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TWO EXPERIMENTS YIELDING LUNAR SURFACE INFORMATION EMPLOYING POLARIZED RADAR WAVES

Donald E. Barrick

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THE OHIO STATE UNIVERSITY
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Investigation of	Theoretical and Experimental Analysis of the Electromagnetic Scattering and Radiative Properties of Terrain, with Emphasis on Lunar-like Surfaces
Subject of Report	Two Experiments Yielding Lunar Surface Information Employing Polarized Radar Waves
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ABSTRACT

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Two experiments are discussed which permit the determination of the local statistical backscattering properties of the lunar surface or the surface of another planet. The first experiment employs the return from a "range ring" of the moon, i.e., the fact that a transmitted radar pulse of the proper length illuminates only a certain well-defined annular portion of the lunar surface at a time. The second experiment makes use of the doppler spreading of a discrete CW incident wave upon the lunar surface which is moving (rotating and translating) in a predetermined manner. In this manner, the power density spectrum of the returned signal at a given frequency near the center of the carrier corresponds directly to a given strip of surface area having a definite velocity component in the line of sight. This strip of surface area is called here a "doppler strip".

The values in obtaining these local statistical backscattering properties is that they can be compared directly to similar properties of a variety of surface samples from the Earth. From this comparison one can learn more about lunar surface composition, roughness, and average dimensions of surface features without having to rely upon the assumption of a certain model or theory of scattering in the formulation of the problem.

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TWO EXPERIMENTS YIELDING LUNAR SURFACE INFORMATION EMPLOYING POLARIZED RADAR WAVES

I. INTRODUCTION

Little is known about the processes which form the lunar surface. Therefore mathematical attempts to determine statistical properties of a section of the local lunar surface from reflected electromagnetic radiation by initially assuming a certain statistical scattering model generally agree poorly with measured data over a wide range of frequencies. Any mathematical model is at best a combination of many assumptions and approximations; the justification for many of these at times seems to lie somewhere between fact and wishful thinking. Therefore it seems justifiable to attempt to compare the ability of the lunar surface to scatter electromagnetic waves of various polarizations with similar measurements made on a variety of surface samples in the laboratory, rather than basing all predictions solely upon the assumption of some dubious scattering theory and model. One of the assumptions made in reducing any scattering theory to useful results is the restriction of the frequency range either to the high (where wavelength is much shorter than lunar surface features) or to the low end (where the converse is true). It is believed that much information about average surface dimensions can be found precisely in the intermediate region.

At present, radar transmission is not directive enough that a well-defined and easily locatable lunar surface area can be separately illuminated. Thus, one difficulty arises in designing an experiment which can determine local lunar statistical surface properties involving the familiar backscattering cross-sections per unit area rather than a lumped cross-section for the entire illuminated lunar hemisphere. Another difficulty lies in obtaining average properties and in determining the type of averaging to be done. In this report, two experiments are discussed, each yielding average values of backscattering cross-sections per unit area of the local lunar surface of various transmitting and receiving polarizations. Also found are average values of combinations of these cross-sections with the phase differences between the elementary scattering matrix elements.[1] All of these average quantities can then be compared with similar, easily obtainable averages made on various surface samples on the Earth. These comparisons can yield dimensional statistical information, such as mean height and slope as well as composition of the lunar surface and its electircal reflective properties.

Assuming the mean lunar surface is spherical, the two experiments described here employ methods of planetary mapping discussed by others in the literature involving return from a "range ring" and return from a "doppler strip". The situation to be examined here is more involved than those previously discussed, because polarization effects are included. Thus a given linear transmitted polarized wave does not strike all points of a range ring or of a doppler strip with the polarized field vector oriented in the same direction. This is not an issue where one ignores polarization, as in the case of acoustical waves or where light waves of all polarizations (random polarization) are present. The inclusion of these polarization effects naturally results in a more complex situation to be analyzed; however, in return, this added complexity yields more statistical surface information.

The first experiment involves the transmission of a well-defined radar pulse of a determined polarization. The energy in this pulse in the far field then propagates in a wall enclosed by two planes of fixed separation. Upon striking the forward lunar hemisphere, this "wall" illuminates a definite annular area, called a "range ring", which moves toward the limbs of the moon as the wall propagates forward. The return from this "range ring" can then be used to obtain certain information about the local surface backscattering properties. The experiment is therefore a study in the time domain. The transmitting and receiving antennas are to be various combinations of linearly and circularly polarizing antennas.

The second experiment involves the study of the power density spectrum of the return from the moon after illumination by a CW carrier. This is possible because the moon has a determinable angular velocity at any given time, which results in a doppler spreading of the transmitted frequency related to the axis orientation and angular velocity of the moon. The surface regions farther away from the axis of rotation shift the incoming frequency by the greatest amount. An incremental surface area which results in a uniform shift in frequency is called a "doppler strip". The return from such a doppler strip at a given frequency can then be used to obtain local surface backscattering properties. This experiment is therefore a study in the frequency domain. Again, various combinations of polarization will be transmitted and received. It will be shown that much more local surface information can be obtained from this second experiment involving a doppler strip than from the first experiment involving a range ring, although the analysis in the second experiment is more involved.

The averaging processes used in obtaining these local statistical properties from each of the above experiments will be briefly discussed here. More than likely, averages made of local samples of surfaces on the Earth will be ensemble averages. Averages made of the moon's surface, however, will employ the fact that the orientation of the lunar surface with respect to the Earth changes with time. Thus for the first experiment, pulses transmitted at different times, over as long a period as several hours or even days, should give rise to markedly different returns, since the surface included in a given range ring changes in orientation over a period of time. When these returned pulses of power are averaged, they represent an ensemble average in time. In the second experiment, if the CW signal is transmitted for an extended period of time, such as several hours, the orientation of the actual lunar surface included in a given doppler strip will change enough so that a time average of the returned signal should provide a meaningful average of local surface properties. Whether one is justified in assuming that the averaging methods discussed above yield the same result as averages made on the Earth on an ensemble of different surfaces will not be defended or explored here. Suffice it to say that since the lunar surface changes its aspect quite slowly and since measurements can be made over long periods of time, the above-mentioned averages should not be markedly different from a surface ensemble average.

The local statistical backscattering cross-sections and scattering matrix phase differences which will be considered here relate local vertically and horizontally polarized radiation. Local vertical polarization is defined to be in the plane of incidence upon a given local surface area, while horizontal is perpendicular to the plane of incidence. These average backscattering cross-sections and scattering matrix phase differences should and will be functions only of the angle of incidence, ν, with respect to the local surface element.

II. BACKSCATTERING CROSS SECTION OF AN ARBITRARY SURFACE ELEMENT

A. Linear Transmitting, Linear Receiving Antennas

Throughout this section, the plane of incidence is taken to be the y-z plane, the vertical polarization direction is defined as the $\hat{\theta}$ direction, and the horizontal polarization direction is defined as the $\hat{\phi}$ (in this case, $\hat{x}$) direction. With the linear polarizing antennas oriented in the directions shown, and employing the scattering matrix[1] relating scattered vertical and horizontal to incident vertical and horizontal (the subscript "1" refers to vertical, "2" to horizontal), the desired incident and back-scattered fields are written, respectively, as follows: (see Figs. 1 and 2):

$$(1) \quad \overline{E}_{\nu}^{i} = k \begin{bmatrix} \cos\nu \\ \sin\nu \end{bmatrix} \; ; \; \therefore \; \overline{E}_{\nu-\zeta}^{s} = \frac{k}{\sqrt{4\pi r^2}} \left[\cos(\nu-\zeta), \; \sin(\nu-\zeta) \right] \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} \cos\nu \\ \sin\nu \end{vmatrix} .$$

The back-scattering cross-section for linear incident polarization in the ν direction and linear scattered polarization in the $\nu-\zeta$ direction from the surface element dA is defined as

$$d\sigma_{\nu-\zeta,\,\nu} = \frac{4\pi r^2 \left| \overline{E}_{\nu-\zeta}^{s} \right|^2}{\left| \overline{E}_{\nu}^{i} \right|^2} .$$

In terms of the incident and scattered fields of Eq. (1), and employing the fact that $a_{12} = a_{21}$ for backscattering, this cross-section becomes,

$$\begin{aligned} d\sigma_{\nu-\zeta,\,\nu} = {} & |a_{11}|^2 \cos^2\nu \cos^2(\nu-\zeta) + 2|a_{11}|\,|a_{12}| \cos(\underline{/a_{11}} - \underline{/a_{12}}) \\ & \cdot \cos\nu \cos(\nu-\zeta) \sin(2\nu-\zeta) \\ & + 2|a_{11}|\,|a_{22}| \cos(\underline{/a_{11}} - \underline{/a_{22}}) \cos\nu \sin\nu \cos(\nu-\zeta) \sin(\nu-\zeta) \\ & + |a_{12}|^2 \sin^2(2\nu-\zeta) \\ & + 2|a_{22}|\,|a_{12}| \cos(\underline{/a_{12}} - \underline{/a_{22}}) \sin\nu \sin(\nu-\zeta) \sin(2\nu-\zeta) \\ & + |a_{22}|^2 \sin^2\nu \sin^2(\nu-\zeta). \end{aligned}$$

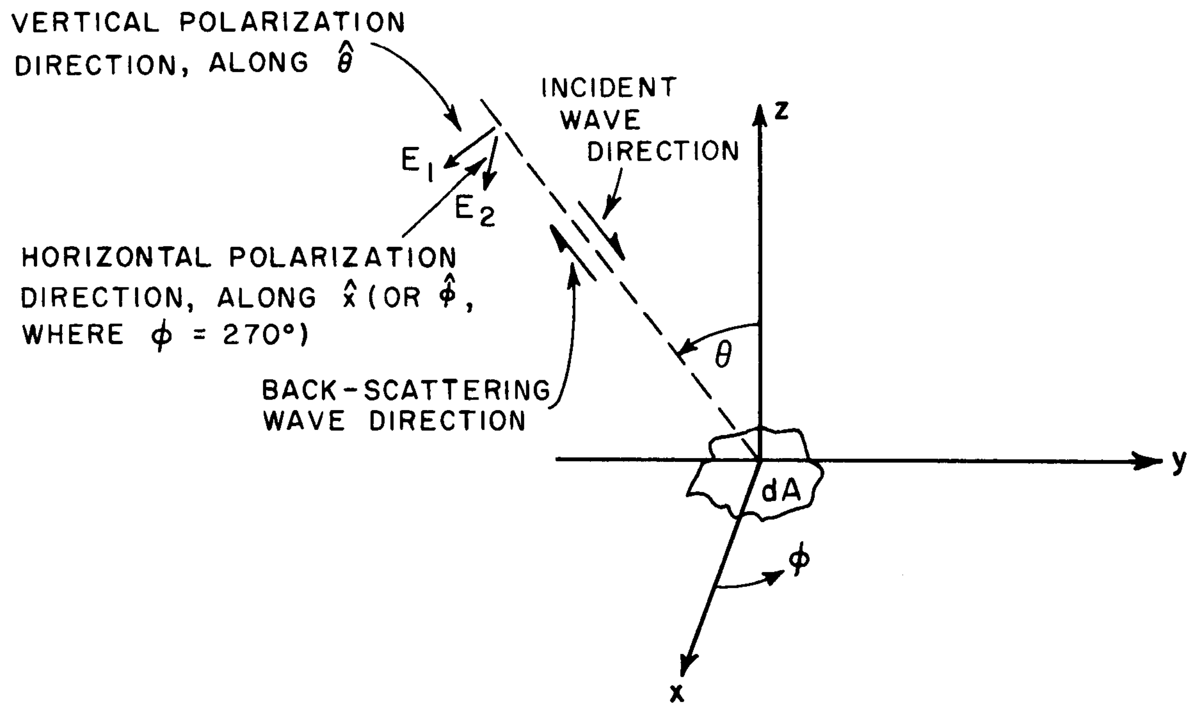

Fig. 1. Back-scattering from a surface element, dA.

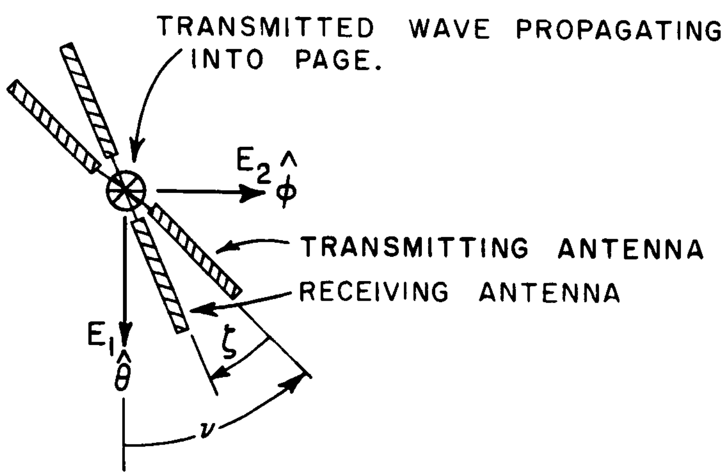

Fig. 2. Linear-to-linear antenna arrangement.

However, the squared magnitudes of the scattering matrix elements are the respective backscattering cross sections. Therefore the above equation becomes,

$$
\begin{aligned}
d\sigma_{\nu-\zeta,\,\nu} = {} & \sigma_{11}\cos^2\nu\,\cos^2(\nu-\zeta) + 2\sigma_{11}^{\frac{1}{2}}\sigma_{12}^{\frac{1}{2}}(\underline{/a_{11}}-\underline{/a_{12}})\cos\nu\,\cos(\nu-\zeta)\,\sin(2\nu-\zeta) \\
& + 2\sigma_{11}^{\frac{1}{2}}\sigma_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}}-\underline{/a_{22}})\cos\nu\,\sin\nu\,\cos(\nu-\zeta)\,\sin(\nu-\zeta) + \sigma_{12}\sin^2(2\nu-\zeta) \\
& + 2\sigma_{22}^{\frac{1}{2}}\sigma_{12}^{\frac{1}{2}}\cos(\underline{/a_{12}}-\underline{/a_{22}})\sin\nu\,\sin(\nu-\zeta)\,\sin(2\nu-\zeta) + \sigma_{22}\sin^2\nu\,\sin^2(\nu-\zeta).
\end{aligned}
$$

The above cross sections are the total scattering cross sections for the area element dA; they have the dimension of area. From them one can define a dimensionless scattering cross section per unit area as follows:

$$\eta_{\alpha\beta} = \frac{d\sigma_{\alpha\beta}}{dA} .$$

Writing the last equation in terms of this dimensionless cross section, one obtains

$$
\begin{aligned}
(2)\quad \eta_{\nu-\zeta,\,\nu} = {} & \eta_{11}\cos^2\nu\,\cos^2(\nu-\zeta) + 2\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{11}}-\underline{/a_{12}})\cos\nu\,\cos(\nu-\zeta)\,\sin(2\nu-\zeta) \\
& + 2\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}}-\underline{/a_{22}})\cos\nu\,\sin\nu\,\cos(\nu-\zeta)\,\sin(\nu-\zeta) + \eta_{12}\sin^2(2\nu-\zeta) \\
& + 2\eta_{22}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{12}}-\underline{/a_{22}})\sin\nu\,\sin(\nu-\zeta)\,\sin(2\nu-\zeta) + \eta_{22}\sin^2(\nu-\zeta).
\end{aligned}
$$

From reciprocity considerations, it is evident that $\eta_{\nu-\zeta,\,\nu} = \eta_{\nu,\,\nu-\zeta}$ for backscattering. (The first subscript always refers to the polarization state of the receiving antenna, while the second refers to the transmitting antenna.)

B. <u>Linear Transmitting, Circular Receiving Antennas</u>

In the case considered, the transmitted incident wave is linearly polarized in a direction which makes an angle ν with the vertical, and the left circularly polarized component present in the scattered wave is to be received. The incident wave and desired scattered wave in this case are given by[1], (see Fig. 3),

$$(3) \qquad \overline{E}^{i}_{\nu} = k \begin{bmatrix} \cos\nu \\ \sin\nu \end{bmatrix} \; ; \qquad \overline{E}^{s}_{L} = k \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{j}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\nu \\ \sin\nu \end{bmatrix} .$$

The backscattering cross section from the surface element dA in this case is defined as

$$d\sigma_{L\nu} = \frac{4\pi r^2 \left|\overline{E}^{s}_{L}\right|^2}{\left|\overline{E}^{i}_{\nu}\right|^2} .$$

Upon substitution of Eq. (3) into the above expression, expanding, and converting to a dimensionless backscattering cross section, one obtains

$$(4) \qquad \begin{aligned} \eta_{L\nu} = \frac{1}{2}\Big[& \eta_{11}\cos^2\nu + 2\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\angle a_{11} - \angle a_{12})\cos\nu\sin\nu \\ & + 2\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\angle a_{12} - \angle a_{22})\cos\nu\sin\nu + 2\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\angle a_{11} - \angle a_{12})\cos^2\nu \\ & + 2\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\angle a_{12} - \angle a_{22})\sin^2\nu + 2\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\sin(\angle a_{11} - \angle a_{22})\cos\nu\sin\nu \\ & + \eta_{12} + \eta_{22}\sin^2\nu \Big] . \end{aligned}$$

From reciprocity considerations, it is true that in this case of backscattering, $\eta_{L\nu} = \eta_{\nu L}$ and $\eta_{R\nu} = \eta_{\nu R}$.

If the right circularly polarized component in the scattered wave is to be received, the "j" in Eq. (3) is preceded by a minus sign and the backscattering cross section is

$$(5) \qquad \begin{aligned} \eta_{R\nu} = \eta_{\nu L} = \frac{1}{2}\Big[& \eta_{11}\cos^2\nu + 2\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\angle a_{11} - \angle a_{12})\cos\nu\sin\nu \\ & + 2\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\angle a_{12} - \angle a_{22})\cos\nu\sin\nu - 2\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\angle a_{11} - \angle a_{12})\cos^2\nu \\ & - 2\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\angle a_{12} - \angle a_{22})\sin^2\nu - 2\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\sin(\angle a_{11} - \angle a_{22})\cos\nu\sin\nu \\ & + \eta_{12} + \eta_{22}\sin^2\nu \Big] . \end{aligned}$$

C. Circular Transmitting, Circular Receiving Antennas

From Ref. 1, the backscattering cross sections sought here can be written immediately in terms of the horizontal and vertical cross sections and matrix elements;

(6) $$\eta_{RL} = \eta_{LR} = \frac{1}{4}\left[\eta_{11} + \eta_{22} + 2\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{22}})\right] ,$$

(7) $$\eta_{LL} = \frac{1}{4}\left[\eta_{11} + \eta_{22} + 4\eta_{12} - 2\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{22}}) + 4\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{11}} - \underline{/a_{12}}) + 4\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{12}} - \underline{/a_{22}})\right] ,$$

and

(8) $$\eta_{RR} = \frac{1}{4}\left[\eta_{11} + \eta_{22} + 4\eta_{12} - 2\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{22}}) - 4\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{11}} - \underline{/a_{12}}) - 4\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{12}} - \underline{/a_{22}})\right] .$$

All of the dimensionless backscattering cross sections considered are functions of frequency and of the actual surface element and its orientation. However, upon making an ensemble average over statistically similar surface elements, the average backscattering cross sections per unit area are functions only of frequency and the angle of incidence, θ, thus, if one makes an ensemble average of $\eta_{\nu-\zeta,\nu}$, for example, and averages term by term, Eq. (2) becomes (the brackets < > denote ensemble average)

(9) $$\begin{aligned}\langle\eta_{\nu-\zeta,\nu}\rangle = {} & \langle\eta_{11}\rangle\cos^2\nu\cos^2(\nu-\zeta) + 2\langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{12}})\rangle\cos\nu\cos(\nu-\zeta)\sin(2\nu-\zeta) \\ & + 2\langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{12}} - \underline{/a_{22}})\rangle\cos\nu\sin\nu\cos(\nu-\zeta)\sin(\nu-\zeta) + \langle\eta_{12}\rangle\sin^2(2\nu-\zeta) \\ & + 2\langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{12}} - \underline{/a_{22}})\rangle\sin\nu\sin(\nu-\zeta)\sin(2\nu-\zeta) + \langle\eta_{22}\rangle\sin^2\nu\sin^2(\nu-\zeta).\end{aligned}$$

Similar results are obtained for the other cross sections. All averages are functions of θ, incidence angle. In all, the following averages appear from the measurements discussed in this section:

(10) $$\langle\eta_{11}\rangle,\ \langle\eta_{12}\rangle,\ \langle\eta_{22}\rangle,\ \langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{22}})\rangle,\ \langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\sin(\underline{/a_{11}} - \underline{/a_{22}})\rangle,$$

(10) cont. $<\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{12}})>$, $<\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{11}} - \underline{/a_{12}})>$,

$<\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{12}} - \underline{/a_{22}})>$, and $<\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{12}} - \underline{/a_{22}})>$.

Since the backscattering cross sections η_{11}, η_{12}, η_{22}, and the various phase differences are all random variables over an ensemble of surfaces, there is no justification, in general, for assuming statistical independence among any of them. Thus averages like $<\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{12}})>$ generally cannot be factored into the individual averages $<\eta_{11}^{\frac{1}{2}}>$ $<\eta_{22}^{\frac{1}{2}}>$ $<\cos(\underline{/a_{11}} - \underline{/a_{12}})>$. It appears therefore as though there are nine averages listed in Eq. (10) above which completely describe the scattering properties of any statistically similar ensemble of surfaces. By knowing all of these averages as a function of the incident frequency and angle of incidence much should be revealed about the nature of a surface, and it is indeed these quantities which are sought from the experiments described in the next two sections.

The statistical quantities listed in Eq. (10) can all be easily measured in the laboratory for ensembles of sample surfaces having various statistical, dimensional, and electrical properties and at various angles of incidence. This can be done by arranging the sample surfaces on a mean planar bed and using Eqs. (2), (4), (5), (6), (7), (8), along with the techniques and tables given in Ref. 2. The angles ν and ζ can, of course, be chosen judiciously so that the desired quantities fall out very simply. Once these ensemble averages in Eq. (10) have been determined for a [illegible] be compared with similar quantities from the moon and other planets in order to learn more about the surfaces of these bodies.

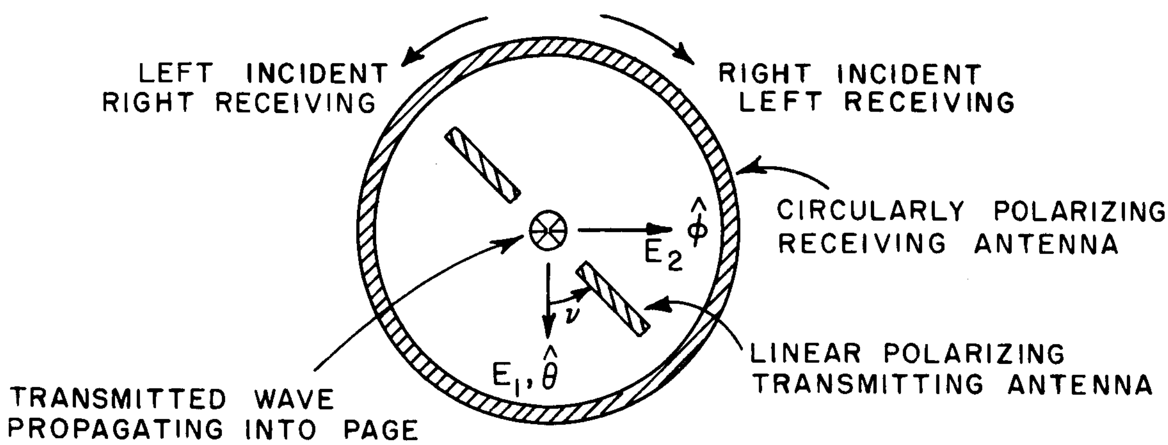

Fig. 3. Circular-to-linear antenna arrangement.

In the next two sections, the formulas derived in this section for the various backscattering cross sections will be applied to the lunar surface. In such a case the element of area, dA, will be a surface element on a sphere, and the angle of incidence will vary for different area elements on the surface.

III. RETURN FROM A RANGE RING (See Fig. 4)

A radar pulse of sufficiently small width upon reaching the moon will illuminate an annular ring, or range ring, on the forward hemisphere of the moon's surface. Thus if the return is studied as a function of time, it will correspond to the scattering properties of a progressively changing range ring, starting with the forward-most point and proceeding toward the ring of maximum diameter. The angle of incidence, α, therefore is a constant at all points on a given range ring, and increases with time as the range ring illuminated moves progressively to the rear with the incoming pulse. The resulting power received should then be a measure of the back-scattering cross section per range ring, and can be converted from a function of time to a function of the angle of incidence, α (see Appendix A for this derivation).

If $\eta_{\sigma\tau}$ denotes the backscattering cross section per unit area from the moon's surface, as discussed in the previous section (the σ and τ refer to the received and transmitted polarization states, respectively, under consideration), then the backscattering cross section for an area element, dA, on a given range ring is given by $d\sigma_{\sigma\tau} = \eta_{\sigma\tau} dA$, where dA is the area of the element of actual reflecting surface of the moon.

The average backscattering cross section from an ensemble of area elements, dA, in the same position on the same range ring but with different rough surface samples present in the area element, is therefore a function only of the angle of incidence, α, and the average area element; i.e., the area of an element on a perfect sphere of radius R. Denote this cross section by $\langle d\sigma_{\sigma\tau}\rangle_\alpha$:

$$(11) \qquad \langle d\sigma_{\sigma\tau}\rangle_\alpha = \langle\eta_{\sigma\tau}\rangle_\alpha \, dA_{SPHERE} = \langle\eta_{\sigma\tau}\rangle_\alpha \, R^2 \sin\alpha \, d\alpha \, d\nu .$$

The subscript α on the ensemble average bracket indicates that the averages are functions of angle of incidence, which is to be denoted by α here. This subscript may be omitted at various places for brevity. The average backscattering cross section for the entire range ring is therefore found by integrating Eq. (11) over the entire range ring, and is denoted by $\langle d\sigma_{\sigma T}{}^{RR}\rangle_\alpha$:

$$(12) \qquad \langle d\sigma_{\sigma T}{}^{RR}\rangle_\alpha = R^2 \sin\alpha \, d\alpha \int_0^{2\pi} \langle \eta_{\sigma T}\rangle_\alpha \, d\nu .$$

Again it is assumed that the quantity on the left side of Eq. (12) is measurable from the described experiment as a function of α (or can be computed from measurable quantities with Appendix A).

From Eq. (12) define an average backscattering cross section per unit area for a range ring by dividing Eq. (12) by the area of a range ring on a sphere; i.e., $dA^{RR} = 2\pi R^2 \sin\alpha \, d\alpha$,

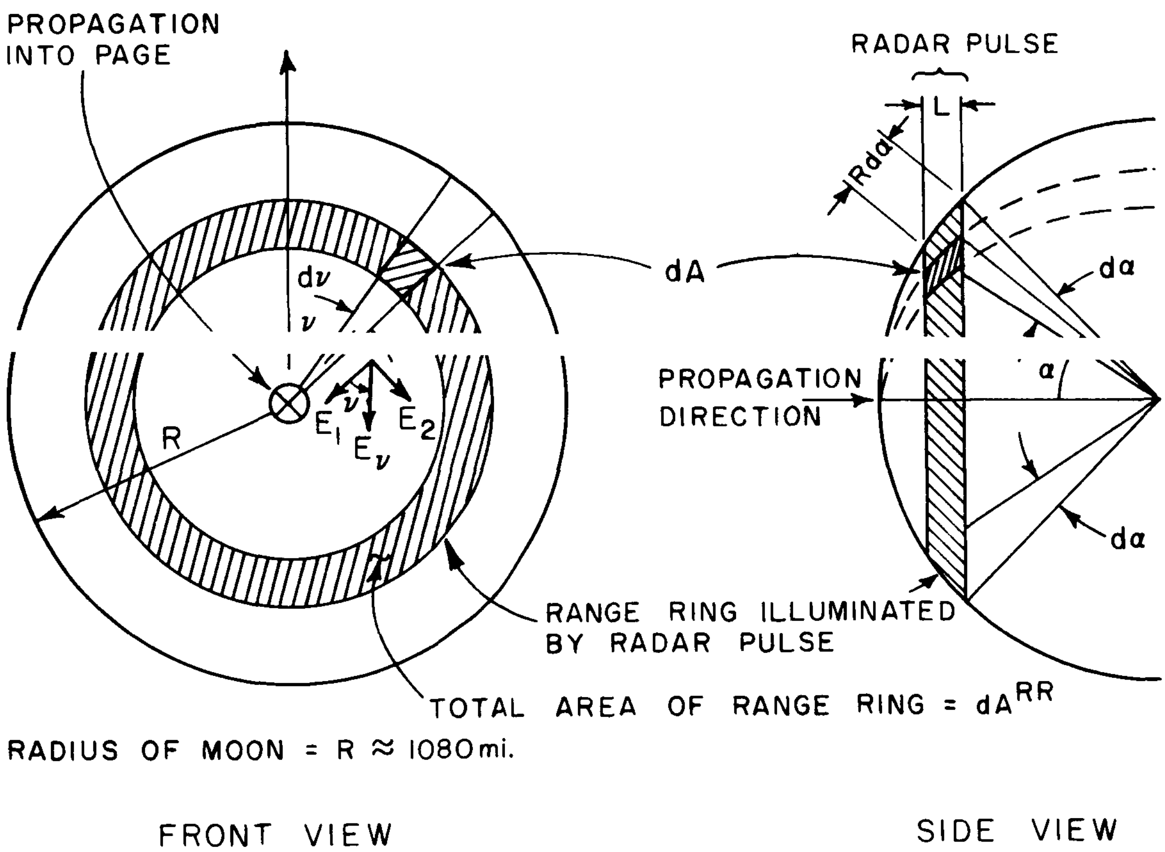

Fig. 4. Diagram showing radar pulse and illuminated range ring.

$$<\eta_{\sigma\tau}^{RR}>_\alpha = \frac{<d\sigma_{\sigma\tau}^{RR}>_\alpha}{dA^{RR}} \quad .$$

This results in

(13) $$<\eta_{\sigma\tau}^{RR}>_\alpha = \frac{1}{2\pi}\int_0^{2\pi} <\eta_{\sigma\tau}>_\alpha \, d\nu \quad .$$

A. <u>Linear Transmitting, Linear Receiving Antennas</u>

In this case, it is assumed that a linearly polarized wave, E_ν, (see Fig. 4) is transmitted. At the area element shown (dA) on a given range ring, the electric vector makes an angle ν with the direction of vertical polarization for that particular area element. The direction of vertical polarization varies with the angle, ν, of the particular area element, dA, on the selected range ring and is always directed radially inward in planes where ν = constant. If the linear polarization to be received is in the direction $\nu - \zeta$, the average backscattering cross section per unit area which is applicable on the entire range ring is found by taking the average of Eq. (2), as given in Eq. (9), as the element to be used for $<\eta_{\sigma\tau}>_\alpha$; thus for $<\eta_{\sigma\tau}>_\alpha = <\eta_{\nu-\zeta,\nu}>_\alpha$, Eq. (13) integrates to

(14) $$<\eta^{RR}_{\nu-\ ,\nu}>_\alpha = \frac{1}{4}[<\eta_{11}>_\alpha \times (1 + \frac{1}{2}\cos 2\zeta)$$

$$+ <\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\angle a_{11} - \angle a_{22})>_\alpha \cos 2\zeta + 2<\eta_{12}>_\alpha + <\eta_{22}>_\alpha \times (1+\frac{1}{2}\cos 2\zeta)] \, .$$

Using particular angular antenna separations, ζ, the above relationship can be simplified for three special cases; i.e.,

(14a) $$<\eta^{RR}_{\nu,\nu}>_\alpha = \frac{1}{8}[\, 3<\eta_{11}>_\alpha + 4<\eta_{12}>_\alpha + 3<\eta_{22}>_\alpha$$

$$+ 2<\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\angle a_{11} - \angle a_{22})>_\alpha] \, , \text{ for } \zeta = 0;$$

(14b) $$<\eta^{RR}_{\nu-\pi/2,\,\nu}>_\alpha = \frac{1}{8}[<\eta_{11}>_\alpha + 4<\eta_{12}>_\alpha + <\eta_{22}>_\alpha - 2<\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{22}})>_\alpha], \quad \text{for } \zeta = \pm\pi/2;$$

and

(14c) $$<\eta^{RR}_{\nu-\pi/4,\,\nu}>_\alpha = \frac{1}{8}[2<\eta_{11}>_\alpha + 4<\eta_{12}>_\alpha + 2<\eta_{22}>_\alpha], \quad \text{for } \zeta = \pm\pi/4.$$

Notice in the above formulas that certain quantities cannot be determined separately from such a measurement; from Eq. (14) it is seen that $<\eta_{11}>_\alpha$ cannot be separated from $<\eta_{22}>_\alpha$ no matter how many measurements are made at various angles, ζ. This separability will be examined later.

B. Linear Transmitting, Circular Receiving Antennas

In this case, the incident wave is linearly polarized in the direction E_ν, as shown in Fig. 4, and the circularly polarized components in the backscattered wave are desired. The average backscattering cross section per unit area to be employed here is found by taking the average of Eqs. (4) or (5) and substituting the result into Eq. (13) for $<\eta_{\sigma\tau}>_\alpha$, i.e., $<\eta_{\sigma\tau}>_\alpha = <\eta_{{L \atop R}\nu}>_\alpha$. Upon integration Eq. (13) becomes

(15a, b) $$<\eta^{RR}_{{L \atop R}\nu}>_\alpha = \frac{1}{4}[<\eta_{11}>_\alpha + 2<\eta_{12}>_\alpha + <\eta_{22}>_\alpha \pm 2<\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{11}} - \underline{/a_{12}})>_\alpha \pm 2<\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{12}} - \underline{/a_{22}})>_\alpha].$$

The upper sign applies when the left circularly polarized component is received, i.e., when the "L" in the subscript of the left side of Eq. (15) is employed. The lower sign applies to the right circular component. Therefore, Eq. (15) is two equations representing two different measurements.

From reciprocity considerations discussed in the previous sections, identical results can be obtained by transmitting circular and receiving linear at an angle ν. In this case, $<\eta^{RR}_{{L \atop R}\nu}>_\alpha = <\eta^{RR}_{\nu{R \atop L}}>_\alpha$.

In this subsection and in subsection A also, it is interesting to note that averages may be made from measurements in two different ways. One way was mentioned previously: this is to transmit pulses at different times so that the orientation of the moon's surface will have shifted slightly between pulse transmissions. An ensemble average can be made in this manner by averaging the returned powers at the various times, assuming a sufficiently long averaging period. However, even if the surface were perfectly stationary, an average could be made by rotating the receiving and transmitting antennas together between transmissions. This causes a different combination of linear states to be incident upon the same area element for successive transmissions. The same result would be accomplished by keeping the antennas fixed but rotating the moon about the axis along the direction of propagation between transmissions; this results in an ensemble average for a variety of surface orientations. This process is possible for the above two subsections because there are an infinite number of independent linear polarization states and because the surface is rotationally symmetrical in a mean sense. This type of averaging is not possible in the following subsection involving strictly circular states because only two independent circular states are possible.

C. Circular Transmitting, Circular Receiving Antennas

In this case, the average backscattering cross section per unit area to be employed is found by taking the average of Eqs. (6), (7), and (8) and substituting into Eq. (13) for $\langle\eta_{\sigma\tau}\rangle_\alpha$; e.g., $\langle\eta_{\sigma\tau}\rangle_\alpha = \langle\eta_{RR}\rangle_\alpha$, etc. Notice that the various cross sections given in Eqs. (6), (7), and (8) are independent of ν, the angular position on a given range ring, contrary to the situation in the former two subsections. Therefore

(16a) $$\langle\eta_{RL}^{RR}\rangle_\alpha = \langle\eta_{LR}^{RR}\rangle_\alpha = \frac{1}{4}[\langle\eta_{11}\rangle_\alpha + 2\langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{22}})\rangle_\alpha + \langle\eta_{22}\rangle_\alpha]$$

and

(16b, c) $$\langle\eta_{\substack{LL\\RR}}^{RR}\rangle_\alpha = \frac{1}{4}[\langle\eta_{11}\rangle_\alpha + 4\langle\eta_{12}\rangle_\alpha + \langle\eta_{22}\rangle_\alpha - 2\langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{22}})\rangle_\alpha \pm 4\langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{11}} - \underline{/a_{12}})\rangle_\alpha \pm 4\langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{12}} - \underline{/a_{22}})\rangle_\alpha].$$

These three equations add no information which is not already present in Eqs. (15) and (14), but in certain cases these measurements may be simpler to make.

The statement may also be made that essentially no further information can be obtained from employing any other general elliptical polarization states, because all other states are a combination of linear and circular states.

D. Independent Information Obtainable from these Measurements

From the measurements discussed in the above three subsections employing range ring scattering, the following appear to be a summary of the total amount of separate information obtainable from various algebraic combinations of Eqs. (14), (15), and (16):

(17a) $\langle\eta_{11}\rangle_\alpha + 2\langle\eta_{12}\rangle_\alpha + \langle\eta_{22}\rangle_\alpha$,

(17b) $\langle\eta_{11}\rangle_\alpha + 2\langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{22}})\rangle_\alpha + \langle\eta_{22}\rangle_\alpha$,

(17c) $\langle\eta_{12}\rangle_\alpha - \langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{22}})\rangle_\alpha$,

and

(17d) $\langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{11}} - \underline{/a_{12}})\rangle_\alpha + \langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{12}} - \underline{/a_{22}})\rangle_\alpha$.

Even these four quantities are not all independent, since Eq. (17c) can be obtained from Eqs. (17a) and (17b). Just three independent polarization parameters can be determined from all possible range ring experiments. These do not seem like a significant amount of information from the measurements and equations developed in this section; however, they are better than no information at all. The actual value of this information when compared to similar information measured from sample surfaces on the Earth at various incidence angles seems worthy of serious experimental investigation.

IV. RETURN FROM A DOPPLER STRIP (See Fig. 5)

It has been shown by Compton[3] and others that the component of velocity of the moon's surface in the "x" direction resulting from rotation of the moon for a given doppler strip located at "z" is given by

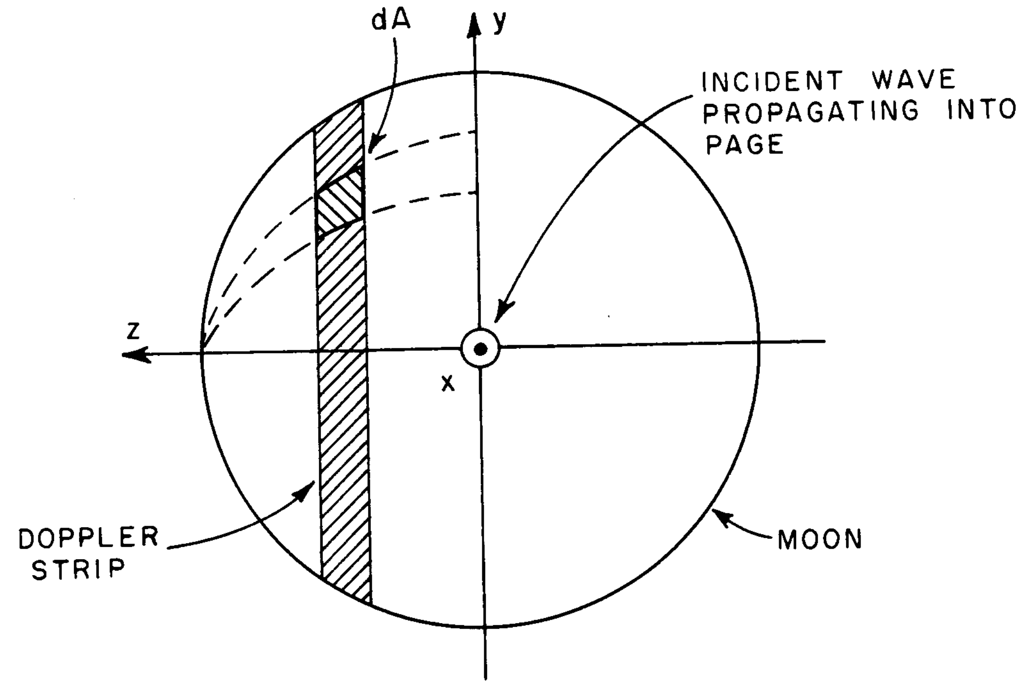

Fig. 5. Diagram showing doppler strip.

$$v_x = \Omega_y z$$

and

$$v_x = \Omega \; z \cos \xi ,$$

where ξ is the angle between the "y" axis and the moon's angular velocity vector, $\overline{\Omega}$, having magnitude Ω. The coordinate system is oriented so that $\hat{x}$ points in the direction of the reflected wave (toward the Earth) and $\overline{\Omega}$ is located in the x-y plane. Thus the component of angular velocity along the y axis, $\Omega_y = \Omega \cos \xi$, causes all points of given "x" coordinate (on the same doppler strip) to have the same doppler shift since they all have the same "x" component of velocity.

Therefore, the shift in frequency of radiation returning from a given doppler strip at z is

$$\Delta\omega = \frac{2\omega_o}{c} \, v_z = \frac{2\omega_o}{c} \Omega \quad \cos \xi .$$

The frequency of the returning wave is $\omega_R = \omega_o + Kz$; $K = \frac{2\omega_o}{c} \Omega \cos \xi$; ω_o = carrier frequency (frequency of the returning signal reflected from the portion at $z = 0$).

In polar coordinates, where $z = R \cos \theta$, $x = R \cos \phi \sin \theta$, and $y = R \sin \phi \sin \theta$ on the surface of the moon (assumed spherical in the mean), this returned frequency is a function of θ; $\omega_R = \omega_o + KR \cos \theta$.

The returning time average power, $\overline{S}_r$, can be expressed in terms of the one-sided power density spectrum as

$$\overline{S}_r = \frac{1}{2\pi} \int_0^\infty P(\omega_r) d\omega_r \quad .$$

If there is no noise present in the returning signal,* then $P(\omega_r)$ is zero outside a certain finite band resulting from the doppler shift of the carrier. This integral can therefore be expressed in terms of z or θ:

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* If the returning signal contains noise, this noise component must be subtracted out of the power density spectrum; otherwise the scattering cross section near the limbs (at $z \rightarrow \pm R$) will appear to become infinite, as pointed out by Compton in Ref. 3. This noise spectrum is flat over the narrow doppler spread of the returning signal and may be subtracted out with no difficulty.

$$(18) \qquad \overline{S}_{r_{\sigma,\tau}} = \frac{1}{2\pi}\int_{\omega_o - KR}^{\omega_o + KR} P_{\sigma\tau}(\omega_r)d\omega_r = \frac{K}{2\pi}\int_{-R}^{R} P_{\sigma\tau}(\omega_o + Kz)dz$$

$$= \frac{KR}{2\pi}\int_0^{\pi} P_{\sigma\tau}(\omega_o + KR\cos\theta)\sin\theta d\theta .$$

It is assumed that the moon is illuminated by a CW signal of a discrete frequency. If it were not for the rotation of the moon (i.e., for translation only), then the returning signal would consist of a discrete frequency also, ω_o. It is assumed that the power density spectrum, $P_{\sigma\tau}(\omega_r)$, of the returning signal of a given polarization can be measured. This quantity will be used to obtain information about average polarization scattering properties of the surface. In Eq. (18), the σ and τ therefore stand for the polarization properties of the receiving and transmitting antennas used to make these measurements. The polarization states to be considered in this section are linear and circular and combinations of the two. The time average, $\overline{S}_{r_{\sigma,\tau}}$, of the returning signal power as the lunar surface rotates slowly over a long period of time is assumed in this report to be equivalent to an ensemble average over surfaces statistically similar to the lunar surface. The power density spectrum, $P_{\sigma\tau}(\omega_r)$, at a particular returning frequency, $\omega_o + KR\cos\theta$, can be integrated with respect to time in order to obtain an average power density spectrum at that particular frequency and value of θ. From this point, therefore, $P_{\sigma\tau}(\omega_r)$ is assumed to be an average value over time, equivalent to an ensemble average.

This average returned power, $\overline{S}_{r_{\sigma\tau}} = \langle S_{r_{\sigma\tau}} \rangle$,* may also be written in terms of the backscattering cross section as discussed in the preceding sections.

$$(19) \qquad \langle S_{r_{\sigma\tau}} \rangle = S_{inc_\tau} \times \frac{\langle \sigma_{\sigma\tau} \rangle}{4\pi R^2} = S_{inc_\tau} \times \frac{1}{4\pi R^2}\iint_{\text{Illuminated Surface}} \langle \eta_{\sigma\tau} \rangle_\alpha \, dA,$$

where α is the angle of incidence at a particular area element dA on the sphere. S_{inc_τ} is the power in the incident signal in the polarization state τ.

* The brackets $\langle S \rangle$ are used to mean ensemble average, while the bar $\overline{S}$ is taken to mean time average.

Since the mean lunar surface is a sphere, the area element dA for this mean surface is $dA = R^2 \sin\theta d\theta d\varphi$. Therefore, equating Eq. (18) to Eq. (19), one obtains

$$\frac{KR}{2\pi}\int_0^{\pi} P_{\sigma\tau}(\omega_0 + KR\cos\theta)\sin\theta d\theta$$
$$= S_{inc_\tau} * \frac{R^2}{4\pi R^2}\int_0^{\pi}\left[\int_{-\pi/2}^{\pi/2} \langle\eta_{\sigma\tau}\rangle_\alpha d\varphi\right]\sin\theta d\theta \ .$$

Dropping the integration over θ on both sides, this expression gives

(20) $$\int_{-\pi/2}^{\pi/2} \langle\eta_{\sigma\tau}\rangle\alpha \, d\phi = \frac{2KR}{S_{inc_\tau}} P_{\sigma\tau}(\omega_0 + KR\cos\theta) \equiv F_{\sigma\tau}(\theta) \ .$$

For brevity $F_{\sigma\tau}(\theta)$ is defined here as the doppler function and is easily determinable from the average power density spectrum as a function of θ. Thus from this measurable doppler function for a given set of transmitted and received polarization states, the goal is to determine the various averages in Eq. (10) from the left side of Eq. (20).

First it is necessary to convert Eq. (20) from an integral over ϕ to an integral over α, the angle of incidence at an area element dA on a given doppler strip at θ. The mean plane of incidence at dA is the plane containing $\hat{x}$ and the normal to the surface at dA, i.e., $\hat{r}$.

(21) $$\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta.$$

The angle of incidence, α, is therefore defined by

(22) $$\cos\alpha = \hat{x}\cdot\hat{r} = \sin\theta\cos\phi, \quad \therefore \ \cos\varphi = \frac{\cos\alpha}{\sin\theta}$$

and $$\therefore \ d(\cos\alpha) = -\sin\alpha d\alpha = -\sin\theta\sin\phi\, d\varphi \ ;$$

$$d\phi = \frac{\sin\alpha d\alpha}{\sin\theta\sin\phi} \ ; \text{ but } \sin\phi = \pm\sqrt{1-\cos^2\phi} = \pm\sqrt{1 - \frac{\cos^2\alpha}{\sin^2\theta}} \ .$$

The upper sign applies in the upper half of the doppler strip, where φ is positive, and the lower sign applies in the lower half. The limits on the left side of Eq. (20) over α are determined by breaking Eq. (20) up into two integrals,

$$\int_{-\pi/2}^{\pi/2} \langle\eta_{\sigma\tau}\rangle_\alpha d\phi = \int_0^{\pi/2} \langle\eta_{\sigma\tau}\rangle_\alpha^u \, d\phi + \int_{-\pi/2}^{0} \langle\eta_{\sigma\tau}\rangle_\alpha^\ell \, d\phi \, .$$

Here the superscript u signifies the function over the upper half only and the ℓ signifies the function over the lower half only. The limits in ϕ may now be replaced by the limits in α;

$$\therefore \int_{-\pi/2}^{\pi/2} \langle\eta_{\sigma\tau}\rangle_\alpha \, d\phi = \int_{-\pi/2-\theta}^{\pi/2} \langle\eta_{\sigma\tau}\rangle_\alpha^u \frac{\sin\alpha \, d\alpha}{\sqrt{\sin^2\theta - \cos^2\alpha}}$$

$$- \int_{\pi/2}^{\pi/2-\theta} \langle\eta_{\sigma\tau}\rangle_\alpha^\ell \frac{\sin\alpha \, d\alpha}{\sqrt{\sin^2\theta - \cos^2\alpha}} \, .$$

This process is necessary because α, angle of incidence, is positive over the entire doppler strip. By interchanging limits on the second integral and noting that

$$\sin^2\theta - \cos^2\alpha = \sin^2\alpha - \cos^2\theta \quad ,$$

the above expression can be rewritten as

$$F_{\sigma\tau}(\theta) = \int_{\pi/2-\theta}^{\pi/2} \left[\langle\eta_{\sigma\tau}\rangle_\alpha^u + \langle\eta_{\sigma\tau}\rangle_\alpha^\ell\right] \frac{\sin\alpha \, d\alpha}{\sqrt{\sin^2\alpha - \cos^2\theta}} \, . \tag{23}$$

Equation (23) is a form of the familiar Abel integral equation; the functions $\langle\eta_{\sigma\tau}\rangle_\alpha^u$ and $\langle\eta_{\sigma\tau}\rangle_\alpha^\ell$ and both functions of α in general, and must be evaluated for the particular polarization states being considered.

A. Linear Transmitting, Linear Receiving Antennas

Here again, as in the preceding section, the applicable average backscattering cross section per unit area to be employed in Eq. (23) is found by taking an ensemble average of Eq. (2), resulting in Eq. (9); thus

$<\eta_{\sigma\tau}>_\alpha = <\eta_{\nu-\zeta,\nu}>_\alpha$. However, the angle ν must be determined in this case in terms of α and θ for a given area element dA. The angle ν, as defined, is the angle between the plane of incidence and the direction of polarization of the incident wave; it is positive in a counter-clockwise sense, as shown in Figs. 1 and 2. However, in this case the plane of incidence rotates for each area element along a given doppler strip; it rotates with the projection of the radial vector $\hat{r}$, normal to dA, on the y-z plane. This projection in the y-z plane, looking in the direction of the -x axis, i.e., the view of Fig. 5, is given as $\bar{r}' = \hat{y} \sin\theta \sin\phi + \hat{z} \cos\theta$. Assume that the direction of linear incident polarization makes an angle, γ, with $\hat{z}$, positive clockwise. Then the angle ν for a given dA is the difference between them, i.e.,

$$\nu = \underline{/\bar{r}', \hat{z}}\, - \gamma .$$

But the angle between $\bar{r}'$ and $\hat{z}$ is found from their dot product:

$$\cos \underline{/\bar{r}', z} = \frac{\bar{r}' \cdot \hat{z}}{|\bar{r}'| \cdot |\hat{z}|} = \frac{\cos\theta}{\sqrt{\sin^2\theta \sin^2\phi + \cos^2\theta}} .$$

However, $\sin^2\phi = 1 - \cos^2\phi = 1 - \dfrac{\cos^2\alpha}{\sin^2\theta}$, from Eq. (22);

$$\therefore \sqrt{\sin^2\theta \sin^2\phi + \cos^2\theta} = \sqrt{\sin^2\theta - \cos^2\alpha + \cos^2\theta} = \sqrt{1 - \cos^2\alpha} = \sin\alpha .$$

Therefore, the angle ν may be expressed as

(24a) $$\nu = \pm \cos^{-1}\left[\frac{\cos\theta}{\sin\alpha}\right] - \gamma .$$

The choice of signs in Eq. (24a) exists because an angle with a given cosine can be either positive or negative. The upper sign is used therefore for the upper half of the doppler strip.

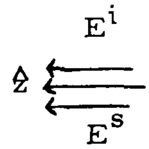
The angle ζ in Eq. (9) is defined here to be the angle between the direction of incident polarization, γ, and the direction of the desired linearly polarized component in the scattered field; as in Fig. 2, it is positive when the scattered linear direction is counter clockwise from the incident linear direction. The angle ν-ζ can then be expressed as

(24b) $$\nu - \zeta = \pm \cos^{-1}\left[\frac{\cos\theta}{\sin\alpha}\right] - \gamma + \zeta .$$

At this point, one can substitute Eqs. (24a) and (24b) into Eq. (9) and then substitute Eq. (9) into Eq. (23), using the upper sign wherever the choice appears for the function $\langle\eta_{\nu-\zeta,\nu}\rangle_\alpha^u$, and the lower sign for the function $\langle\eta_{\nu-\zeta,\nu}\rangle_\alpha^\ell$. The procedure is quite straightforward, but upon attempting to expand the algebra becomes extremely cumbersome. This seems to be the proper point, therefore, to make a judious choice for angles, γ and ζ, and obtain specific sets of integral equations. The particular angles are chosen with two aims in mind: (a) to make the resulting mathematical expressions not too involved, and (b) to permit the experimental equipment to be set up with certain easily established orientations.

Even so, the algebraic reduction of the resulting expressions is cumbersome. For this reason, only the resulting integral equations are listed here and the details are shown in Appendix B. The accompanying figures show the orientation of the incident and desired linearly polarized scattered component when looking toward the moon.

1. $\underline{\gamma = 0,\ \zeta = 0}$



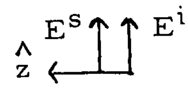
(25a)
$$F_{\gamma=0,\zeta=0}(\theta) = 2\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}\rangle_\alpha \frac{\cos^4\theta\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$+\ 8\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{12}\rangle_\alpha \frac{\cos^2\theta(\sin^2\alpha-\cos^2\theta)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$+\ 4\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\angle a_{11}-\angle a_{22})\rangle_\alpha \frac{\cos^2\theta(\sin^2\alpha-\cos^2\theta)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos\theta}}$$

$$+\ 2\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{22}\rangle_\alpha \frac{(\sin^2\alpha-\cos^2\theta)^2 d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

2. $\underline{\gamma = \pm\,\pi/2,\ \zeta = 0}$



(25b)
$$F_{\gamma=\pi/2,\zeta=0}(\theta) = 2\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}\rangle_\alpha \frac{(\sin^2\alpha-\cos^2\theta)^2 d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

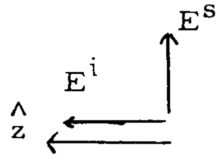
(25b) cont.

$$+\ 8\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{12}\rangle_\alpha \frac{\cos^2\theta(\sin^2\alpha-\cos^2\theta)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$+\ 4\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}}-\underline{/a_{22}})\rangle_\alpha \frac{\cos^2\theta(\sin^2\alpha-\cos^2\theta)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$+\ 2\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{22}\rangle_\alpha \frac{\cos^4\theta\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

3. $\underline{\gamma = 0, \zeta = \pi/2 \text{ or } \gamma = \pi/2, \zeta = \pi/2}$



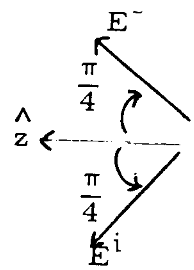
$$\text{(25c)}\quad F_{\gamma=0,\zeta=\pi/2}(\theta) = 2\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}\rangle_\alpha \frac{\cos^2\theta(\sin^2\alpha-\cos^2\theta)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$+\ 2\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{12}\rangle_\alpha \frac{(2\cos^2\theta-\sin^2\alpha)^2\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$-4\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}}-\underline{/a_{22}})\rangle_\alpha \frac{\cos^2\theta(\sin^2\alpha-\cos^2\theta)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$+\ 2\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{22}\rangle_\alpha \frac{\cos^2\theta(\sin^2\alpha-\cos^2\theta)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

4. $\underline{\gamma = \mp\pi/4, \zeta = \pm\pi/2}$



$$\text{(25d)}\quad F_{\gamma=-\pi/4,\zeta=\pi/2}(\theta) = \frac{1}{2}\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}\rangle_\alpha \frac{(2\cos^2\theta-\sin^2\alpha)^2\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

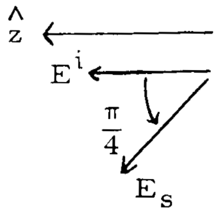
$$+\ 8\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{12}\rangle_\alpha \frac{\cos^2\theta(\sin^2\alpha-\cos^2\theta)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$-\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}}-\underline{/a_{22}})\rangle_\alpha \frac{(2\cos^2\theta-\sin^2\alpha)^2\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

(25d) cont.

$$+\frac{1}{2}\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{22}\rangle_\alpha \frac{(2\cos^2\theta - \sin^2\alpha)^2\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha - \cos^2\theta}} \;.$$

5. $\underline{\gamma = 0, \zeta = \pi/4 \text{ or } \gamma = -\pi/4, \zeta = -\pi/4}$



(25e) $$F_{\gamma=0,\zeta=\pi/4}(\theta) = \int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}\rangle_\alpha \frac{\cos^2\theta\sin^2\alpha\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

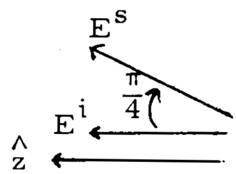
$$+ 2\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{12}})\rangle_\alpha \frac{\cos^2\theta(4\cos^2\theta-3\sin^2\alpha)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$+ \int_{\pi/2-\theta}^{\pi/2} \langle\eta_{12}\rangle_\alpha \frac{\sin^4\alpha\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$+ 2\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{12}} - \underline{/a_{22}})\rangle_\alpha \frac{(\sin^2\alpha-\cos^2\theta)(4\cos^2\theta-\sin^2\alpha)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$+ \int_{\pi/2-\theta}^{\pi/2} \langle\eta_{22}\rangle_\alpha \frac{\sin^2\alpha(\sin^2\alpha-\cos^2\theta)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}} \;.$$

6. $\underline{\gamma = 0, \zeta = -\pi/4 \text{ or } \gamma = \pi/4, \zeta = \pi/4}$



(25f) $$F_{\gamma=0,\zeta=-\pi/4}(\theta) = \int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}\rangle_\alpha \frac{\cos^2\theta\sin^2\alpha\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$- 2\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{12}})\rangle_\alpha \frac{\cos^2\theta(4\cos^2\theta-3\sin^2\alpha)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$+ \int_{\pi/2-\theta}^{\pi/2} \langle\eta_{12}\rangle_\alpha \frac{\sin^4\alpha\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$- 2\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{12}} - \underline{/a_{22}})\rangle_\alpha \frac{(\sin^2\alpha-\cos^2\theta)(4\cos^2\theta-\sin^2\alpha)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

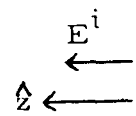
$$\text{(25f)}\ \text{(cont)} \qquad + \int_{\pi/2-\theta}^{\pi/2} \langle\eta_{22}\rangle_\alpha \frac{\sin^2\alpha(\sin^2\alpha - \cos^2\theta)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha - \cos^2\theta}} \quad .$$

There is an infinite number of measurements consisting of various combinations of linear polarization, but any of these other combinations will yield no further independent information than that already contained in the above six equations; six averages which appeared in Eq. (9) must be determined, therefore these six equations will suffice.

B. Linear Transmitting, Circular Receiving Antennas

The applicable backscattering cross section per unit area is found in this case by taking the averages of Eqs. (4) and (5); thus $\langle\eta_{\sigma_T}\rangle_\alpha$ = $\langle\eta_{L\nu}\rangle_\alpha$ or $\langle\eta_{R\nu}\rangle_\alpha$. (As shown previously, these quantities are identical to using circular transmitting and linear receiving antennas; in this latter case, the order of subscripts are interchanged, i.e., $\langle\eta_{L\nu}\rangle_\alpha = \langle\eta_{\nu L}\rangle_\alpha$ and $\langle\eta_{R\nu}\rangle_\alpha = \langle\eta_{\nu R}\rangle_\alpha$.) Since the transmitting antenna is linear, emitting a wave polarized in a direction making an angle ν with the plane of incidence for an area element dA, this much of the problem is identical to the preceding case. Therefore, the angle ν, as defined in Eq. (24a), is applicable here also, where the angle γ is defined in the same way. Again it is less cumbersome to choose particular transmitting antenna angles, ... than to develop a general expression. The details are contained in Appendix B.

1.
2. $\gamma = 0$



$$\text{(26a, b)} \qquad F_{\substack{L\\R},\gamma=0}(\theta) = \int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}\rangle_\alpha \frac{\sin^2\alpha\cos^2\theta\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha - \cos^2\theta}}$$

$$\pm 2\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{11}} - \underline{/a_{12}})\rangle_\alpha \frac{\sin^2\alpha\cos^2\theta\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha - \cos^2\theta}}$$

$$\pm 2\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{12}} - \underline{/a_{22}})\rangle_\alpha \frac{(\sin^2\alpha - \cos^2\theta)\sin^2\alpha\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha - \cos^2\theta}}$$

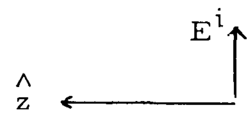
(26a, b)
(cont)

$$+\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{12}\rangle_\alpha \frac{\sin^4\alpha \, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$+\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{22}\rangle_\alpha \frac{(\sin^2\alpha-\cos^2\theta)\sin^2\alpha \, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}} \quad .$$

The above expression is actually two equations, the upper sign throughout going with the upper subscript and referring to the left circularly polarized component in the scattered wave.

3.
4. $\underline{\gamma = \pm\pi/2}$



(26c, d)

$$F_{\substack{L\\R}\,\gamma=\pi/2}(\theta) = \int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}\rangle_\alpha \frac{(\sin^2\alpha-\cos^2\theta)\sin^2\alpha \, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

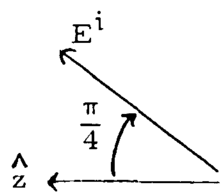
$$\pm 2\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{11}}-\underline{/a_{12}})\rangle_\alpha \frac{(\sin^2\alpha-\cos^2\theta)\sin^2\alpha \, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$\pm 2\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{12}}-\underline{/a_{22}})\rangle_\alpha \frac{\sin^2\alpha\cos^2\theta \, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$+\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{12}\rangle_\alpha \frac{\sin^4\alpha \, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$+\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{22}\rangle_\alpha \frac{\sin^2\alpha\cos^2\theta \, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}} \quad .$$

5.
6. $\underline{\gamma = \pi/4}$



(26e, f)

$$F_{\substack{L\\R}\,\gamma=\pi/4}(\theta) = \frac{1}{2}\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}\rangle_\alpha \frac{\sin^4\alpha \, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

(26e, f)
(cont)

$$\pm \int_{\pi/2-\theta}^{\pi/2} \langle \eta_{11}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \sin(\underline{/a_{11}} - \underline{/a_{12}}) \rangle_\alpha \frac{\sin^4 \alpha \, d\alpha}{\sin^3 \alpha \sqrt{\sin^2 \alpha - \cos^2 \theta}}$$

$$\pm \int_{\pi/2-\theta}^{\pi/2} \langle \eta_{22}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \sin(\underline{/a_{12}} - \underline{/a_{22}}) \rangle_\alpha \frac{\sin^4 \alpha \, d\alpha}{\sin^3 \alpha \sqrt{\sin^2 \alpha - \cos^2 \theta}}$$

$$+ \int_{\pi/2-\theta}^{\pi/2} \langle \eta_{12} \rangle_\alpha \frac{\sin^4 \alpha \, d\alpha}{\sin^3 \alpha \sqrt{\sin^2 \alpha - \cos^2 \theta}}$$

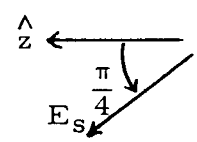
$$+ \frac{1}{2} \int_{\pi/2-\theta}^{\pi/2} \langle \eta_{22} \rangle_\alpha \frac{\sin^4 \alpha \, d\alpha}{\sin^3 \alpha \sqrt{\sin^2 \alpha - \cos^2 \theta}}$$

$$- \int_{\pi/2-\theta}^{\pi/2} \langle \eta_{11}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{11}} - \underline{/a_{12}}) \rangle_\alpha \frac{\sin^2 \alpha (2\cos^2 \theta - \sin^2 \alpha) d\alpha}{\sin^3 \alpha \sqrt{\sin^2 \alpha - \cos^2 \theta}}$$

$$- \int_{\pi/2-\theta}^{\pi/2} \langle \eta_{22}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{22}} - \underline{/a_{12}}) \rangle_\alpha \frac{\sin^2 \alpha (2\cos^2 \theta - \sin^2 \alpha) d\alpha}{\sin^3 \alpha \sqrt{\sin^2 \alpha - \cos^2 \theta}}$$

$$\mp \int_{\pi/2-\theta}^{\pi/2} \langle \eta_{11}^{\frac{1}{2}} \eta_{22}^{\frac{1}{2}} \sin(\underline{/a_{11}} - \underline{/a_{22}}) \rangle_\alpha \frac{\sin^2 \alpha (2\cos^2 \theta - \sin^2 \alpha) d\alpha}{\sin^3 \alpha \sqrt{\sin^2 \alpha - \cos^2 \theta}}$$

7.
8. $\underline{\gamma = -\pi/4}$



(26g, h)

$$F_{\substack{L \\ R}\gamma=-\pi/4}(\theta) = \frac{1}{2} \int_{\pi/2-\theta}^{\pi/2} \langle \eta_{11} \rangle_\alpha \frac{\sin^4 \alpha \, d\alpha}{\sin^3 \alpha \sqrt{\sin^2 \alpha - \cos^2 \theta}}$$

$$\pm \int_{\pi/2-\theta}^{\pi/2} \langle \eta_{11}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \sin(\underline{/a_{11}} - \underline{/a_{12}}) \rangle_\alpha \frac{\sin^4 \alpha \, d\alpha}{\sin^3 \alpha \sqrt{\sin^2 \alpha - \cos^2 \theta}}$$

$$\pm \int_{\pi/2-\theta}^{\pi/2} \langle \eta_{22}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \sin(\underline{/a_{12}} - \underline{/a_{22}}) \rangle_\alpha \frac{\sin^4 \alpha \, d\alpha}{\sin^3 \alpha \sqrt{\sin^2 \alpha - \cos^2 \theta}}$$

$$+ \int_{\pi/2-\theta}^{\pi/2} \langle \eta_{12} \rangle_\alpha \frac{\sin^4 \alpha \, d\alpha}{\sin^3 \alpha \sqrt{\sin^2 \alpha - \cos^2 \theta}}$$

(26g, h) (cont)

$$+\frac{1}{2}\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{22}\rangle_\alpha \frac{\sin^4\alpha\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$+\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{11}}-\underline{/a_{12}})\rangle_\alpha \frac{\sin^2\alpha(2\cos^2\theta-\sin^2\alpha)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$+\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{22}}-\underline{/a_{12}})\rangle_\alpha \frac{\sin^2\alpha(2\cos^2\theta-\sin^2\alpha)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$\pm\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\sin(\underline{/a_{11}}-\underline{/a_{22}})\rangle_\alpha \frac{\sin^2\alpha(2\cos^2\theta-\sin^2\alpha)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}} .$$

Not all of the above set of measurements need necessarily be made as long as all of those in Eq. (25) are made. Several pieces of information, obtainable from Eq. (26), cannot be obtained using only combinations of linear polarization, as in Eq. (25).

C. Circular Transmitting, Circular Receiving Antennas

The equations in this case are simplest of all because they are not dependent upon any antenna rotation, ν. Taking averages of Eqs. (6), (7), and (8) and substituting them into Eq. (23) yields three more measurements immediately:

(27a)

$$F_{RL}(\theta) = F_{LR}(\theta) = \frac{1}{2}\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}\rangle_\alpha \frac{\sin^4\alpha\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$+\frac{1}{2}\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{22}\rangle_\alpha \frac{\sin^4\alpha\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$+\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}}-\underline{/a_{22}})\rangle_\alpha \frac{\sin^4\alpha\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

and

$$\text{(27b, c)} \quad F_{\substack{LL\\RR}}(\theta) = \frac{1}{2}\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}\rangle_\alpha \frac{\sin^4\alpha \, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha - \cos^2\theta}}$$

$$+ \frac{1}{2}\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{22}\rangle_\alpha \frac{\sin^4\alpha \, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha - \cos^2\theta}}$$

$$- \int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{22}})\rangle_\alpha \frac{\sin^4\alpha \, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha - \cos^2\theta}}$$

$$+ 2\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{12}\rangle_\alpha \frac{\sin^4\alpha \, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha - \cos^2\theta}}$$

$$\pm 2\int_{\pi/>-\theta}^{\pi/2} \langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{11}} - \underline{/a_{12}}\rangle_\alpha \frac{\sin^4\alpha \, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha - \cos^2\theta}}$$

$$\pm 2\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{12}} - \underline{/a_{22}})\rangle_\alpha \frac{\sin^4\alpha \, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha - \cos^2\theta}} \quad .$$

Again, the last expression is two separate equations representing two separate measurements using the two possible permutations of right and left circular antennas.

D. Determination of the Averages of Eq. (10) from the Measurements of this Section

From the measurements discussed thus far in this section, all the averages found in Eq. (10) can be found. In separating each of these averages, three basic forms of an integral equation must be solved. The solutions of these integral equations are discussed later.

In this section, a brief outline will be presented for the determination of the averages of Eq. (10) from the measurements discussed thus far. There are several methods of finding these averages, and the method illustrated here is certainly not unique. One could, for example, solve Eq. (25a) by merely grouping everything under one integral sign and solving by brute force; the result would be cumbersome and trigonometrically involved. By proper and judicious algebraic manipulations before actually solving the integral equations, the complexity and detail in their solution can be greatly reduced beforehand, and the results will be merely linear combinations of the desired averages.

1. Determination of $\langle\eta_{11}\rangle_\alpha$, $\langle\eta_{22}\rangle_\alpha$, $\langle\eta_{12}\rangle_\alpha$, and $\langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}}-\underline{/a_{22}})\rangle_\alpha$

Add Eq. (27b) to Eq. (27c) and add the result to two times Eq. (27a):

(28a) $$2\int_{\pi/2-\theta}^{\pi/2}\left[\langle\eta_{11}\rangle_\alpha + 2\langle\eta_{12}\rangle_\alpha + \langle\eta_{22}\rangle_\alpha\right]\frac{\sin^4\alpha\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$= F_{LL}(\theta) + F_{RR}(\theta) + 2F_{RL}(\theta)\,.$$

Add Eq. (27b) to Eq. (27c) and subtract from it two times Eq. (27a):

(28b) $$2\int_{\pi/2-\theta}^{\pi/2}\left[\langle\eta_{12}\rangle_\alpha - \langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}}-\underline{/a_{22}})\rangle_\alpha\right]\frac{\sin^4\alpha\, d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$= F_{LL}(\theta) + F_{RR}(\theta) - 2F_{RL}(\theta)\,.$$

Now add Eq. (26a) to Eq. (26b) and subtract from this the sum of Eqs. (26c) and (26d):

(28c) $$2\int_{\pi/2-\theta}^{\pi/2}\left[\langle\eta_{11}\rangle_\alpha - \langle\eta_{22}\rangle_\alpha\right]\frac{\sin^2\alpha(2\cos^2\theta-\sin^2\alpha)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$

$$= F_{L,\gamma=0}(\theta) + F_{R,\gamma=0}(\theta) - F_{L,\gamma=\pi/2}(\theta) - F_{R,\gamma=\pi/2}(\theta)\,.$$

Subtract Eq. (25c) from Eq. (25d) to obtain the following:

(28d) $$\frac{1}{2}\int_{\pi/2-\theta}^{\pi/2}\left[\langle\eta_{11}\rangle_\alpha - 4\langle\eta_{12}\rangle_\alpha - 2\langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}}-\underline{/a_{22}})\rangle_\alpha + \langle\eta_{22}\rangle_\alpha\right]$$

$$\text{(28d) (cont)} \qquad \frac{(6\cos^4\theta - 8\cos^2\theta\sin^2\alpha + \sin^4\alpha)\,d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha - \cos^2\theta}}$$

$$= F_{\gamma=-\pi/4,\ \zeta=\pi/2}(\theta) - F_{\gamma=0,\ \zeta=\pi/2}(\theta)\ .$$

The left-hand sides of the above equations could have been obtained in the same form by other combinations of Eqs. (25), (26), and (27). The above method is merely one of several.

At this point, the above four integral equations can be solved as functions of α, the angle of incidence. The resulting solutions are the bracketed quantities in the integrands. These are linear combinations of the desired averages; they are four linearly independent equations in four unknowns,

$$\langle\eta_{11}\rangle_\alpha,\ \langle\eta_{22}\rangle_\alpha,\ \langle\eta_{12}\rangle_\alpha,\ \text{and}\ \langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{22}})\rangle_\alpha\ ,$$

and may be readily solved.

2. Determination of $\langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\sin(\underline{/a_{11}} - \underline{/a_{22}})\rangle_\alpha$

Subtract Eq. (26e) from Eq. (26f), subtract Eq. (26h) from Eq. (26g), and then add these two differences to obtain

$$\text{(29)} \qquad 4\int_{\pi/2-\theta}^{\pi/2} \langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\sin(\underline{/a_{11}} - \underline{/a_{22}})\rangle_\alpha \frac{\sin^2\alpha(2\cos^2\theta - \sin^2\alpha)\,d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha - \cos^2\theta}}$$

$$= F_{R,\gamma=\pi/2}(\theta) - F_{L,\gamma=\pi/4}(\theta) + F_{L,\gamma=-\pi/4}(\theta) - F_{R,\gamma=-\pi/4}(\theta)\ .$$

From the solution of this integral equation, $\langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\sin(/a_{11} - \underline{/a_{22}})\rangle_\alpha$ is obtained.

3. Determination of $\langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{11}} - \underline{/a_{12}})\rangle_\alpha$ and $\langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{11}} - \underline{/a_{22}})\rangle_\alpha$

Subtract Eq. (26b) from Eq. (26a); subtract Eq. (26d) from Eq. (26c), and then subtract the latter difference from the former:

(30a)
$$4\int_{\pi/2-\theta}^{\pi/2}[\langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{11}}-\underline{/a_{12}})\rangle_\alpha - \langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{12}}-\underline{/a_{22}})\rangle_\alpha]$$
$$\frac{\sin^2\alpha(2\cos^2\theta-\sin^2\alpha)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$
$$= F_{L,\gamma=0}(\theta) - F_{R,\gamma=0}(\theta) - F_{L,\gamma=\pi/2}(\theta) + F_{R,\gamma=\pi/2}(\theta) .$$

Subtract Eq. (26f) from Eq. (26e), subtract Eq. (26h) from Eq. (26g), and then add these two differences:

(30b)
$$4\int_{\pi/2-\theta}^{\pi/2}[\langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{11}}-\underline{/a_{12}})\rangle_\alpha + \langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\sin(\underline{/a_{12}}-\underline{/a_{22}})\rangle_\alpha]$$
$$\frac{\sin^4\alpha\ d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$
$$= F_{L,\gamma=\pi/4}(\theta) - F_{R,L=\pi/4}(\theta) + F_{L,\gamma=-\pi/4}(\theta) - F_{R,\gamma=-\pi/4}(\theta) .$$

Upon solution of these two integral equations, the bracketed quantities in the integrands yield the desired averages.

4. Determination of $\langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{11}}-\underline{/a_{12}})\rangle_\alpha$ and $\langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{12}}-\underline{/a_{22}})\rangle_\alpha$

Subtract Eq. (26e) from Eq. (26g), subtract Eq. (26f) from Eq. (26h), and then add these two differences:

(31a)
$$4\int_{\pi/2-\theta}^{\pi/2}[\langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{11}}-\underline{/a_{12}})\rangle_\alpha + \langle\eta_{22}\eta_{12}\cos(\underline{/a_{12}}-\underline{/a_{22}})\rangle_\alpha]$$
$$\frac{\sin^2\alpha(2\cos^2\theta-\sin^2\alpha)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha-\cos^2\theta}}$$
$$= F_{L,\gamma=-\pi/4}(\theta) - F_{L,\gamma=\pi/4}(\theta) + F_{R,\gamma=-\pi/4}(\theta) - F_{R,\gamma=\pi/4} .$$

Now subtract Eq. (25f) from Eq. (25e) and multiply the result by two; subtract from this Eq. (31a) to obtain the following

$$(31b) \qquad 4\int_{\pi/2-\theta}^{\pi/2} \Big[\langle \eta_{11}\eta_{12}\cos(\underline{/a_{11}}-\underline{/a_{12}})\rangle_\alpha - \langle \eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{12}}-\underline{/a_{22}})\rangle_\alpha\Big] \frac{(8\cos^4\theta - 8\cos^2\theta\sin^2\alpha + \sin^4\alpha)d\alpha}{\sin^3\alpha\sqrt{\sin^2\alpha - \cos^2\theta}}$$

$$2F_{\gamma=0,\zeta=+\pi/4}(\theta) - 2F_{\gamma=0,\zeta=-\pi/4}(\theta) - F_{L,\gamma=-\pi/4}(\theta) + F_{L,\gamma=\pi/4}(\theta) - F_{R,\gamma=-\pi/4}(\theta) + F_{R,\gamma=\pi/4}(\theta) .$$

Thus upon sloving these last two integral equations, the two desired averages are obtained from the bracketed terms in the integrands.

All the nine averages of Eq. (10) have been determined at this point. This method employing the doppler strip therefore offers much more complete information than that of the last section employing a range ring.

It should be noted that throughout this outline, only three different forms of integrand have appeared in all of the integral equations. The solution of these three integral equations is quite straightforward and its existence is guaranteed, even with the radical in the denominator. As a matter of fact, one of these forms reduces to the well-known Abel's integral equation. In any event, several books on integral equations describe methods for finding exact, closed-form solutions to all these equations. However, in any practical application it seems much more convenient to avoid such exact mathematical solutions in favor of constructing numerical solutions employing a computer. One obvious reason for this is that the data found from experiment will be in the form of isolated points and will not be representable in general by any simple mathematical expression. Thus rather than be forced to plot data graphically and resort to curve fitting, it seems much simpler to construct three programs for their solutions and read the measured data into the machine as input points. At any rate, the manner of solution chosen is better left to the user and to the facilities available to him. Suffice it to say that the solutions do exist.

V. CONCLUSIONS

This report has discussed two experiments for measuring easily definable planar backscattering averages from a large spherical surface, such as the moon or planets. Normal radar scattering from a surface such as the moon yields information about the scattering properties of the moon as a whole, and since radar antennas are not always directive enough to illuminate only a small well-defined portion at a time, the determination of local average backscattering lunar surface properties is complicated. The two experiments overcome this difficulty and enable one to determine nine separate backscattering averages, listed in Eq. (10), as a function of the angle of incidence. These nine averages can easily be made for a variety of observable sample surfaces of different roughness and composition on the Earth, so that these same averages for the moon might be compared with those of a surface with known characteristics. This should yield much comparative information about the nature of the lunar surface.

The first experiment employs a narrow radar pulse which illuminates only a small band, or range ring, on the moon at a time. The second experiment uses a CW carrier and depends upon a spreading of this discrete frequency when scattered, because of the rotation and libration of the moon. The return at a given frequency near the carrier corresponds to points on the moon with the same velocity toward or away from the Earth. The first experiment is simpler in both measurement and measured data reduction, but does not yield all of the nine averages separately. The second experiment depends upon an exact knowledge of the moon's angular velocity and its axis. Knowing these, the transmitting and receiving antennas must be oriented accordingly. After measurement as a function of frequency, several integral equations must be solved in the reduction of measured data; this can all be done on a computer, however, eliminating much plotting and the graphical determination of areas and slopes. In the long run, the second experiment yields every one of the nine averages separately.

One would expect that all of these average backscattering properties would decrease to nearly zero as the angle of incidence increased (i.e., moving away from normal incidence). One would also intuitively predict that as the surface roughness becomes large in comparison to wavelength, the averages containing a phase difference would all go to zero. Another way of saying the same thing is that for a rough surface, one might expect a random and therefore uniform distribution of phase differences from 0 to 2π. For a perfectly smooth surface at normal incidence, one expects $\langle\eta_{12}\rangle$ to be zero, since a smooth surface does not change the direction of incident polarization. However, as the roughness approaches the same order of magnitude as wavelength, one expects $\langle\eta_{12}\rangle$ to become larger, approaching some constant value (at the same angle of incidence), while in turn he expects $\langle\eta_{11}\rangle$ and $\langle\eta_{22}\rangle$ to decrease to some constant value. Thus, one can verify these facts on a set of sample surfaces on the Earth and then attempt to deduce similar facts about surface roughness of the moon by varying the wavelength. Frequency scaling can be used on the sample surfaces on the Earth so that the same results can be applied to surfaces with a much larger roughness scale.

Several significant points can be noted here. For one, the resulting statistical data found in Eq. (17a, b, c, d) can all be found solely from the employment of circular polarizing antennas; linear antennas need not be employed at all. The reason that measurements involving linear polarization are discussed at all is that in certain cases it may be desirable or more convenient to use linear antennas. Also, it is significant to note that this same information in Eq. (17a, b, c, d) from the range ring experiment, which is found strictly by using circular polarizing antennas, can also be found in the same identical form from the doppler strip experiment when only circular polarizing antennas are used. This is evident after comparing Eq. (16a, b, c) with Eq. (27a, b, c). Probably if one were forced to choose a minimum amount of statistical surface information, that found in Eq. (17a, b, c, d) would be both the most important and the simplest to obtain from the standpoint of the physical measurements required. Any measurements involving strictly circular polarization are always easy to make because they involve no special antenna orientations either with respect to each other or with respect to the moon. Either the range ring or the doppler strip experiment could then be used, depending upon which is more convenient to set up.

APPENDIX A

The return from a radar pulse incident upon the moon can be displayed on the oscilloscope as a function of time. In order to relate this time to the angle of incidence, α, on the moon (see Fig. 4), it is necessary to consider the effect of pulse length in free space, L, to the area in a given range ring, i. e., $dA^{RR} = 2\pi R^2 \sin\alpha\, d\alpha$. Obviously, as the pulse length, L, becomes too large, the actual angle of α varies significantly inside the element $d\alpha$; this reduces accuracy proportionately. The actual angular increment, $d\alpha$, is not a constant as the pulse of length L moves past the lunar surface, but is greatest at the portion of the moon hit first. Since the signal returned from this part will be strongest, it seems important to obtain as much undistorted, accurate information from this forward portion as possible.

On the other hand, if the pulse is too small, besides system bandwidth problems, the area illuminated on the moon will have dimensions which are less than prominent surface features. It is doubtful if this situation would yield significant average backscattering data about the overall surface.

If, for example, $T = 10^{-3}$ sec = pulse width in time, the pulse length in free space, L, is approximately 200 miles. With an average lunar radius of 1080 miles, the maximum incidence angle increment, $d\alpha$, at the forward region is about 35°, which is intolerable. With a pulse width of about $T = 0.25 \times 10^{-4}$ sec, $L \approx 5$ miles and $d\alpha \approx 5°$, which seem satisfactory since a much smaller pulse length would be of the same order of magnitude as the surface roughness. With the latter pulse width, system bandwidth requirements are of the order of 40 kc.

With a value of L such as the latter, the portion of the return occuring after the leading edge of the pulse has struck the forward edge of the moon (but before the trailing edge has struck this part) is insignificant in comparison to the total length of the return pulse. Neglecting this portion, the area of the range ring, $dA^{RR} = 2\pi R^2 \sin\alpha\, d\alpha = 2\pi RL$, is a constant at all points along the illuminated area (from purely geometrical considerations). Therefore, from Eq. (13),

$$\langle \eta_{\sigma\tau}^{RR} \rangle_\alpha \sim \langle d\sigma_{\sigma\tau}^{RR} \rangle_\alpha \; .$$

The latter quantity is directly proportional to the average power in the returned pulse, except that the power return is a function of time, whereas the former is a function of angle. As long as the increment, $d\alpha$, is small, the relationship between angle of incidence, α, and time as observed on an oscilloscope is

$$(32) \qquad \alpha = \cos^{-1}\left[1 - \frac{c(t-t_o)}{2R}\right]$$

or

$$t = t_o + \frac{2R}{c}\left[1 - \cos\alpha\right],$$

where c = speed of light and t_o = time L/c seconds after the start of the pulse on the oscilloscope (this is the time that the middle of the pulse strikes the moon).

With the above relationships, time can be converted to angle of incidence, and vice-versa. Thus average power return can be plotted directly in terms of angle of incidence in order that the quantities in Eq. (17) can be computed.

APPENDIX B

The following formulas are helpful in the algebraic reduction of the integral equations:

(33) $$\cos\left\{\cos^{-1}\left[\frac{\cos\theta}{\sin\alpha}\right]\right\} = \frac{\cos\theta}{\sin\alpha}$$

(34) $$\sin\left\{\cos^{-1}\left[\frac{\cos\theta}{\sin\alpha}\right]\right\} = \pm\frac{1}{\sin\alpha}\sqrt{\sin^2\alpha-\cos^2\theta}$$

(35) $$\cos\left\{2\cos^{-1}\left[\frac{\cos\theta}{\sin\alpha}\right]\right\} = \cos^2\left\{\cos^{-1}\left[\frac{\cos\theta}{\sin\alpha}\right]\right\} - \sin^2\left\{\cos^{-1}\left[\frac{\cos\theta}{\sin\alpha}\right]\right\}$$

$$\therefore \left\{\cos\ 2\cos^{-1}\left[\frac{\cos\theta}{\sin\alpha}\right]\right\} = \frac{1}{\sin^2\alpha}\left[2\cos^2\theta-\sin^2\alpha\right]$$

(36) $$\sin\left\{2\cos^{-1}\left[\frac{\cos\theta}{\sin\alpha}\right]\right\} = 2\sin\left\{\cos^{-1}\left[\frac{\cos\theta}{\sin\alpha}\right]\right\}$$

$$\times\cos\left\{\cos^{-1}\left[\frac{\cos\theta}{\sin\alpha}\right]\right\}$$

$$\therefore \sin\left\{2\cos^{-1}\left[\frac{\cos\theta}{\sin\alpha}\right]\right\} = \frac{\pm 2\cos\theta}{\sin^2\alpha}\sqrt{\sin^2\alpha-\cos^2\theta}\ .$$

(25a) $\underline{\gamma = 0,\ \zeta = 0}$

$$\therefore \cos\nu = \frac{\cos\theta}{\sin\alpha} = \cos(\nu-\zeta);\quad \sin\nu = \pm\frac{1}{\sin\alpha}\sqrt{\sin^2\alpha-\cos^2\theta} = \sin(\nu-\zeta)$$

$$\cos(2\nu-\zeta) = \frac{1}{\sin^2\alpha}\left[2\cos^2\theta-\sin^2\alpha\right];$$

$$\sin(2\nu-\zeta) = \frac{\pm 2\cos\theta}{\sin^2\alpha}\sqrt{\sin^2\alpha-\cos^2\theta}\ .$$

Substitute these into Eq. (9).

$$\therefore \quad \langle \eta_{\nu-\zeta,\nu} \rangle_{\alpha}^{u} = \langle \eta_{11} \rangle_{\alpha} \frac{\cos^4\theta}{\sin^4\alpha}$$

$$+ 2\langle \eta_{11}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{11}} - \underline{/a_{12}}) \rangle_{\alpha} \left[\frac{2\cos^3\theta}{\sin^4\alpha} \sqrt{\sin^2\alpha - \cos^2\theta} \right]$$

$$+ 2\langle \eta_{11}^{\frac{1}{2}} \eta_{22}^{\frac{1}{2}} \cos(\underline{/a_{11}} - \underline{/a_{22}}) \rangle_{\alpha} \left[\frac{\cos^2\theta}{\sin^4\alpha} (\sin^2\alpha - \cos^2\theta) \right]$$

$$+ \langle \eta_{12} \rangle_{\alpha} \left[\frac{4\cos^2\theta}{\sin^4\alpha} (\sin^2\alpha - \cos^2\theta) \right]$$

$$+ 2\langle \eta_{22}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{12}} - \underline{/a_{22}}) \rangle_{\alpha} \left[\frac{2\cos\theta}{\sin^4\alpha} (\sin^2\alpha - \cos^2\theta)^{\frac{3}{2}} \right]$$

$$+ \langle \eta_{22} \rangle_{\alpha} \left[\frac{(\sin^2\alpha - \cos^2\theta)^2}{\sin^4\alpha} \right]$$

$$\langle \eta_{\nu-\zeta,\nu} \rangle_{\alpha}^{\ell} = \langle \eta_{11} \rangle_{\alpha} \frac{\cos^4\theta}{\sin^4\alpha}$$

$$- 2\langle \eta_{11}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{11}} - \underline{/a_{12}}) \rangle_{\alpha} \left[\frac{2\cos^3\theta}{\sin^4\alpha} \sqrt{\sin^2\alpha - \cos^2\theta} \right]$$

$$+ 2\langle \eta_{11}^{\frac{1}{2}} \eta_{22}^{\frac{1}{2}} \cos(\underline{/a_{12}} - \underline{/a_{22}}) \rangle_{\alpha} \left[\frac{\cos^2\theta}{\sin^4\alpha} (\sin^2\alpha - \cos^2\theta) \right]$$

$$+ \langle \eta_{12} \rangle_{\alpha} \left[\frac{4\cos^2\theta}{\sin^4\alpha} (\sin^2\alpha - \cos^2\theta) \right]$$

$$- 2\langle \eta_{22}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{12}} - \underline{/a_{22}}) \rangle_{\alpha} \left[\frac{2\cos\theta}{\sin^4\alpha} (\sin^2\alpha - \cos^2\theta)^{\frac{3}{2}} \right]$$

$$+ \langle \eta_{22} \rangle_{\alpha} \left[\frac{\sin^2\alpha - \cos^2\theta)^2}{\sin^4\alpha} \right] .$$

Equation (25a) is obtained by using the above equations.

(25b). $\underline{\gamma = \pi/2, \zeta = 0}$

$$\cos \nu = \pm \frac{1}{\sin \alpha} \sqrt{\sin^2 \alpha - \cos^2 \theta} = \cos(\nu - \zeta) ;$$

$$\sin \nu = - \frac{\cos \theta}{\sin \alpha} = \sin(\nu - \zeta)$$

$$\cos(2\nu - \zeta) = - \frac{1}{\sin^2 \alpha} [2\cos^2 \theta - \sin^2 \alpha] ;$$

$$\sin(\nu - \zeta) = \mp \frac{2\cos \theta}{\sin^2 \alpha} \sqrt{\sin^2 \alpha - \cos^2 \theta}$$

$$\therefore \quad <\eta_{\nu-\zeta,\nu}>^u_\alpha = <\eta_{11}>_\alpha \left[\frac{\sin^2 \alpha - \cos^2 \theta)^2}{\sin^4 \alpha} \right]$$

$$- 2<\eta_{11}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{11}} - \underline{/a_{12}})>_\alpha \left[\frac{2\cos \theta}{\sin^4 \alpha} (\sin^2 \alpha - \cos^2 \theta)^{\frac{3}{2}} \right]$$

$$+ 2<\eta_{11}^{\frac{1}{2}} \eta_{22}^{\frac{1}{2}} \cos(\underline{/a_{11}} - \underline{/a_{22}})>_\alpha \left[\frac{\cos^2 \theta}{\sin^4 \alpha} (\sin^2 \alpha - \cos^2 \theta) \right]$$

$$+ <\eta_{12}>_\alpha \left[\frac{4\cos^2 \theta}{\sin^4 \alpha} (\sin^2 \alpha - \cos^2 \theta) \right]$$

$$- 2<\eta_{22}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{12}} - \underline{/a_{22}})>_\alpha \left[\frac{2\cos^3 \theta}{\sin^4 \alpha} \sqrt{\sin^2 \alpha - \cos^2 \theta} \right]$$

$$+ <\eta_{22}>_\alpha \frac{\cos^4 \theta}{\sin^4 \alpha}$$

$$<\eta_{\nu-\zeta,\nu}>^\ell_\alpha = <\eta_{11}>_\alpha \left[\frac{(\sin^2 \alpha - \cos^2 \theta)^2}{\sin^4 \alpha} \right]$$

$$+ 2<\eta_{11}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{11}} - \underline{/a_{12}})>_\alpha \left[\frac{2\cos \theta}{\sin^4 \alpha} (\sin^2 \alpha - \cos^2 \theta)^{\frac{3}{2}} \right]$$

$$+ 2\langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}}-\underline{/a_{22}})\rangle_\alpha \left[\frac{\cos^2\theta}{\sin^4\alpha}\ (\sin^2\alpha-\cos^2\theta)\right]$$

$$+ \langle\eta_{12}\rangle_\alpha \left[\frac{4\cos^2\theta}{\sin^4\alpha}(\sin^2\alpha-\cos^2\theta)\right]$$

$$+ 2\langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{12}}-\underline{/a_{22}})\rangle_\alpha \left[\frac{2\cos^3\theta}{\sin^4\alpha}\ \sqrt{\sin^2\alpha-\cos^2\theta}\right]$$

$$+ \langle\eta_{22}\rangle_\alpha\ \frac{\cos^4\theta}{\sin^4\alpha} \quad .$$

By using these above equations, Eq. (25b) is obtained. The same results for $\langle\eta_{\nu-\zeta,\nu}\rangle_\alpha^u$ and $\langle\eta_{\nu-\zeta,\nu}\rangle_\alpha^\ell$ are obtained if $\gamma = -\pi/2$ and $\zeta = 0$.

(25c). $\underline{\gamma = 0,\ \zeta = \pi/2}$

$$\cos\nu = \frac{\cos\theta}{\sin\alpha} = \sin(\nu-\zeta);\quad \sin\nu = \pm\frac{1}{\sin\alpha}\sqrt{\sin^2\alpha-\cos^2\theta} = -\cos(\nu-\zeta)$$

$$\cos(2\nu-\zeta) = \frac{\pm 2\cos\theta}{\sin^2\alpha}\ \sqrt{\sin^2\alpha-\cos^2\theta}$$

$$\sin(2\nu-\zeta) = -\frac{1}{\sin^2\alpha}\ [2\cos^2\theta-\sin^2\alpha]$$

$$\therefore\ \langle\eta_{\nu-\zeta,\nu}\rangle_\alpha^u = \langle\eta_{11}\rangle_\alpha \left[\frac{\cos^2\theta}{\sin^4\alpha}\ (\sin^2\alpha-\cos^2\theta)\right]$$

$$+ 2\langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{11}}-\underline{/a_{12}})\rangle_\alpha \left[\frac{\cos\theta}{\sin^4\alpha}\ (2\cos^2\theta-\sin^2\alpha)\ \sqrt{\sin^2\alpha-\cos^2\theta}\right]$$

$$- 2\langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}}-\underline{/a_{22}})\rangle_\alpha \left[\frac{\cos^2\theta}{\sin^4\alpha}\ (\sin^2\alpha-\cos^2\theta)\right]$$

$$+ \langle\eta_{12}\rangle_\alpha \left[\frac{1}{\sin^4\alpha}\ (2\cos^2\theta-\sin^2\alpha)^2\right]$$

$$- 2\langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{12}}-\underline{/a_{22}})\rangle_\alpha \left[\frac{\cos\theta}{\sin^4\alpha}(2\cos^2\theta-\sin^2\alpha)\sqrt{\sin^2\alpha-\cos^2\theta}\right]$$

$$+ \langle\eta_{22}\rangle_\alpha\left[\frac{\cos^2\theta}{\sin^4\alpha}(\sin^2\alpha-\cos^2\theta)\right].$$

$$\langle\eta_{\nu-\zeta,\nu}\rangle_\alpha^{\ell} = \langle\eta_{11}\rangle_\alpha\left[\frac{\cos^2\theta}{\sin^4\alpha}(\sin^2\alpha-\cos^2\theta)\right]$$

$$- 2\langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{11}}-\underline{/a_{12}})\rangle_\alpha\left[\frac{\cos\theta}{\sin^4\alpha}(2\cos^2\theta-\sin^2\alpha)\sqrt{\sin^2\alpha-\cos^2\theta}\right]$$

$$- 2\langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}}-\underline{/a_{22}})\rangle_\alpha\left[\frac{\cos^2\theta}{\sin^4\alpha}(\sin^2\alpha-\cos^2\theta)\right]$$

$$+ \langle\eta_{12}\rangle_\alpha\left[\frac{1}{\sin^4\alpha}(2\cos^2\theta-\sin^2\alpha)^2\right]$$

$$+ 2\langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{12}}-\underline{/a_{22}})\rangle_\alpha\left[\frac{\cos\theta}{\sin^4\alpha}(2\cos^2\theta-\sin^2\alpha)\sqrt{\sin^2\alpha-\cos^2\theta}\right]$$

$$+ \langle\eta_{22}\rangle_\alpha\left[\frac{\cos^2\theta}{\sin^4\alpha}(\sin^2\alpha-\cos^2\theta)\right] .$$

The same expressions for $\langle\eta_{\nu-\zeta,\nu}\rangle_\alpha^u$ and $\langle\eta_{\nu-\zeta,\nu}\rangle_\alpha^\ell$ are obtained if $\gamma = 0$ and $\zeta = -\pi/2$ or $\gamma = \pi/2$ and $\zeta = \pi/2$. These expressions can then be substituted into Eq. (23) to give Eq. (25c).

(25d) $\underline{\gamma = -\pi/4,\ \zeta = \pi/2}$

$$\cos\nu = \sin(\nu-\zeta) = \frac{1}{\sqrt{2}\times\sin\alpha}\left(\cos\theta\mp\sqrt{\sin^2\alpha-\cos^2\theta}\right)$$

$$\sin\nu = -\cos(\nu-\zeta) = \frac{1}{\sqrt{2}\times\sin\alpha}\left(\cos\theta\pm\sqrt{\sin^2\alpha-\cos^2\theta}\right)$$

$$\sin(2\nu-\zeta) = \frac{\mp 2\cos\theta}{\sin^2\alpha}\sqrt{\sin^2\alpha-\cos^2\theta}$$

$$\therefore \quad \langle\eta_{\nu-\zeta,\nu}\rangle_\alpha^u = \frac{1}{4}\langle\eta_{11}\rangle_\alpha \left[\frac{(2\cos^2\theta - \sin^2\alpha)^2}{\sin^4\alpha}\right]$$

$$+ 2\langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{12}})\rangle_\alpha \left[\frac{\cos\theta(2\cos^2\theta - \sin^2\alpha)\sqrt{\sin^2\alpha - \cos^2\theta}}{\sin^4\alpha}\right]$$

$$- \frac{1}{2}\langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{22}})\rangle_\alpha \left[\frac{(2\cos^2\theta - \sin^2\alpha)^2}{\sin^4\alpha}\right]$$

$$+ 4\langle\eta_{12}\rangle_\alpha \left[\frac{\cos^2\theta(\sin^2\alpha - \cos^2\theta)}{\sin^4\alpha}\right]$$

$$-2\langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{12}} - \underline{/a_{22}})\rangle_\alpha \left[\frac{\cos\theta(2\cos^2\theta - \sin^2\alpha)\sqrt{\sin^2\alpha - \cos^2\theta}}{\sin^4\alpha}\right]$$

$$+ \frac{1}{4}\langle\eta_{22}\rangle_\alpha \left[\frac{(2\cos^2\theta - \sin^2\alpha)^2}{\sin^4\alpha}\right]$$

$$\langle\eta_{\nu-\zeta,\nu}\rangle_\alpha^\ell = \frac{1}{4}\langle\eta_{11}\rangle_\alpha \left[\frac{(2\cos^2\theta - \sin^2\alpha)^2}{\sin^4\alpha}\right]$$

$$- 2\langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{12}})\rangle_\alpha \left[\frac{\cos\theta(2\cos^2\theta - \sin^2\alpha)\sqrt{\sin^2\alpha - \cos^2\theta}}{\sin^4\alpha}\right]$$

$$- \frac{1}{2}\langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{22}})\rangle_\alpha \left[\frac{(2\cos^2\theta - \sin^2\alpha)^2}{\sin^4\alpha}\right]$$

$$+ 4\langle\eta_{12}\rangle_\alpha \left[\frac{\cos^2\theta(\sin^2\alpha - \cos^2\theta)}{\sin^4\alpha}\right]$$

$$+ 2\langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{12}} - \underline{/a_{22}})\rangle_\alpha \left[\frac{\cos\theta(2\cos^2\theta - \sin^2\alpha)\sqrt{\sin^2\alpha - \cos^2\theta}}{\sin^4\alpha}\right]$$

$$+ \frac{1}{4}\langle\eta_{22}\rangle_\alpha \left[\frac{(2\cos^2\theta - \sin^2\alpha)^2}{\sin^4\alpha}\right].$$

The same expressions for $\langle\eta_{\nu-\zeta,\nu}\rangle_\alpha^u$ and $\langle\eta_{\nu-\zeta,\nu}\rangle_\alpha^\ell$ are obtained if $\gamma = +\pi/4$ and $\zeta = \pm\pi/2$. These expressions are then substituted into Eq. (23) to give Eq. (25d).

(25e) $\underline{\gamma = 0,\ \zeta = \pi/4}$

$$\therefore \quad \cos\nu = \frac{\cos\theta}{\sin\alpha}\ ;\quad \sin\nu = \pm\frac{1}{\sin\alpha}\sqrt{\sin^2\alpha - \cos^2\theta}$$

$$\cos(\nu-\zeta) = \frac{1}{\sqrt{2}\sin\alpha}\left[\cos\theta \mp \sqrt{\sin^2\alpha - \cos^2\theta}\right];$$

$$\sin(\nu-\zeta) = \frac{1}{\sqrt{2}\sin\alpha}\left[\cos\theta \pm \sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$\sin(2\nu-\zeta) = \frac{1}{\sqrt{2}\cdot\sin^2\alpha}\left[(2\cos^2\theta - \sin^2\alpha) \pm 2\cos\theta\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$\therefore \quad \langle\eta_{\nu-\zeta,\nu}\rangle^{u}_{\alpha} = \langle\eta_{11}\rangle_{\alpha}\frac{\cos^2\theta}{2\sin^4\alpha}\left[\sin^2\alpha - 2\cos\theta\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$+ \langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{12}})\rangle_{\alpha}\frac{\cos^2\theta}{\sin^4\alpha}\left[4\cos^2\theta - 3\sin^2\alpha + \frac{\sin^2\alpha}{\cos\theta}\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$+ \langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{22}})\rangle_{\alpha}\frac{\cos\theta}{\sin^4\alpha}(2\cos^2\theta - \sin^2\alpha)\sqrt{\sin^2\alpha - \cos^2\theta}$$

$$+ \frac{1}{2}\langle\eta_{12}\rangle_{\alpha}\frac{1}{\sin^4\alpha}\left[\sin^4\alpha + 2\cos\theta(2\cos^2\theta - \sin^2\alpha)\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$+ \langle\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{12}} - \underline{/a_{22}})\rangle_{\alpha}\frac{1}{\sin^4\alpha}\left[-4\cos^4\theta + 5\cos^2\theta\sin^2\alpha - \sin^4\alpha\right.$$

$$\left. + \cos\theta\sin^2\alpha\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$+ \langle\eta_{22}\rangle_{\alpha}\frac{1}{2\sin^4\alpha}\left[\sin^2\alpha - \cos^2\theta\right]\left[\sin^2\alpha + 2\cos\theta\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$\langle\eta_{\nu-\zeta,\nu}\rangle^{\ell}_{\alpha} = \langle\eta_{11}\rangle_{\alpha}\frac{\cos^2\theta}{2\sin^4\alpha}\left[\sin^2\alpha + 2\cos\theta\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$+ \langle\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{12}})\rangle_{\alpha}\frac{\cos^2\theta}{\sin^4\alpha}\left[4\cos^2\theta - 3\sin^2\alpha - \frac{\sin^2\alpha}{\cos\theta}\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$- \langle\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{22}})\rangle_{\alpha}\frac{\cos\theta}{\sin^4\alpha}(2\cos^2\theta - \sin^2\alpha)\sqrt{\sin^2\alpha - \cos^2\theta}$$

$$+\frac{1}{2}<\eta_{12}>_\alpha \frac{1}{\sin^4\alpha}\left[\sin^4\alpha - 2\cos\theta(2\cos^2\theta - \sin^2\alpha)\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$+<\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{12}} - \underline{/a_{22}})>_\alpha \frac{1}{\sin^4\alpha}\left[-4\cos^4\theta + 5\cos^2\theta\sin^2\alpha\right.$$

$$\left.-\sin^4\alpha - \cos\theta\sin^2\alpha\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$+<\eta_{22}>_\alpha \frac{1}{2\sin^4\alpha}(\sin^2\alpha - \cos^2\theta) * \left[\sin^2\alpha - 2\cos\theta\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

The same results can be arrived at for $\gamma = -\pi/4$ and $\zeta = -\pi/4$. Upon substitution of the above expressions into Eq. (23), Eq. (25e) is obtained.

(25f) $\underline{\gamma = 0,\ \zeta = -\pi/4}$

$$\cos\nu = \frac{\cos\theta}{\sin\alpha}\ ;\ \sin\nu = \pm\frac{1}{\sin\alpha}\sqrt{\sin^2\alpha - \cos^2\theta}$$

$$\cos(\nu-\zeta) = \frac{1}{\sqrt{2}\cdot\sin\alpha}\left[\cos\theta \pm \sqrt{\sin^2\alpha - \cos^2\theta}\right]\ ;$$

$$\sin(\nu-\zeta) = \frac{1}{\sqrt{2}\cdot\sin\alpha}\left[-\cos\theta \pm \sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$\sin(2\nu-\zeta) = \frac{1}{\sqrt{2}\cdot\sin^2\alpha}\left[-(2\cos^2\theta - \sin^2\alpha) \pm 2\cos\theta\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$\therefore\ <\eta_{\nu-\zeta,\nu}>_\alpha^u = <\eta_{11}>_\alpha \frac{\cos^2\theta}{2\sin^4\alpha}\left[\sin^2\alpha + 2\cos\theta\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$+<\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{12}})>_\alpha \frac{\cos^2\theta}{\sin^4\alpha}\left[-4\cos^2\theta + 3\sin^2\alpha\right.$$

$$\left.+\frac{\sin^2\alpha}{\cos\theta}\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$-<\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{22}})>_\alpha \frac{\cos\theta}{\sin^4\alpha}(2\cos^2\theta - \sin^2\alpha)\sqrt{\sin^2\alpha - \cos^2\theta}$$

$$+ \frac{1}{2} <\eta_{12}>_\alpha \frac{1}{\sin^4 \alpha} \left[\sin^4 \alpha - 2\cos\theta(2\cos^2\theta - \sin^2\alpha)\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$+ <\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{12}} - \underline{/a_{22}})>_\alpha \frac{1}{\sin^4\alpha} *$$

$$\left[4\cos^4\theta - 5\cos^2\theta\sin^2\alpha + \sin^4\alpha + \cos\theta\sin^2\alpha\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$+ <\eta_{22}>_\alpha \frac{1}{2\sin^4\alpha}\left[\sin^2\alpha - \cos^2\theta\right] *$$

$$\left[\sin^2\alpha - 2\cos\theta\sqrt{\sin^2\alpha - \cos^2\theta}\right] .$$

$$<\eta_{\nu-\zeta,\nu}>_\alpha^\ell = <\eta_{11}>_\alpha \frac{\cos^2\theta}{2\sin^4\alpha}\left[\sin^2\alpha - 2\cos\theta\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$+ <\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{12}})>_\alpha \frac{\cos^2\theta}{\sin^4\alpha}\left[-4\cos^2\theta + 3\sin^2\alpha\right.$$

$$\left. - \frac{\sin^2\alpha}{\cos\theta}\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$+ <\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}}\cos(\underline{/a_{11}} - \underline{/a_{22}})>_\alpha \frac{\cos\theta}{\sin^4\alpha}(2\cos^2\theta - \sin^2\alpha)\sqrt{\sin^2\alpha - \cos^2\theta}$$

$$+ \frac{1}{2} <\eta_{12}>_\alpha \frac{1}{\sin^4\alpha}\left[\sin^4\alpha + 2\cos\theta(2\cos^2\theta - \sin^2\theta)\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$+ <\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}}\cos(\underline{/a_{12}} - \underline{/a_{22}})>_\alpha \frac{1}{\sin^4\alpha} *$$

$$\left[4\cos^2\theta - 5\cos^2\theta\sin^2\alpha + \sin^4\theta - \cos\theta\sin^2\alpha\sqrt{\sin^2\alpha - \cos^2\theta}\right]$$

$$+ <\eta_{22}>_\alpha \frac{1}{2\sin^4\alpha}\left[\sin^2\alpha - \cos^2\theta\right] * \left[\sin^2\alpha + 2\cos\theta\sqrt{\sin^2\alpha - \cos^2\theta}\right] .$$

The same results could have been obtained for $\gamma = \pi/4$ and $\zeta = \pi/4$. Upon substitution of the above expressions into Eq. (23), Eq. (25f) is obtained.

(26a, b) $\underline{\gamma = 0}$

$$\therefore \cos\nu = \frac{\cos\theta}{\sin\alpha} \quad ; \quad \therefore \sin\nu = \pm \frac{1}{\sin\alpha}\sqrt{\sin^2\alpha - \cos^2\theta}$$

$$\sin 2\nu = \pm \frac{2\cos\theta}{\sin^2\alpha} \sqrt{\sin^2\alpha - \cos^2\theta}$$

Substitute these into Eqs. (4) and (5).

$$<\eta_{L_\nu \atop R}>^u_\alpha = \frac{1}{2} <\eta_{11}>_\alpha \frac{\cos^2\theta}{\sin^2\alpha}$$

$$+ <\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{11}} - \underline{/a_{12}})>_\alpha \frac{\cos\theta}{\sin^2\alpha}\sqrt{\sin^2\alpha - \cos^2\theta}$$

$$+ <\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{12}} - \underline{/a_{22}})>_\alpha \frac{\cos\theta}{\sin^2\alpha}\sqrt{\sin^2\alpha - \cos^2\theta}$$

$$\pm <\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}} \sin(\underline{/a_{11}} - \underline{/a_{12}})>_\alpha \frac{\cos^2\theta}{\sin^2\alpha}$$

$$\pm <\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}} \sin(\underline{/a_{12}} - \underline{/a_{22}})>_\alpha \frac{(\sin^2\alpha - \cos^2\theta)}{\sin^2\alpha}$$

$$\pm <\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}} \sin(\underline{/a_{11}} - \underline{/a_{22}})>_\alpha \frac{\cos\theta}{\sin^2\alpha}\sqrt{\sin^2\alpha - \cos^2\theta}$$

$$+ \frac{1}{2} <\eta_{12}>_\alpha + \frac{1}{2} <\eta_{22}>_\alpha \frac{(\sin^2\alpha - \cos^2\theta)}{\sin^2\alpha}$$

$$<\eta_{L_\nu \atop R}>^\ell_\alpha = \frac{1}{2} <\eta_{11}>_\alpha \frac{\cos^2\theta}{\sin^2\alpha}$$

$$- <\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{11}} - \underline{/a_{12}})>_\alpha \frac{\cos\theta}{\sin^2\alpha}\sqrt{\sin^2\alpha - \cos^2\theta}$$

(26a, b) cont.

$$- \langle \eta_{22}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \cos(\angle a_{12} - \angle a_{22}) \rangle_\alpha \frac{\cos\theta}{\sin^2\alpha} \sqrt{\sin^2\alpha - \cos^2\theta}$$

$$\pm \langle \eta_{11}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \sin(\angle a_{11} - \angle a_{12}) \rangle_\alpha \frac{\cos^2\theta}{\sin^2\alpha}$$

$$\pm \langle \eta_{22}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \sin(\angle a_{12} - \angle a_{22}) \rangle_\alpha \frac{(\sin^2\alpha - \cos^2\theta)}{\sin^2\alpha}$$

$$\mp \langle \eta_{11}^{\frac{1}{2}} \eta_{22}^{\frac{1}{2}} \sin(\angle a_{12} - \angle a_{22}) \rangle_\alpha \frac{\cos\theta}{\sin^2\alpha} \sqrt{\sin^2\alpha - \cos^2\theta}$$

$$+ \frac{1}{2} \langle \eta_{12} \rangle_\alpha + \frac{1}{2} \langle \eta_{22} \rangle_\alpha \frac{(\sin^2\alpha - \cos^2\theta)}{\sin^2\alpha} .$$

Upon substitution of these results into Eq. (23), Eq. (26a, b) is obtained.

(26c, d) $\underline{\gamma = \pi/2}$

$$\therefore \cos\nu = \pm \frac{1}{\sin\alpha} \sqrt{\sin^2\alpha - \cos^2\theta} ; \quad \sin\nu = -\frac{\cos\theta}{\sin\alpha}$$

$$\sin 2\nu = \mp \frac{2\cos\theta}{\sin^2\alpha} \sqrt{\sin^2\alpha - \cos^2\theta}$$

$$\therefore \langle \eta_{\substack{L \\ R}\nu} \rangle_\alpha^u = \frac{1}{2} \langle \eta_{11} \rangle_\alpha \frac{(\sin^2\alpha - \cos^2\theta)}{\sin^2\alpha}$$

$$- \langle \eta_{11}^{\frac{1}{2}} \eta_{22}^{\frac{1}{2}} \cos(\angle a_{11} - \angle a_{12}) \rangle_\alpha \frac{\cos\theta}{\sin^2\alpha} \sqrt{\sin^2\alpha - \cos^2\theta}$$

$$- \langle \eta_{22}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \cos(\angle a_{12} - \angle a_{22}) \rangle_\alpha \frac{\cos\theta}{\sin^2\alpha} \sqrt{\sin^2\alpha - \cos^2\theta}$$

$$\pm \langle \eta_{11}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \sin(\angle a_{11} - \angle a_{22}) \rangle_\alpha \frac{(\sin^2\alpha - \cos^2\theta)}{\sin^2\alpha}$$

$$\pm \langle \eta_{22}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \sin(\angle a_{12} - \angle a_{22}) \rangle_\alpha \frac{\cos^2\theta}{\sin^2\alpha}$$

$$\mp \langle \eta_{11}^{\frac{1}{2}} \eta_{22}^{\frac{1}{2}} \sin(\angle a_{11} - \angle a_{22}) \rangle_\alpha \frac{\cos\theta}{\sin^2\alpha} + \frac{1}{2} \langle \eta_{12} \rangle_\alpha + \frac{1}{2} \langle \eta_{22} \rangle_\alpha \frac{\cos^2\theta}{\sin^2\alpha}$$

(26c, d) cont.

$$<\eta_{\substack{L\\R}\nu}>_\alpha^{\ell} = \frac{1}{2} <\eta_{11}>_\alpha \frac{(\sin^2\alpha - \cos^2\theta)}{\sin^2\alpha}$$

$$+ <\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{11}} - \underline{/a_{12}})>_\alpha \frac{\cos\theta}{\sin^2\alpha} \sqrt{\sin^2\alpha - \cos^2\theta}$$

$$+ <\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{12}} - \underline{/a_{22}})>_\alpha \frac{\cos\theta}{\sin^2\alpha} \sqrt{\sin^2\alpha - \cos^2\theta}$$

$$\pm <\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}} \sin(\underline{/a_{11}} - \underline{/a_{12}})>_\alpha \frac{(\sin^2\alpha - \cos^2\theta)}{\sin^2\alpha}$$

$$\pm <\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}} \sin(\underline{/a_{12}} - \underline{/a_{22}})>_\alpha \frac{\cos^2\theta}{\sin^2\alpha}$$

$$\pm <\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}} \sin(\underline{/a_{11}} - \underline{/a_{22}})>_\alpha \frac{\cos\theta}{\sin^2\alpha} \sqrt{\sin^2\alpha - \cos^2\theta}$$

$$+ \frac{1}{2} <\eta_{12}>_\alpha + \frac{1}{2} <\eta_{22}>_\alpha \frac{\cos^2\theta}{\sin^2\alpha} \quad .$$

Upon substitution of the above into Eq. (23), Eq. (26c, d) result.

(26e, f) $\underline{\gamma = \pi/4}$

$$\cos\nu = \frac{1}{\sqrt{2}\cdot\sin\alpha} (\cos\theta \pm \sqrt{\sin^2\alpha - \cos^2\theta});$$

$$\sin\nu = \frac{1}{\sqrt{2}\cdot\sin\alpha} (-\cos\theta \pm \sqrt{\sin^2\alpha - \cos^2\theta})$$

$$\sin 2\nu = -\frac{1}{\sin^2\alpha} [2\cos^2\theta - \sin^2\alpha]$$

(26e, f) cont.

$$\therefore \langle \eta_{L_{R}\nu} \rangle_\alpha^u = \frac{1}{4} \langle \eta_{11} \rangle_\alpha \frac{1}{\sin^2\alpha} \left[\sin^2\alpha + 2\cos\theta \sqrt{\sin^2\alpha - \cos^2\theta} \right]$$

$$- \frac{1}{2} \langle \eta_{11}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{11}} - \underline{/a_{12}}) \rangle_\alpha \frac{1}{\sin^2\alpha} \left[2\cos^2\theta - \sin^2\alpha \right]$$

$$- \frac{1}{2} \langle \eta_{22}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{12}} - \underline{/a_{22}}) \rangle_\alpha \frac{1}{\sin^2\alpha} \left[2\cos^2\theta - \sin^2\alpha \right]$$

$$\pm \frac{1}{2} \langle \eta_{11}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \sin(\underline{/a_{11}} - \underline{/a_{12}}) \rangle_\alpha \frac{1}{\sin^2\alpha} \left[\sin^2\alpha + 2\cos\theta \sqrt{\sin^2\alpha - \cos^2\theta} \right]$$

$$\mp \frac{1}{2} \langle \eta_{11}^{\frac{1}{2}} \eta_{22}^{\frac{1}{2}} \sin(\underline{/a_{11}} - \underline{/a_{22}}) \rangle_\alpha \frac{1}{\sin^2\alpha} \left[2\cos^2\theta - \sin^2\alpha \right]$$

$$\pm \frac{1}{2} \langle \eta_{22}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \sin(/a_{12} - /a_{22}) \rangle_\alpha \frac{1}{\sin^2\alpha} \left[\sin^2\alpha - 2\cos\theta \sqrt{\sin^2\alpha - \cos^2\theta} \right]$$

$$\pm \frac{1}{2} \langle \eta_{12} \rangle_\alpha + \frac{1}{4} \langle \eta_{22} \rangle_\alpha \frac{1}{\sin^2\alpha} \left[\sin^2\alpha + 2\cos\theta \sqrt{\sin^2\alpha - \cos^2\theta} \right].$$

$$\langle \eta_{L_{R}\nu} \rangle_\alpha^\ell = \frac{1}{4} \langle \eta_{11} \rangle_\alpha \frac{1}{\sin^2\alpha} \left[\sin^2\alpha - 2\cos\theta \sqrt{\sin^2\alpha - \cos^2\theta} \right]$$

$$- \frac{1}{2} \langle \eta_{11}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{11}} - \underline{/a_{12}}) \rangle_\alpha \frac{1}{\sin^2\alpha} \left[2\cos^2\theta - \sin^2\alpha \right]$$

$$- \frac{1}{2} \langle \eta_{22}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{12}} - \underline{/a_{22}}) \rangle_\alpha \frac{1}{\sin^2\alpha} \left[2\cos^2\theta - \sin^2\alpha \right]$$

$$\pm \frac{1}{2} \langle \eta_{11}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \sin(\underline{/a_{11}} - \underline{/a_{12}}) \rangle_\alpha \frac{1}{\sin^2\alpha} \left[\sin^2\alpha - 2\cos\theta \sqrt{\sin^2\alpha - \cos^2\theta} \right]$$

$$\mp \frac{1}{2} \langle \eta_{11}^{\frac{1}{2}} \eta_{22}^{\frac{1}{2}} \sin(\underline{/a_{11}} - \underline{/a_{22}}) \rangle_\alpha \frac{1}{\sin^2\alpha} \left[2\cos^2\theta - \sin^2\alpha \right]$$

(26e, f) cont.

$$\pm \frac{1}{2} \langle \eta_{22}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \sin(\underline{/a_{12}} - \underline{/a_{22}}) \rangle_\alpha \frac{1}{\sin^2\alpha} [\sin^2\alpha + 2\cos\theta \sqrt{\sin^2\alpha - \cos^2\theta}]$$

$$+ \frac{1}{2} \langle \eta_{12} \rangle_\alpha + \frac{1}{4} \langle \eta_{22} \rangle_\alpha \frac{1}{\sin^2\alpha} [\sin^2\alpha + 2\cos\theta \sqrt{\sin^2\alpha - \cos^2\theta}] .$$

(26g, h) $\underline{\gamma = -\pi/4}$

$$\cos\nu = \frac{1}{\sqrt{2}\sin\alpha} (\cos\theta \mp \sqrt{\sin^2\alpha - \cos^2\theta});$$

$$\sin\nu = \frac{1}{\sqrt{2}\sin\alpha} (\cos\theta \pm \sqrt{\sin^2\alpha - \cos^2\theta})$$

$$\sin 2\nu = + \frac{1}{\sin^2\alpha} [2\cos^2\theta - \sin^2\alpha]$$

$$\therefore \langle \eta_{{L \atop R}\nu} \rangle_\alpha^u + \frac{1}{4} \langle \eta_{11} \rangle_\alpha \frac{1}{\sin^2\alpha} [\sin^2\alpha - 2\cos\theta \sqrt{\sin^2\alpha - \cos^2\theta}]$$

$$+ \frac{1}{2} \langle \eta_{11}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{11}} - \underline{/a_{\cdot 2}}) \rangle_\alpha \frac{1}{\sin^2\alpha} [2\cos^2\theta - \sin^2\alpha]$$

$$+ \frac{1}{2} \langle \eta_{22}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{12}} - \underline{/a_{22}} \rangle_\alpha \frac{1}{\sin^2\alpha} [2\cos^2\theta - \sin^2\alpha]$$

$$\pm \frac{1}{2} \langle \eta_{11}^{\frac{1}{2}} \eta_{22}^{\frac{1}{2}} \sin(\underline{/a_{11}} - \underline{/a_{22}}) \rangle_\alpha \frac{1}{\sin^2\alpha} [2\cos\ \theta - \sin\ \alpha]$$

$$\pm \frac{1}{2} \langle \eta_{11}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \sin(\underline{/a_{11}} - \underline{/a_{12}}) \rangle_\alpha [\sin^2\alpha - 2\cos\theta \sqrt{\sin^2\alpha - \cos^2\theta}]$$

$$\pm \frac{1}{2} \langle \eta_{22}^{\frac{1}{2}} \eta_{12}^{\frac{1}{2}} \sin(\underline{/a_{12}} - \underline{/a_{22}}) \rangle_\alpha [\sin^2\alpha + 2\cos\theta \sqrt{\sin^2\alpha - \cos^2\theta}]$$

$$+ \frac{1}{2} \langle \eta_{12} \rangle_\alpha + \frac{1}{4} \langle \eta_{22} \rangle_\alpha \frac{1}{\sin^2\alpha} [\sin^2\alpha + 2\cos\theta \sqrt{\sin^2\alpha - \cos^2\theta}]$$

(26g, h) cont.

$$<\eta_{\substack{L\\R}\nu}>^{\ell}_{\alpha} = \frac{1}{4}<\eta_{11}>_{\alpha} \frac{1}{\sin^2\alpha} [\sin^2\alpha + 2\cos\theta \sqrt{\sin^2\alpha - \cos^2\theta}]$$

$$+ \frac{1}{2}<\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{11}} - \underline{/a_{12}})>_{\alpha} \frac{1}{\sin^2\alpha} [2\cos^2\theta - \sin^2\alpha]$$

$$+ \frac{1}{2}<\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}} \cos(\underline{/a_{12}} - \underline{/a_{22}})>_{\alpha} \frac{1}{\sin^2\alpha} [2\cos^2\theta - \sin^2\alpha]$$

$$\pm \frac{1}{2}<\eta_{11}^{\frac{1}{2}}\eta_{22}^{\frac{1}{2}} \sin(\underline{/a_{12}} - \underline{/a_{22}})>_{\alpha} \frac{1}{\sin^2\alpha} [2\cos^2\theta - \sin^2\alpha]$$

$$\pm \frac{1}{2}<\eta_{11}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}} \sin(\underline{/a_{11}} - \underline{/a_{12}})>_{\alpha} [\sin^2\alpha + 2\cos\theta \sqrt{\sin^2\alpha - \cos^2\theta}]$$

$$\pm \frac{1}{2}<\eta_{22}^{\frac{1}{2}}\eta_{12}^{\frac{1}{2}} \sin(\underline{/a_{12}} - \underline{/a_{22}})>_{\alpha} [\sin^2\alpha - 2\cos\theta \sqrt{\sin^2\alpha - \cos^2\theta}]$$

$$+ \frac{1}{2}<\eta_{12}>_{\alpha} + \frac{1}{4}<\eta_{22}>_{\alpha} \frac{1}{\sin^2\alpha} [\sin^2\alpha - 2\cos\theta \quad \sin^2\alpha - \cos^2\theta] .$$

The above two quantities are then substituted in Eq. (23) to give Eqs. (26g) and (26h).

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