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ARRAY SYNTHESIS - A LEAST INTEGRAL SQUARE ERROR METHOD

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REPORT
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ABSTRACT

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This report presents a method of array synthesis based on a least integral square error criterion. After certain assumptions about array symmetry, it is shown how the array coefficients may be chosen to minimize the integral square error between the antenna pattern and some "desired" pattern. The method is applicable to nonuniformly spaced arrays. Two numerical examples are given. Author

TABLE OF CONTENTS

ARRAY SYNTHESIS - A LEAST INTEGRAL SQUARE ERROR METHOD

I. INTRODUCTION

This report discusses a least integral square error method of array synthesis. The array is assumed to be one-dimensional, with the elements spaced symmetrically about the array center. The amplitude coefficients are assumed to be complex, but to satisfy certain symmetry properties. With these assumptions, the voltage pattern of the antenna is a real function, and the even and odd parts of the antenna pattern result from the real and imaginary parts of the amplitude coefficients, respectively. Hence the synthesis problem is split into two parts. The real parts of the amplitude coefficients are chosen to make the even part of the pattern approximate the even part of some "desired" pattern. The imaginary parts are chosen to make the odd part of the pattern approximate the odd part of the desired pattern. In each case, the coefficients are chosen so that the integral square error between the actual and desired patterns is minimized.

One advantage of this method of synthesis is that it may be used with nonuniformly spaced arrays. Although numerous methods are known for synthesizing uniformly spaced arrays, most of them offer no help when the spacing is not uniform. Since very little is known in general about the properties of nonuniform arrays, it is hoped that the method presented here may be helpful for studying their behavior.

II. FORMULATION OF THE METHOD

Consider a linear array of 2N isotropic elements. Let x_i, $i = \pm 1, \pm 2, --- \pm N$ denote the position of the i^{th} element on the x-axis, as shown in Fig. 1.

Let $I_i = A_i + jB_i$ be the complex excitation coefficient of the i^{th} element. And let θ be the angle between the normal to the x-axis and the field observation point, as shown in Fig. 1.

We shall restrict the discussion at once to the case in which the element positions and excitation coefficients satisfy certain symmetry properties. Namely, we assume

(1) The element positions are symmetrical about $x = 0$, i.e.,

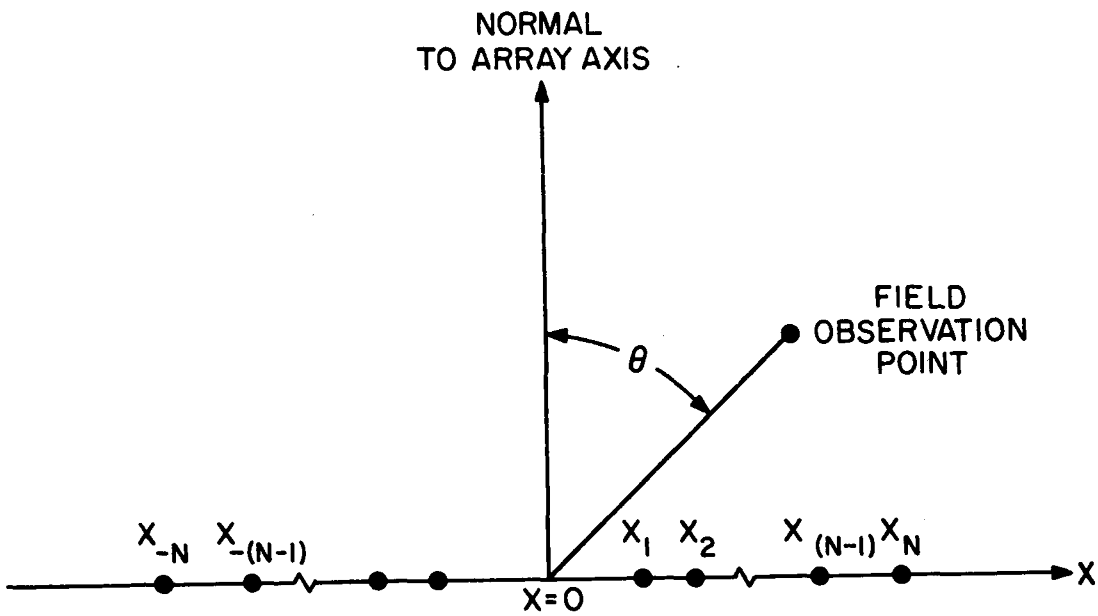

Fig. 1. Array geometry.

(1) $x_{-i} = -x_i$;

(2) $Re(I_i) = A_i$ is an even function of i, i.e.,

(2) $A_{-i} = A_i$;

and

(3) $Im(I_i) = B_i$ is an odd function of i, i.e.,

(3) $B_{-i} = -B_i$.

As will be seen below, these assumptions cause the voltage pattern of the array to be a real function of θ, which will be necessary for the derivation of the synthesis method.

It should be pointed out that although the positions of the elements are assumed symmetrical about x = 0, there is no requirement of uniform spacing in this method. The method can be used for synthesizing nonuniformly spaced arrays.

The voltage pattern of the array, F(θ), is given by

(4) $$F(\theta) = \sum_{i=1}^{N} \left[I_i \, e^{j\psi_i(\theta)} + I_{-i} \, e^{j\psi_{-i}(\theta)} \right] ,$$

where

(5) $$\psi_i(\theta) = k_o \, x_i \sin\theta \; .$$

We consider separately the patterns which result from the real and imaginary parts of I_i. First suppose $B_i = 0$ for all elements. Then

(6) $$F(\theta) = \sum_{i=1}^{N} \left[A_i \, e^{j\psi_i(\theta)} + A_{-i} \, e^{+j\psi_{-i}(\theta)} \right] .$$

Because of the symmetry properties, Eqs. (1) and (2), Eq. (6) may be written

(7) $$F(\theta) = \sum_{i=1}^{N} 2 \, A_i \cos\left[\psi_i(\theta)\right] .$$

Equation (7) has two important properties. First, $F(\theta)$ is a real function of θ; and second, $F(\theta)$ is an even function of θ about $\theta = 0$. For this reason, we will call this pattern (resulting from the A_i terms) $F_e(\theta)$;

(8) $$F_e(\theta) = \sum_{i=1}^{N} 2 \, A_i \cos\left[\psi_i(\theta)\right] .$$

Next consider the case when $A_i = 0$ for all elements. Then Eq. (4) gives

(9) $$F(\theta) = \sum_{i=1}^{N} \left[j \, B_i \, e^{j\psi_i(\theta)} + j \, B_{-i} \, e^{j\psi_{-i}(\theta)} \right] ,$$

which, in view of Eqs. (1) and (3), may be written as

(10) $$F(\theta) = \sum_{i=1}^{N} -2\ B_i \sin[\psi_i(\theta)] \ .$$

The pattern which results from the B_i terms is also real, but it is an odd function of θ about $\theta = 0$. Hence, we denote this pattern as $F_o(\theta)$;

(11) $$F_o(\theta) = \sum_{i=1}^{N} -2\ B_i \sin[\psi_i(\theta)] \ .$$

By superposition the pattern of the array with coefficients $I_i = A_i + B_i$ is given by

(12) $$F(\theta) = F_e(\theta) + F_o(\theta) \ .$$

Now suppose $F_d(\theta)$ is some "desired" pattern which we wish $F(\theta)$ to approximate. (It will be assumed that $F_d(\theta)$ is a real function of θ.) $F_d(\theta)$ may be split into a sum of two terms; i.e.,

(13) $$F_d(\theta) = F_{de}(\theta) + F_{do}(\theta) \ ,$$

where $F_{de}(\theta)$ is the even part of $F_d(\theta)$, given by

(14) $$F_{de}(\theta) = \frac{1}{2}\ [F_d(\theta) + F_d(-\theta)] \ ,$$

and $F_{do}(\theta)$ is the odd part of $F_d(\theta)$, given by

(15) $$F_{do}(\theta) = \frac{1}{2}[F_d(\theta) - F_d(-\theta)] \ .$$

If $F(\theta)$ is to be made to approximate $F_d(\theta)$, it is clear that we should use the even part of the pattern to approximate the even part of the desired pattern and the odd part of the pattern to approximate the odd part of the desired pattern.

Hence the A_i coefficients are to be chosen so that $F_e(\theta)$ is a "best" approximation to $F_{de}(\theta)$. As the criterion of "best", we choose the A_i to minimize the weighted integral square error between the actual pattern and the desired pattern

$$(16) \qquad I_e = \int_{-\pi/2}^{\pi/2} g(\theta) \left[F_e(\theta) - F_{de}(\theta) \right]^2 d\theta \ .$$

Here $g(\theta)$ is an arbitrary weighting function which may be used, for example, to attach different significance to the error in different ranges of θ.

The values of A_i which minimize I_e are easily found. Substituting Eq. (8) in Eq. (16) gives

$$(17) \qquad I_e = \int_{-\pi/2}^{\pi/2} g(\theta) \left[\sum_{i=1}^{N} \sum_{j=1}^{N} 4 A_i A_j \cos[\psi_i(\theta)] \cos[\psi_j(\theta)] - 2 F_{de}(\theta) \sum_{i=1}^{N} 2 A_i \cos[\psi_i(\theta)] + F_{de}^2(\theta) \right] d\theta \ .$$

Then setting

$$(18) \qquad \frac{\partial I_e}{\partial A_i} = 0 : i = 1, 2, ---, N$$

gives the system of equations

$$(19) \qquad \sum_{j=1}^{N} \beta_{ij} A_j = \alpha_i, \ i = 1, 2, ---, N,$$

where

$$(20) \qquad \beta_{ij} = 2 \int_{-\pi/2}^{\pi/2} g(\theta) \cos[\psi_i(\theta)] \cos[\psi_j(\theta)] \, d\theta$$

and

$$(21) \qquad \alpha_i = \int_{-\pi/2}^{\pi/2} g(\theta)\, F_{de}(\theta) \cos[\psi_i(\theta)]\, d\theta \ .$$

Similarly, to find the "best" B_i, we define

$$(22) \qquad I_o = \int_{-\pi/2}^{\pi/2} g(\theta)\, [F_o(\theta) - F_{do}(\theta)]^2\, d\theta ,$$

and set

$$(23) \qquad \frac{\partial I_o}{\partial B_i} = 0 : i = 1, 2, \text{---}, N.$$

This gives the system of equations

$$(24) \qquad \sum_{j=1}^{N} \gamma_{ij} B_j = \delta_i , \quad i = 1, 2, \text{---}, N ,$$

where

$$(25) \qquad \gamma_{ij} = 2 \int_{-\pi/2}^{\pi/2} g(\theta) \sin[\psi_i(\theta)] \sin[\psi_j(\theta)]\, d\theta$$

and

$$(26) \qquad \delta_i = - \int_{-\pi/2}^{\pi/2} g(\theta)\, F_{do}(\theta) \sin[\psi_i(\theta)]\, d\theta .$$

This completes the formal solution for the coefficeints A_i and B_i. Once $F_d(\theta)$ is specified, Eqs. (19) - (21) and (24) - (26) can be solved for the A_i and B_i. In general, however, the evaluation of the integrals in Eqs. (20), (21), (25), and (26) is a tedious affair. For this reason it is helpful to point out certain possible simplifications.

First, if the weighting function $g(\theta)$ is chosen to be

(27) $g(\theta) = \cos\theta$,

then the coefficients β_{ij} and γ_{ij} may be evaluated in closed form. This choice of function for $g(\theta)$ imposes a heavier penalty for errors in the vicinity of $\theta = 0$ than in the vicinity of $\theta = \pm 90°$. With this $g(\theta)$, Eq. (20) yields

(28) $$\beta_{ij} = 2\left\{\frac{\sin[k_o(x_i - x_j)]}{k_o(x_i - x_j)} + \frac{\sin[k_o(x_i + x_j)]}{k_o(x_i + x_j)}\right\}$$

and Eq. (25) yields

(29) $$\gamma_{ij} = 2\left\{\frac{\sin[k_o(x_i - x_j)]}{k_o(x_i - x_j)} - \frac{\sin[k_o(x_i + x_j)]}{k_o(x_i + x_j)}\right\} .$$

Also, Eqs. (21) and (26) become

(30) $$\alpha_i = \int_{-\pi/2}^{\pi/2} F_{de}(\theta) \cos[k_o x_i \sin\theta] \cos\theta \, d\theta$$

and

(31) $$\delta_i = -\int_{-\pi/2}^{\pi/2} F_{do}(\theta) \sin[k_o x_i \sin\theta] \cos\theta \, d\theta .$$

By defining

(32) $u = \sin\theta$,

(33) $f_e(u) = F_{de}(\sin^{-1} u)$,

and

(34) $f_o(u) = F_{do}(\sin^{-1} u)$,

Eqs. (30) and (31) may be written

$$\alpha_i = \int_{-\infty}^{\infty} f_e(u) \cos (k_o x_i u)\, du \tag{35}$$

and

$$\delta_i = - \int_{-\infty}^{\infty} f_o(u) \sin (k_o x_i u)\, du, \tag{36}$$

where we have used the additional assumption that $F_{de}(\theta)$ and $F_{do}(\theta)$ are identically zero for $|\theta| \geq 90°$ so that the limits of integration may be extended to infinity.

Equations (35) and (36) are recognized as the even and odd Fourier Transforms, respectively, of $f_e(u)$ and $f_o(u)$. There is a method discussed by Guillemin[1] for evaluating these integrals by inspection. First, we rewrite Eqs. (35) and (36) in terms of the derivatives of $f_e(u)$ and $f_o(u)$; i.e.,

$$\alpha_i = \frac{-1}{(k_o x_i)^2} \int_{-\infty}^{\infty} f_e''(u) \cos (k_o x_i u)\, du \tag{37}$$

and

$$\delta_i \quad \frac{1}{(k_o x_i)^2} \int_{-\infty}^{\infty} f_o''(u) \sin (k_o x_i u)\, du, \tag{38}$$

where

$$f_e''(u) = \frac{d^2 f_e(u)}{du^2} \tag{39}$$

and

$$f_o''(u) = \frac{d^2 f_o(u)}{du^2} \quad . \tag{40}$$

Then we approximate $f_e(u)$ and $f_o(u)$, each by a continuous curve consisting of a series of linear slopes. The second derivative of such a curve is a series of impulse functions. The impulses occur at the positions on the u-axis where the piecewise linear function changes slope, and the area of each impulse is equal to the change in slope at that point. Hence $f_e''(u)$ and $f_o''(u)$ are each a finite sum of impulses, and Eqs. (37) and (38) may then be evaluated by inspection. This method is used in the two examples given in the next section.

III. TWO EXAMPLES

In this section we give two examples to illustrate the method.

Example 1

Suppose we wish to synthesize the voltage pattern shown in Fig. 2,

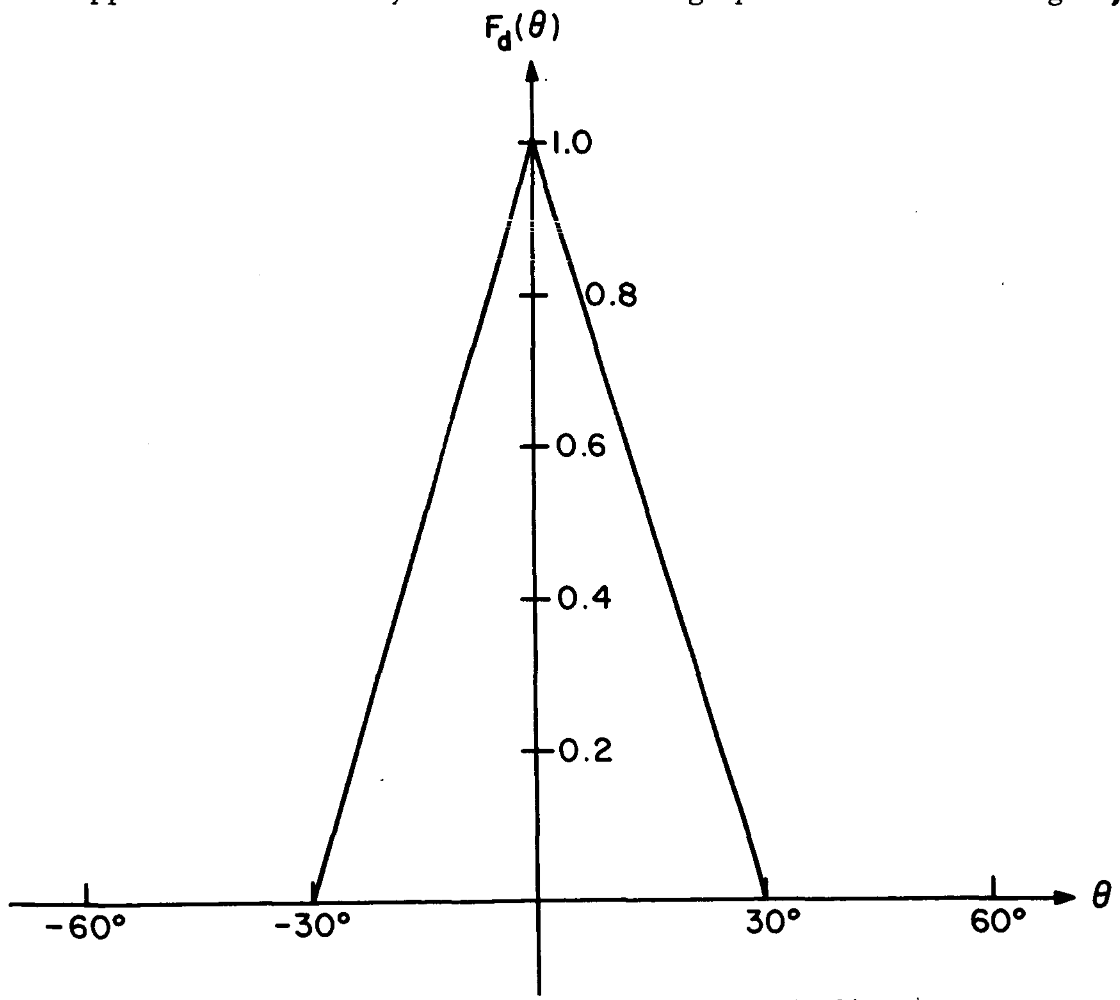

Fig. 2. The desired pattern (Example 1).

with an array of six elements, located as follows:

$$x_1 = \frac{\lambda}{4} \quad \left(x_{-1} = -\frac{\lambda}{4}\right) ,$$

(41) $$x_2 = \frac{\lambda}{2} \quad \left(x_{-2} = -\frac{\lambda}{2}\right) ,$$

and

$$x_3 = \lambda \quad (x_{-3} = -\lambda) ,$$

as shown in Fig. 3.

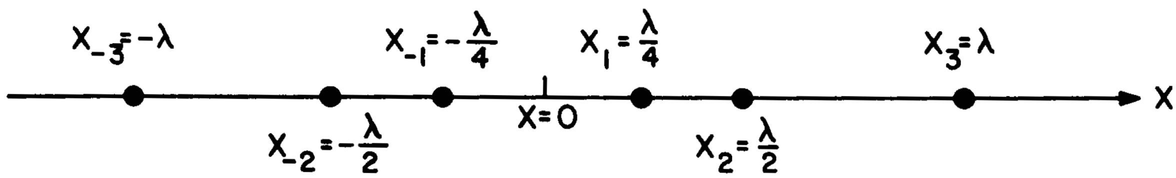

Fig. 3. Position of elements in 6-element array.

Since the "desired" pattern of Fig. 2 is itself an even function of θ, we will need only the A_i coefficients. Hence $B_j = 0$ for all j, and $F_{de}(\theta) = F_d(\theta)$.

First, we have

$$k_o x_1 = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \frac{\pi}{2} ,$$

(42) $$k_o x_2 = \pi ,$$

and

$$k_o x_3 = 2\pi ,$$

and then Eq. (20) yields for the β matrix:

$$(43)\qquad (\beta) = \begin{pmatrix} 2 & 0.85 & -0.17 \\ 0.85 & 2 & 0 \\ -0.17 & 0 & 2 \end{pmatrix} .$$

To compute the α_i, we first replot $F_{de}(\theta)$ versus $u = \sin\theta$ to obtain $f_e(u)$. Using Table I, we plot the curve shown in Fig. 4 for $f_e(u)$.

TABLE I

θ	$u = \sin\theta$	$F_{de}(\theta)$
$\pm$ 0°	0	1.000
$\pm$ 5°	$\pm$0.087	0.833
$\pm$10°	$\pm$0.174	0.667
$\pm$15°	$\pm$0.259	0.500
$\pm$20°	$\pm$0.342	0.333
$\pm$25°	$\pm$0.423	0.167
$\pm$30°	$\pm$0.500	0

$f_e(u)$ is so close to being linear in the regions $-0.5 \leq u \leq 0$ and $0 \leq u \leq 0.5$ that we may choose for our piecewise linear approximation

$$(44)\qquad f_e(u) = \begin{cases} 0: & u < -0.5 \\ 1 + 2u: & -0.5 \leq u \leq 0 \\ 1 - 2u: & 0 \leq u \leq 0.5 \\ 0: & 0.5 < u \end{cases} .$$

Hence the second derivative of $f_e(u)$ is

$$(45)\qquad f''_e(u) = 2\,\delta(u+0.5) - 4\,\delta(u) + 2\,\delta(u-0.5),$$

where $\delta(z)$ is an impulse function occurring at $z = 0$. Equation (37) then gives for α_i

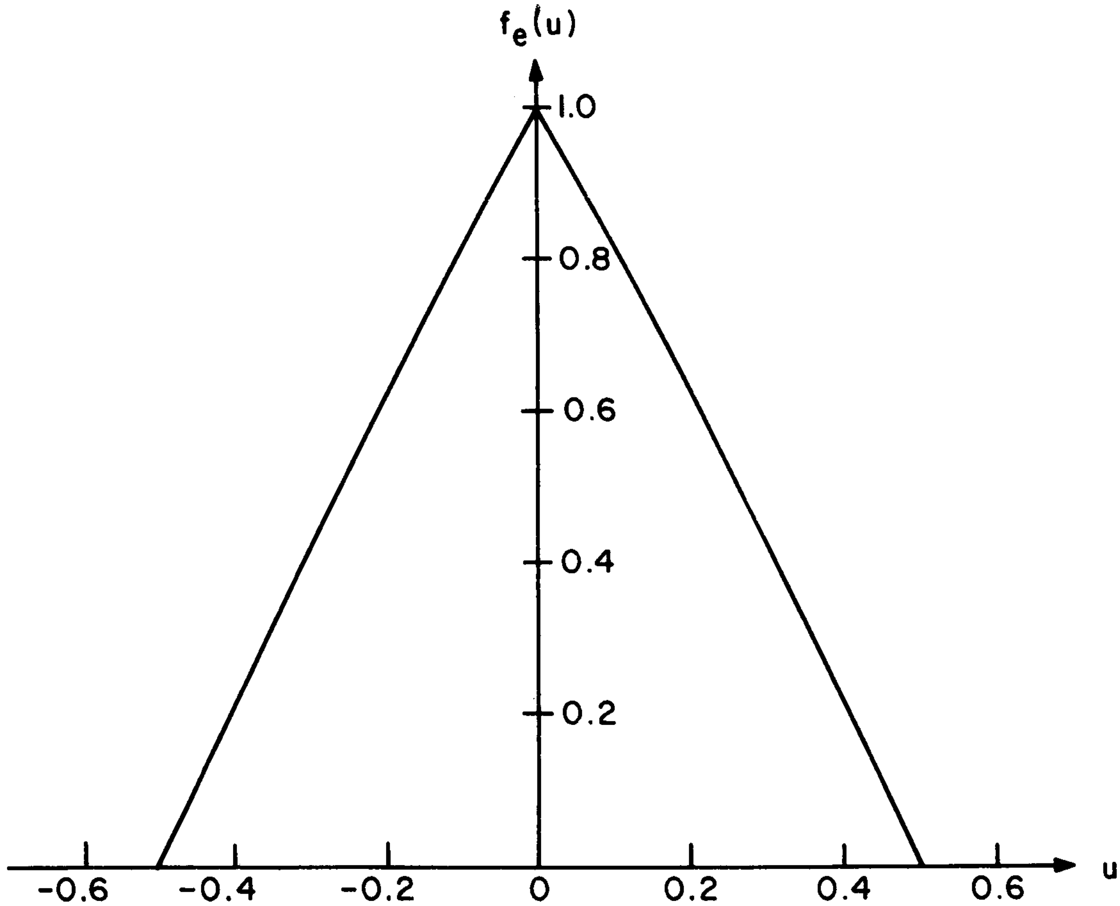

Fig. 4. The function $f_e(u)$.

$$(46) \qquad \alpha_i = \frac{-1}{(k_o x_i)^2}\left[2\cos(-0.5k_o x_i) - 4 + 2\cos(+0.5\,k_o x_i)\right]$$

$$= \frac{4}{(k_o x_i)^2}\left[1 - \cos\left(\frac{k_o x_i}{2}\right)\right] ,$$

and Eq. (42) gives

$$(47) \qquad (\alpha) = \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} = \begin{pmatrix}0.473\\ 0.404\\ 0.202\end{pmatrix} .$$

Notice that the use of Guillemin's impulse method here allows us to find a single formula from which all α_i may be calculated; we only need to evaluate the integral in Eq. (37) once.

Finally, using the results for (β) and (α) given in Eqs. (43) and (47), we may solve Eq. (19) to find the coefficients A_i. The result is

$$(48) \qquad \begin{aligned} A_1 &= 0.196, \\ A_2 &= 0.119, \text{ and} \\ A_3 &= 0.118 \quad . \end{aligned}$$

From Eq. (8), the pattern resulting from this set of coefficients is given by

$$(49) \qquad F_e(\theta) = 0.392 \cos\left(\frac{\pi}{2} \sin\theta\right) + 0.238 \cos\ (\pi \sin\theta) + 0.236 \cos\ (2\pi \sin\theta),$$

which is shown, along with $F_d(\theta)$, in Fig. 5. It may be seen that the agreement is quite good.

Example 2:

Next suppose we wish to synthesize the voltage pattern shown in Fig. 6. We will use the same element positions as for Example 1 above. (These are shown in Fig. 3).

Since $F_d(\theta)$ is neither even nor odd, both coefficients A_i and B_i will be needed. Using Eqs. (14) and (15), we find $F_{de}(\theta)$ and $F_{do}(\theta)$ to be given by the curves shown in Figs. 7 and 8.

First, consider the A_i. With the element positions the same as in the preceding example, the β matrix will be the same (see Eq. (43)), i.e.,

$$(50) \qquad (\beta) = \begin{pmatrix} 2 & .85 & -.17 \\ .85 & 2 & 0 \\ -.17 & 0 & 2 \end{pmatrix} \quad .$$

Next $f_e(u)$ may be obtained by making a table similar to Table I above. The result is shown in Fig. 9.

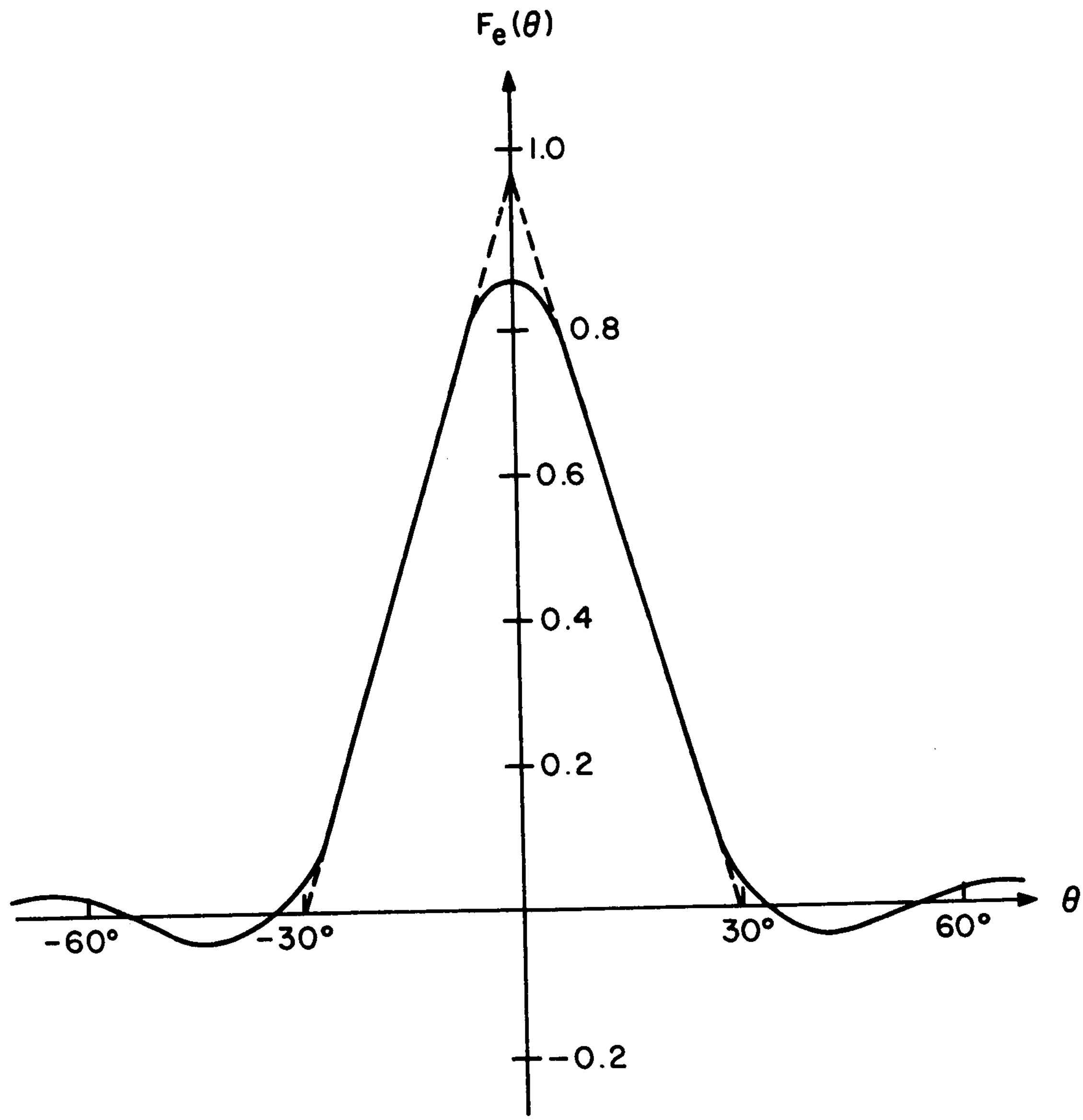

Fig. 5. The optimum pattern $F_e(\theta)$. (Example 1).

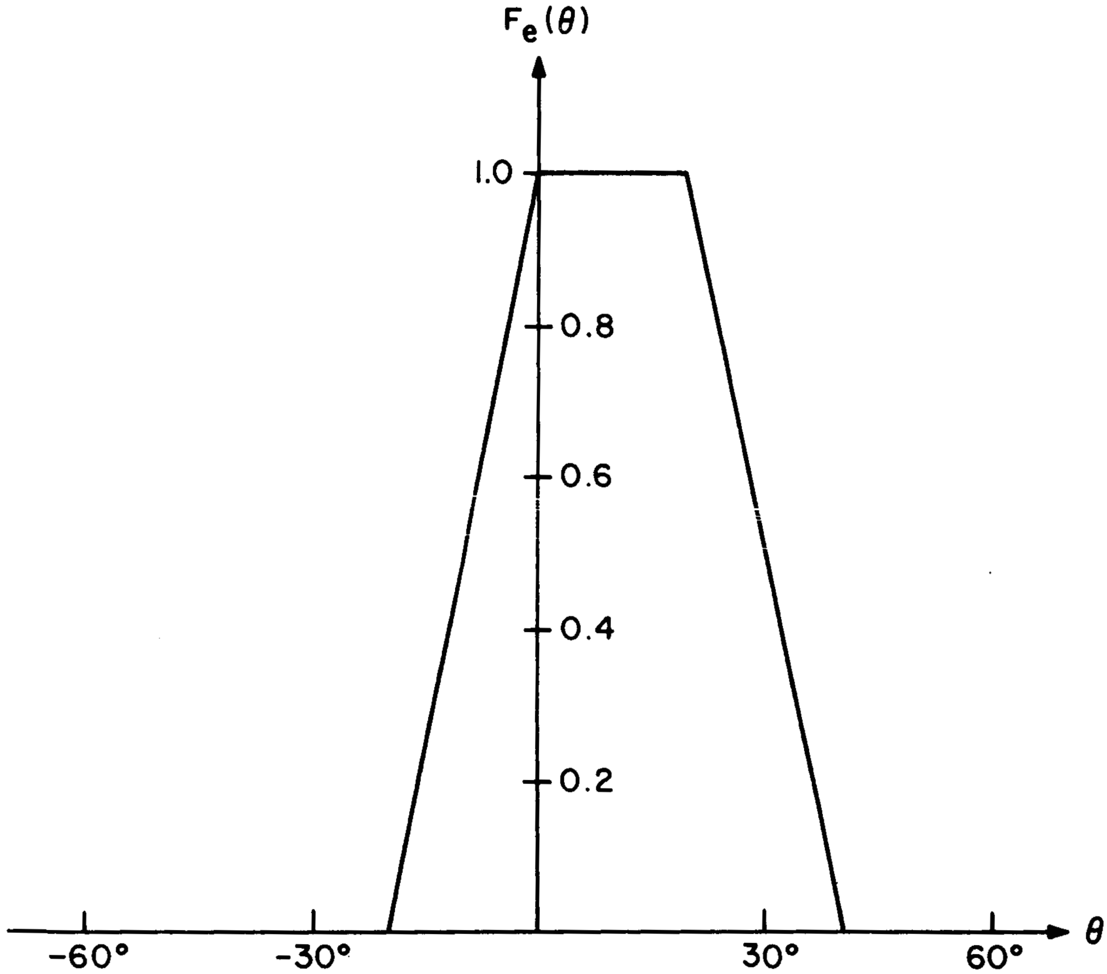

Fig. 6. The desired pattern (Example 2).

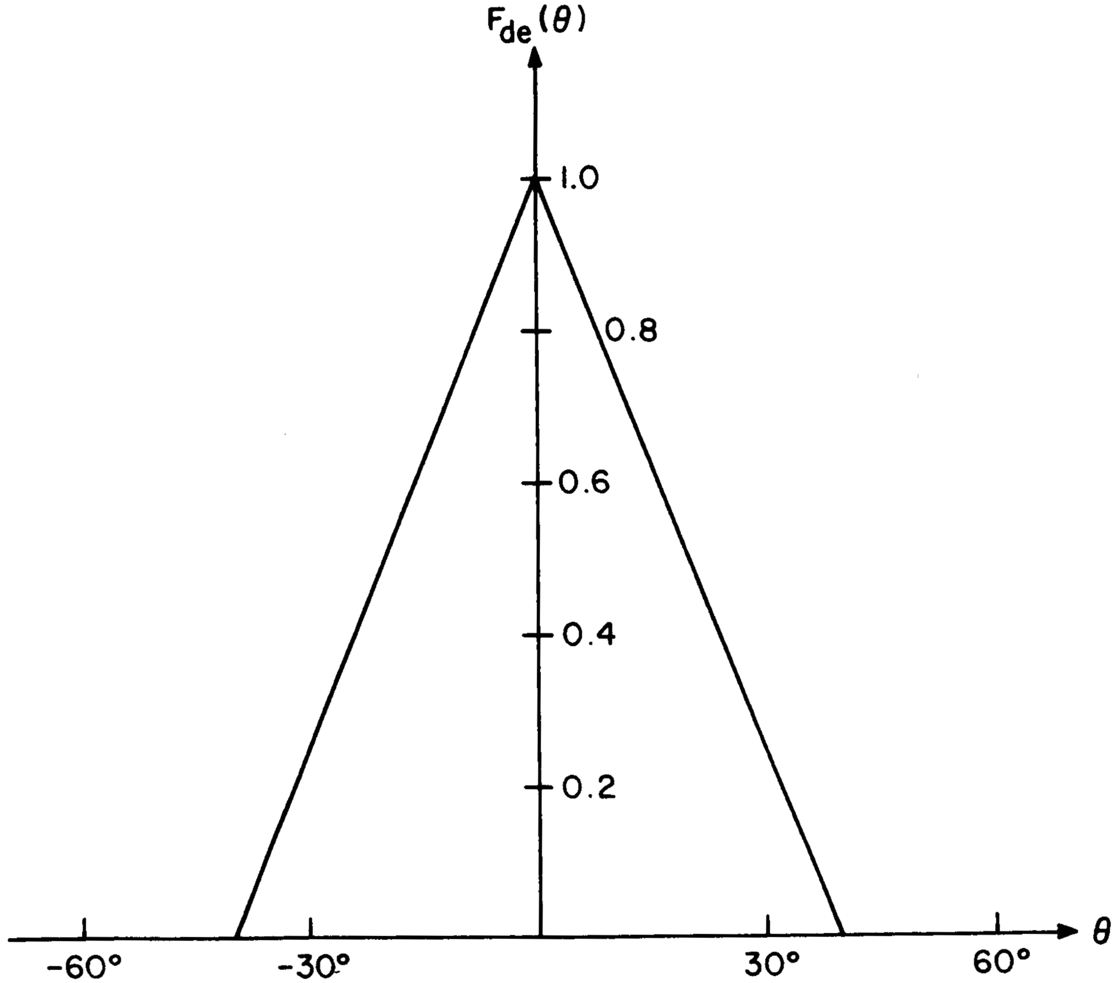

Fig. 7. The even part of the desired pattern.

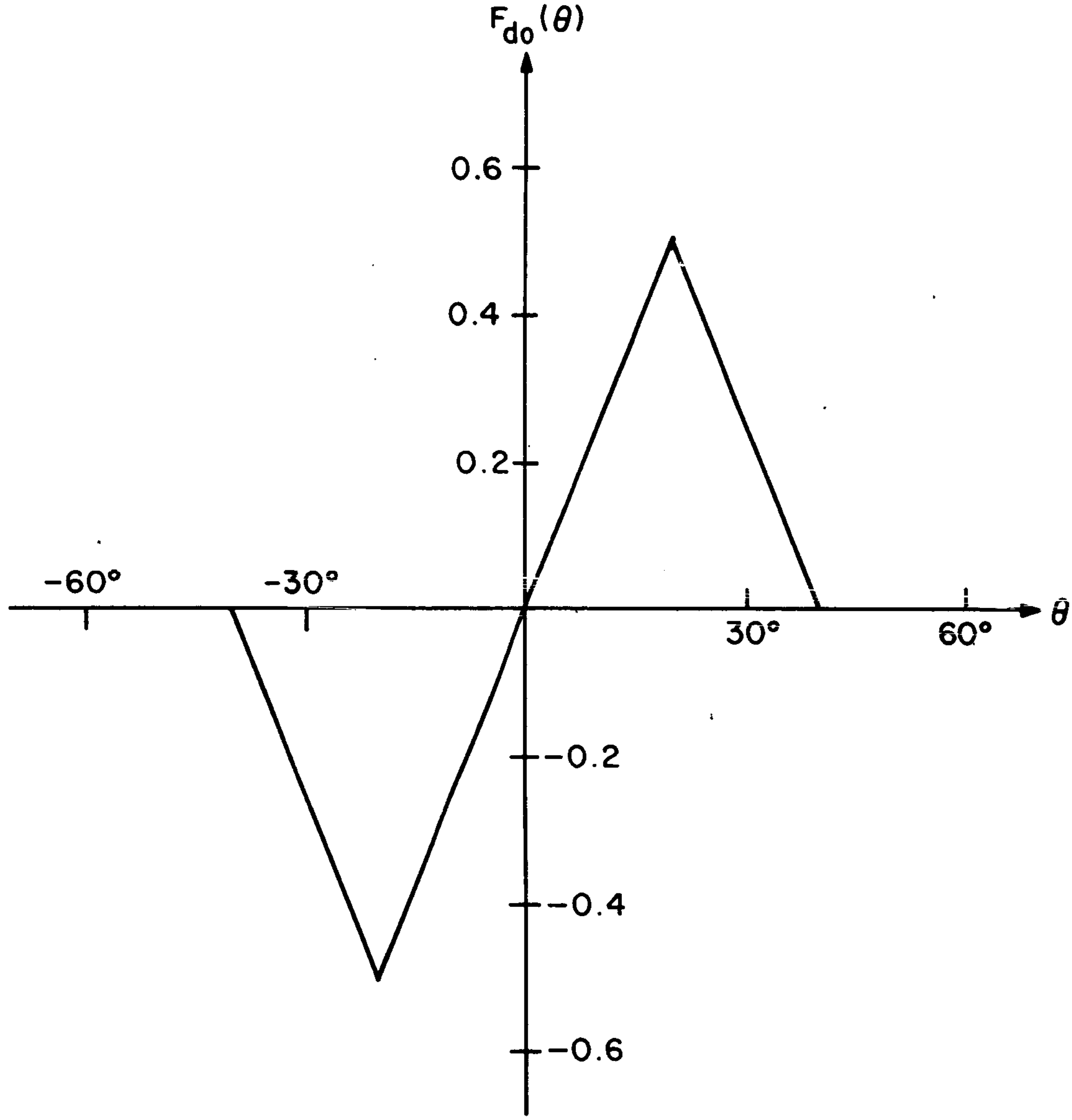

Fig. 8. The odd part of the desired pattern.

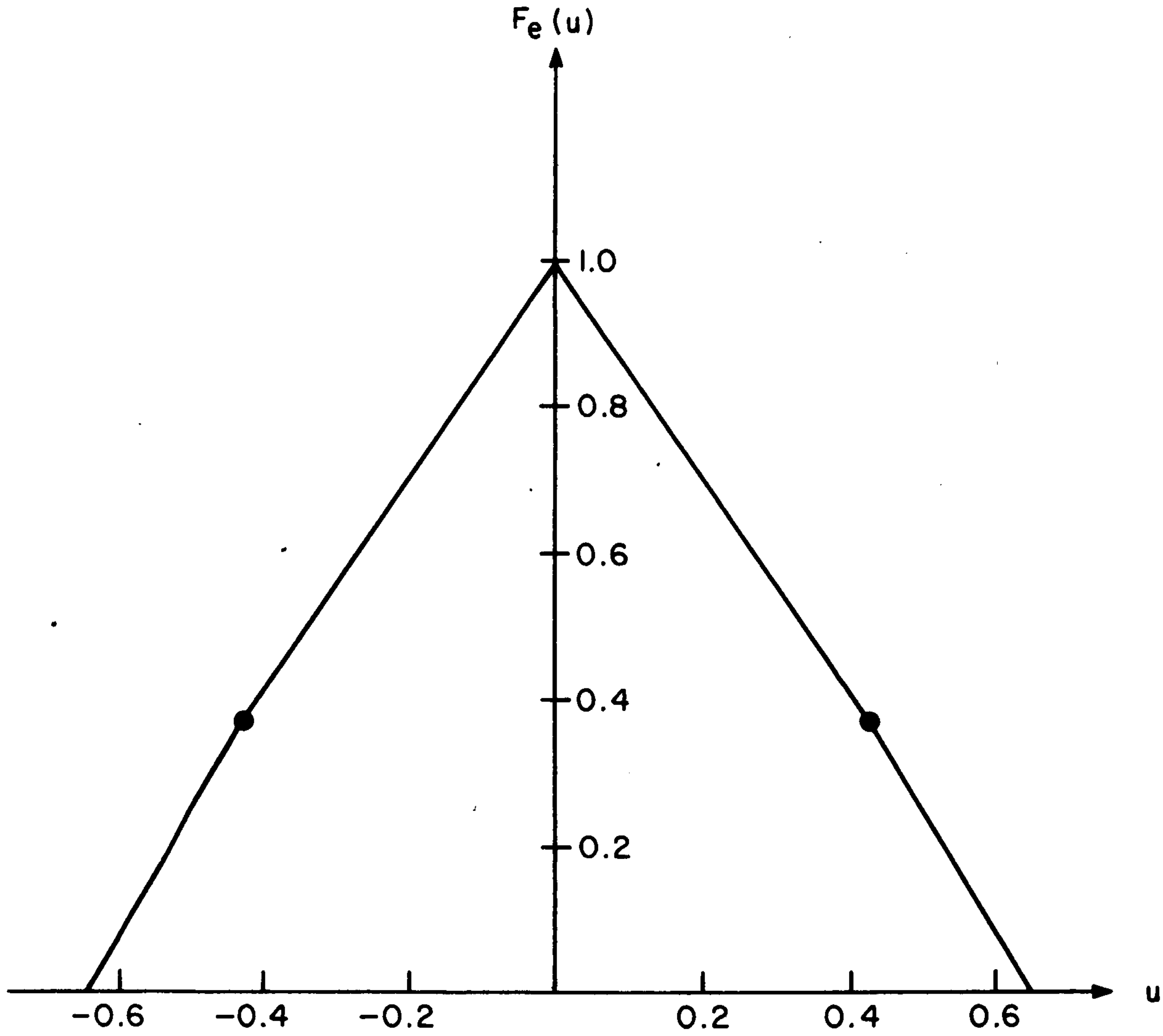

Fig. 9. The function $f_e(u)$.

A good piecewise linear approximation is given by the formula

$$
(51)\qquad f_e(u) = \begin{cases} 0 & : \quad u \leq -0.643 \\ 1.705(u + 0.643): & -0.643 \leq u \leq -0.423 \\ 1.478(u + 0.423): & -0.423 \leq u \leq 0 \\ -1.478(u - 0.423): & 0 \leq u \leq 0.423 \\ -1.705(u - 0.643): & 0.423 \leq u \leq 0.643 \\ 0 & : \quad 0.643 \leq u \end{cases} .
$$

Therefore

$$
(52)\qquad f_e''(u) = 1.705\ \delta(u + 0.643) - 0.227\ \delta(u + 0.423)
$$

$$
- 2.956\ \delta(u) - 0.227\ \delta(u - 0.423) + 1.705\ \delta(u - 0.643);
$$

and, from Eq. (37),

$$
(53)\qquad \alpha_i = \frac{-1}{(k_o x_i)^2} [1.705 \cos(-0.643\ k_o x_i) - 0.227 \cos(-0.423\ k_o x_i)
$$

$$
- 2.956 - 0.227 \cos(0.423\ k_o x_i) + 1.705 \cos(0.643\ k_o x_i)]
$$

$$
= \frac{+1}{(k_o x_i)^2} [2.956 + 0.454 \cos(0.423\ k_o x_i) - 3.41 \cos(0.643\ k_o x_i)] .
$$

Evaluating Eq. (53) by using Eq. (42) gives

$$
(54)\qquad (\alpha) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0.608 \\ 0.458 \\ 0.118 \end{pmatrix}
$$

and then the solution to Eq. (19) is found to be

$$
(55)\qquad \begin{aligned} A_1 &= 0.260 \\ A_2 &= 0.119, \text{ and} \\ A_3 &= 0.081 \end{aligned} .
$$

Equation (8) gives, for the even part of the pattern,

$$F_e(\theta) = 0.520 \cos\left(\frac{\pi}{2}\sin\theta\right) + 0.238\cos(\pi\sin\theta) + 0.162\cos(2\pi\sin\theta), \tag{56}$$

which is shown in Fig. 10.

Next, consider the B_i. From Eqs. (42) and (29), the γ matrix is found to be

$$\gamma = \begin{pmatrix} 2 & 1.7 & -0.68 \\ 1.7 & 2 & 0 \\ -0.68 & 0 & 2 \end{pmatrix}. \tag{57}$$

$f_o(u)$ is easily determined from $F_{do}(\theta)$ (Fig. 8) and is shown in Fig. 11.

A good approximation to $f_o(u)$ is given by

$$f_o(u) = \begin{cases} 0 & : \quad u \leq -0.643 \\ -3.32(u + 0.643) & : \quad -0.643 \leq u \leq -0.342 \\ 2.93\,u & : \quad -0.342 \leq u \leq +0.342 \\ -3.32(u - 0.643) & : \quad 0.342 \leq u \leq 0.643 \\ 0 & : \quad 0.643 \leq u \end{cases} \tag{58}$$

and therefore

$$f_o''(u) = -3.32\,\delta(u + 0.643) + 6.25\,\delta(u + 0.342) - 6.25\,\delta(u - 0.342) + 3.32\,\delta(u - 0.643). \tag{59}$$

Equation (38) then gives

$$\delta_i = \frac{1}{(k_o x_i)^2}\left[6.64\sin(0.643\,k_o x_i) - 12.5\sin(0.342\,k_o x_i)\right], \tag{60}$$

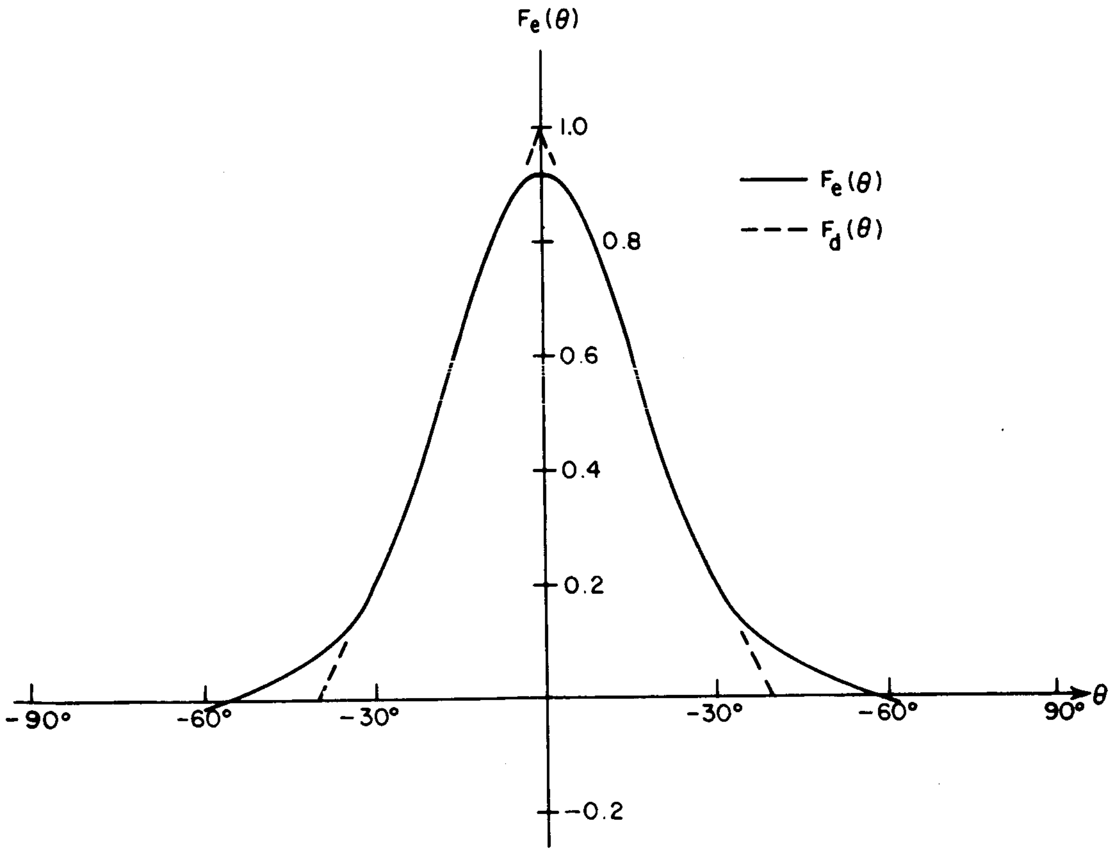

Fig. 10. The even part of the antenna pattern.

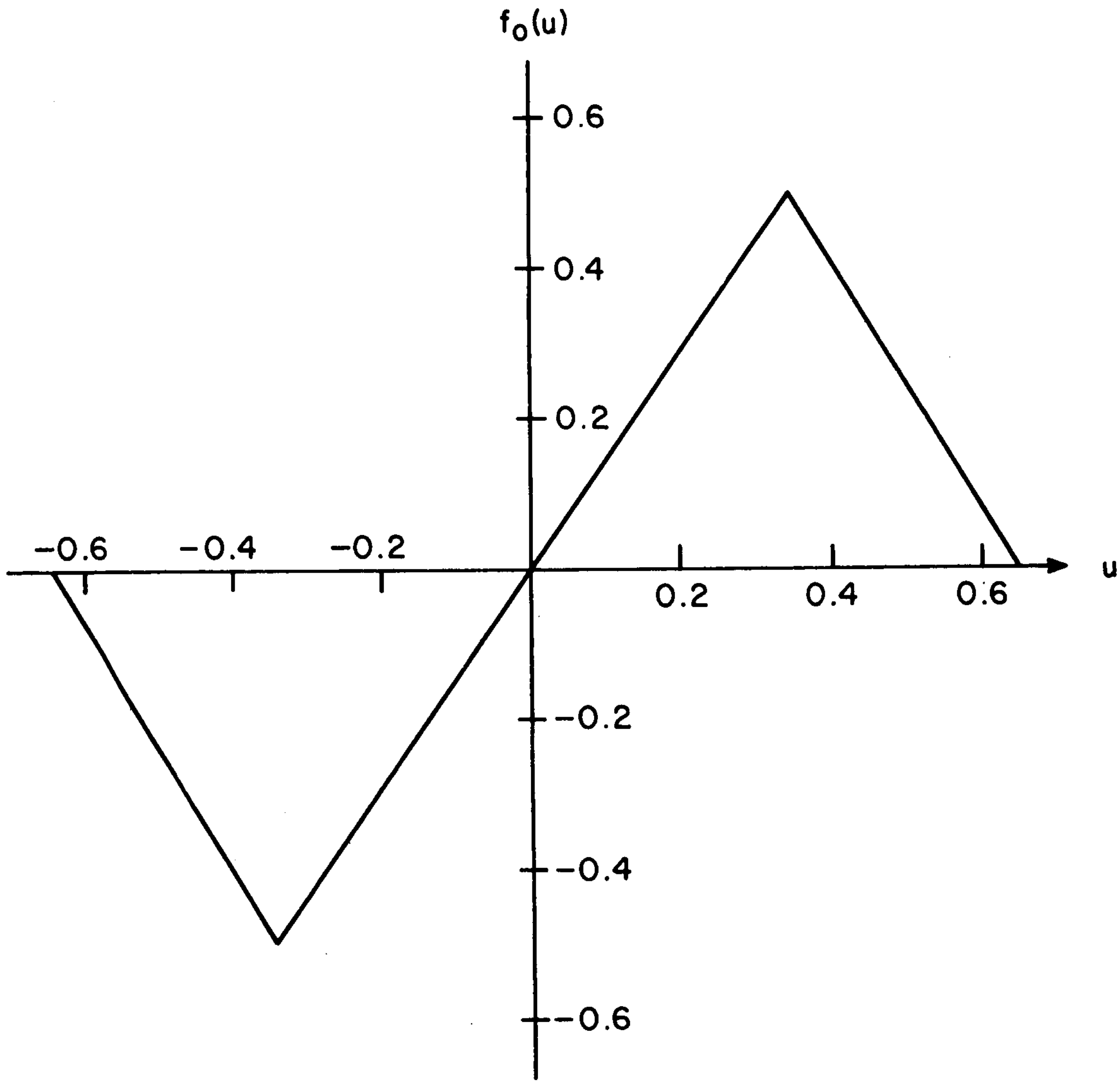

Fig. 11. The function $f_o(u)$.

which gives, with Eq. (42),

$$(61) \qquad (\delta) = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} -0.315 \\ -0.501 \\ -0.400 \end{pmatrix} \quad .$$

Equations (24) then yield for the B_i

$$(62) \qquad \begin{aligned} B_1 &= -0.076, \\ B_2 &= -0.185, \text{ and} \\ B_3 &= -0.226 \quad . \end{aligned}$$

Substituting these in Eq. (10) gives, for the optimum odd pattern,

$$(63) \qquad F_o(\theta) = 0.152 \sin\left(\frac{\pi}{2} \sin\theta\right) + 0.370 \sin(\pi \sin\theta) + 0.452 \sin(2\pi \sin\theta),$$

which is plotted in Fig. 12.

Finally, the total excitation coefficients for the array are given by $I_i = A_i + jB_i$; i.e.,

$$(64) \qquad \begin{aligned} I_1 &= 0.260 - j\,0.076 & I_{-1} &= 0.260 + j\,0.076, \\ I_2 &= 0.119 - j\,0.185 & I_{-2} &= 0.119 + j\,0.185, \text{ and} \\ I_3 &= 0.081 - j\,0.226 & I_{-3} &= 0.081 + j\,0.226 \,, \end{aligned}$$

and the resulting pattern $F(\theta) = F_e(\theta) + F_o(\theta)$ is shown in Fig. 13.

It may be seen that this pattern is not quite as close to the desired pattern as the result was in the first example. Both the even and odd parts are somewhat off near $\theta = \pm 90°$. Although the errors of the even and odd parts tend to cancel near $\theta = +90°$, they add near $\theta = -90°$ to cause a significant total error. However, the weighting function, $g(\theta)$, which was used de-emphasizes errors in the vicinity of $\theta = \pm 90°$, so we might expect the results to be poor in these regions. The choice of a function not going to zero at $\theta = \pm 90°$ should give better results there. Otherwise, the resulting pattern shown in Fig. 13 is a good approximation to the desired curve.

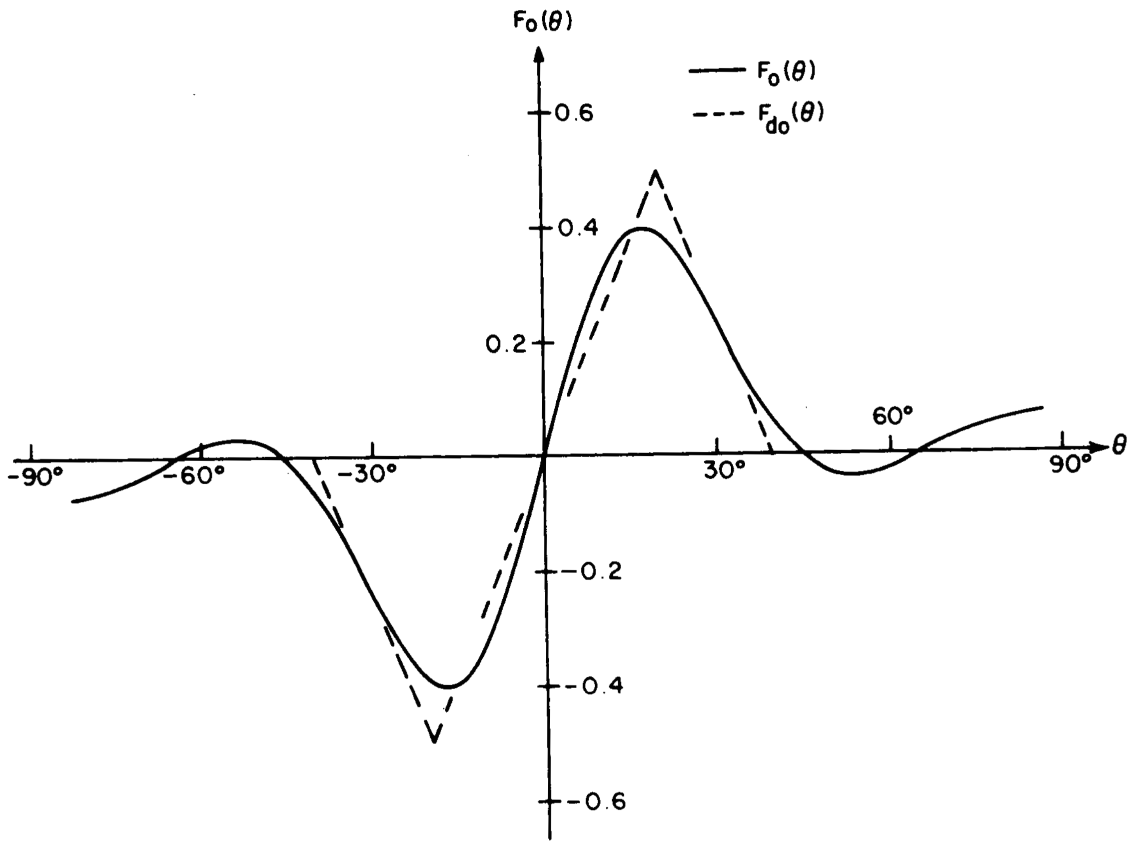

Fig. 12. The odd part of the antenna pattern.

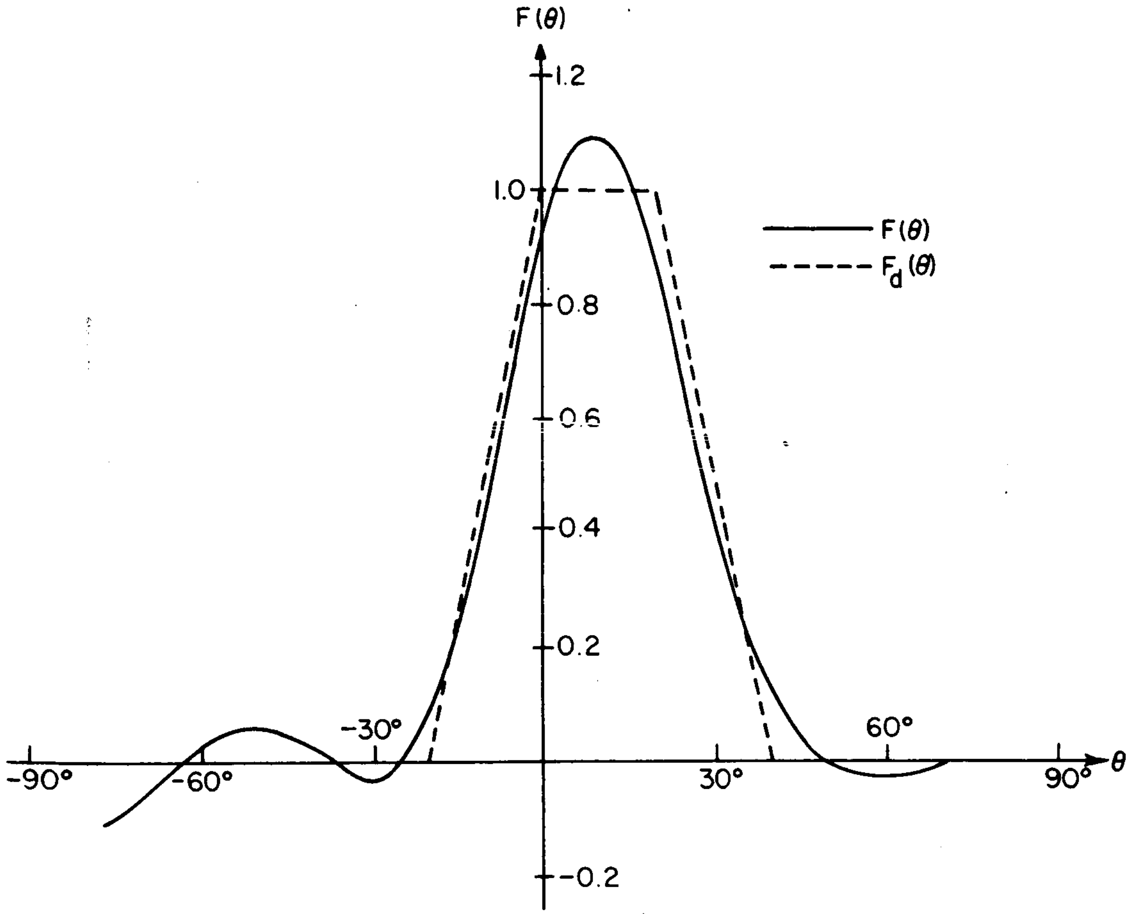

Fig. 13. The total antenna pattern.

IV. CONCLUSIONS

A method of array synthesis based on a least integral square error criterion has been given. It has been shown that the proper choice of symmetry for the element positions and the excitation coefficients causes the antenna pattern to be a real function. Also, with these symmetry conditions, the synthesis problem is split into two parts. The real parts of the amplitude coefficients are found from the even portion of the pattern, and the imaginary part from the odd portion. The coefficients are chosen to give the least integral square error between the antenna pattern and some desired pattern.

For one special choice of integral weighting function, the evaluation of the array coefficients was found to be simple, because the integrals which must be calculated may be done in closed form. Two examples, using a nonuniformly spaced array, have been worked numerically.

Finally, it is noted that the synthesis procedure may be used with arbitrary element spacing. There is no requirement for uniform spacing.

REFERENCE

1. Guillemin, E. A., Synthesis of Passive Networks, John Wiley and Sons, Inc., (1957), p. 663.