1546-FTR 3

Honeywell Inc.

OPTIMAL STABILIZATION OF

NONLINEAR DYNAMICAL SYSTEMS

October 1965

Prepared under Contract No. NAS8-5222

RESEARCH AND STUDY IN SYSTEM OPTIMIZATION

National Aeronautics and Space Administration
Office of Astrodynamics and Guidance Theory
Aero and Astrodynamics Division,
George C. Marshall Space Flight Center
Huntsville, Alabama

SYSTEMS AND RESEARCH DIVISION, RESEARCH DEPARTMENT

2345 WALNUT STREET, ST. PAUL, MINNESOTA 55113

1546-FTR 3

OPTIMAL STABILIZATION OF NONLINEAR DYNAMICAL SYSTEMS

by D. L. Lukes

October 1965

Prepared under Contract No. NAS8-5222

RESEARCH AND STUDY IN SYSTEM OPTIMIZATION

National Aeronautics and Space Administration
Office of Astrodynamics and Guidance Theory
Aero and Astrodynamics Division,
George C. Marshall Space Flight Center
Huntsville, Alabama

Approved by: O. Hugo Schuck
O. Hugo Schuck
Director of Research

HONEYWELL INC.

SYSTEMS AND RESEARCH DIVISION, RESEARCH DEPARTMENT

St. Paul, Minnesota

E.L.R.

ABSTRACT

A mathematical formulation of a problem of optimizing nonlinear closed-loop dynamical control systems is presented. It is shown that an integral performance functional induces a partial ordering of the closed-loop controls which stabilize the system. Definitions of minimal and optimal controls are made.

It is proved that for certain conditions on the nonlinearities in the equation of the system any minimal control is optimal and a technique is presented for constructing a minimal control. Thus the existence and construction of optimal closed-loop controls for many systems has been established.

FOREWORD

This document is the final report on one phase of the study carried on under contract NAS8-5222, sponsored by the Office of Astrodynamics and Guidance Theory, Aero and Astrodynamics Division of NASA at the George C. Marshall Space Flight Center, Huntsville, Alabama. Mr. C. C. Dearman was the project monitor for this program. The purpose of this portion of the work was to develop an optimal synthesis technique for closed-loop systems using Liapunov techniques. The other portion of the work was presented in Honeywell Report 1546-FTR 2, "Predictive P-Guidance." In this a simplified guidance technique for possible back-up systems was evaluated. Project personnel were D. L. Lukes, R. G. Johnson; principal investigators. The work was supervised by E. R. Rang.

OPTIMAL STABILIZATION OF
NONLINEAR DYNAMICAL SYSTEMS

by D. L. Lukes

INTRODUCTION AND SUMMARY

A mathematical formulation of a problem of optimizing nonlinear closed-loop dynamical control systems is presented. It is shown that an integral performance functional induces a partial ordering of the closed-loop controls which stabilize the system. A natural definition of an optimal control is that control which represents a zero of the partially ordered system. The minimal elements in this ordering are analogous to extremal controls in open-loop optimization problems.

It is proved that any minimal control is optimal if the nonlinearities in the equation of the system satisfy certain boundedness conditions, and a technique is presented for constructing a minimal control. Thus, the existence and construction of optimal closed-loop controls for many systems has been established.

Letov and Kirillova studied the problem of optimal stabilization for linear systems with quadratic index-integrand in References 2,3 and 5, respectively. The existence of an optimal control for the case in which the equations of the system are linear in the control but possibly nonlinear in the dependent variable was investigated by Al'brekht, Reference 6. His technique was based upon the construction of a Liapunov function. The work presented here outlines the extension of these problems to cases in which the control function may appear nonlinearly in the equations of the system and the state variables may occur in a general manner in the performance index. The proof that the formal calculation of the closed-loop control law converges will be completed in subsequent work.

DEFINITIONS AND ASSUMPTIONS

We assume that the uncontrolled process is described by the system equation

$$\dot{x} = f(x,u) = Ax + Bu + h(x,u),$$

where $h \epsilon C^{\omega}$ and $\|h(x,u)\| = O(\|(x,u)\|^2)$.*

For the choice of allowable feedback controls, we restrict our attention to the control space

$$\mathcal{U} = \{u(x) \epsilon C^{\omega},\ u(x) = Dx + O(\|x\|^2),\ \mathrm{Re}\lambda[A + BD] < o\}.$$

We shall assume that $\mathcal{U} \neq \phi$. A sufficient condition for this is that rank $[B, AB, A^2B,\ldots,A^{n-1}B] = n$. This assumption on the linear part of the system equation and control functions ensures that the origin is asymptotically stable for each choice of control function from $\mathcal{U}$.

In order to compare the performance of the controls in $\mathcal{U}$ we define a performance functional. Let us assume as given

$$G(x,u) = (x, \mathcal{O}\!\mathcal{L}\, x) + (u, \mathcal{B}\, u) + H(x),$$

where $\mathcal{O}\!\mathcal{L}$ and $\mathcal{B}$ are symmetric and positive definite matrices, $H(x) \epsilon C^{\omega}$ and $|H(x)| = O(\|x\|^3)$. Then we can define the performance integral

$$J(x,u) = \int_0^{\infty} G\, dt.$$

The integration is done along a trajectory of the system with control $u \epsilon \mathcal{U}$; x is the

---

*C^{ω} denotes the class of functions analytic in a neighborhood of the origin. We let dim(x) = n and dim(u) = r. A and B are constant matrices.

initial condition. It can be shown that $J(x,u)$ is finite-valued and analytic for all x in some neighborhood of $x = o$ for each $u \in \mathcal{U}$.

We shall say that $u_2 \leq u_1$ for u_1 and u_2 in $\mathcal{U}$ if $J(x,u_2) \leq J(x,u_1)$ on some neighborhood of $x = o$.

Lemma $(\mathcal{U}, \leq)$ is a partially ordered system.*

Proof: That $u \leq u$ is trivial for each $u \in \mathcal{U}$.

Suppose that $u_1 \leq u_2$ and $u_2 \leq u_3$ where u_1, u_2 and u_3 are in $\mathcal{U}$. Then $J(x,u_1) \leq$ $\leq J(x,u_2)$ and $J(x,u_2) \leq J(x,u_3)$ on neighborhoods θ_1 and θ_2, respectively, of the origin. Thus $J(x,u_1) \leq J(x,u_3)$ on $\theta_3 = \theta_1 \cap \theta_2$. But this says $u_1 \leq u_3$.

Q. E. D.

Definitions

If $u_o \in \mathcal{U}$ and $u_o \leq \mathcal{U}$, then u_o is called an optimal control for the system.

If $u_1 \leq u_2$ but it is not true that $u_2 \leq u_1$ for u_1 and u_2 in $\mathcal{U}$, then we say $u_1 < u_2$.

For each pair of control elements u_1 and u_2 in $\mathcal{U}$ we define an "order function" of x by $O_x(u_1,u_2) \equiv f(x,u_1(x)) \cdot \nabla_x J(x,u_2) + G(x,u_1(x))$ on a sufficiently small deleted neighborhood of the origin.

Lemma

For each pair u_1 and u_2 in $\mathcal{U}$,

(1) $O_x(u_1,u_2) < o$ implies $u_1 < u_2$

---

*A partially ordered system is a set S with a relation $\leq$ satisfying the conditions:

(1) $a \leq b$ and $b \leq c$ imply $a \leq c$
(2) and $a \leq a$ for all a, b and c in S.

(2) $O_x(u_1,u_2) \leq o$ implies $u_1 \leq u_2$

(3) $O_x(u_1,u_2) \equiv o$ implies $u_1 \leq u_2$ and $u_2 \leq u_1$

(4) $O_x(u_1,u_2) \geq o$ implies $u_1 \geq u_2$

(5) $O_x(u_1,u_2) > o$ implies $u_1 > u_2$

(6) $O_x(u_1,u_2) \leq o$ implies $J(x,u_2)$ is a Liapunov function for the system with control u_1.

Proof: We first prove (2). Let u_1 and u_2 be in $\mathcal{U}$ and suppose $O_x(u_1,u_2) \leq o$ on some deleted neighborhood of $x = o$. Then on some sufficiently small deleted neighborhood of $x = o$ (namely a neighborhood in which both $J(x,u_1)$ is finite and $O_x(u_1,u_2) \leq o$) we can integrate the inequality

$$O_x(u_1,u_2) = f(x,u_1(x)) \cdot \nabla_x J(x,u_2) + G(x,u_1(x)) \leq o$$

along the trajectories of the system with control $u_1(x)$, $X(t)$ with $X(o) = x$, to get

$J(X,u_2) - J(x,u_2) + \int_o^t G(X,u_1(X))d\sigma \leq o$ for $t \geq o$. Then letting $t \to \infty$, and using the fact that $X(t) \to o$, we get $J(x,u_2) \geq \int_o^\infty G(X,u_1(X))d\sigma \equiv J(x,u_1)$ on a neighborhood of $x = o$ so $u_1 \leq u_2$. This proves (2).

The proofs of (1), (3), (4) and (5) are obtained in a similar manner by replacing the inequality signs by the appropriate alternatives in the above calculations.

Now we prove (6). For $u_2 \epsilon \mathcal{U}$ we can easily see that $J(x,u_2)$ satisfies most of the requirements of a Liapunov function; namely, $J(o,u_2) = o$, $J(x,u_2) > o$ on a deleted neighborhood of the origin, and $J(x,u_2) \epsilon C^\omega$. Then the added assumption that

$O_x(u_1,u_2) \leq o$ on a deleted neighborhood of the origin requires

$$f(x,u_1(x)) \cdot \nabla_x J(x,u_2) + G(x,u_1(x)) \leq o.$$

This in turn implies that

$$f(x,u_1(x)) \cdot \nabla_x J(x,u_2) \leq -G(x,u_1(x)) < o$$

on a deleted neighborhood of $x = o$ since $\mathcal{Q}$ and $\mathcal{R}$ are positive definite. Thus $J(x,u_2)$ is a Liapunov function for the system with control u_1.

Q. E. D.

We now prove a theorem which states that in at least some cases proving the existence of an optimal control reduces to proving the existence of a minimal element* in $(\mathcal{U}, \leq)$. Of course, every optimal control is minimal.

Lemma

If $u_1 \in \mathcal{U}$ is minimal and there is a $u_2 \in \mathcal{U}$ so that

$$O_x(u_2,u_1) = \min_u [f(x,u) \cdot \nabla_x J(x,u_1) + G(x,u)]$$

on a neighborhood of $x = o$, then u_1 is optimal.

Proof: Assume the hypothesis. But $O_x(u_2,u_1) \leq o$ since $O_x(u_1,u_1) = o$. Thus $u_2 \leq u_1$.

Now if $O_x(u_2,u_1) = o$ on a neighborhood of $x = o$ then u_1 is optimal since then $O_x(u,u_1) \geq o$ for every $u(x) \in \mathcal{U}$ on a neighborhood of $x = o$. Now apply (4) of the previous lemma.

The contrary case, namely that $O_x(u_2,u_1)$ is negative on a sequence of points

---

*If $(S,\leq)$ is a partially ordered system then an element $a \in S$ is called minimal if $b \in S$ and $b \leq a$ implies $a \leq b$.

converging to the origin, violates the minimality of u_1. This last fact can be seen from the same calculation used to prove (1).

Q.E.D.

Theorem

Let $J^{(2)}_{(x)}$ denote the positive definite quadratic form determined by the optimum solution of the linear problem.*

If both:

(1) $u_1 \in \mathcal{U}$ is minimal and

(2) $\min_u [(Bu + h(x,u)) \cdot \nabla_x J^{(2)}(x) + (u, \mathcal{B} u)]$ has an analytic solution $u(x)$ on a neighborhood of $x = o$,

then u_1 is optimal.

Proof:

We assume the hypotheses. We first note that $J^{(2)}(x,u_1)$, the quadratic term in the expansion of $J(x,u_1)$, depends upon only the linear part of u_1. Also the minimality of u_1 implies that its linear part is the optimum solution to the linear problem. Thus $J^{(2)}(x,u_1) = J^{(2)}(x)$.

Therefore,

$$\min_u [(Bu + h(x,u)) \cdot \nabla_x J^{(2)}(x,u_1) + (u, \mathcal{B} u)]$$

$$= [(Bu(x)+h(x,u(x))) \cdot \nabla_x J^{(2)}(x,u_1) + (u(x), \mathcal{B} u(x))]$$

*By the linear problem we mean the case $H(x) \equiv o$ and $f(x,u) = Ax + Bu$. The solution is determined by the equations

$$u_o(x) = -\tfrac{1}{2}B^* \nabla_x J^{(2)}(x)$$

$$(Ax + Bu_o(x)) \cdot \nabla_x J^{(2)}(x) = -(x, \mathcal{A} x) - (u_o(x), \mathcal{B} u_o(x))$$

(see reference 5)

on a neighborhood of $x = o$. Therefore, on a neighborhood of $x = o$, by adding terms in x, we get

$$\min_u [f(x,u) \cdot \nabla_x J(x,u_1) + G(x,u)]$$

$$= f(x,u(x)) \cdot \nabla_x J(x,u_1) + G(x,u(x))$$

$$\equiv O_x(u,u_1).$$

Thus all that remains before we can apply the lemma is to show that $u(x) \in \mathcal{U}$. (Note that $u(o) = o$). Thus we must show that its linear term makes the linear part of the closed loop system stable.

But since $u(x)$ satisfies the minimizing condition (2) it is necessary that

$$\frac{\partial}{\partial u}[(Bu + h(x,u)) \cdot \nabla_x J^{(2)}(x) + (u, \mathcal{B} u)]\Big|_{u = u(x)} = o$$

on a neighborhood of $x = o$. That is

$$u(x) = -\tfrac{1}{2} B^* \nabla_x J^{(2)}(x) + R(x)$$

where $\|R(x)\| = O(\|x\|^2)$. This says that the linear part of $u(x)$ is the solution to the linear problem. Therefore $u \in \mathcal{U}$. Thus, applying the lemma by using $u_2(x) = u(x)$ we conclude that u_1 is optimal.

Q.E.D.

An example of a nonlinear system satisfying the hypotheses of the theorem is

Example

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{u}{1+u^2} + x_1 u - x_1^3.$$

THE FORMAL CONSTRUCTION

We now develop a procedure for constructing a minimal control.

Let $u \in \mathcal{U}$ be fixed and let $X = X(x,t)$ be the solution of the corresponding closed loop system equation where $X(x,o) = x$. We restrict x to the domain of asymptotic stability about the origin. Then we calculate

$$J(X(x,t),u) = \int_o^\infty G[X(X(x,t),\sigma),\ u(X(X(x,t),\sigma))]d\sigma$$

$$= \int_o^\infty G[X(x,\sigma + t),\ u(X(x,\sigma + t))]d\sigma$$

$$= \int_t^\infty G[X(x,\sigma),\ u(X(x,\sigma))]d\sigma.$$

Differentiating with respect ot t yields

$$\frac{dJ(X,u)}{dt} = -G[X,u(X)], \text{ for } t \geq o.$$

But using the chain rule,

$$\frac{dJ(X,u)}{dt} = \nabla_X J(X,u) \cdot \frac{dX}{dt}$$

$$= f(X,u(X)) \cdot \nabla_X J(X,u).$$

Then setting $t = o$ we get $f(x,u(x)) \cdot \nabla_x J(x,u) = -G(x,u(x))$.

This last equation is an identity in x on the domain of asymptotic stability. Since all functions are analytic, in a neighborhood of the origin we can expand all terms in the identity in convergent power series. Let

$$J(x,u) = J^{(2)}(x) + J^{(3)}(x) + \ldots.$$

$$u(x) = u^{(1)}(x) + u^{(2)}(x) + \ldots.$$

$$H(x) = H^{(3)}(x) + H^{(4)}(x) + \ldots.$$

where the superscript denotes the degree of the homogeneous forms.

Then the identity may be rewritten as

$$\{[Ax + Bu^{(1)}] + [B(u-u^{(1)}) + h(x,u)]\} \cdot \nabla_x [J^{(2)} + J^{(3)} + \dots]$$

$$= -(x,\mathcal{O}\!\mathcal{L}\, x) - (u, \mathcal{Q}\, u) - [H^{(3)} + H^{(4)} + \dots].$$

Equating terms of similar degree, we get the system of equations

$$(Ax + Bu^{(1)}) \cdot \nabla_x J^{(2)} = -(x,\mathcal{O}\!\mathcal{L}\, x) - (u^{(1)}, \mathcal{Q}\, u^{(1)})$$

$$(Ax + Bu^{(1)}) \cdot \nabla_x J^{(3)} = -[Bu^{(2)} + h^{(2)}] \cdot \nabla_x J^{(2)} - 2(u^{(1)}, \mathcal{Q}\, u^{(2)}) - H^{(3)}$$

$$(Ax + Bu^{(1)}) \cdot \nabla_x J^{(m)} = -\sum_{k=2}^{m-1} [Bu^{(m-k+1)} + h^{(m-k+1)}] \cdot \nabla_x J^{(k)}$$

$$-2 \sum_{k=1}^{\left[\frac{m-1}{2}\right]^{*}} (u^{(k)}, \mathcal{Q}\, u^{(m-k)})$$

$$- (u^{(\frac{m}{2})}, \mathcal{Q}\, u^{(\frac{m}{2})})^{**} - H^{(m)}$$

$m = 3,4,5, \dots$

---

*In the limits of summation the symbol [M] denotes the integer part of M.

**This term is omitted for m odd.

Here the symbol $h^{(k)}$ denotes the k^{th}-order terms in the variables $x_1, x_2, \ldots x_n$ obtained from the formal power series expansion of $h(x, u^{(1)}(x) + u^{(2)}(x) + \ldots)$. Thus it is an n-vector whose components $h_i^{(k)}$ are homogeneous forms of degree k.

We now examine the way the $J^{(k)}(x)$ depend upon the choice of u. Notice that the general form of the equations is

$$\hat{A}x : \nabla_x Z^{(k)}(x) = Y^{(k)}(x),$$

where $\hat{A}$ is a stability matrix $(\mathrm{Re}\lambda[\hat{A}] < 0)$ and $Z^{(k)}(x)$ and $Y^{(k)}(x)$ are k^{th}-degree homogeneous forms. In this equation how the choice of $Y^{(k)}(x)$ affects the solution $Z^{(k)}(x)$ and the converse are questions about solutions of linear equations. A theorem proved by Malkin, Reference 1 , states that each choice of $Y^{(k)}(x)$ results in a unique solution for $Z^{(k)}$ and the converse also holds. Also we note that $u^{(m-1)}$ is the highest ordered term upon which $h^{(m)}$ depends so $J^{(m)}$ depends upon no term higher than $u^{(m-1)}$. $J^{(2)}$ depends upon only $u^{(1)}$.

Since we can restrict our attention to an arbitrarily small neighborhood of the origin in discussing most questions we would expect that in selecting u so as to achieve a minimal control the lower ordered $J^{(k)}$ would have highest priority. With this motivation we now develop a method of determining an infinite series

$$\sum_{k=1}^{\infty} u^{(k)}.$$

The discussion of its minimality and convergence is momentarily postponed.

We shall make use of the formula

$$\frac{\partial h^{(m)}}{\partial u^{(k)}} = \left[\frac{\partial h}{\partial u}\right]^{(m-k)} \qquad \begin{array}{l} m = 2,3,4,\ldots \\ k = 1,2,3,\ldots \\ m \geq k. \end{array}$$

<u>1st Step</u> We begin the selection by choosing $u^{(1)}$ to be the solution of the truncated optimization problem $(f(x,u) = Ax + Bu,\ H(x) \equiv 0)$. That is, according to the equations

$$(Ax + Bu^{(1)}) \cdot \nabla_x J^{(2)} = -(x, \mathcal{C} x) - (u^{(1)}, \mathcal{Q} u^{(1)})$$

$$u^{(1)} = -\tfrac{1}{2} \mathcal{Q}^{-1} B^* \nabla_x J^{(2)}$$

and $J^{(2)}(x)$ is positive definite.

<u>2nd Step</u> For the determination of $J^{(3)}$ we consider the equation

$$(Ax + Bu^{(1)}) \cdot \nabla_x J^{(3)} = -[Bu^{(2)} + h^{(2)}] \cdot \nabla_x J^{(2)} - 2(u^{(1)}, \mathcal{Q} u^{(2)}) - H^{(3)}$$

But in view of our choice of $u^{(1)}$ the equation reduces to

$$(Ax + Bu^{(1)}) \cdot \nabla_x J^{(3)} = -h^{(2)} \cdot \nabla_x J^{(2)} - H^{(3)}$$

and since all terms have been determined, except $J^{(3)}$, we appeal to Malkin's theorem and define $J^{(3)}$ to be the unique solution to the equation.

<u>3rd Step</u> In this step we make the determination of $J^{(4)}$ and $u^{(2)}$ by considering the equation

$$(Ax + Bu^{(1)}) \cdot \nabla_x J^{(4)} = -h^{(2)} \cdot \nabla_x J^{(3)} - h^{(3)} \cdot \nabla_x J^{(2)} - Bu^{(2)} \cdot \nabla_x J^{(3)} - Bu^{(3)} \nabla_x J^{(2)} - 2(u^{(1)}, \mathcal{Q} u^{(3)}) - (u^{(2)}, \mathcal{Q} u^{(2)}) - H^{(4)}$$

By our choice of $u^{(1)}$ the terms containing $u^{(3)}$ cancel and the right hand side depends upon $u^{(2)}$ as the only undetermined variable. Since $u^{(2)}$ enters linearly in $h^{(3)}$ and $\mathcal{Q}$ is positive definite, by integrating both sides of the equation along the trajectories of the linear system $\dot{x} = Ax + Bu^{(1)}$ maximizing the right hand side with respect to $u^{(2)}$ minimizes $J^{(4)}(x)$ for every choice of x. Thus $u^{(2)}$ is determined by the formula

$$u^{(2)}(x) = -\tfrac{1}{2}\mathcal{Q}^{-1}[B^*\nabla_x J^{(3)} + (\frac{\partial h^{(3)}}{\partial u^{(2)}})^* \nabla_x J^{(2)}]$$

Using our previously mentioned formula this can be written as

$$u^{(2)}(x) = -\tfrac{1}{2}\mathcal{Q}^{-1}[B^*\nabla_x J^{(3)} + (\frac{\partial h}{\partial u})^{*(1)} \nabla_x J^{(2)}]$$

Then with $u^{(2)}$ determined, $J^{(4)}$ is determined via Malkin.

<u>Inductive Step</u> We now define the $u^{(k)}$ and $J^{(k)}$ by induction. Let $m > 3$ be even and fixed. Let

$$u^{(k)}(x) = -\tfrac{1}{2}\mathcal{Q}^{-1}[B^*\nabla_x J^{(k+1)} + \sum_{j=1}^{k-1} (\frac{\partial h}{\partial u})^{(j)*} \nabla_x J^{(k-j+1)}]$$

$$k = 1,2,3,\ldots,\frac{m}{2} - 1$$

in which the $J^{(p)}(x)$ have been determined recursively so that the identity equations have been satisfied for $p = 2,3,\ldots, m-1$ and so that

$$J^{(m-2)}(x) \le J^{(m-2)}(x,\tilde{u}) \text{ for all } x \qquad \textbf{(P)}$$

whenever

$$\tilde{u} \in \{u \mid u \in \mathcal{U} \text{ and } J^{(s)}(x,\tilde{u}) = J^{(s)}(x) \text{ for all } x,\ 2 \le s \le m-3\}.$$

(We assume that in the previous step the right hand side of the equation used to determine $J^{(m-1)}$ had been shown to be independent of all undetermined functions).

Now we examine the right hand side of the identity equation for $J^{(m)}$.

Differentiating the right hand side with respect to $u^{(p)}$ for $p = m-1, m-2, \ldots, \frac{m}{2}+1$, we get

$$-\sum_{k=2}^{m-1} \frac{\partial u^{(m-k+1)}}{\partial u^{(p)}} B^* \nabla_x J^{(k)} - \sum_{k=2}^{m-1} \left(\frac{\partial h^{(m-k+1)}}{\partial u^{(p)}}\right)^* \nabla_x J^{(k)} - 2\mathcal{Q}u^{(m-p)}$$

$$= -B^* \nabla_x J^{(2)} - \sum_{k=2}^{m-p} \left(\frac{\partial h}{\partial u}\right)^{*(m-k+1-p)} \nabla_x J^{(k)} - 2\mathcal{Q}u^{(m-p)}.$$

Making a change of variable of summation $s = m-p-k+1$, we get

$$-B^* \nabla_x J^{(2)} - \sum_{s=1}^{m-p-1} \left(\frac{\partial h}{\partial u}\right)^{*(s)} \nabla_x J^{(m-p-s+1)} - 2\mathcal{Q}u^{(m-p)}.$$

This is identically zero for the range of p considered because this is another way of writing

$$u^{(k)}(x) = -\tfrac{1}{2}\mathcal{Q}^{-1}\left[B^* \nabla_x J^{(k+1)} + \sum_{j=1}^{k-1} \left(\frac{\partial h}{\partial u}\right)^{(j)*} \nabla_x J^{(k-j+1)}\right]$$

for $k = 1,2,3,\ldots,\frac{m}{2}-1$, which was assumed above.

Thus the right hand side depends upon no $u^{(k)}$ of order higher than $\frac{m}{2}$.

By the same reasoning used in the third step we can use Malkin's theorem to define $J^{(m)}(x)$ as the solution of the equation where $J^{(m)}(x)$ is minimized for all x by choosing $u^{(\frac{m}{2})}$ so as to maximize the right hand side. Thus $u^{(\frac{m}{2})}$ is determined by differentiation as the solution to the equation

$$-2\mathcal{Q}u^{(\frac{m}{2})} - \sum_{k=2}^{m-1} \frac{\partial u^{(m-k+1)}}{\partial u^{(\frac{m}{2})}} B^* \nabla_x J^{(k)} - \sum_{k=2}^{m-1} \left(\frac{\partial h^{(m-k+1)}}{\partial u^{(\frac{m}{2})}}\right)^* \nabla_x J^{(k)} =$$

$$= -2\mathcal{Q}u^{(\frac{m}{2})} - B^*\nabla_x J^{(\frac{m}{2}+1)} - \sum_{k=2}^{\frac{m}{2}} (\frac{\partial h}{\partial u})^{*(\frac{m}{2}-k+1)} \nabla_x J^{(k)}$$

$$= -2\mathcal{Q}u^{(\frac{m}{2})} - B^*\nabla_x J^{(\frac{m}{2}+1)} - \sum_{k=1}^{\frac{m}{2}-1} (\frac{\partial h}{\partial u})^{*(k)} \nabla_x J^{(\frac{m}{2}-k+1)}$$

$$= o.$$

But this formula agrees with the one used to define $u^{(k)}(x)$ $k = 1,2,\ldots,\frac{m}{2}-1$.

With this determination of $u^{(\frac{m}{2})}(x)$ and $J^{(m)}(x)$ all that remains to complete the proof of the induction is to show that the right hand side of the equation for $J^{(m+1)}$ is independent of $u^{(p)}$, $p = \frac{m}{2}+1, \frac{m}{2}+2,\ldots,m$. Again by differentiating we get

$$-\sum_{k=2}^{m} \frac{\partial u^{(m-k+2)}}{\partial u^{(p)}} B^*\nabla_x J^{(k)} - \sum_{k=2}^{m} (\frac{\partial h^{(m-k+2)}}{\partial u^{(p)}})^* \nabla_x J^{(k)}$$

$$-2\sum_{k=1}^{[\frac{m}{2}]} \frac{\partial}{\partial u^{(p)}} (u^{(k)}, \mathcal{Q}u^{(m+1-k)})$$

$$= -B^*\nabla_x J^{(m-p+2)} - \sum_{k=2}^{m-p+1} (\frac{\partial h}{\partial u})^{*(m-k+2-p)} \nabla_x J^{(k)} - 2\mathcal{Q}u^{(m-p+1)}$$

(Letting $j = m-k+2-p$),

$$= -B^*\nabla_x J^{(m-p+2)} - \sum_{j=1}^{m-p} (\frac{\partial h}{\partial u})^{*(j)} \nabla_x J^{(m-p-j+2)} - 2\mathcal{Q}u^{(m-p+1)} = o$$

for the range of p being considered. This completes the induction.

Thus we have shown that it is possible to generate infinite series $\sum_{k=1}^{\infty} u^{(k)}(x)$ and $\sum_{k=2}^{\infty} J^{(k)}(x)$ for which the identity equations hold and so the property (P) holds. Note that if the infinite series $u(x) = \sum_{k=1}^{\infty} u^{(k)}(x)$ has a non-zero radius of convergence than $u(x) \in \mathcal{U}$.

We summarize the construction by a theorem.

Theorem If $u(x) = \sum_{k=1}^{\infty} u^{(k)}(x)$ constructed above has a non-zero radius of convergence then $u \in \mathcal{U}$ and $u(x)$ is a minimal control.

Proof: Assume the hypothesis. Thus $u(x) = \sum_{k=1}^{\infty} u^{(k)}(x)$ as constructed has a non-zero radius of convergence. Also $J^{(k)}(x) = J^{(k)}(x,u)$, $k = 2,3,4,\ldots$ because of the uniqueness of the solutions of the equations used to define the $J^{(k)}(x)$.

To prove u is minimal we shall use the property (P). Let C be a chain in $(\mathcal{U}, \leq)$ containing u and let u_α be an arbitrary fixed element in C.

Thus either $J(x,u_\alpha) \leq J(x,u)$ on a neighborhood of $x = o$ or else $J(x,u) \leq J(x,u_\alpha)$ on a neighborhood of $x = o$. In the latter case we have nothing to prove.

Therefore suppose $J(x,u_\alpha) \leq J(x,u)$ in a neighborhood of $x = o$. It can be chosen so that the expansions

$$J(x,u_\alpha) = J^{(2)}(x,u_\alpha) + J^{(3)}(x,u_\alpha) + \ldots.$$

$$J(x,u) = J^{(2)}(x,u) + J^{(3)}(x,u) + \ldots.$$

are valid there.

$$J^{(2)}(x,u_\alpha) \leq J^{(2)}(x,u) \text{ for all } x$$

because

$$J(\lambda x,u_\alpha) - J(\lambda x,u) = \lambda^2[J^{(2)}(x,u_\alpha) - J^{(2)}(x,u)] + \lambda^3 o(\|x\|^3)$$

for every x for all λ in some open interval (possibly depending upon x) about $\lambda = o$.

But also

$$J^{(2)}(x,u_\alpha) \geq J^{(2)}(x,u)$$

for all x by the property (P) resulting from the way u was constructed. Therefore, $J^{(2)}(x,u_\alpha) = J^{(2)}(x,u)$ for all x. But this implies that $J^{(3)}(x,u_\alpha) = J^{(3)}(x,u)$ for all x too because

$$J(\lambda x,u_\alpha) - J(\lambda x,u) = \lambda^3[J^{(3)}(x,u_\alpha) - J^{(3)}(x,u)] + \lambda^4\, O(\|x\|^4)$$

for every x for all λ in some open interval about $\lambda = o$.

We now proceed by induction. Suppose that m is even and that $J^{(k)}(x,u_\alpha) = J^{(k)}(x,u)$ for all x for k = 2,3,..., m-1. Thus

$$J(\lambda x,u_\alpha) - J(\lambda x,u) = \lambda^m[J^{(m)}(x,u_\alpha) - J^{(m)}(x,u)] + \lambda^{m+1}\, O(\|x\|^m)$$

for all x for all λ in some open interval about $\lambda = o$ so $J^{(m)}(x,u_\alpha) - J^{(m)}(x,u) \leq o$ for otherwise we would contradict the fact that $J(x,u_\alpha) \leq J(x,u)$ about $x = o$.

But also $J^{(m)}(x,u_\alpha) - J^{(m)}(x,u) \geq o$ for all x by (P). Therefore $J^{(m)}(x,u_\alpha) = J^{(m)}(x,u)$ for all x. Therefore,

$$J(\lambda x,u_\alpha) - J(\lambda x,u) = \lambda^{m+1}[J^{(m+1)}(x,u_\alpha) - J^{(m+1)}(x,u)] + \lambda^{m+2} O(\|x\|^{m+2})$$

where m+1 is odd. This shows that $J^{(m+1)}(x,u_\alpha) = J^{(m+1)}(x,u)$ for all x. This completes the induction.

Therefore $J(x,u) = J(x,u_\alpha)$ on a neighborhood of $x = o$ so we have shown $u_\alpha \leq u$ implies $u \leq u_\alpha$. But u_α was arbitrary in C and C was an arbitrary chain containing u. Thus u is a minimal control.

Q.E.D.

BIBLIOGRAPHY

1. Malkin, I. G., Theory of Stability of Motion, AEC Translation 3352.

2. Letov, A. M., "The Analytical Design of Control Systems," Automation and Remote Control, Vol. 21, No. 4, 1960.

3. Letov, A. M., "The Analytical Design of Control Systems," Automation and Remote Control, Vol. 22, No. 4, 1961 pp. 363-372.

4. Bellman, R., Dynamic Programming, Princeton University Press, Princeton, N. J., 1957.

5. Kirillova, F. M., "On the Problem of Analytical Construction of Controls," PMM, Vol. 25, No. 3, 1961, pp. 433-439.

6. Al'Brekht, E. G., "On the Optimal Stabilization of Nonlinear Systems," PMM, Vol. 25, No. 5, 1961, pp. 836-844.