\hline
-\int_{\xi_{30}}^{\xi_{31}}\left(r\cdot r_1\phi_o\ \cos\ \phi\{u(\xi)\}\right)_{a3} d\xi_{a3} \\
-\int_{\xi_{30}}^{\xi_{31}}\left(r\cdot r_1\phi_o\ \sin\ \phi\{v(\xi)\}\right)_{a3} d\xi_{a3}
\end{array}\right\}
$$

$$\left\{\frac{\left(r_{a2}^2 - r_{a3}^2\right)}{2}\hat{u}(x)\right\}_3 = \left\{\begin{array}{c} 0 \\ \hdashline + \int_{\xi_{30}}^{\xi} \left(r \cot \phi_o \{u(\xi)\}\right)_{a2} \left(r_1 \phi_o \sin \phi\right)_{a3} d\,\xi_{a3} \\ + \int_{\xi_{30}}^{\xi} \left(r\{v(\xi)\}\right)_{a2} \left(r_1 \phi_o \sin \phi\right)_{a3} d\,\xi_{a3} \\ \hdashline + \int_{\xi_{30}}^{\xi} \left(r \cdot r_1 \phi_o \cos \phi \{u(\xi)\}\right)_{a3} d\,\xi_{a3} \\ - \int_{\xi_{30}}^{\xi} \left(r \cdot r_1 \phi_o \sin \phi \{v(\xi)\}\right)_{a3} d\,\xi_{a3} \end{array}\right\}$$

$\left\{r\hat{v}(x, r)\right\}_1$ and $\left\{r\hat{v}(x, r)\right\}_2$ are same as Case I.

$$\left\{\hat{v}(x, r)\right\}_{3,3} = \left\{\begin{array}{c} 0 \\ \hdashline 0 \\ \hdashline \left(\cot \phi \{u(\xi)\}\right)_{a3} \\ \{v(\xi)\}_{a3} \end{array}\right\} - \frac{2(\cot \phi)_{a3}}{\left(r_{a2}^2 - r_{a3}^2\right)} \left\{\frac{\left(r_{a2}^2 - r_{a3}^2\right)}{2}\hat{u}(x)\right\}_3$$

$$
\left\{ \hat{v}(x, r) \right\}_{3,2} = \left\{ \begin{array}{c} 0 \\ \hline \left(\cot \phi_o \left\{ u(\xi) \right\} \right)_{a2} \\ \left\{ v(\xi) \right\}_{a2} \\ \hline 0 \end{array} \right\} - \frac{2 (\cot \phi_o)_{a2}}{\left(r_{a2}^2 - r_{a3}^2 \right)} \left\{ \frac{\left(r_{a2}^2 - r_{a3}^2 \right)}{2} \; \hat{u}(x) \right\}_3
$$

E.5.2 Integration Limit Data

Fluid Surface in Section 1

$$\xi_{11} = \left(\overline{\phi}/\phi_o\right)_{a1}$$

$$\xi_{20} = (|b|)_{a3}/(|L|)_{a2}$$

$$\xi_{31} = 0$$

$$\xi_{10} = \xi_{21} = \xi_{30} = 1$$

$$\overline{\gamma}_1 = \overline{\gamma}_2 = \overline{\gamma}_3 = \gamma$$

Fluid Surface in Section 2

$$\xi_{10} = \xi_{11} = \xi_{30} = 1$$

$$\xi_{21} = (H/|L|)_{a2}$$

$$\xi_{20} = (|b|)_{a3}/(|L|)_{a2}$$

$$\xi_{31} = 0$$

$$\overline{\gamma}_1 = 0$$

$$\overline{\gamma}_2 = \overline{\gamma}_3 = \gamma$$

Fluid Surface in Section 3

$$\xi_{10} = \xi_{11} = 1$$

$$\xi_{20} = \xi_{21} = \left(\overline{H}_3/|L|\right)_{a2}$$

$$\xi_{30} = 1$$

$$\xi_{31} = \left(\overline{\phi}/\phi_o\right)_{a3}$$

$$\overline{\gamma}_1 = \overline{\gamma}_2 = 0$$

$$\overline{\gamma}_3 = \gamma$$

E.5.3 Specific Matrix Data

$$(K1)_i = -4\pi\overline{\gamma}_1\left(\phi_o\right)_{a1}\left\{\left(\frac{r_1 \sin \phi}{r^2}\right)_{a1}\right\}_{a1,i} \cdot (\xi_{11} - \xi_{10})^*$$

$$(K2)_i = +4\pi\overline{\gamma}_2(|L|)_{a2}\left\{\left(\frac{1}{r^2}\right)_{a2}\right\}_{a2,i} \cdot (\xi_{21} - \xi_{20})^*$$

$$(K3)_i = -4\pi\overline{\gamma}_3\left(\phi_o\right)_{a3}\left\{\frac{\left(r_1 \sin \phi\right)_{a3}}{r_{a2}^2 - r_{a3}^2}\right\}_{a3,i} \cdot (\xi_{31} - \xi_{30})$$

$$(C1)_i = -\frac{\pi\overline{\gamma}_1}{2}\left(\phi_o\right)_{a1}\left\{\left(r_1 \sin \phi\right)_{a1}\right\}_{a1,i} \cdot (\xi_{11} - \xi_{10})^*$$

$$(C2)_i = (C3)_i = 0^*$$

$$(C4)_i = +\frac{\pi\overline{\gamma}_2}{2}(|L|)_{a2}\left\{1\right\}_{a2,i} \cdot (\xi_{21} - \xi_{20})^*$$

$$(C5)_i = -\pi\overline{\gamma}_3\left(\phi_o\right)_{a3}\left\{F_1(r_{a3}, r_{a2}) \cdot \left(r_1 \sin \phi\right)_{a3}\right\}_{a3,i} \cdot (\xi_{31} - \xi_{30})$$

$$(C6)_i = -\pi\overline{\gamma}_3\left(\phi_o\right)_{a3}\left\{F_2(r_{a3}, r_{a2}) \cdot \left(r_1 \sin \phi\right)_{a3}\right\}_{a3,i} \cdot (\xi_{31} - \xi_{30})$$

$$(C7)_i = +\pi\overline{\gamma}_3\left(\phi_o\right)_{a3}\left\{F_1(r_{a2}, r_{a3}) \cdot \left(r_1 \sin \phi\right)_{a3}\right\}_{a3,i} \cdot (\xi_{31} - \xi_{30})$$

---

*Same as for Case I

$$(KU_1)_i = +\left(\phi_o\right)_{a1}\left\{\left(r \cdot r_1 \cos\phi\right)_{a1}\right\}_{a1,i} \cdot (\xi_{11} - \xi_{10})^*$$

$$(KV_1)_i = +\left(\phi_o\right)_{a1}\left\{\left(r \cdot r_1 \sin\phi\right)_{a1}\right\}_{a1,i} \cdot (\xi_{11} - \xi_{10})^*$$

$$(KU2)_i = -(|L|\cot\phi_o)_{a2}\left\{(r)_{a2}\right\}_{a2,i} \cdot (\xi_{21} - \xi_{20})^*$$

$$(KV_2)_i = -(|L|)_{a2}\left\{(r)_{a2}\right\}_{a2,i} \cdot (\xi_{21} - \xi_{20})^*$$

$$(\overline{KU}_2)_i = (\overline{KV}2) = 0^*$$

$$(\overline{\overline{KU}}_2)_i = +(\cot\phi_o)_{a2}\left(\phi_o\right)_{a3}\left\{(r)_{a2}\left(r_1 \sin\phi\right)_{a3}\right\}_{a3,i} \cdot (\xi_{31} - \xi_{30})$$

$$(\overline{\overline{KV}}_2)_i = +\left(\phi_o\right)_{a3}\left\{(r)_{a2}\left(r_1 \sin\phi\right)_{a3}\right\}_{a3,i} \cdot (\xi_{31} - \xi_{30})$$

$$(KU_3)_i = -\left(\phi_o\right)_{a3}\left\{\left(r \cdot r_1 \cos\phi\right)_{a3}\right\}_{a3,i} \cdot (\xi_{31} - \xi_{30})$$

$$(KV_3)_i = -\left(\phi_o\right)_{a3}\left\{\left(r \cdot r_1 \sin\phi\right)_{a3}\right\}_{a3,i} \cdot (\xi_{31} - \xi_{30})$$

$$(KU_4)_i = +\left\{(r \cot\phi)_{a1}\right\}^*_{a1,i}$$

$$(KV_4)_i = +\left\{(r)_{a1}\right\}^*_{a1,i}$$

$$(\overline{KU}_5)_i = (\overline{KV}5) = 0^*$$

---

*Same as for Case I

$$(KU_5)_i = +(\cot\phi_o)_{a2}\left\{(r)_{a2}\right\}^{*}_{a2,i}$$

$$(KV_5)_i = +\left\{(r)_{a2}\right\}^{*}_{a2,i}$$

$$(KU_6)_i = +\left\{(\cot\phi)_{a3}\right\}_{a3,i}$$

$$(KV_6)_i = +1$$

$$(\overline{\overline{KU}}_5)_i = +\left\{(\cot\phi_o)_{a3}\right\}_{a3,i}$$

$$(\overline{\overline{KV}}_5)_i = +1$$

$$(K_4)_i = -\left\{\left(\frac{2\cot\phi}{r}\right)_{a1}\right\}^{*}_{a1,i}$$

$$(\overline{K}_5)_i = 0^{*}$$

$$(K_5)_i = -2(\cot\phi_o)_{a2}\left\{\left(\frac{1}{r}\right)_{a2}\right\}^{*}_{a2,i}$$

$$(\overline{\overline{K}}_5)_i = -2(\cot\phi_o)_{a2}\left\{\frac{1}{\left(r^2_{a2} - r^2_{a3}\right)}\right\}_{a3,i}$$

$$(K_6)_i = -2\left\{(\cot\phi)_{a3}\cdot\frac{1}{\left(r^2_{a2} - r^2_{a3}\right)}\right\}_{a3,i}$$

---

* Same as for Case I

E.6 LAGRANGIAN WEIGHTING MATRIX TABLES

DATA FOR $\lceil W \rfloor$			DATA* FOR $[\bar{\bar{W}}]_{21x21}$		
W_1	0.01341	70742	W_1	0.01341	70742
W_2	0.08876	79707	W_2	0.08876	79707
W_3	-0.04052	17853	W_3	-0.04052	17853
W_4	0.22747	31442	W_4	0.22747	31442
W_5	-0.21757	75613	W_5	-0.21757	75613
W_6	0.35688	23152	W_6	0.35688	23152
W_7	-0.21757	75613	W_7	-0.21757	75613
W_8	0.22747	31442	W_8	0.22747	31442
W_9	-0.04052	17853	W_9	-0.04052	17853
W_{10}	0.08876	79707	W_{10}	0.08876	79707
W_{11}	0.02683	41484	W_{11}	0.02683	41484
W_{12}	0.08876	79707	W_{12}	0.08876	79707
W_{13}	-0.04052	17853	W_{13}	-0.04052	17853
W_{14}	0.22747	31442	W_{14}	0.22747	31442
W_{15}	-0.21757	75613	W_{15}	-0.21757	75613
W_{16}	0.35688	23152	W_{16}	0.35688	23152
W_{17}	-0.21757	75613	W_{17}	-0.21757	75613
W_{18}	0.22747	31442	W_{18}	0.22747	31442
W_{19}	-0.04052	17853	W_{19}	-0.04052	17853
W_{20}	0.08876	79707	W_{20}	0.08876	79707
W_{21}	0.01341	70742	W_{21}	0.01341	70742

*This column is identical for all 21 columns of the matrix.

DATA FOR $[\overline{W}]$

0	0.01400 94748	0.01367 01873	0.01372 55225	0.01370 76586	0.01371 71384	0.01370 94156	0.01371 88954	0.01370 10314	0.01375 63666	0.01341 70742	0.01341 70742	0.01341 70742	0.01341 70742	0.01341 70742	0.01341 70742	0.01341 70742	0.01341 70742	0.01341 70742	0.01341 70742	0.01341 70742
0	0.06849 51420	0.08623 68393	0.08528 88596	0.08554 06980	0.08541 85560	0.08551 29870	0.08540 09862	0.08560 69692	0.08498 04181	0.08876 79707	0.08876 79707	0.08876 79707	0.08876 79707	0.08876 79707	0.08876 79707	0.08876 79707	0.08876 79707	0.08876 79707	0.08876 79707	0.08876 79707
0	-0.09291 98931	-0.04308 58386	-0.02230 07052	-0.02423 12009	-0.02345 79722	-0.02400 48702	-0.02338 90489	-0.02448 35658	-0.02125 41467	-0.04052 17853	-0.04052 17853	-0.04052 17853	-0.04052 17853	-0.04052 17853	-0.04052 17853	-0.04052 17853	-0.04052 17853	-0.04052 17853	-0.04052 17853	-0.04052 17853
0	0.15096 03601	0.11402 37294	0.15472 74756	0.17846 01572	0.17496 54908	0.17701 29870	0.17490 19185	0.17846 52878	0.16824 04626	0.22747 31442	0.22747 31442	0.22747 31442	0.22747 31442	0.22747 31442	0.22747 31442	0.22747 31442	0.22747 31442	0.22747 31442	0.22747 31442	0.22747 31442
0	-0.19032 41624	-0.15133 03271	-0.17000 63413	-0.13519 76912	-0.10833 66684	-0.11437 98701	-0.10920 40325	-0.11721 01972	-0.09538 62114	-0.21757 75613	-0.21757 75613	-0.21757 75613	-0.21757 75613	-0.21757 75613	-0.21757 75613	-0.21757 75613	-0.21757 75613	-0.21757 75613	-0.21757 75613	-0.21757 75613
0	0.17857 71204	0.14500 60927	0.15843 50650	0.14801 21854	0.17844 11576	0.20887 01298	0.19844 72503	0.21187 62226	0.17830 51948	0.35688 23152	0.35688 23152	0.35688 23152	0.35688 23152	0.35688 23152	0.35688 23152	0.35688 23152	0.35688 23152	0.35688 23152	0.35688 23152	0.35688 23152
0	-0.12219 13499	-0.10036 73641	-0.10837 35288	-0.10319 76912	-0.10924 08930	-0.08237 98702	-0.04757 12200	-0.06624 72342	-0.02725 33990	-0.21757 75613	-0.21757 75613	-0.21757 75613	-0.21757 75613	-0.21757 75613	-0.21757 75613	-0.21757 75613	-0.21757 75613	-0.21757 75613	-0.21757 75613	-0.21757 75613
0	0.05923 26815	0.04900 78564	0.05257 12256	0.05046 01571	0.05250 76534	0.04901 29870	0.07274 56685	0.11344 94148	0.07651 27841	0.22747 31442	0.22747 31442	0.22747 31442	0.22747 31442	0.22747 31442	0.22747 31442	0.22747 31442	0.22747 31442	0.22747 31442	0.22747 31442	0.22747 31442
0	-0.01928 76386	-0.01603 82195	-0.01713 27364	-0.01651 69152	-0.01706 38131	-0.01629 05844	-0.01822 10802	0.00256 40532	0.05239 81078	-0.04052 17853	-0.04052 17853	-0.04052 17853	-0.04052 17853	-0.04052 17853	-0.04052 17853	-0.04052 17853	-0.04052 17853	-0.04052 17853	-0.04052 17853	-0.04052 17853
0	0.00378 75527	0.00316 10016	0.00336 69846	0.00325 49837	0.00334 94147	0.00322 72728	0.00347 91112	0.00253 11314	0.02027 28288	0.08876 79707	0.08876 79707	0.08876 79707	0.08876 79707	0.08876 79707	0.08876 79707	0.08876 79707	0.08876 79707	0.08876 79707	0.08876 79707	0.08876 79707
0	-0.00033 92925	-0.00028 39573	-0.00030 18212	-0.00029 23414	-0.00030 00642	-0.00029 05844	-0.00030 84484	-0.00025 31132	-0.00059 24056	0.01341 70742	0.02742 65540	0.02708 72615	0.02714 25967	0.02712 47328	0.02713 42126	0.02712 64898	0.02713 59696	0.02711 81056	0.02717 34408	0.02683 41484
0	0	0	0	0	0	0	0	0	0	0	0.06849 51420	0.08623 68393	0.08528 88596	0.08554 06980	0.08541 85560	0.08551 29870	0.08540 09862	0.08560 69692	0.08498 04181	0.08876 79707
0	0	0	0	0	0	0	0	0	0	0	-0.09291 98931	-0.04308 58386	-0.02230 07052	-0.02423 12009	-0.02345 79722	-0.02400 48702	-0.02338 90489	-0.02448 35658	-0.02123 41467	-0.04052 17853
0	0	0	0	0	0	0	0	0	0	0	0.15096 03601	0.11402 37294	0.15472 74756	0.17846 01572	0.17496 54908	0.17701 29870	0.17490 19185	0.17846 52878	0.16824 04626	0.22747 31442
0	0	0	0	0	0	0	0	0	0	0	-0.19032 41624	-0.15133 03271	-0.17000 63413	-0.13519 76912	-0.10833 66684	-0.11437 98701	-0.10920 40325	-0.11721 01972	-0.09538 62114	-0.21757 75613
0	0	0	0	0	0	0	0	0	0	0	0.17857 71204	0.14500 60927	0.15843 50650	0.14801 21854	0.17844 11576	0.20887 01298	0.19844 72503	0.21187 62226	0.17830 51948	0.35688 23152
0	0	0	0	0	0	0	0	0	0	0	-0.12219 13499	-0.10036 73641	-0.10837 35288	-0.10319 76912	-0.10924 08930	-0.08237 98702	-0.04757 12200	-0.06624 72342	-0.02725 33990	-0.21757 75613
0	0	0	0	0	0	0	0	0	0	0	0.05923 26815	0.04900 78564	0.05257 12256	0.05046 01571	0.05250 76534	0.04901 29870	0.07274 56685	0.11344 94148	0.07651 27841	0.22747 31442
0	0	0	0	0	0	0	0	0	0	0	-0.01928 76386	-0.01603 82195	-0.01713 27364	-0.01651 69152	-0.01706 38131	-0.01629 05844	-0.01822 10802	0.00256 40532	0.05239 81078	-0.04052 17853
0	0	0	0	0	0	0	0	0	0	0	0.00378 75527	0.00316 10016	0.00336 69846	0.00325 49837	0.00334 94147	0.00322 72728	0.00347 91112	0.00253 11314	0.02027 28288	0.08876 79707
0	0	0	0	0	0	0	0	0	0	0	-0.00033 92925	-0.00028 39573	-0.00030 18212	-0.00029 23414	-0.00030 00642	-0.00029 05844	-0.00030 84484	-0.00025 31132	-0.00059 24056	0.01341 70742

APPENDIX F

SAMPLE INPUT DATA

Sample input data for a typical one stage launch vehicle (Figure F. 1) is presented to illustrate the basic data requirements. For simplicity all structural components are constructed of aluminum. The vehicle oxidizer and fuel tanks are both assumed to be simple tanks according to the terminology used in Section 2.0. To demonstrate the capability of handling stringers and ribs, two shell sections are given orthotropic properties as indicated in Figure F. 1.

The physical model is first subdivided into a consistent set of shell, fluid and mass-spring components as shown in Figure F. 2. In this example the vehicle is represented by eleven (11) shell components, two (2) fluid components and four (4) spring-mass components to account for the payload, engine and equipment. The displacement coordinate locations are then selected and numbered according to the requirements discussed in Section 8.0. For this sample problem the vehicle is assumed to be unsupported, i. e., no fixed coordinates are specified.

The input data load sheets have been prepared for the launch vehicle illustrated in Figure F. 1 and are included in this appendix to illustrate the input format. The system data appears first followed in order by the data specifications for each of the shell, fluid and spring-mass components.

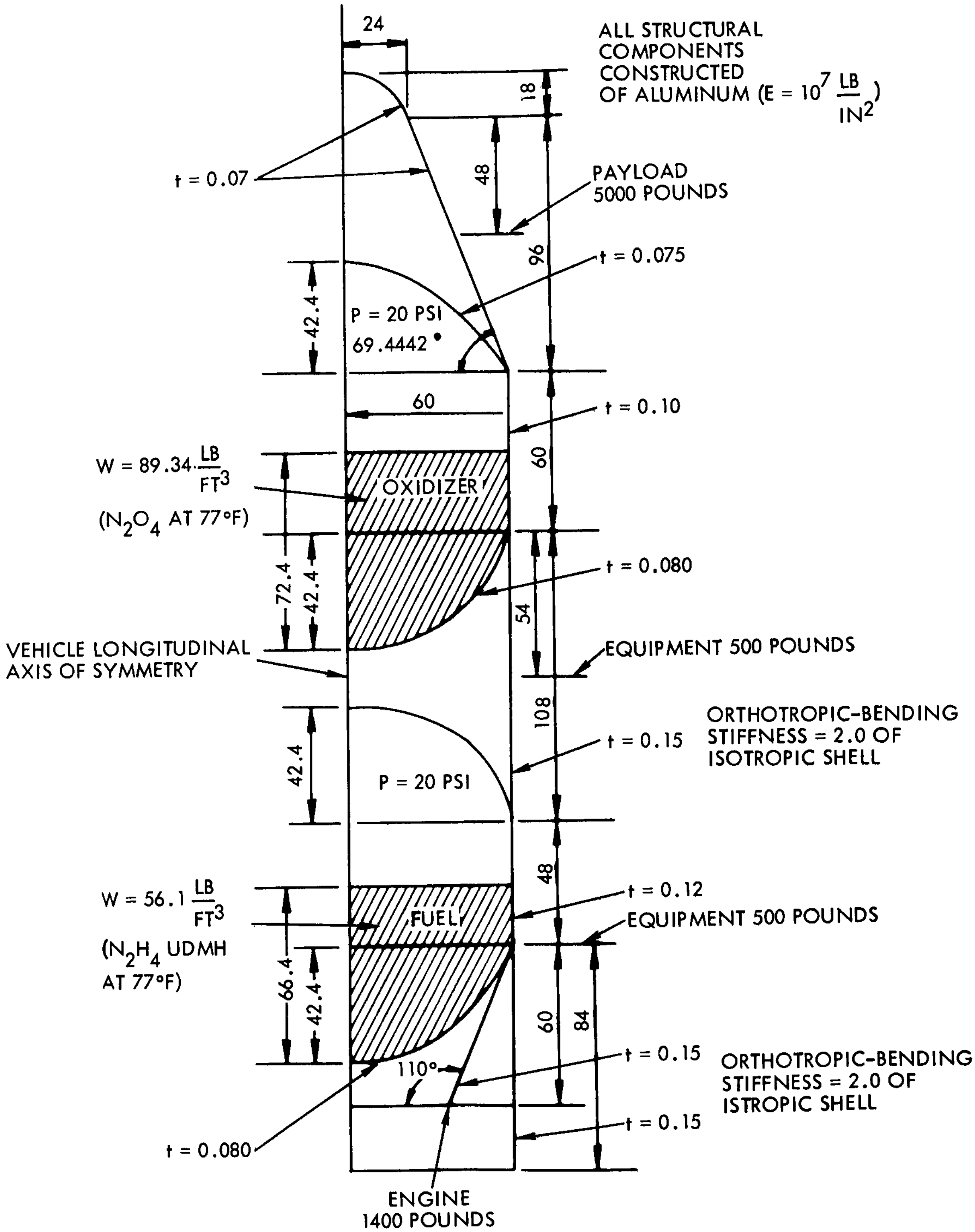


Figure F. 1. Typical One-Stage Launch Vehicle

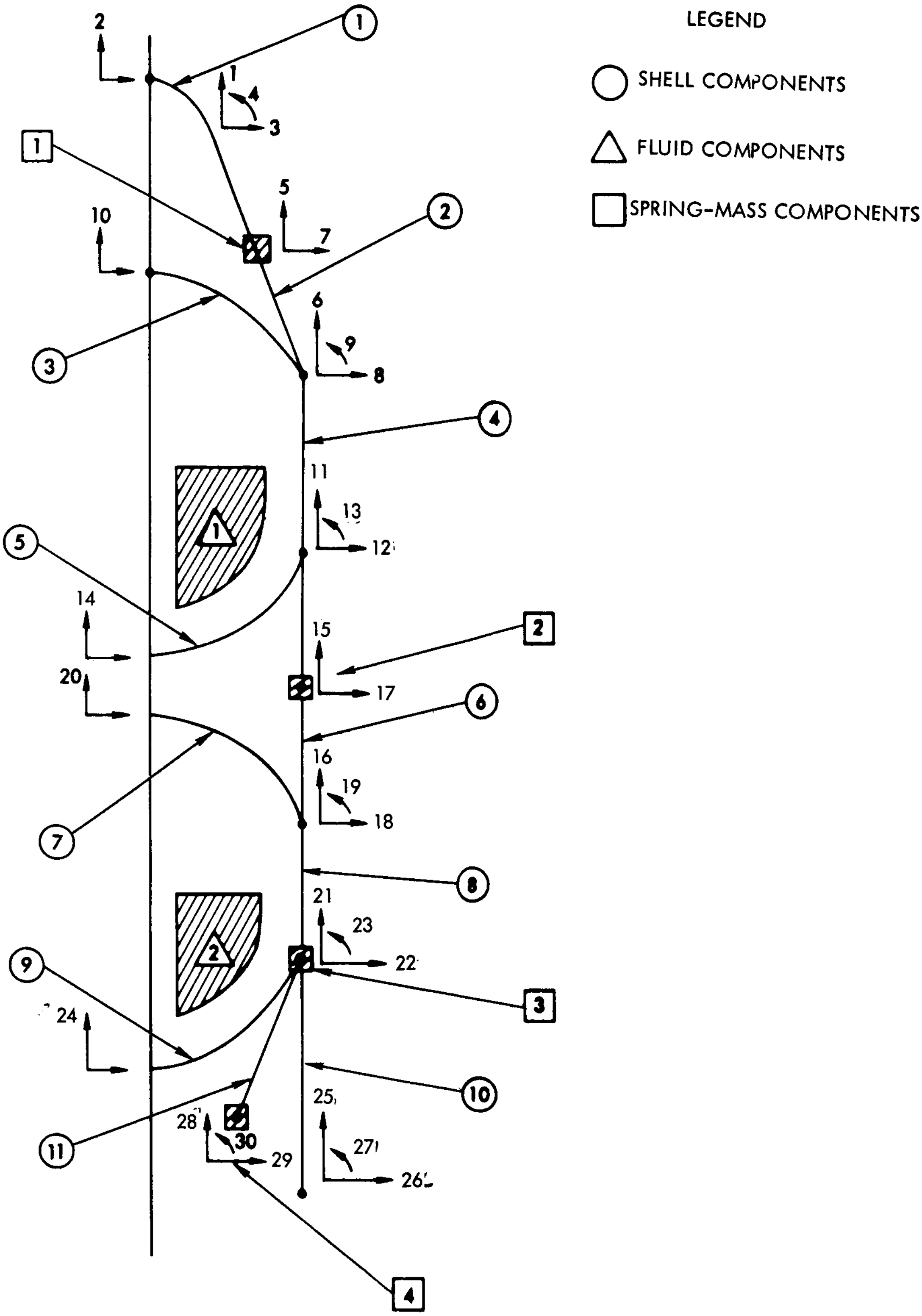


Figure F. 2. System Components and Coordinates

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DATE
NAME
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X1

SYMBOL	PRE	LOC.	VALUE	EXP.
	I	NC	30	
	I	NS	11	
	I	NE	2	
	I	NM	4	
	I	NØ	0	
		G	1.0	
	I	INS	-1	
	I	IØPT1	1	
	I	IØPT2	1	
	I	IØPT3	0	
	I	IØPT4	0	
	I	NEL	30	
	I	INP	3	
	E	ND		
1	I	UBAR	2	
	I	VBAR	2	
	M	APØLY	11,11	
		01,01	1.0	
		02,03	1.0	
	M	BPØLY	11,11	
		01,02	1.0	
		02,03	1.0	
	E	ND		
2	I	UBAR	2	
	I	VBAR	4	
	M	APØLY	11,11	
		01,01	1.0	
		02,02	1.0	
	M	BPØLY	11,11	
		01,01	1.0	
		02,02	1.0	
		03,03	1.0	

SYMBOL	PRE	LOC.	VALUE	EXP
		04,04	1.0	
	E	ND		
3	I	UBAR	3	
	I	VBAR	5	
	M	APØLY	11,11	
		01,01	1.0	
		02,02	1.0	
		03,03	1.0	
	M	BPØLY	11,11	
		01,01	1.0	
		02,02	1.0	
		03,03	1.0	
		04,04	1.0	
		05,05	1.0	
	E	ND		

X1

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SYMBOL	PRE	LOC.	VALUE	EXP.
	I	INA	-1	
	I	IU	2	
	I	IV	2	
	I	IUB	2	
	I	IVB	2	
	I	IDVT	1	
	I		2	
	I		3	
	I		4	
	I	INK	1	
		PHIN	69.444	
		XL	0.	
		R2IN	0.	
		BBAR	18.0	
		ABAR	24.0	
		C11I	.7692307	06
			.7692307	06
		C12I	.2307692	06
			.2307692	06
		C22I	.7692307	06
			.7692307	06
		C33I	.31410256	03
			.31410256	03
			.31410256	03
			.31410256	03
		C34I	.94230769	02
			.94230769	02
			.94230769	02
			.94230769	02
		C44I	.31410256	03
			.31410256	03
			.31410256	03
			.31410256	03
		THI	.07	
			.07	
		DESTA	.2588	-3
		AM	0.	
		HI	0.	
		DESTI	0.	
		PRI	0.	

SYMBOL	PRE	LOC.	VALUE	EXP.
		HE	0.	
		DESTE	0.	
		PRE	0.	
		WEØRC	0.	
	E	ND		

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NAME ______ PRIORITY ______

PROBLEM NO. ______ KEYPUNCHED BY ______

NO. OF CARDS ______ VERIFIED BY ______

SYMBOL	PRE	LOC.	VALUE	EXP.
	I	INA	2	
	I	IU	3	
	I	IV	3	
	I	IUB	3	
	I	IVB	5	
	I	IDVT	1	
	I		5	
	I		6	
	I		3	
	I		7	
	I		8	
	I		4	
	I		9	
	I	INK	3	
		PHIN	69.444	
		XL	96.	
		R2IN	60.	
		BBAR	0.	
		ABAR	0.	
		C11I	.7692307	06
			.7692307	06
		C12I	.2307692	06
			.2307692	06
		C22I	.7692307	06
			.7692307	06
		C33I	.31410256	03
			.31410256	03
			.31410256	03
			.31410256	03
		C34I	.94230769	02
			.94230769	02
			.94230769	02
			.94230769	02
		C44I	.31410256	03
			.31410256	03
			.31410256	03
			.31410256	03
		THI	.07	
			.07	
		DESTH	.2555	-3

SYMBOL	PRE	LOC.	VALUE	EXP.
		AM	.490146	
		HI	0.	
		DESTI	0.	
		PRI	0.	
		HE	0.	
		DESTE	0.	
		PRE	0.	
		WEARC	-19.6	
	E	ND		

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SYMBOL	PRE	LOC.	VALUE	EXP.
	I	INA	-3	
	I	IU	2	
	I	IV	2	
	I	IUB	2	
	I	IVB	2	
	I	IDVT	6	
	I		10	
	I		8	
	I		9	
	I	INK	1	
		PHIN	69.444	
		XL	0.	
		R2IN	0.	
		BBAR	42.4	
		ABAR	60.	
		C11I	.8241758	06
			.8241758	06
		C12I	.2472527	06
			.2472527	06
		C22I	.8241759	06
			.8241759	06
		C33I	.38633241	03
			.38633241	03
			.38633241	03
			.38633241	03
		C34I	.11589772	03
			.11589772	03
			.11589772	03
			.11589772	03
		C44I	.38633241	03
			.38633241	03
			.38633241	03
			.38633241	03
		THI	.075	
			.075	
		DESTA	.2588	-3
		AM	0.	
		HI	0.	
		DESTI	0.	
		PRI	0.	

SYMBOL	PRE	LOC.	VALUE	EXP
		HE	0.	
		DESTE	0.	
		PRE	0.	
		WFORC	0.	
	E	ND		

X1

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SYMBOL	R	LOC.	VALUE	EXP.
	I	INA	4	
	I	IU	2	
	I	IV	2	
	I	IUB	2	
	I	IVB	4	
	I	IDVT	6	
	I		11	
	I		8	
	I		12	
	I		9	
	I		13	
	I	INK	2	
		PHIN	90.	
		XL	60.	
		R2IN	60.	
		BBAR	0.	
		ABAR	0.	
		C11I	1.098901	06
			1.098901	06
		C12I	.3296703	06
			.3296703	06
		C22I	1.098901	06
			1.098901	06
		C33I	.915750	03
			.915750	03
			.915750	03
			.915750	03
		C34I	.2747250	03
			.2747250	03
			.2747250	03
			.2747250	03
		C44I	.915750	03
			.915750	03
			.915750	03
			.915750	03
		THI	.10	
			.10	
		DESTA	.2588	-3
		AM	.585389	
		HI	30.	

SYMBOL	R	LOC.	VALUE	EXP
		DESTI	.51701388	-1
		PRI	20.	
		HE	0.	
		DESTE	0.	
		PRE	0.	
		WFØRC	-5347.022	
	E	ND		

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PRIORITY

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X1

SYMBOL	PRE	LOC.	VALUE	EXP.
	I	INA	-5	
	I	IU	2	
	I	IV	2	
	I	IUB	2	
	I	IVB	2	
	I	IDVT	11	
	I		14	
	I		12	
	I		13	
	I	INK	1	
		PHIN	90.	
		XL	0.	
		R2IN	0.	
		BBAR	-42.4	
		ABAR	60.	
		C11I	.87912	06
			.87912	06
		C12I	.263736	06
			.263736	06
		C22I	.87912	06
			.87912	06
		C33I	.46886447	03
			.46886447	03
			.46886447	03
			.46886447	03
		C34I	.14065934	03
			.14065934	03
			.14065934	03
			.14065934	03
		C44I	.46886447	03
			.46886447	03
			.46886447	03
			.46886447	03
		THI	.08	
			.08	
		DESTA	.2588	-3
		AM	.380503	
		HI	-30.	
		DESTI	.51701388	-1
		PRI	20.	

SYMBOL	PRE	LOC.	VALUE	EXP
		HE	0.	
		DESTE	0.	
		PRE	0.	
		WEARC	0.	
	E	ND		

X1

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SYMBOL	PRE	LOC.	VALUE	EXP.
	I	INA	6	
	I	IU	3	
	I	IV	3	
	I	IUB	3	
	I	IVB	5	
	I	IOVT	11	
	I		15	
	I		16	
	I		12	
	I		17	
	I		18	
	I		13	
	I		19	
	I	INK	3	
		PHIN	90.	
		XL	108.	
		R2IN	60.	
		BBAR	0.	
		ABAR	0.	
		C11I	1.64835	06
			1.64835	06
		C12I	.494505	06
			.494505	06
		C22I	1.64835	06
			1.64835	06
		C33I	.618	04
			.618	04
			.618	04
			.618	04
		C34I	.185	04
			.185	04
			.185	04
			.185	04
		C44I	.618	04
			.618	04
			.618	04
			.618	04
		THI	.15	
			.15	
		DESTA	.2588	-3

SYMBOL	PRE	LOC.	VALUE	EXP.
		AM	1.580557	
		HI	0.	
		DESTI	0.	
		PRI	0.	
		HE	0.	
		DESTE	0.	
		PRE	0.	
		WEARC	-39479.357	
	E	ND		

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X1

SYMBOL	PRE	LOC.	VALUE	EXP.
	I	INA	-7	
	I	IU	2	
	I	IV	2	
	I	IUB	2	
	I	IVB	2	
	I	IDVT	16	
	I		20	
	I		18	
	I		19	
	I	INK	1	
		PHIN	90	
		XL	0.	
		R2IN	0.	
		BBAR	42.4	
		ABAR	60.	
		C11I	.8241758	06
			.8241758	06
		C12I	.2472527	06
			.2472527	06
		C22I	.8241758	06
			.8241758	06
		C33I	.38633241	03
			.38633241	03
			.38633241	03
			.38633241	03
		C34I	.11589972	03
			.11589972	03
			.11589972	03
			.11589972	03
		C44I	.38633241	03
			.38633241	03
			.38633241	03
			.38633241	03
		THI	.075	
			.075	
		DESTA	.2598	-3
		AM	.356721	
		HI	0.	
		DESTI	0.	
		PRI	20.	

SYMBOL	PRE	LOC.	VALUE	EXP
		HE	0.	
		DESTE	0.	
		PRE	0.	
		WFARC	0.	
	END			

X1

SPACE TECHNOLOGY LABORATORIES
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DATE ______

NAME ______ PRIORITY ______

PROBLEM NO. ______ KEYPUNCHED BY ______

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X

SYMBOL	PRE	LOC.	VALUE	EXP.
	I	INA	8	
	I	IU	2	
	I	IV	2	
	I	IUB	2	
	I	IVB	4	
	I	IDVT	16	
	I		21	
	I		18	
	I		22	
	I		17	
	I		23	
	I	INK	2	
		PHIN	90.	
		XL	48.	
		R2IN	60.	
		BBAR	0.	
		ABAR	0.	
		C11I	1.31868	06
			1.31868	06
		C12I	.395604	06
			.395604	06
		C22I	1.31868	06
			1.31868	06
		C33I	1.5824176	03
			1.5824176	03
			1.5824176	03
			1.5824176	03
		C34I	.47472528	03
			.47472528	03
			.47472528	03
			.42472528	03
		C44I	1.5824176	03
			1.5824176	03
			1.5824176	03
			1.5824176	03
		THI.	.12	
			.12	
		DESTA	.2588	-3
		AM	.561973	
		HI	24.	

SYMBOL	PRE	LOC.	VALUE	EXP
		DESTI	.32465277	-1
		PRI	20.	
		HE	0.	
		DESTE	0.	
		PRE	0.	
		WFARC	40720.379	
	E	ND		

X

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DATE
NAME
PROBLEM NO.
NO. OF CARDS

PRIORITY
KEYPUNCHED BY
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X1

SYMBOL	PRE	LOC.	VALUE	EXP.
	I	INA	-9	
	I	IU	2	
	I	IV	2	
	I	IUB	2	
	I	IVB	2	
	I	IDVT	21	
	I		24	
	I		22	
	I		23	
	I	INK	1	
		PHIN	110.	
		XL	0	
		R2IN	0	
		BBAR	-42.4	
		ABAR	60.	
		C11I	.87912	06
			.87912	06
		C12I	.263736	06
			.263736	06
		C22I	.87912	06
			.87912	06
		C33I	.46886447	03
			.46886447	03
			.46886447	03
			.46886447	03
		C34I	.14065934	03
			.14065934	03
			.14065934	03
			.14065934	03
		C44I	.46886447	03
			.46886447	03
			.46886447	03
			.46886447	03
		THI	.08	
			.08	
		DESTA	.2588	-3
		AM	0	
		HI	-24.	
		DESTI	.32465277	-1
		PRI	20.	

SYMBOL	PRE	LOC.	VALUE	EXP
		HE	0.	
		DESTE	0.	
		PRE	0.	
		WEBPC	0.	
	E	ND		

X1

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NAME

PROBLEM NO.

NO. OF CARDS

PRIORITY

KEYPUNCHED BY WD

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SYMBOL	F_{R_E}	LOC.	VALUE	EXP.
	I	INA	10	
	I	IU	2	
	I	IV	2	
	I	IUE	2	
	I	IVB	4	
	I	IOUT	21	
	I		25	
	I		22	
	I		26	
	I		23	
	I		27	
	I	INK	2	
		PHIN	90.	
		XL	54.	
		R2IN	60.	
		BBAR	0.	
		ABAR	0.	
		C11I	1.64835	06
			1.64835	06
		C12I	.494505	06
			.494505	06
		C22I	1.64835	06
			1.64835	06
		C33I	.618	04
			.618	04
			.618	04
			.618	04
		C34I	.185	04
			.185	04
			.185	04
			.185	04
		C44I	.618	04
			.618	04
			.618	04
			.618	04
		THI	.15	
			.15	
AM		DELTA	.2588	-3
		AM	1.227317	
		PE	0.	

SYMBOL	F_{R_E}	LOC.	VALUE	EXP.
		DESTI	0.	
		PRI	0.	
		HE	0.	
		DESTE	0.	
		PEE	0.	
		WEREC	-62607.878	
	E	ND		

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NAME

PROBLEM NO.

NO. OF CARDS

PRIORITY

KEYPUNCHED BY WK

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X

SYMBOL	PRE	LOC.	VALUE	EXP.
	I	INA	11	
	I	IU	2	
	I	IV	2	
	I	IUB	2	
	I	IVB	4	
	I	IDVT	21	
	I		28	
	I		22	
	I		29	
	I		23	
	I		30	
	I	INK	2	
		PHIN	110.	
		XL	-60.	
		R2IN	38.1618	
		RBAR	0.	
		ABAR	0.	
		C11I	1.64835	06
			1.64835	06
		C12I	.494505	06
			.494505	06
		C22I	1.64835	06
			1.64835	06
		C33I	3.0906593	03
			3.0906593	03
			3.0906593	03
			3.0906593	03
		C34I	.9271978	03
			.9271978	03
			.9271978	03
			.9271978	03
		C44I	3.0906593	03
			3.0906593	03
			3.0906593	03
			3.0906593	03
		THI	.15	
			.15	
		DESTA	.2588	-3
		AM	.764387	
		HI	0.	

SYMBOL	PRE	LOC.	VALUE	EXP
		DESTI	0.	
		PRI	0.	
		HE	0.	
		DESTE	0.	
		PRE	0.	
		WFPRC	11.95.3252	
	E	ND		

SPACE TECHNOLOGY LABORATORIES
COMPUTATION AND DATA REDUCTION CENTER

DATE

NAME

PROBLEM NO.

NO. OF CARDS

PRIORITY

KEYPUNCHED BY

VERIFIED BY

X1

SYMBOL	PRE	LOC.	VALUE	EXP.
b	I	INB	1	
	I	IDA1	-3	
	I	IDA2	4	
	I	IDA3	-5	
		HF	30.	
		DESTF	.13380276	-3
		FM	87.368307	
	END			
b	I	INB	2	
	I	IDA1	-7	
	I	IDA2	8	
	I	IDA3	-9	
		HF	24.	
		DESTF	.84019868	-4
		FM	0.	
	END			

X1

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DATE
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PRIORITY
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VERIFIED BY

SYMBOL	PRE	LOC.	VALUE	EXP.
1	I	INC	1	
	I	IN	2	
	M	CK	10,10	
		02,02	1.0	3
	M	CM	10,10	
		01,01	12.939958	
	I	IDVTC	5	
	I		7	
	E	ND		
2	I	INC	2	
	I	IN	2	
	M	CK	10,10	
		02,02	1.0	3
	M	CM	10,10	
		01,01	1.293995	
	I	IDVTC	15	
	I		17	
	E	ND		
3	I	INC	3	
	I	IN	2	
	M	CK	10,10	
		02,02	1.0	3
	M	CM	10,10	
		01,01	1.293995	
	I	IDVTC	21	
	I		22	
	E	ND		
4	I	INC	4	
	I	IN	3	
	M	CK	10,10	
		02,02	1.0	3

SYMBOL	PRE	LOC.	VALUE	EXP.
		03,03	1.0	4
	M	CM	10,10	
		01,01	3.623188	
	I	IDVTC	28	
	I		27	
	I		30	
	E	ND		