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DEFINITE AND INDEFINITE FORMS OF MAXWELL'S EQUATIONS FOR MOVING MEDIA +

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Introduction

In a recent article reviewing the electrodynamics of moving media(1) the author compared different formulations with Minkowski's theory and pointed out the importance of the constitutive relations in providing a complete theory. In discussing the constitutive relations, he used merely the first order theory to show the main theme of that paper. In view of the interest and, perhaps, some confusion expressed by several writers(2, 3, 4) on this subject it seems desirable that a more thorough comparison based upon the exact relativistic transformation should be given. Such a comparison may reveal more clearly the intimate relations between various formulations. We emphasize once more that all the latter formulations can be derived from Minkowski's theory by making use of the relativistic transformation of the field vectors. We demonstrate this fact by using the same technique previously illustrated by the method of motional flux, but in a more precise manner. Finally, the exact constitutive relations for various formulations are derived from Minkowski's theory for moving isotropic media. author

Maxwell-Minkowski Equations and the Transformation of the Field Vectors

As reviewed in detail by Sommerfeld(5), Maxwell's equations are invariant in all inertial systems which are moving uniformly with respect to each other. The

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three independent equations are:

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \tag{1}$$

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \tag{2}$$

$$\nabla \cdot \bar{J} = -\frac{\partial \rho}{\partial t} \tag{3}$$

These are supplemented by the two auxiliary, but not independent, equations:

$$\nabla \cdot \bar{D} = \rho \tag{4}$$

$$\nabla \cdot \bar{B} = 0 \tag{5}$$

The nomenclature is given by

$\bar{E}$, $\bar{D}$ = electric field vectors

$\bar{B}$, $\bar{H}$ = magnetic field vectors

$\bar{J}$, ρ = free-current and free-charge densities.

Equations (1) to (3) contain a total of sixteen scalar functions, i.e., five vector functions and one scalar function, and these are defined in seven scalar differential equations. Thus, nine more scalar equations or relations are needed to form a complete system of equations. These additional equations are described by the constitutive relations between the field vectors. For convenience, we shall designate Eqs. (1) to (3) as Maxwell's equations in the "indefinite" form as long as the constitutive relations are unknown or unspecified. They become "definite" when the constitutive relations are stated.

The Relativistic Transformation of the Field Variables

When two inertial systems are moving with respect to each other in the z-direction as shown in Fig. 1, the field variables defined in these two systems transform according to the relations

$$\bar{E}' = \bar{\bar{\gamma}} \cdot (\bar{E} + \bar{v} \times \bar{B}) \tag{6}$$

$$\bar{B}' = \bar{\bar{\gamma}} \cdot (\bar{B} - \frac{\bar{v} \times \bar{E}}{c^2}) \tag{7}$$

$$\bar{H}' = \bar{\bar{\gamma}} \cdot (\bar{H} - \bar{v} \times \bar{D}) \tag{8}$$

$$\bar{D}' = \bar{\bar{\gamma}} \cdot (\bar{D} + \frac{\bar{v} \times \bar{H}}{c^2}) \tag{9}$$

$$\bar{J}' = \frac{1}{\sqrt{1-\beta^2}} \bar{\bar{\gamma}}^{-1} \cdot (\bar{J} - \rho \bar{v}) \tag{10}$$

$$\rho' = \frac{1}{\sqrt{1-\beta^2}} (\rho - \frac{\bar{v} \cdot \bar{J}}{c^2}) \tag{11}$$

where

$$c = (\mu_o \epsilon_o)^{-1/2}$$

$$\beta = v/c$$

$$\bar{\bar{\gamma}} = \begin{bmatrix} (1-\beta^2)^{-1/2} & 0 & 0 \\ 0 & (1-\beta^2)^{-1/2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\bar{\bar{\gamma}}^{-1}$ denotes the reciprocal of $\bar{\bar{\gamma}}$. If one introduces two material field vectors $\bar{P}$ and $\bar{M}$ such that

$$\bar{D} = \epsilon_o \bar{E} + \bar{P} \tag{12}$$

$$\bar{B} = \mu_o (\bar{H} + \bar{M}) \tag{13}$$

and similarly

$$\bar{D}' = \epsilon_o \bar{E}' + \bar{P}' \quad (14)$$

$$\bar{B}' = \mu_o(\bar{H}' + \bar{M}') \quad (15)$$

then, by substituting (12) through (15) into (6) through (9), he finds that the material field vectors transform according to the relations

$$\bar{P}' = \bar{\bar{\gamma}} \cdot (\bar{P} - \frac{1}{c^2} \bar{v} \times \bar{M}) \quad (16)$$

$$\bar{M}' = \bar{\bar{\gamma}} \cdot (\bar{M} + \bar{v} \times \bar{P}) \quad (17)$$

Various Indefinite Forms of Maxwell's Equations

As shown previously by the method of motional flux[6], there are many indefinite forms of the Maxwell's equations. One may describe these equations without altering their invariant property. We shall derive these forms once more by making use of the relations stated in the previous section. This new approach will reveal more clearly the role played by the field vectors in EPHMv formulation discussed later. We, first, write the Maxwell's equations in the EPHM form, i.e.,

$$\nabla \times \bar{E} = -\frac{\partial}{\partial t} \mu_o(\bar{H} + \bar{M}) \quad (18)$$

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial}{\partial t} (\epsilon_o \bar{E} + \bar{P}) \quad (19)$$

Since the equation of continuity relating $\bar{J}$ and ρ is the same for all the indefinite forms, we do not have to consider it again. By solving (16) and (17) for $\bar{P}$ and $\bar{M}$ and substituting them into (18) and (19), one obtains

$$\nabla \times \bar{E} = -\frac{\partial}{\partial t}\,\mu_o \left[\bar{H} + \bar{\bar{\gamma}} \cdot (\bar{M}' - \bar{v} \times \bar{P}')\right] \tag{20}$$

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial}{\partial t}\left[\epsilon_o \bar{E} + \bar{\bar{\gamma}} \cdot (\bar{P}' + \frac{1}{c^2}\,\bar{v} \times \bar{M}')\right] \tag{21}$$

Equations (20) and (21) had previously been presented by Born[7]. Now, let us define

$$\bar{\bar{\gamma}} \cdot \bar{P}' = \bar{P}_c \tag{22}$$

$$\bar{\bar{\gamma}} \cdot \bar{M}' = \bar{M}_c \tag{23}$$

$$\epsilon_o \bar{E} + \frac{1}{c^2}\,\bar{\bar{\gamma}} \cdot (\bar{v} \times \bar{M}') = \epsilon_o \bar{E} + \frac{1}{c^2}\bar{v} \times \bar{M}_c = \epsilon_o \bar{E}_c \tag{24}$$

$$\bar{H} - \bar{\bar{\gamma}} \cdot (\bar{v} \times \bar{P}') = \bar{H} - \bar{v} \times \bar{P}_c = \bar{H}_c \tag{25}$$

then (20) and (21) can be written in the form

$$\nabla \times (\bar{E}_c - \mu_o \bar{v} \times \bar{M}_c) = \bar{J} + \frac{\partial}{\partial t}\,\mu_o (\bar{H}_c + \bar{M}_c) \tag{26}$$

$$\nabla \times (\bar{H}_c + \bar{v} \times \bar{P}_c) = \bar{J} + \frac{\partial}{\partial t}\,(\epsilon_o \bar{E}_c + \bar{P}_c) \tag{27}$$

Equations (26) and (27) have been previously derived by Chu[8] using a kinematic method. The undesirable feature of that derivation is that it postulates the so-called magnetic charge model. As we see from the present derivation such a fictitious model is not necessary. We can trace the origin of the field quantities E_c, P_c, H_c and M_c from (22) to (25) which define them.

If we write (20) in the form

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \tag{28}$$

Eq. (21) can be changed to

$$\nabla \times (\frac{\bar{B}}{\mu_o} - \bar{M}_c + \bar{v} \times \bar{P}_c) = \bar{J} + \frac{\partial}{\partial t}\,(\epsilon_o \bar{E} + \bar{P}_c + \frac{1}{c^2}\bar{v} \times \bar{M}_c)\,. \tag{29}$$

Equations (28) and (29) correspond to the EPBMv form. They were first described by Panofsky and Phillips[9] for the special case when $\bar{M}_c = 0$. They have been designated by Fano-Chu-Adler[8] as the Amperian model. It should be pointed out that the material vectors $\bar{M}_c$ and $\bar{P}_c$ appearing in the EPHMv form and in the EPBMv form are the same and are related to the material vectors defined in the primed system by (22) and (23). Table I summarizes the various indefinite forms.

TABLE I: SOME INDEFINITE FORMS OF MAXWELL'S EQUATIONS

Form	Equivalent Quantities			
EDBH	$\bar{E}$	$\bar{D}$	$\bar{B}$	$\bar{H}$
EPHM	$\bar{E}$	$\epsilon_o \bar{E} + \bar{P}$	$\mu_o(\bar{H} + \bar{M})$	$\bar{H}$
EPBM	$\bar{E}$	$\epsilon_o \bar{E} + \bar{P}$	$\bar{B}$	$\frac{\bar{B}}{\mu_o} - \bar{M}$
EPHMv	$\bar{E}_c - \mu_o \bar{v} \times \bar{M}_c$	$\epsilon_o \bar{E}_c + \bar{P}_c$	$\mu_o(\bar{H}_c + \bar{M}_c)$	$\bar{H}_c + \bar{v} \times \bar{P}_c$
EPBMv	$\bar{E}$	$\epsilon_o \bar{E} + \bar{P}_c + \frac{1}{c^2} \bar{v} \times \bar{M}_c$	$\bar{B}$	$\frac{\bar{B}}{\mu_o} - \bar{M}_c + \bar{v} \times \bar{P}_c$
Remark:	$\bar{P}_c = \bar{\bar{\gamma}} \cdot \bar{P}' = \bar{\bar{\gamma}} \cdot \bar{\bar{\gamma}} \cdot (\bar{P} - \frac{1}{c^2} \bar{v} \times \bar{M})$ $\bar{M}_c = \bar{\bar{\gamma}} \cdot \bar{M}' = \bar{\bar{\gamma}} \cdot \bar{\bar{\gamma}} \cdot (\bar{M} + \bar{v} \times \bar{P})$			

Before we discuss the constitutive relations we would like to call attention to the apparently different expression of the Lorentz force. From Table I, it is evident that

$$\bar{E} + \bar{v} \times \bar{B} = \bar{E}_o + \mu_o \bar{v} \times \bar{H}_c \tag{30}$$

Chu[8] postulates $\bar{E}_c + \mu_o \bar{v} \times \bar{H}_c$ as the force exerted on a unit charge in his derivation of the EPHMv form. According to our presentation, this is just an alternative expression of the Lorentz force. An analogous expression which plays an equally important role in the EPHMv formulation is

$$\bar{H} - \bar{v} \times \bar{D} = \bar{H}_c - \epsilon_o \bar{v} \times \bar{E}_c \tag{31}$$

Except for a relativistic correction factor, we merely treat this as the H-field defined in the primed system. In particular, there is no need to identify it as the force exerted on a unit "magnetic charge," as most of us must be convinced by now that the science of electromagnetism is just the science of electricity. Electric charges and electric currents are the sole agencies, which are responsible for the electromagnetic field and their existence has been experimentally verified.

Constitutive Relations for Uniformly Moving Isotropic Media

Minkowski, recognizing the invariant property of Maxwell's equation for material media, was the first to apply the special theory of relativity to determine the constitutive relations for a uniformly moving medium, provided its properties at rest are known. If one identities the primed system as being at rest with respect to an isotropic medium, then

$$\bar{D}' = \epsilon' \bar{E}' \tag{32}$$

$$\bar{B}' = \mu' \bar{H}' \tag{33}$$

$$\bar{J}' = \sigma' \bar{E}' \tag{34}$$

In a lossy medium, $\sigma' \neq 0$, because of the relaxation phenomenon. We then have

$$\rho' = 0 \tag{35}$$

By substituting these relations for (6) through (11) and simplifying the results one finds

$$\bar{D} + \frac{1}{c^2} \bar{v} \times \bar{H} = \epsilon'(\bar{E} + \bar{v} \times \bar{B}) \tag{36}$$

$$\bar{B} - \frac{1}{c^2} \bar{v} \times \bar{E} = \mu'(\bar{H} - \bar{v} \times \bar{D}) \tag{37}$$

$$\bar{J} = \frac{\sigma'}{\sqrt{1-\beta^2}} (\bar{E} + \bar{v} \times \bar{B}) \tag{38}$$

By solving for $\bar{B}$ and $\bar{D}$ in terms of $\bar{E}$ and $\bar{H}$ from (36) and (37) one obtains

$$\bar{D} = \epsilon' \bar{\bar{\alpha}} \cdot \bar{E} + \Omega \times \bar{H} \tag{39}$$

$$\bar{B} = \mu' \bar{\bar{\alpha}} \cdot \bar{H} - \Omega \times \bar{E} \tag{40}$$

where

$$\bar{\Omega} = \frac{(n^2-1)\beta}{(1-n^2\beta^2)c} \hat{z}$$

$$\beta = \frac{v}{c}, \qquad n = \left(\frac{\mu\epsilon}{\mu'\epsilon'}\right)^{1/2}$$

$$\bar{\bar{\alpha}} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad a = \frac{1-\beta^2}{1-n^2\beta^2} .$$

It is to be observed that the conduction current defined by (38) does not have the same connotation as the "conduction current" defined by Sommerfeld[5]. In fact, Sommerfeld did use the quotation marks to identify his current, with perhaps some good reasons. We believe our expression is the proper one to use. The result can also be derived by careful examination of the realistivistic transformation of the convection current, but we will not elaborate upon that here.

By making use of Minkowski's relations as given by (39) and (40), the constitutive relations for the field vectors defined in other indefinite forms can be derived, particularly, the ones associated with the EPHMv form. In the first place, we have

$$\begin{aligned}\bar{P}' &= (\epsilon' - \epsilon_o)\bar{E}' \\ &= (\epsilon' - \epsilon_o)\bar{\bar{\gamma}} \cdot (\bar{E} + \bar{v} \times \bar{B}) \\ &= (\epsilon' - \epsilon_o)\bar{\bar{\gamma}} \cdot \bar{\bar{\alpha}} \cdot (\bar{E} + \mu' \bar{v} \times \bar{H}) \qquad (41)\end{aligned}$$

and

$$\begin{aligned}\mu_o \bar{M}' &= (\mu' - \mu_o)\bar{H}' \\ &= (\mu' - \mu_o)\bar{\bar{\gamma}} \cdot (\bar{H} - \bar{v} \times \bar{D}) \\ &= (\mu' - \mu_o)\bar{\bar{\gamma}} \cdot \bar{\bar{\alpha}} \cdot (\bar{H} - \epsilon' \bar{v} \times \bar{E}) \qquad (42)\end{aligned}$$

The constitutive relations for the material vectors involved in the EPHM and EPBM formulations are given by

$$\begin{aligned}\bar{P} &= \bar{D} - \epsilon_o \bar{E} \\ &= (\epsilon' \bar{\bar{\alpha}} - \epsilon_o \bar{\bar{I}}) \cdot \bar{E} + \bar{\Omega} \times \bar{H} \qquad (43)\end{aligned}$$

and

$$\begin{aligned}\mu_o \bar{M} &= \bar{B} - \mu_o \bar{H} \\ &= (\mu' \bar{\bar{\alpha}} - \mu_o \bar{\bar{I}}) \cdot \bar{H} - \bar{\Omega} \times \bar{E} \qquad (44)\end{aligned}$$

The constitutive relations for the material vectors involved in the EPHMv formulation are given by

$$
\begin{aligned}
\bar{P}_c &= \bar{\bar{\gamma}} \cdot \bar{P}' \\
&= (\epsilon' - \epsilon_o) \bar{\bar{\gamma}} \cdot \bar{\bar{\gamma}} \cdot (\bar{E} + \bar{v} \times \bar{B}) \\
&= (\epsilon' - \epsilon_o) \bar{\bar{\gamma}} \cdot \bar{\bar{\gamma}} \cdot (\bar{E}_c + \mu_o \bar{v} \times \bar{H}_c)
\end{aligned} \tag{45}
$$

and

$$
\begin{aligned}
\mu_o \bar{M}_c &= \bar{\bar{\gamma}} \cdot (\mu_o \bar{M}') \\
&= (\mu' - \mu_o) \bar{\bar{\gamma}} \cdot \bar{\bar{\gamma}} \cdot (\bar{H} - \bar{v} \times \bar{D}) \\
&= (\mu' - \mu_o) \bar{\bar{\gamma}} \cdot \bar{\bar{\gamma}} \cdot (\bar{H}_c - \epsilon_o \bar{v} \times \bar{E}_c)
\end{aligned} \tag{46}
$$

Equations (45) and (46) had previously been given by Fano-Chu-Adler[3] based upon a rather complicated scheme as discussed in their four dimensional formulation. We observe, further, in view of (6), (8), (30) and (31), that

$$\bar{E}_c + \mu_o \bar{v} \times \bar{H}_c = \bar{\bar{\gamma}}^{-1} \cdot \bar{E}' \tag{47}$$

$$\bar{H}_c - \epsilon_o \bar{v} \times \bar{E}_c = \bar{\bar{\gamma}}^{-1} \cdot \bar{H}' \tag{48}$$

By solving (47) and (48) for $\bar{E}_c$ and $\bar{H}_c$ in terms of $\bar{E}'$ and $\bar{H}'$, we obtain

$$\bar{E}_c = \bar{\bar{\gamma}} \cdot (\bar{E}' - \mu_o \bar{v} \times \bar{H}') \tag{49}$$

$$\bar{H}_c = \bar{\bar{\gamma}} \cdot (\bar{H}' + \epsilon_o \bar{v} \times \bar{E}') \tag{50}$$

Equations (49) and (50) together with (22) and (23) perhaps reveal more clearly the real nature of the EPHMv formulation from the point of view of Minkowski's theory. These four relations, of course, are independent of the constitutive relations of the material medium.

Conclusion

A diagnostic presentation of various indefinite forms of Maxwell's equations has been given in this article in which we have used Minkowski's theory to coordinate all the different formulations. In particular, it has been shown that the magnetic charge model is not necessary in order to establish the EPHMv formulations; all the field vectors appearing in various alternative forms can properly be identified in terms of the field vectors defined in two initial systems one of which is stationary with respect to medium while the other one is moving with respect to the medium. Finally, the constitutive relations for a moving isotropic medium as appearing in various formulations can be immediately obtained by making use of Minkowski's result.

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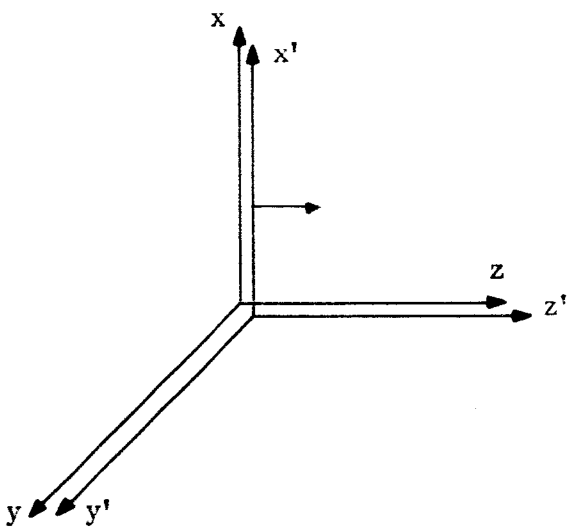

FIG. 1: TWO INITIAL SYSTEMS IN RELATIVE MOTION WITH A VELOCITY $\bar{v}$ IN THE z-DIRECTION.

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