ORNL-3746

Contract No. W-7405-eng-26

Neutron Physics Division

# A GENERAL CATEGORY OF SOLUBLE NUCLEON-MESON CASCADE EQUATIONS

F. S. Alsmiller

Note:

This Work Partially Supported by
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
Under Order R-104

DECEMBER 1965

OAK RIDGE NATIONAL LABORATORY
Oak Ridge, Tennessee
operated by
UNION CARBIDE CORPORATION
for the
U.S. ATOMIC ENERGY COMMISSION

# TABLE OF CONTENTS

# A GENERAL CATEGORY OF SOLUBLE NUCLEON-MESON CASCADE EQUATIONS

F. S. Alsmiller

## ABSTRACT

The set of coupled equations describing a one-dimensional nucleon-meson cascade fed by arbitrary space and energy-dependent sources is reduced to quadrature under the following assumptions: (1) charged-particle stopping, meson decay, and deviation of the nonelastic collision cross sections from a single constant value are neglected for the secondary particles produced in the cascade; (2) the secondary particle differential production spectra, $F_{jk}(E,E')$, which give the number per unit energy, E, of j-type particles produced in a nonelastic collision of a k-type particle of energy E' with a target nucleus are taken to be of the general form:

$$F_{jk}(E,E') = \frac{g_k(E')}{g_j(E)} \left\{K_j(E,E') + h(E)\ k(E')\ \delta_{jk}\right\}$$

with the restriction,

$$\sum_j K_j(E,E') + h(E)\ k(E') = H(E)\ L(E') \quad .$$

The functions $g_j(E)$, H(E), and L(E') are arbitrary. Examples are given.

Solutions are given for a variety of assumptions concerning the form of the primary particle spectra.

# I. INTRODUCTION

Analytic solutions to the nucleon-meson cascade equations in the one-dimensional or straight-ahead approximation have been obtained by Passow,[1] Fisher,[2] and Alsmiller[3] for special examples of the energy distribution functions of the secondary particles. In all these cases, charged-particle stopping, meson decay, and energy variation of the collision cross sections were neglected. In addition, a solution for a two-component cascade, when one component is assumed to be charged and to slow down with a constant stopping power, has been obtained by Alsmiller.[4]

All of these solutions have been obtained for secondary-particle production kernels which are separable in the energy variables of the incident and emergent particles.

In this paper, a certain category of kernels which are in some sense separable is assumed and formal solutions given for a variety of primary particle spectra.

In Sections II, III, and IV we write down the cascade equations to be solved and discuss the kernels used. Sections V and VI present relatively simple cases which generalize the solutions of Passow and Fisher. In Section VII the general case is treated.

---

[1]C. Passow, Deutches Elektronen-Synchrotron, Hamburg, Germany, Phanomenologishe Theory Zur Berechnung einer Kaskade aus Schweren Teilchen (Nukleonkascade) in der Materie," DESY-Notiz A2.85 (February 1962).

[2]C. M. Fisher, "Notes on the Nuclear Cascade in Shielding Materials," Report of the Shielding Conference Held at Rutherford Laboratory on September 26-27, 1962, edited by R. H. Thomas; NIRL/R/40.

[3]F. S. Alsmiller and R. G. Alsmiller, Jr., Neutron Phys. Space Radiation Shielding Research Ann. Progr. Rept. for Period Ending August 31, 1962, ORNL-CF-62-10-29, p. 138.

[4]R. G. Alsmiller, Jr., A Solution to the Nucleon-Meson Cascade Equations Under Very Special Conditions, ORNL-3570 (March 1964).

## II. GENERAL CASCADE EQUATIONS

We write the coupled one-dimensional transport equations for the nucleon and meson secondary-particle intensities, $\phi_{sj}(E,r)$, in the form[5,6]

$$\frac{\partial \phi_{sj}}{\partial r} + Q\phi_{sj} = \mathcal{S}_j(E,r) + Q \int_E^{E_o^+} dE' \sum_k F_{jk}(E,E') \left[ \phi_{ik}(E',r) + \phi_{sk}(E',r) \right] \quad (1)$$

$$\phi_{sj}(E,0) = 0 \quad (2)$$

$$\phi_{sj}(E_o^+,r) = \phi_{ik}(E_o^+,r) = \mathcal{S}_j(E_o^+,r) = 0$$

where

$r$ = dimensionless distance variable, measured along the axis of the cascade;

$E_o^+ = E_o + \epsilon$, where $\epsilon$ is arbitrarily small; this limit is used when $\phi_{ik}(E',r)$ may be a delta function with a peak at $E_o$;

$Q$ = constant inverse mean free path for nonelastic collisions, assumed the same for all particles;

$F_{jk}(E,E')$ = the number of secondary particles of type j per unit energy E produced by incident particles of type k and energy E';

$\mathcal{S}_j(E,r)$ = arbitrary source spectrum.

In this approximation, we have neglected meson decay, charged-particle slowing down, and the variation of the nonelastic cross section with energy and particle type, in the equations for the secondary-particle intensities. To be consistent, one uses for the primary-particle intensities

---

[5]R. G. Alsmiller, Jr., F. S. Alsmiller, and J. E. Murphy, Nucleon-Meson Cascade Calculations: Transverse Shielding for a 45-GeV Electron Accelerator (Part 1), ORNL-3289 (December 1962).

[6]R. G. Alsmiller, Jr., F. S. Alsmiller, and J. E. Murphy, Proceedings of the Symposium on the Protection Against Radiation Hazards in Space, held in Gatlinburg, Tennessee, November 5-7, 1962, paper E-3, p. 698.

$$\phi_{ij}(E,r) = \phi_j(E,0)\ e^{-Qr} \tag{3}$$

with

$$\phi_j(E,0) = N_{jo}\ \delta(E_o - E)$$

for monoenergetic primaries.

Alternatively, one can include these neglected effects in the source term $\mathscr{S}_j(E,r)$, treating them as small perturbations; i.e.,

$$\mathscr{S}_j(E,r) = c_j \frac{\partial}{\partial E} S_j(E)\ \phi_{sj}(E,r) + [Q - Q_j(E) - Q_{jD}(E)]\ \phi_{sj}(E,r) \tag{4}$$

where

$c_j$ = 0,1 if particles are uncharged, charged;

$S_j(E)$ = energy loss per unit distance, - (dE/dr), for j-type particles;

$Q_{jD}(E)$ = inverse energy-dependent mean free path for decay of j-type particles;

$Q_j(E)$ = nonelastic inverse mean free path for j-type particles.

The corresponding primary intensities are

$$\phi_{ij}(E,r) = \phi_j(E_j,0)\ \frac{S_j(E_j)}{S_j(E)}\ e^{-\int_E^{E_j} \frac{Q_j + Q_{jD}}{S_j} d\bar{E}} \tag{5}$$

where $E_j(E,r)$ is defined by

$$r = \int_E^{E_j(E,r)} \frac{d\bar{E}}{S_j(\bar{E})} \quad ,$$

for charged particles. For monoenergetic charged primaries of energy $E_o$,

$$\phi_j(E_j,0) = N_{jo}\,\delta[E_o - E_j(E,r)] \quad . \tag{6}$$

For neutral particles, e.g., neutrons,

$$\phi_{ij}(E,r) = \phi_j(E,0)\, e^{-Q_j(E)r} \quad . \tag{7}$$

The advantage of separating primary and secondary intensities in Eq. 1 lies wholly in the fact that one can use as exact an expression for the primaries as desired, at the expense, usually, of an extra integration in the final result.

## III. ASSUMPTIONS CONCERNING THE KERNELS, $F_{jk}(E,E')$

We shall initially assume for the kernel $F_{jk}(E,E')$ the general form:

$$F_{jk}(E,E') = \frac{g_k(E')}{g_j(E)} \left\{K_j(E,E') + h(E,E')\,\delta_{jk}\right\} \theta(E' - E) \tag{8}$$

$$\theta(E' - E) = 1, \quad \text{if } E' > E; \quad = 0, \quad \text{if } E' < E \quad .$$

where $g_j(E)$ are arbitrary functions. The additive term, $h(E,E')\,\delta_{jk}$, allows for a larger multiplicity and varied energy spectrum for emergent secondaries of the same kind as the incident particle in a collision. The restrictions to the single index, j, on the functions $K_j(E,E')$ and no indices at all on $h(E,E')$ are fairly severe limitations which sometimes make it possible to obtain uncoupled equations for a suitable linear combination of the functions $\phi_{sj}$, and then for each $\phi_{sj}$ separately.

The general definition of the multiplicity or number of j-type secondaries with energy $E > E_L$ produced in a nonelastic collision by an incident k-type particle of energy E' is

$$\nu_{jk}(E') = \frac{Q}{Q_k(E')} \int_{E_L}^{E'} dE\, F_{jk}(E,E') \quad . \tag{9}$$

The kernels are expected to satisfy this integral condition, as well as an energy conservation condition at each collision.

Formal solutions of the cascade equations have been found for a certain category of kernels "almost separable" in E and E'. These are:

$$F_{jk}(E,E') = \frac{g_k(E')}{g_j(E)} \left\{ K_j(E,E') + k(E')\, h(E)\, \delta_{jk} \right\} \theta(E' - E) \quad ; \tag{10}$$

i.e., $h(E,E')$ is separable into functions of E and E'; also $K_j(E,E')$ must satisfy the further restriction,

$$\sum_j K_j(E,E') + h(E)\, k(E') = H(E)\, L(E') \quad . \tag{11}$$

One can remove a factor $h(E)/h(E')$ from the curly brackets and incorporate it in $g_k(E')/g_j(E)$, redefining the remaining functions also, without violating Eq. 11. Hence it is just as general to set

$$h(E) = 1 \quad .$$

It is easy to see that Eq. 11 is a major restriction in that the very simple separable kernel

$$F_{jk}(E,E') = \frac{Q_k(E')}{Q_j(E)} \left[ \frac{B_j(E)\, f_j(E')}{Q} \right]$$

does not generally satisfy it. One example, used in much previous numerical work on nucleon-pion cascades,[5] is

$$f_j(E') = \frac{\nu_j(E')}{\int_{E_L}^{E'} dE \, \frac{B_j(E)}{Q_j(E)}}$$

where $\nu_j(E') = \nu_{jk}(E')$ for all k.

Examples of kernels which do satisfy Eq. 11 are given below.

1. $k(E') = 0;$ $\qquad L(E') = 1$ (12)

$$K_j(E,E') = \alpha_j(E); \qquad H(E) = \sum_j \alpha_j(E) = \alpha(E) \quad .$$

This gives the simplest possible solution. If $k(E') \neq 0$, Eq. 11 is violated unless $\alpha(E) = \sum_j \alpha_j(E) = h(E)$. Hence, we would have a special case of Example 2, putting $h(E) = 1$. Solutions for this case are given later in Eqs. 30, 39, 40, and 41.

2. $k(E') \neq 0$, or $= 0;$ $\qquad L(E') = \alpha(E') + k(E')$ (13)

$$K_j(E,E') = \alpha_j(E'); \qquad H(E) = 1$$

$$\alpha(E') = \sum_j \alpha_j(E'); \qquad h(E) = 1$$

3. $k(E') \neq 0$, or $= 0;$ $\qquad L(E') = \alpha(E') + k(E')$ (14)

$$K_j(E,E') = \frac{\gamma_j}{\gamma} \alpha(E'); \qquad H(E) = 1$$

$$\sum_j \gamma_j = \gamma \; ; \qquad h(E) = 1$$

Example 3 is a special case of 2, but is listed separately because it provides a simpler solution. So far, the examples are all truly separable in E and E'. Solutions for Examples 1 and 3 have been given previously.[3]

4. $k(E') \neq 0$, or $= 0$; $h(E) \neq 1$, or 1; $L(E') = \alpha(E') + k(E')$ (15)

$$\sum_j K_j(E,E') = \alpha(E')\ h(E); \qquad H(E) = h(E)$$

$K_j(E,E')$ may be nonseparable for some j.

Example 2 is a trivial case of Example 4; nontrivial cases might involve a subtraction procedure such as

$$K_N(E,E') = \alpha(E')\ h(E) - K_\pi(E,E') \quad ,$$

with $K_\pi(E,E')$ arbitrary.

5. $k(E') \neq 0$, or $= 0$; $h(E) = 1$

$$K_j(E,E') = \alpha_j(E') + [\alpha(E') + k(E')]\ \beta_j(E)$$

$$+ \sum_{n=0}^{\infty} [\alpha(E')\ c_{jn}(E') + k(E')\ b_{jn}(E')]\ f_n(E) \qquad (16)$$

where

$$\sum_j c_{jn}(E') = 1; \quad \sum_j b_{jn}(E') = 1; \quad \alpha(E') = \sum_j \alpha_j(E');$$

$\alpha_j(E')$, $\beta_j(E)$, and $f_n(E)$ are arbitrary functions:

$$H(E) = 1 + \sum_j \beta_j(E) + \sum_n f_n(E); \quad L(E') = \alpha(E') + k(E') \quad .$$

Example 5 is the most general case, and includes all the preceding examples, except possibly Example 4. A simple example under 5 is

$$F_{jk}(E,E') = \frac{g_k(E')}{g_j(E)} \; [\alpha_j(E) + h(E)\,\delta_{jk}] \; I(E') \tag{17}$$

$$h(E) \neq 0; \quad H(E) = \alpha(E) + h(E); \quad L(E') = I(E') \quad .$$

This can be transformed into

$$F_{jk}(E,E') = \frac{\bar{g}_k(E')}{\bar{g}_j(E)} \left[\alpha_j(E)\,\frac{h(E')}{h(E)} + h(E')\,\delta_{jk}\right] \; I(E') \tag{18}$$

$$\bar{g}_j(E) = \frac{g_j(E)}{h(E)} \; ; \quad H(E) = \frac{\alpha(E)}{h(E)} + 1; \quad \alpha(E) = \sum_j \alpha_j(E)$$

$$k(E') = L(E') = h(E')\; I(E') \quad .$$

Then

$$F_{jk}(E,E') = \frac{\bar{g}_k(E')}{\bar{g}_j(E)} \; [\beta_j(E) + \delta_{jk}] \; k(E')$$

$$K_j(E,E') = k(E')\;\beta_j(E); \quad \beta_j(E) = \frac{\alpha_j(E)}{h(E)} \quad .$$

6. $$F_{jk}(E,E') = \frac{g_k(E')}{g_j(E)} \tag{19}$$

$$\left\{\alpha_j(E') + \alpha(E')\; c_j(E')\; f(E) + [1 + f(E)]\; k(E')\;\delta_{jk}\right\}$$

$$\sum_j c_j(E') = 1; \quad \sum_j \alpha_j(E') = \alpha(E')$$

$$H(E) = 1 + f(E); \quad L(E') = \alpha(E') + k(E') \quad .$$

If $C_j(E') \neq \frac{\alpha_j(E')}{\alpha(E')}$ , this case cannot quite be transformed into the form of Example 5.

## IV. APPLICATION OF LAPLACE TRANSFORMATION WITH RESPECT TO POSITION

Define

$$\chi_j(E,r) = g_j(E)\ \phi_{sj}(E,r)\ e^{Qr} \tag{20}$$

$$\chi_{jo}(E,r) = g_j(E)\ \phi_{ij}(E,r)\ e^{Qr}$$

$$G_j(E,r) = g_j(E)\ \mathcal{S}_j(E,r)\ e^{Qr}$$

$$\chi(E,r) = \sum_j \chi_j(E,r)$$

$$\chi_o(E,r) = \sum_j \chi_{jo}(E,r)$$

$$G(E,r) = \sum_j G_j(E,r)$$

Substituting these definitions in Eq. 1 gives:

$$\frac{\partial \chi_j}{\partial r} = G_j(E,r) + \int_E^{E_o^+} dE'\ Q\ \Big\{K_j(E,E')\,[\chi(E',r) + \chi_o(E',r)] + h(E,E')\,[\chi_j(E',r) + \chi_{jo}(E',r)]\Big\} \tag{21}$$

Adding:

$$\frac{\partial \chi}{\partial r} = G(E,r) + \int_E^{E_o^+} dE'\ Q\ \Big[\sum_j K_j(E,E') + h(E,E')\Big]\ \Big[\chi(E,r) + \chi_o(E',r)\Big] \tag{22}$$

This is the set of uncoupled equations to be solved for various choices of the kernel functions. Note the apparent necessity for the restriction of indices on the functions $K_j(E,E')$ and $h(E,E')$. Laplace transform with respect to r to obtain, if $\overline{\chi}(E,\lambda) = \int_0^\infty e^{-\lambda r} \chi(E,r)$, etc.,

$$\lambda \overline{\chi}_j = \overline{G}_j + \int_E^{E_o^+} dE' \, Q \left\{ K_j(E,E') \left[\overline{\chi}(E',\lambda) + \overline{\chi}_o(E',\lambda)\right] + h(E,E') \left[\overline{\chi}_j(E',\lambda) + \overline{\chi}_{jo}(E',\lambda)\right] \right\} \tag{23}$$

$$\lambda \; \overline{\chi} = \overline{G} + \int_E^{E_o} dE' \, Q \left[ \sum_j K_j(E,E') + h(E,E') \right] \left[ \overline{\chi}(E',\lambda) + \overline{\chi}(E',\lambda) \right] \quad . \tag{24}$$

If the incident primary particles are monoenergetic protons of energy $E_o$, Eqs. 5 and 6 give

$$\overline{\chi}_o(E',\lambda) = N_{Po} \frac{g_P(E')}{S_P(E')} e^{-\lambda \overline{r} + \int_{E'}^{E_o} \frac{[Q - Q_P(\overline{E})]}{S_P(\overline{E})} d\overline{E}} \tag{25}$$

where

$$\overline{r}(E_o,E') = \int_{E'}^{E_o} \frac{d\overline{E}}{S_P(\overline{E})}$$

Also, we will use the Laplace inversion

$$\mathcal{L}^{-1} \; e^{-\lambda \overline{r}} = \delta(r - \overline{r}) \quad .$$

If primary proton slowing down is ignored, and $Q_P(E') = Q$,

$$\overline{\chi}_o(E',\lambda) = N_{P_o} \frac{g_P(E')}{\lambda} \delta(E_o - E') \quad . \tag{26}$$

We will make repeated use of Laplace inversions leading to hyperbolic Bessel functions:

$$\mathcal{L}^{-1} \frac{e^{B/\lambda}}{\lambda} = I_0 (2\sqrt{Br})$$

$$\mathcal{L}^{-1} \frac{e^{B/\lambda}}{\lambda^2} = \left[\frac{r}{B}\right]^{1/2} I_1 (2\sqrt{Br})$$

$$\mathcal{L}^{-1} \frac{e^{B/\lambda}}{\lambda^\nu} = \left[\frac{r}{B}\right]^{(\nu-1)/2} I_{\nu-1} (2\sqrt{Br})$$

$$\mathcal{L}^{-1} e^{B/\lambda} = \delta(r) + \left[\frac{B}{r}\right]^{1/2} I_1 (2\sqrt{Br}) \quad . \tag{27}$$

In addition, the convolution theorem gives

$$\mathcal{L}^{-1} \frac{e^{B/\lambda - \lambda\overline{r}}}{\lambda^\nu} = \int_0^r dr' \left[\frac{r'}{B}\right]^{(\nu-1)/2} I_{\nu-1} [2\sqrt{Br'}] \, \delta(r - r' - \overline{r})$$

$$= \left[\frac{r - \overline{r}}{B}\right]^{(\nu-1)/2} I_{\nu-1}[2\sqrt{B(r - \overline{r})}] \; ;$$

and more generally,

$$\mathcal{L}^{-1} \frac{e^{B/\lambda}}{\lambda^\nu} \overline{\chi}_0(E',\lambda) = \int_0^r dr' \left[\frac{r'}{B}\right]^{(\nu-1)/2} I_{\nu-1}(2\sqrt{Br'}) \, \chi_0(E',r - r') \quad .$$

Also,

$$\left(\frac{r}{B}\right)^{\nu/2} I_\nu(2\sqrt{Br}\,) = \frac{\partial}{\partial r}\left[\left(\frac{r}{B}\right)^{(\nu+1)/2} I_{\nu+1}[2\sqrt{Br}\,]\right] \tag{28}$$

$$\frac{\partial B}{\partial E}\left(\frac{r}{B}\right)^{(\nu+1)/2} I_{\nu+1}(2\sqrt{Br}\,) = \frac{\partial}{\partial E}\left[\left(\frac{r}{B}\right)^{\nu/2} I_\nu[2\sqrt{Br}\,]\right] .$$

## V. SOME SIMPLE SOLUTIONS FOR $F_{jk}(E,E') = \frac{g_k(E')}{g_j(E)}\,\alpha_j(E)$

In this section we will derive as special cases some slightly generalized versions of a solution published previously by Passow.[1]

Let

$$F_{jk}(E,E') = \frac{g_k(E')}{g_j(E)}\,\alpha_j(E) \quad . \tag{29}$$

This is the first example satisfying Eqs. 10 and 11. For monoenergetic incident protons of energy $E_0$ with $Q = Q_p$ and slowing down neglected, the secondary-particle intensity is

$$\phi_{sj}(E,r) = e^{-Qr}\,\frac{\alpha_j(E)}{g_j(E)}\,QN_{P_0}\,g_P(E_0)\left[\frac{r}{B(E_0,E)}\right]^{1/2} I_1[2\sqrt{rB(E_0,E)}\,] \tag{30}$$

$$B(E_0,E) = Q\sum_k \int_E^{E_0} \alpha_k(\bar{E})\,d\bar{E} = \sum_k B_k(E,E_0) \quad . \tag{31}$$

The general solution for arbitrary primary spectra is given in Eq. 39. A particular choice of kernel in this form is the function

$$F_{jk}(E,E') = \alpha_j\,\frac{E'^{\ell}}{E^n}\,\theta(E' - \eta_k) \tag{32}$$

for which the solution was carried out by Passow.[1] The use of the step function is an attempt to account very roughly for charged-particle energy losses by ionization; $\eta_k$ is an energy limit for the kth kind of particle above which the energy loss is assumed to be zero and below which it is infinite, so that no further production of secondaries takes place. Here

$$g_k(E') = E'^{\ell}\,\theta(E' - \eta_k); \quad g_j(E) = E^{\ell}\,\theta(E - \eta_j)$$

$$\alpha_j(E) = \alpha_j\,\theta(E - \eta_j)\,E^{\ell-n}$$

and

$$\phi_{sj}(E,r) = \alpha_j\,e^{-Qr}\,QN_{P_0}\,\frac{E_0^{\ell}}{E^{n-\ell}}\,\theta(E_0 - \eta_P)\left[\frac{r}{B(E_0,E)}\right]^{1/2} I_1\left[2\sqrt{rB(E_0,E)}\right] \quad (33)$$

with

$$B(E_0,E) = Q\sum_k \alpha_k \int_E^{E_0} d\bar{E}\,\bar{E}^{\ell-n}\,\theta(\bar{E} - \eta_k) \quad .$$

To derive these results, let

$$\alpha(E) = \sum_j \alpha_j(E); \quad B(E',E) = Q\int_E^{E'} \alpha(\bar{E})\,d\bar{E} \quad (34)$$

$$\psi(E,r) = \chi(E,r)/\alpha(E) = \sum_j \chi_j(E,r)/\alpha(E) \quad .$$

Substitution of Eq. 34 in Eq. 24 for $\bar{\chi}$ gives, after dividing through by $\alpha(E)$ and then differentiating with respect to E:

$$\lambda \frac{\partial}{\partial E}\bar{\Psi}(E) = \frac{\partial}{\partial E}\left[\frac{\bar{G}(E,\lambda)}{\alpha(E)}\right] - \alpha(E)\,Q\left[\bar{\Psi}(E,\lambda) + \psi_0(E,\lambda)\right] \quad . \quad (35)$$

Then,

$$\bar{\psi}(E,\lambda) = \int_E^{E_o^+} dE' \, \frac{e}{\lambda}^{\frac{B(E',E)}{\lambda}} \left[ Q \bar{\chi}_o(E',\lambda) - \frac{\partial}{\partial E'} \, \frac{\bar{G}(E',\lambda)}{\alpha(E')} \right] . \qquad (36)$$

Substituting Eq. 36 in Eq. 23 for $\bar{\chi}_j$ gives directly,

$$\bar{\chi}_j(E,\lambda) = \frac{\bar{G}_j}{\lambda} + Q \frac{\alpha_j(E)}{\lambda} \int_E^{E_o^+} dE' \left\{ \bar{\chi}_o(E',\lambda) + \alpha(E') \int_{E'}^{E_o^+} dE'' \, \frac{e}{\lambda}^{\frac{B(E'',E')}{\lambda}} \right.$$

$$\left. \cdot \left[ Q \bar{\chi}_o(E'',\lambda) - \frac{\partial}{\partial E''} \, \frac{\bar{G}(E'',\lambda)}{\alpha(E'')} \right] \right\} \qquad (37)$$

Interchanging integrations,

$$\bar{\chi}_j(E,\lambda) = \frac{\bar{G}_j}{\lambda} + Q \, \alpha_j(E) \int_E^{E_o^+} \frac{dE'}{\lambda} \left\{ \bar{\chi}_o(E',\lambda) \left[ 1 + \int_E^{E'} dE'' \, \frac{\alpha(E'')Q}{\lambda} \, e^{\int_E^{E''} Q \frac{\alpha}{\lambda} d\bar{E}} \right] \right.$$

$$\left. - \left[ \frac{\partial}{\partial E'} \left( \frac{\bar{G}}{\alpha} \right) \right] \int_E^{E'} dE'' \, \frac{\alpha(E'')}{\lambda} \, e^{\int_E^{E''} Q \frac{\alpha}{\lambda} d\bar{E}} \right\}$$

$$= \frac{\bar{G}_j(E,\lambda)}{\lambda} + Q \, \alpha_j(E) \int_E^{E_o^+} dE' \, \frac{e}{\lambda}^{\frac{B(E',E)}{\lambda}} \left[ \bar{\chi}_o(E',\lambda) + \frac{\bar{G}}{\lambda}(E',\lambda) \right] . \qquad (38)$$

The general solution for arbitrary primary spectra, $\phi_{ik}(E,r)$, is

$$\phi_{sj}(E,r) = e^{-Qr}\, Q\, \frac{\alpha_j(E)}{g_j(E)} \int_0^r dr' \int_E^{E_o^+} dE' \sum_k g_k(E')\, \phi_{ik}(E',r')\, e^{Qr'}\, I_o\left[2\sqrt{B(E',E)(r-r')}\,\right]$$

$$+\, e^{-Qr} \int_0^r dr'\, e^{Qr'} \left\{ \mathcal{S}_j(E,r') + \frac{Q\,\alpha_j(E)}{g_j(E)} \int_E^{E_o^+} dE' \sum_k g_k(E')\, \mathcal{S}_k(E',r') \right.$$

$$\left. \cdot \left[\frac{(r - r')}{B(E',E)}\right]^{1/2} I_1\left[2\sqrt{B(E',E)(r - r')}\,\right] \right\} . \qquad (39)$$

Use of Eq. 26 for $\overline{\chi}_o$ reduces the first term to Eq. 30.

If $\phi_{ik}(E',r') = \phi_k(E',0)\, e^{-Qr'}$, i.e., arbitrary initial spectra at $r = 0$, with no slowing down, and $Q_k = Q$, use Eq. 28 to carry out the integration in the first term.

$$\phi_{sj}(E,r) = e^{-Qr}\, Q\, \frac{\alpha_j(E)}{g_j(E)} \int_E^{E_o} dE' \sum_k \phi_k(E',0) \left[\frac{r}{B(E',E)}\right]^{1/2} I_1\left[2\sqrt{B(E',E)r}\,\right]$$

$$+ \text{ terms in } \mathcal{S} . \qquad (40)$$

The solution in this form has been given earlier, with $\alpha_j(E)/g_j(E) = f_j(E)$.[3]

If slowing down of monoenergetic primary protons is to be included, use Eq. 25 for $\overline{\chi}_o$, and invert Eq. 38 using the convolution theorem to obtain

$$\phi_{sj}(E,r) = e^{-Qr}\, \frac{\alpha_j(E)}{g_j(E)}\, QN_{P_o} \int_{E_L(E_o,r),\ \text{or } E}^{E_o^+} dE'\, \frac{g_P(E')}{S_P(E')}\, I_o\left[2\sqrt{B(E',E)(r - \bar{r}')}\,\right] \Theta(r - \bar{r})$$

$$\cdot \exp\left[\int_{E'}^{E_o} \frac{[Q - Q_P(\bar{E})]\, d\bar{E}}{S_P(\bar{E})}\right] + \text{ terms in } \mathcal{S} . \qquad (41)$$

Again, from Eq. 25,

$$\bar{r}(E_0,E') = \int_{E'}^{E_0} d\bar{E}/S_P(\bar{E}) \quad .$$

The lower limit on the integration over E' is then the larger of E or $E_L(E_0,r)$, defined by

$$r = \bar{r}(E_0,E_L) = \int_{E_L(E_0,r)}^{E_0} d\bar{E}/S_P(\bar{E}) \quad . \tag{42}$$

Equation 30 is obtained by integration of Eq. 41 if we first put

$$Q = Q_P; \quad g_P(E') \rightarrow g_P(E_0); \; B(E',E) \rightarrow B(E_0,E) \quad ;$$

$$\frac{dE'}{S_P(E')} = - d\bar{r} \quad ,$$

and use Eq. 28.

In essence, the importance of primary proton slowing down depends on the sensitivity of $g_P(E')$ and $B(E',E)$ to changes in E'.

A second particular choice of kernel in Example 1 is

$$F_{jk}(E,E') = e^{\nu(E'-E)} \frac{\alpha_k}{\alpha_j} \beta_j \quad . \tag{43}$$

Assuming a constant stopping power, S , for charged particles, the exact solution for a two-component cascade of charged and uncharged particles was carried out by R. G. Alsmiller, Jr.[4]

## VI. GENERALIZATION OF A SOLUTION BY FISHER

Consider a two-component cascade of nucleons and pions only, and neglect the presumably low-energy nucleons produced by pion-nucleus collisions. Let

$$F_{nn}(E,E') = \frac{g_n(E')}{g_n(E)} \alpha_n(E'); \quad F_{n\pi} = 0$$

$$F_{\pi\pi} = \frac{g_\pi(E')}{g_\pi(E)} \alpha_\pi(E'); \quad F_{\pi n} = \frac{f_n(E')\ \alpha_\pi(E')}{g_\pi(E)} \quad . \qquad (45)$$

This case was solved by Fisher,[2] for kernels corresponding to

$$g_n(E) = g_n(E') = 1$$

$$g_\pi(E') = E'^2; \quad g_\pi(E) = E^2$$

$$\alpha_n(E') = \alpha_\pi(E') = \frac{1}{E'} \qquad (46)$$

and

$$f_n(E') = g_\pi(E') = E'^2; \text{ i.e.,}$$

$$F_{\pi\pi} = F_{\pi n}$$

This choice of kernel actually does not fit the form of Eqs. 9, 10, and 11, unless we consider only pions in the cascade, treating the nucleon term, which can be derived separately, as a source term. We then have the case of Example 2, Eq. 13. Neglecting external source terms and slowing down, the solutions for monoenergetic protons are

$$\phi_{sn}(E,r) = N_{P_O} Q\, e^{-Qr} \frac{g_n(E_O)\ \alpha_n(E_O)}{g_n(E)} \left(\frac{r}{B_n(E_O,E)}\right)^{1/2} I_1[2\sqrt{B_n(E_O,E)r}\ ] \qquad (47)$$

where

$$B_n(E_0,E) = Q \int_E^{E_0} \alpha_n(\bar{E})\ d\bar{E} \quad ;$$

$$\phi_{s\pi}(E,r) = \frac{N_{P_0} Q\ e^{-Qr}}{g_\pi(E)} \quad \alpha_\pi(E_0)\ g_\pi(E_0) \left(\frac{r}{B_n(E_0,E)}\right)^{1/2} I_1[2\sqrt{B_n(E_0,E)\,r}] \qquad (48)$$

$$+ \alpha_n(E_0)\ g_n(E_0) \int_E^{E_0} dE''\ \frac{\alpha_\pi(E'')\ f_n(E'')\ r}{g_n(E'')} \frac{I_2\left(2\sqrt{r[B_n(E_0,E'') + B_\pi(E'',E)]}\right)}{[B_n(E_0,E'') + B_\pi(E'',E)]}$$

where

$$B_\pi(E_0,E'') = Q \int_{E''}^{E_0} \alpha_\pi(\bar{E})\ d\bar{E} \quad .$$

A more general solution is given in Eqs. 52 and 54. For Fisher's kernel, $\alpha_\pi = \alpha_n$, so

$$B_n(E_0,E'') + B_\pi(E'',E) = B(E_0,E) = Q\ \ell n\ \frac{E_0}{E} \quad .$$

The absence of dependence on E" makes it possible to carry out the integral in Eq. 47. Then

$$\phi_{sn}(E,r) = N_{P_0} Q\ e^{-Qr} \frac{1}{E_0} \left(\frac{r}{Q\ \ell n(E_0/E)}\right)^{1/2} I_1[2\sqrt{Q\ r\ \ell n(E_0/E)}\ ] \qquad (49)$$

$$\phi_{s\pi}(E,r) = \frac{N_{P_0} Q\ e^{-Qr}}{E^2} \left\{E_0 \left(\frac{r}{Q\ \ell n(E_0/E)}\right)^{1/2} I_1[2\sqrt{Qr\ell n(E_0/E)}\ ]\right.$$

$$\left. + \frac{E_0^2 - E^2}{2E_0} \left(\frac{r}{Q\ \ell n(E_0/E)}\right) I_2[2\sqrt{Qr\ell n(E_0/E)}\ ] \right\}$$

The derivation proceeds from Eq. 23, which may be written for this special case:

$$\lambda \overline{\chi}_n = \overline{G}_n + \int_E^{E_o^+} dE' \; Q \; \alpha_n(E') \; [\overline{\chi}_n + \overline{\chi}_{no}]$$

$$\lambda \overline{\chi}_\pi = \overline{G}_\pi + \int_E^{E_o^+} dE' \; Q \; \alpha_\pi(E') \left[ \overline{\chi}_\pi + \overline{\chi}_{\pi o} + \frac{f_n(E')}{g_n(E')} (\overline{\chi}_n + \overline{\chi}_{no}) \right] \; . \tag{50}$$

Note, the terms in $\overline{\chi}_n$ and $\overline{\chi}_{no}$ are essentially source terms in the equation for $\overline{\chi}_\pi$.

Differentiate $\overline{\chi}_N$ with respect to E to obtain

$$\lambda \frac{\partial}{\partial E} \overline{\chi}_n = - Q \; \alpha_n(E) \; [\overline{\chi}_n(E,\lambda) + \overline{\chi}_{no}(E,\lambda)] \; + \frac{\partial}{\partial E} \overline{G}_n(E,\lambda) \; .$$

Now integrate:

$$\overline{\chi}_n(E,\lambda) = \int_E^{E_o^+} dE' \; \frac{e^{\frac{B_n(E',E)}{\lambda}}}{\lambda} \left[ Q \; \alpha_n(E') \; \overline{\chi}_{no} - \frac{\partial}{\partial E'} \overline{G}_n \right] \; . \tag{51}$$

Similarly, differentiate $\overline{\chi}_\pi$ with respect to E and integrate:

$$\overline{\chi}_\pi = \int_E^{E_o^+} dE' \; \frac{e^{\frac{B_\pi(E',E)}{\lambda}}}{\lambda} \left\{ Q \; \alpha_\pi(E') \left[ \overline{\chi}_{\pi o} + \frac{f_n(E')}{g_n(E')} (\overline{\chi}_n + \overline{\chi}_{no}) \right] - \frac{\partial}{\partial E'} \overline{G}_\pi(E',\lambda) \right\} \; .$$

Substituting Eq. 51 in for $\overline{\chi}_n$ gives

$$\overline{\chi}_\pi(E,\lambda) = \overline{G}_\pi(E,\lambda) + \int_E^{E_o} dE' \left\{ \frac{e^{\frac{B_\pi(E',E)}{\lambda}}}{\lambda} Q\, \alpha_\pi(E') \right.$$

$$\cdot \left[ \overline{\chi}_{\pi o} + \frac{f_n(E')}{g_n(E')} \overline{\chi}_{no} + \frac{\overline{G}_\pi(E',\lambda)}{\lambda} \right] + \left[ Q\, \alpha_n(E')\, \overline{\chi}_{no}(E',\lambda) - \frac{\partial}{\partial E'} G_n(E',\lambda) \right].$$

$$\left. \cdot \int_E^{E'} dE'' \frac{f_n(E'')\, Q\, \alpha_\pi(E'')}{g_n(E'')} \frac{e^{\frac{B_n(E',E'')}{\lambda} + \frac{B_\pi(E'',E)}{\lambda}}}{\lambda^2} \right\} \qquad (52)$$

Equations 47 and 48 are obtained by inversion of 51 and 52, with

$$G_n = G_\pi = 0 = \chi_{\pi o} = 0; \quad \chi_{no} = N_{Po}\, g_n(E') \frac{\delta(E_o - E')}{\lambda}$$

from Eq. 26.

The formal solutions of Eqs. 51 and 52 for arbitrary initial spectra are

$$\phi_{sn}(E,r) = \frac{e^{-Qr}}{g_n(E)} \int_0^r dr' \int_E^{E_o^+} dE'\, I_0[2\sqrt{B_n(E',E)(r - r')}\,]$$

$$\left[ Q\, \alpha_n(E')\, g_n(E')\, \phi_{in}(E',r')\, e^{Qr'} - \frac{\partial}{\partial E'} [g_n(E') \mathcal{J}_n(E',r')]\, e^{Qr'} \right] . \qquad (53)$$

Here, the source term $\mathcal{J}_n(E',r')$ may include, in some approximation, the production rate per unit distance per unit energy of nucleons from pion collisions.

$$\phi_{s\pi}(E,r) = \frac{e^{-Qr}}{g_\pi(E')} \int_0^r dr' \int_E^{E_0^+} dE' \left\{ I_0[2\sqrt{B_\pi(E',E)(r - r')}\,]\, Q\, \alpha_\pi(E') \right.$$

$$\cdot \left[ g_\pi(E')\, \phi_{i\pi}(E',r') + f_n(E')\, \phi_{in}(E',r') - Q\, \alpha_\pi(E') \frac{\partial}{\partial E'} [g_\pi(E') \not{\phi}_\pi(E',r')] \right] e^{Qr'}$$

$$+ \left[ Q\, \alpha_n(E')\, g_n(E')\, \phi_{in}(E',r')\, e^{Qr'} - \frac{\partial}{\partial E'} [g_n(E') \not{\phi}_n(E',r')] \right] e^{Qr'}$$

$$\cdot \int_E^{E'} dE''\, Q \frac{f_n(E'')\, \alpha_\pi(E'')}{g_n(E'')} \left[ \frac{(r - r')}{B_n(E',E'') + B_\pi(E'',E)} \right]^{1/2}$$

$$\left. \cdot I_1 \left( 2\sqrt{[B_n(E',E'') + B_\pi(E'',E)](r - r')} \right) \right\} \qquad (54)$$

If $\phi_{i\pi} = 0$, and we include slowing down of the initial primary protons, use Eqs. 5 and 6:

$$\phi_{in}(E',r)\, e^{Qr'}\, dE' = dE_P'\, N_{P_0}\, \delta[E_0 - E_P'(E',r')]\, e^{\int_{E'}^{E_P'} \frac{[Q-Q_P(\bar{E})]\, d\bar{E}}{S_P(\bar{E})}} .$$

Then

$$\phi_{sn}(E,r) = \frac{QN_{P_0}\, e^{-Qr}}{g_n(E)} \int_0^r dr'\, \alpha_n(E')\, g_n(E')\, e^{\int_{E'}^{E_0} \frac{(Q-Q_P)\, d\bar{E}}{S_P}}$$

$$I_0[2\sqrt{B_n(E',E)(r - r')}\,] + \text{terms in } \not{\phi}_n \qquad (55)$$

where $E'(E_0,r') = E_L(E_0,r')$ is defined by

$$r' = \int_{E'(E_0,r')}^{E_0} \frac{d\bar{E}}{S_P(\bar{E})} \quad . \tag{55}$$

Alternatively, using Eq. 25 in Eq. 51, and inverting by means of the convolution theorem, we see that Eq. 55 is equivalent to

$$\phi_{sn}(E,r) = \frac{QN_{P_0}\, e^{-Qr}}{g_n(E)} \int_{E_L(E_o,r),\ \text{or } E}^{E_0} dE' \, \frac{\alpha_n(E')\, g_n(E')}{S_P(E')}\, e^{\int_{E'}^{E_0} \frac{(Q-Q_P)}{S_P} d\bar{E}}\, \theta[r-\bar{r}(E_0,E')]$$

$$I_0[2\sqrt{B_n(E',E)(r - \bar{r})}\,] + \text{terms in } \mathcal{J}_n \quad . \tag{56}$$

where $E_L(E_o,r)$ is defined by Eq. 42, $\bar{r}$ by Eq. 25. Similarly,

$$\phi_{s\pi}(E,r) = \frac{N_{P_0} Q\, e^{-Qr}}{g_n(E)} \int_{E_L(E_o,r),\ \text{or } E}^{E_0} dE' \left\{ \frac{\alpha_\pi(E')\, f_n(E')}{S_P(E')}\, e^{\int_{E'}^{E_0} \frac{(Q-Q_P)}{S_P} d\bar{E}} \right.$$

$$I_0[2\sqrt{B_\pi(E',E)(r - \bar{r})}\,] + e^{\int_{E'}^{E_0} \frac{(Q-Q_P)}{S_P} d\bar{E}}\, \frac{\alpha_n(E')\, g_n(E')}{S_P(E')} \int_{E}^{E'} dE''$$

$$\left. \frac{f_n(E'')\, \alpha_\pi(E'')}{g_n(E'')} \left[ \frac{[r - \bar{r}(E_0,E')]}{B_n(E',E'') + B_\pi(E'',E)} \right]^{1/2} I_1\left(2\sqrt{(r-\bar{r})\,[B_n(E',E'')+B_\pi(E'',E)]}\right) \right\}$$

$$\cdot\, \theta[r - \bar{r}(E_0,E')] \qquad + \text{terms in } \mathcal{J}_n \text{ and } \mathcal{J}_\pi . \tag{57}$$

## VII. GENERAL SOLUTION FOR KERNELS "ALMOST SEPARABLE" IN ENERGY VARIABLES

We use

$$F_{jk}(E,E') = \frac{g_k(E')}{g_j(E')} \left\{ K_j(E,E') + \delta_{jk}\ k(E')\ h(E) \right\}$$

where

$$\sum_j K_j(E,E') + k(E)\ h(E') = H(E)\ L(E') \quad .$$

Although it is understood that either k(E') or h(E) can be set equal to unity without loss of generality, we will carry both factors so as to make the final formulas as flexible as possible. For example, the kernel of Eq. 17 could be used as it stands, without transforming to Eq. 18.

Return to Eqs. 23 and 24 and write:

$$\lambda\ \overline{\chi} = \overline{G} + H(E)\ Q \int_E^{E_0} dE'\ L(E')\ [\overline{\chi}(E',\lambda) + \overline{\chi}_0(E',\lambda)] \quad . \qquad (58)$$

Define $\overline{\psi}(E,\lambda) = \overline{\chi}(E,\lambda)/H(E)$

$$\lambda \frac{\partial}{\partial E} \overline{\psi} = \frac{\partial}{\partial E} \frac{\overline{G}}{H(E)} - Q\ H(E)\ L(E')\ \overline{\psi}(E,\lambda) - L(E)\ Q\ \overline{\chi}_0(E,\lambda) \qquad (59)$$

Then

$$\overline{\psi}(E,\lambda) = \int_E^{E_0^+} dE'\ \frac{e^{\int_E^{E'} \frac{Q}{\lambda} H(\overline{E}) L(\overline{E}) d\overline{E}}}{\lambda} \left[ Q L(E')\ \overline{\chi}_0(E',\lambda) - \frac{\partial}{\partial E'} \frac{\overline{G}(E',\lambda)}{H(E)} \right] \quad . \qquad (60)$$

The reason for the restriction, $\sum_j K_j(E,E') + h(E)\ k(E') = H(E)\ L(E')$, lies in the possibility of obtaining a first-order linear differential equation in E for $\overline{\psi}$. Equation 24 now becomes

$$\lambda \overline{\chi}_j = \overline{G}_j + \int_E^{E_o} dE' \, Q \, K_j(E,E') \, [H(E') \, \overline{\psi} + \overline{\chi}_o]$$

$$+ h(E) \, Q \int_E^{E'} dE' \, k(E') \, [\overline{\chi}_j + \overline{\chi}_{jo}] \, . \qquad (61)$$

Define $\overline{N}_j(E,\lambda) = \overline{\chi}_j(E,\lambda)/h(E)$. (62)

Then

$$\lambda \frac{\partial}{\partial E} \overline{N}_j = \frac{\partial}{\partial E} \int_E^{E_o} dE' \, Q \, \frac{K_j(E,E')}{h(E)} \, [H(E') \, \overline{\psi} + \overline{\chi}_o]$$

$$- Q \, k(E) \, [h(E) \, \overline{N}_j(E) + \overline{\chi}_{jo}(E)] + \frac{\partial}{\partial E} (\overline{G}_j/h) \quad ; \qquad (63)$$

and, integrating,

$$\overline{\chi}_j(E,\lambda) = h(E) \int_E^{E_o^+} dE' \, \frac{e^{\int_E^{E'} \frac{Q}{\lambda} h(\overline{E}) k(\overline{E}) d\overline{E}}}{\lambda} \left\{ \left[ Qk(E') \, \overline{\chi}_{jo} - \frac{\partial}{\partial E'} \left( \frac{\overline{G}_j}{h} \right) \right] \right.$$

$$\left. - \frac{\partial}{\partial E'} \int_{E'}^{E_o^+} dE'' \, Q \, \frac{K_j(E',E'')}{h(E')} \, [\overline{\chi}_o(E'',\lambda) + H(E'') \, \overline{\psi}(E'',\lambda)] \right\} . \qquad (64)$$

Here, it is clear that the restriction

$$h(E,E') = h(E) \, k(E')$$

enables one to obtain a first-order linear differential equation for $\overline{N}_j$, provided $\overline{\Psi}$ is known.

The solution for kernels of the type given by Example 1, Eq. 12, has already been presented in Section IV, with Eqs. 36 and 37 corresponding to 60 and 64.

Define:

$$C(E_0,E) = \int_E^{E_0} Q\, L(\overline{E})\, H(\overline{E})\, d\overline{E} = \int_E^{E_0} [\alpha(\overline{E}) + k(\overline{E})]\, H(\overline{E})\, Q\, d\overline{E} \qquad (65)$$

$$D(E_0,E) = \int_E^{E_0} Q\, h(\overline{E})\, k(\overline{E})\, d\overline{E} \quad . \qquad (66)$$

At this point, one can specialize to Example 2, Eq. 13, such that

$$F_{jk}(E,E') = \frac{g_k(E')}{g_j(E)} [\alpha_j(E') + \delta_{jk}\, k(E')]$$

$$K_j(E,E') = \alpha_j(E'); \quad h(E) = 1$$

$$H(E) = 1; \quad L(E') = \sum_j \alpha_j(E') + k(E') = \alpha(E') + k(E') \quad .$$

Then, differentiating the last integral in Eq. 64 at the lower limit gives

$$\overline{\chi}_j(E,\lambda) = \int_E^{E_o^+} dE' \, \frac{e^{\int^{E'} \frac{Qk}{\lambda} d\overline{E}}}{\lambda} \Big\{ Qk(E') \, \overline{\chi}_{jo}(E',\lambda) - \frac{\partial}{\partial E'} \overline{G}_j(E',\lambda)$$

$$+ Q \, \alpha_j(E') \left[ \overline{\chi}_o(E',\lambda) + \int_{E'}^{E_o^+} dE'' \, \frac{e^{\int^{E''} \frac{Q}{\lambda} (\alpha+k) d\overline{E}}}{\lambda} \left[ Q \, L(E'') \, \overline{\chi}_o - \frac{\partial}{\partial E''} \overline{G} \right] \right] \Big\} \qquad (67)$$

This expression could be inverted as it stands; or, we can interchange integrations in E' and E" to obtain

$$\overline{\chi}_j(E,\lambda) = \int_E^{E_o^+} dE' \left\{ \frac{e^{\int^{E'} \frac{Qk}{\lambda} d\overline{E}}}{\lambda} \left[ Q \, k(E') \, \overline{\chi}_{jo} - \frac{\partial}{\partial E'} \overline{G}_j + Q \, \alpha_j(E') \, \overline{\chi}_o \right] \right.$$

$$\left. + \left[ Q \, L(E') \, \overline{\chi}_o - \frac{\partial}{\partial E'} \overline{G} \right] \int_E^{E'} dE'' \, Q \, \alpha_j(E'') \, \frac{e^{\int^{E'} \frac{Qk}{\lambda} d\overline{E} + \int_{E''}^{E'} \frac{Q\alpha}{\lambda} d\overline{E}}}{\lambda^2} \right\} . \qquad (68)$$

If $\alpha_j(E'')$ and $\alpha(E'')$ are proportional to each other, i.e., if $\alpha_j(E'') = \frac{\gamma_j}{\gamma} \alpha(E'')$, with $\sum_j \gamma_j = \gamma$; $L(E') = \alpha(E') + k(E')$, we have Example 3, Eq. 14. The last integral can then be integrated by parts, as follows

$$- \frac{\gamma_j}{\gamma} \frac{e^{\int_E^{E'} \frac{Qk}{\lambda} d\bar{E}}}{\lambda} \int_E^{E'} dE'' \frac{\partial}{\partial E''} e^{\int_{E''}^{E'} \frac{Q\alpha}{\lambda} d\bar{E}}$$

$$= \frac{\gamma_j}{\gamma} \left[ \frac{e^{\int_E^{E'} \frac{Q}{\lambda} (\alpha+k) d\bar{E}}}{\lambda} - \frac{e^{\int_E^{E'} \frac{Qk}{\lambda} d\bar{E}}}{\lambda} \right] .$$

Then Eq. 67 becomes, using $L(E) = \alpha(E) + k(E)$,

$$\bar{X}_j(E,r) = \int_E^{E_o} dE' \left\{ \frac{e^{\int_E^{E'} \frac{Qk}{\lambda} d\bar{E}}}{\lambda} \left[ Q\ k(E')\ (\bar{X}_{jo} - \frac{\gamma_j}{\gamma} \bar{X}_o) - \frac{\partial}{\partial E'} (\bar{G}_j - \frac{\gamma_j}{\gamma} \bar{G}) \right] \right.$$
$$\left. + \frac{e^{\int_E^{E'} \frac{Q}{\lambda} L(\bar{E}) d\bar{E}}}{\lambda} \frac{\gamma_j}{\gamma} \left[ Q\ L(E')\ \bar{X}_o - \frac{\partial}{\partial E'} \bar{G} \right] \right\} . \qquad (69)$$

This is particular simple, since one integration has been eliminated. Using Eq. 26,

$$\bar{X}_o(E',\lambda) = N_{P_o}\ g_P(E_o)\ \frac{\delta(E_o - E')}{\lambda}$$

for monoenergetic primary protons without slowing down, and the definitions of Eqs. 65 and 66. The inversion of Eq. 69 is then:

$$\phi_{sj}(E,r) = \frac{e^{-Qr}}{g_j(E)} N_{P_0}\ g_P(E_0)\ Q \left\{ k(E_0) \left( \delta_{jP} - \frac{\gamma_j}{\gamma} \right) \left[ \frac{r}{D(E_0,E)} \right]^{\frac{1}{2}} \right.$$

$$\left. \cdot\ I_1[2\sqrt{rD(E_0,E)}] + \frac{\gamma_j}{\gamma} L(E_0) \left[ \frac{r}{C(E_0,E)} \right]^{\frac{1}{2}} I_1[2\sqrt{rC(E_0,E)}] \right\}$$

$$- \frac{e^{-Qr}}{g_j(E)} \int_E^{E_0} dR' \int_0^r dr'\ I_0[2\sqrt{D(E',E)(r-r')}]\ \frac{\partial}{\partial E'} [g_j(E')\ \mathcal{J}_j(E',r')\ e^{Qr'}]$$

$$+ \frac{e^{-Qr}}{g_j(E)} \frac{\gamma_j}{\gamma} \int_E^{E_0} dR' \int_0^r dr' \left\{ \left[ I_0[2\sqrt{D(E',E)(r-r')}] - I_0[2\sqrt{C(E',E)(r-r')}] \right] \right.$$

$$\left. \cdot\ \frac{\partial}{\partial E'} \sum_k g_k(E')\ \mathcal{J}_k(E',r')\ e^{Qr'} \right\}. \qquad (70)$$

If primary proton slowing down, and $Q \neq Q_P$ areto be included, use Eq. 25 for $\overline{\chi}_0$ in Eq. 69,

$$\phi_{sj}(E,r) = \frac{e^{-Qr}}{g_j(E)} N_{P_0}\ Q \int_{E_L(E_0,r),\ \text{or } E}^{E_0} dE'\ \frac{g_P(E')}{S_P(E')} \left\{ k(E') \left( \delta_{jP} - \frac{\gamma_j}{\gamma} \right) \right.$$

$$\left. \cdot\ I_0[2\sqrt{(r-\overline{r})D(E',E)}] + \frac{\gamma_j}{\gamma} L(E')\ I_0[2\sqrt{(r-\overline{r})(E_0,E)}] \right\}\ e^{\int_{E'}^{E_0} \frac{(Q-Q_P)}{S_P} d\overline{E}}\ \theta[r-\overline{r}(E_0,E')] \qquad (71)$$

$+$ terms in $\mathcal{J}_j$ and $\mathcal{J}_k$.

The solution for this case for arbitrary initial spectra of the form

$$\phi_{ij}(E,r) = \phi_j(E,0)\ e^{-Qr}$$

has been given earlier.[3]

Similarly, the inversion of Eq. 68 for the more general Example 2, with $Q = Q_P$ and no slowing down for the monoenergetic primary proton, is

$$\phi_{sj}(E,r) =$$

$$\frac{e^{-Qr}}{g_j(E)} Q N_{Po} g_P(E_0) \Bigg\{ [k(E_o)\ \delta_{jP} + \alpha_j(E_o)] \left[\frac{r}{D(E_o,E)}\right]^{\frac{1}{2}} I_1[2\sqrt{rD(E_o,E)}]$$

$$+ L(E_o) \int_E^{E_o} dE' \frac{Q\,\alpha_j(E')\ r}{C(E_o,E') + D(E',E)} I_2 \left[ 2\sqrt{r[C(E_o,E')+D(E',E)]} \right] \Bigg\}$$

$$- \frac{e^{-Qr}}{g_j(E)} \int_E^{E_o} dE' \int_o^r dr' \Bigg\{ I_0[2\sqrt{D(E',E)(r-r')}] \frac{\partial}{\partial E'} [g_j(E')\ \mathcal{J}_j(E',r')\ e^{Qr'}]$$

$$+ \left[ \frac{\partial}{\partial E'} \sum_k g_k(E') \mathcal{J}_k(E',r')\ e^{Qr'} \right] \cdot \int_E^{E'} dE''\ Q\,\alpha_j(E'')$$

$$\cdot \left[ \frac{(r - r')}{C(E',E'') + D(E'',E)} \right]^{\frac{1}{2}} I_1 \left[ 2\sqrt{(r-r')[C(E',E'')+D(E'',E)]} \right] \Bigg\} \tag{72}$$

As expected, Eq. 72 reduces to Eq. 70 when $\alpha_j(E') = \frac{\gamma_j}{\gamma} \alpha(E')$, by use of the second of Eqs. 28.

If primary proton stopping and energy-dependent absorption are to be included, use Eq. 25 in Eq. 68 and proceed in the same manner as used for Eqs. 41, or 56 and 57.

$$\phi_{sj}(E,r) = \frac{e^{-Qr}}{g_j(E)} N_{Po}\ Q \int_{E_L(E_o,r),\ \text{or}\ E}^{E_o} dE' \frac{g_P(E')}{S_P(E')} \Bigg\{ [k(E')\ \delta_{jP} + \alpha_j(E')]$$

$$\cdot I_0[2\sqrt{(r-\bar{r})D(E',E)}] + L(E') \int_E^{E'} dE''\ Q\,\alpha_j(E'')$$

$$\cdot \left[ \frac{[r - \bar{r}(E_o,E')]}{C(E',E'') + D(E'',E)} \right]^{\frac{1}{2}} I_1 \left[ 2\sqrt{(r-\bar{r})[C(E',E'')+D(E'',E)]} \right] \Bigg\}\ \theta[r - \bar{r}(E_o,E')]$$

$$\cdot \exp \int_{E'}^{E_o} \frac{[Q - Q_P(\bar{E})]}{S_P(\bar{E})} d\bar{E} \tag{73}$$

+ terms in $\mathcal{J}_j$ and $\mathcal{J}_k$.

Equation 73 reduces to Eq. 71 if $\alpha_j(E') = \frac{\gamma_j}{\gamma}\alpha(E')$. Returning to Eq. 64 for the general case, we have, by straightforward differentiation of the last integral,

$$\overline{\chi}_j(E,\lambda) = h(E)\int_E^{E_o^+} dE' \; \frac{e^{\int_E^{E'} \frac{Q}{\lambda} hk d\overline{E}}}{\lambda} \Bigg\{ Q\, k(E')\, \overline{\chi}_{jo} - \frac{\partial}{\partial E'}\frac{\overline{G}_j}{h}$$

$$+ Q\,\frac{K_j(E',E')}{h(E')}\left(\overline{\chi}_o(E',\lambda) + H(E')\int_{E'}^{E_o^+} dE'' \; \frac{e^{\int_{E'}^{E''} \frac{Q}{\lambda} HL d\overline{E}}}{\lambda}\left[Q\, L(E'')\, \overline{\chi}_o - \frac{\partial}{\partial E'}\frac{\overline{G}}{\overline{H}}\right]\right)$$

$$- Q\int_{E'}^{E_o^+} dE'' \; \frac{\partial}{\partial E'}\left[\frac{K_j(E',E'')}{h(E')}\right]$$

$$\cdot\left(\overline{\chi}_o(E'',\lambda) + H(E'')\int_{E''}^{E_o^+} dE''' \; \frac{e^{\int_{E''}^{E'''} \frac{Q}{\lambda} HL d\overline{E}}}{\lambda}\left[Q\, L\, \overline{\chi}_o - \frac{\partial}{\partial E''}\frac{\overline{G}}{\overline{H}}\right]\right)\Bigg\} \qquad (74)$$

For $\chi_o(E',\lambda) = N_{Po}\, g_P(E')\, \frac{\delta(E' - E_O)}{\lambda}$; i.e., monoenergetic primary protons, with no slowing down and $Q = Q_P$, the inversion is

$\phi_{sj}$ =

$$\frac{e^{-Qr}}{g_j(E)} h(E)\, N_{P_O}\, Q\, g_P(E_o) \left\{ \left[ k(E_o)\, \delta_{jP} + \frac{K_j(E_o,E_o)}{h(E_o)} \right] \left[\frac{r}{D(E_o,E)}\right]^{\frac{1}{2}} I_1\, 2\sqrt{rD(E_o,E)} \right.$$

$$+ \int_E^{E_o^+} dE' \left[ \frac{K_j(E',E')}{h(E')} \quad \frac{H(E')\, L(E_o)\, r}{[C(E_o,E') + D(E',E)]} I_2 \left[2\sqrt{r[C(E_o,E')+D(E',E)]}\right] \right.$$

$$\left. - \left[\frac{r}{D(E',E)}\right]^{\frac{1}{2}} I_1[2\sqrt{rD(E',E)}]\, \frac{\partial}{\partial E'} \frac{K_j(E',E_o)}{h(E')} \right]$$

$$- \int_E^{E_o^+} dE' \int_{E'}^{E_o^+} dE'' \left[ \frac{\partial}{\partial E'} \frac{K_j(E',E'')}{h(E')} \right] H(E'')\, L(E_o)\, \frac{r}{[C(E_o,E'') + D(E',E)]}$$

$$\left. \cdot\, I_2 \left[2\sqrt{r[C(E_o,E'')+D(E',E)]}\right] \right\} \qquad (75)$$

+ terms in $\mathscr{S}_j$ and $\mathscr{S}_k$.

The source terms are

$$- \frac{e^{-Qr}}{g_j(E)} h(E) \int_o^r dr' \int_E^{E_o^+} dE' \left\{ I_0[2\sqrt{(r-r')D(E',E)}]\, \frac{\partial}{\partial E'} \left[\frac{g_j(E')}{h(E')} \mathscr{S}_j(E',r')\, e^{Qr'}\right] \right.$$

$$+ Q\, \frac{K_j(E',E')}{h(E')} H(E') \int_{E'}^{E_o^+} dE'' \left[\frac{(r - r')}{[C(E'',E') + D(E',E)]}\right]^{\frac{1}{2}} I_1 \left[2\sqrt{(r-r')[C(E'',E')+D(E',E)]}\right]$$

$$\cdot\, \frac{\partial}{\partial E''} \left[ \sum_k \frac{g_k(E'')\mathscr{S}_k(E'',r')\, e^{Qr'}}{H(E'')} \right] - \int_{E'}^{E_o^+} Q\, dE'' \left[\frac{\partial}{\partial E'} \frac{K_j(E',E'')}{h(E')}\right] H(E'')$$

$$\cdot \int_{E''}^{E_o^+} dE''' \left[\frac{(r - r')}{[C(E''',E'') + D(E',E)]}\right]^{\frac{1}{2}} I_1 \left[2\sqrt{(r-r')[C(E''',E'')+D(E',E)]}\right]$$

$$\left. \frac{\partial}{\partial E'''} \sum_k \frac{g_k(E''')\mathscr{S}_k(E''',r')\, e^{Qr'}}{H(E''')} \right\} . \qquad (76)$$

For the sake of completeness, we include the expression when primary proton slowing down is important, but, since this involves a triple integration, it is probably not very useful. Equation 75 is replaced by

$$\phi_{sj}(E,r) =$$

$$\frac{e^{-Qr}}{g_j(E)}\, h(E)\, N_{P_0}\, Q \left\{ \int_{E_L(E_o,r),\,\text{or } E}^{E_o^+} dE' \left\{ \frac{g_P(E')}{S_P(E')}\, e^{\int_{E'}^{E_o} (Q-Q_P)\, d\bar{E}}\, \theta[r - \bar{r}(E_o,E')] \right.\right.$$

$$\cdot \left[ k(E')\,\delta_{jP} + \frac{K_j(E',E')}{h(E')} \right] I_0 \left[ 2\sqrt{[r-\bar{r}(E_o,E')]D(E',E)} \right] \Big\}$$

$$+ \int_E^{E_o^+} dE' \int_{E_L(E_o,r),\,\text{or } E'}^{E_o^+} dE''\, \frac{g_P(E'')}{S_P(E'')}\, e^{\int_{E''}^{E_o} \frac{Q-Q_P}{S_P} d\bar{E}}\, \theta[r - \bar{r}(E_o,E'')]$$

$$\cdot \left\{ \frac{K_j(E',E')}{h(E')}\, H(E')\, L(E'') \left[ \frac{r - \bar{r}(E_o,E'')}{C(E'',E') + D(E',E)} \right]^{\frac{1}{2}} \right.$$

$$\cdot\, I_1 \left[ 2\sqrt{[r-\bar{r}(E_o,E'')][C(E'',E')+D(E',E)]} \right]$$

$$- I_0 \left[ 2\sqrt{[r-\bar{r}(E_o,E'')]D(E',E)} \right] \frac{\partial}{\partial E'} \left[ \frac{K_j(E',E'')}{h(E')} \right] \Big\} \Big\}$$

$$+ \int_E^{E_o^+} dE' \int_{E'}^{E_o^+} dE''\, H(E'') \left[ \frac{\partial}{\partial E'} \frac{K_j(E',E'')}{h(E')} \right] \int_{E_L(E_o,r),\,\text{or } E''}^{E_o^+} dE'''\, L(E''')\, e^{\int_{E'''}^{E_o} \frac{(Q-Q_P)\,d\bar{E}}{S_P}}$$

$$\cdot\, \frac{g_P(E''')}{S_P(E''')}\, \theta[r - \bar{r}(E_o,E''')] \left[ \frac{r - \bar{r}(E_o,E''')}{C(E''',E'') + D(E',E)} \right]^{\frac{1}{2}}$$

$$\cdot\, I_1 \left[ 2\sqrt{[r-\bar{r}(E_o,E''')][C(E''',E'')+D(E',E)]} \right] \quad + \text{ terms in } \mathcal{J}_j \text{ and } \mathcal{J}_k. \qquad (77)$$

$$\bar{r}(E_o,E') = \int_{E'}^{E_o} \frac{d\bar{E}}{S_P(\bar{E})} \quad , \quad \text{etc.};$$

$$r = \int_{E_L(E_o,r)}^{E_o} \frac{d\bar{E}}{S_P(\bar{E})} \quad .$$

The lower limits on the integrals are the larger of $E_L(E_o,r)$ or E, and of $E_L(E_o,r)$ or E", respectively.

The source terms remain the same as in Eq. 76. A general inversion of Eq. 74 for arbitrary initial spectra can be carried out, using the convolution theorem; it will involve quadruple integrations in the absence of a delta function expression for $X_o(E,r)$.

An interesting feature of all solutions in this section is that when $k(E') \neq 0$, the first term in the solution for $\phi_{sj}$ is a term involving $\phi_{ij}$, only. If only proton primaries are considered, this term vanishes for all secondary particles other than the secondary protons. Hence, the $\delta_{jk}\, k(E')$ term in the kernel produces an additive term in the secondary intensities of particles of the same kind as the primary particles.

ORNL-3746
UC-34 — Physics
TID-4500 (46th ed.)

INTERNAL DISTRIBUTION

1. Biology Library
2-4. Central Research Library
5. Reactor Division Library
6-7. ORNL — Y-12 Technical Library
Document Reference Section
8-314. Laboratory Records Department
315. Laboratory Records, ORNL R.C.
316. F. S. Alsmiller
317. R. G. Alsmiller
318. E. P. Blizard
319. F. Clark
320. J. K. Dickens
321. W. H. Jordan
322. C. E. Larson
323. M. Leimdorfer
324. H. G. MacPherson
325. F. C. Maienschein
326. R. W. Peelle
327. R. T. Santoro
328. M. J. Skinner
329. J. E. Turner
330. D. R. Vondy
331. J. W. Wachter
332. A. M. Weinberg
333. H. A. Wright

EXTERNAL DISTRIBUTION

334. Research and Development Division, AEC, ORO
335. V. B. Bhanot, Physics Dept., Panjab University, Chandigarh-3, India
336-704. Given distribution as shown in TID-4500 (46th ed.) under Physics category (75 copies — CFSTI)