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THEORY OF MINIMUM EFFORT CONTROL

by John V. Breakwell, Herbert E. Rauch, and Frank F. Tung

Prepared under Contract No. NAS 1-3777 *by*
LOCKHEED MISSILES AND SPACE COMPANY
Sunnyvale, Calif.
for Langley Research Center

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ABSTRACT

This report is concerned with optimum guidance for interplanetary missions using either a high thrust or a low thrust engine. The guidance problem is formulated as a problem in optimum control theory, and control theory techniques are applied to its solution. For a high thrust engine this involves the minimization of the total average velocity correction during midcourse for specified terminal accuracies in the presence of initial injection errors, state measurement errors, and control mechanization errors. The solution is first presented for the case where (1) only one component of the position at the terminal time is to be specified, (2) the information rate histories are specified in advance, (3) there is negligible engine mechanization error, and (4) the magnitude of the control is linearly related to the predicted miss distance. The solution is then extended to four separate cases, namely, (1) the rms value of more than one terminal component is specified, (2) the "information rate" histories (i.e., the rate of measurement and the type of observations) are to be optimized, (3) engine mechanization errors are taken into account, and (4) nonlinear feedback is allowed.

For the low thrust mission it is assumed that the engine is operated at a constant specific impulse and that it is turned on only in the vicinities of the departure planet and the target planet. Hence, when leaving the departure planet, the optimization problem involves reaching a specified energy and asymptotic angular direction

with minimum mass expenditure (minimum time, in this case). When approaching the target planet, the optimum turn-on time must also be determined. The guidance problems for the low thrust mission are solved by using a neighboring optimum control scheme, which generates a linear feedback control law. For both high and low thrust missions numerical and, in some cases, analytic results are presented to serve as a guide in evaluating the various optimum and sub-optimum guidance techniques.

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1. INTRODUCTION

1.1 Motivation

One phase of research in the field of future manned space flights concerns the problem of guidance. A comprehensive presentation of feasible guidance schemes and much of the work in navigation is contained in a recent book by Battin.[1*] Broadly speaking, the problem of guidance is the determination of a control program which will steer the vehicle to its desired destination. This control program will depend on the particular mission, the measured information concerning the trajectory, and the way in which the control is to be executed. An optimum guidance program is one which will accomplish such a task in a most economical way. For our purpose, this implies the consumption of the least amount of corrective propellant. Hence, the problem of optimum guidance can be visualized as a search for a control program which produces a correction schedule in a "best" fashion. The control program uses a set of measured information concerning the trajectory and it must meet some fixed error criterion. This has the typical form of a problem in optimal control, a field which has received considerable attention over the past decade. It is, therefore, desirable to formulate the problem of optimum guidance as a problem in optimal control and apply control theory techniques for its solution.

The amount of corrective propellant, which is the measure of the performance we have adopted, depends on the type of engines used for guidance. We shall not be concerned with identifying the various types of engines, but will, instead, classify them into two major groups - the high-thrust engine and the low-thrust engine. This report is concerned with the problem of optimum guidance for interplanetary missions using either a high-thrust

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*References used in each chapter are listed at the end of that chapter.

engine or a low-thrust engine. Although both problems belong to the realm of optimal control, the solutions in the two cases are very different. In the high-thrust case, it is assumed that a separate engine is used after boost to make the trajectory corrections. In this case, the propellant is measured by the total amount of velocity correction required. By contrast, in the low-thrust vehicle, the same engine is used for guidance as well as for the actual mission. The propellant, in this case, is measured by the total time during which the engine is on.

1.2 Objective

The objective of this work is to provide:

- A mathematical model for studying the physical problem of optimum guidance.
- A solution of such a problem by formulating it as a problem in optimal control and applying control theory techniques for its solution.
- Some numerical work to evaluate the various optimum and sub-optimum guidance techniques.

It is hoped that the various solutions we have obtained will shed much insight on future investigations in the field of interplanetary guidance.

1.3 Outline

This report is divided into four chapters. Chapter 1 is the Introduction. Chapter 2 considers the problem of optimum midcourse guidance using high-thrust engines. It has ten sections giving the formulation of the problem, the approaches, and the results of the various extensions of an optimum guidance theory developed by Breakwell and Striebel.[2] Chapter 3 considers the problem of optimum guidance using low-thrust engines. Unlike the work on high thrust, the guidance is assumed to take place only in the vicinities of the departure planet and the target planet. No consideration is given to the midcourse guidance in that the midcourse trajectory is assumed to be determined by the energy and the asymptotic direction of the vehicle leaving the vicinity of the departing planet. In view of the distinct difference between the two cases, Chapters 2 and 3 are organized so that they are self-contained. The last chapter gives a summary of the results we have obtained in this study and recommendations for future investigations.

1.4 Summary of Past Work

During the past few years, there have been many published papers on midcourse guidance for lunar and interplanetary missions using high-thrust engines. As representative, we cite the work of Noton, Cutting and Barnes,[3] Gates, Scull and Watkins[4] concerning ground-based trackings, and the recent work of Smith,[5] Stern,[6] Curdendale and Pfeiffer,[7] Battin,[8] and others for proposing feasible guidance schemes allowing arbitrary information rates. Among those who have attempted to optimize the trajectory correction schedule are Lawden,[9] Breakwell,[10] and Pfeiffer.[11] Lawden and Breakwell have found solutions to the timing of the corrective impulses so as to require, on the average, the least fuel expenditure for the special case where each correction is a full correction (a full correction completely nulls out the predicted miss-distance) and where the error in estimating miss-distance is due entirely to an error in estimating the instantaneous velocity vector. The solution of Pfeiffer, which is concerned with minimizing the terminal miss-distance when a fixed amount of fuel is available, shows that the timing of the corrective thrusts depends on the estimated miss-distance.

It also calls for full corrections. Since information prior to correction is not perfect, it may be more economical to undercorrect. The only published work on the problem of optimum guidance which results in an undercorrective strategy and at the same time allows arbitrary information rates appears to be that by Breakwell and Striebel.[2] It was assumed that the magnitude of the control acceleration is linearly related to the predicted miss-distance. Their result shows that, in the absence of mechanization errors, the optimum corrections are continuous instead of discrete and the optimum strategy involves an initial period of no control followed by a period of continuous control and then a period of no control near the end.

In contrast to the work on high-thrust guidance, there has been very little published work on the problem of guidance using low-thrust engines. The first published work appears to be by Battin and Miller[12] who have devised a feasible guidance scheme for a variable thrust vehicle on a lunar mission, assuming that guidance takes place both while spiraling out from Earth and while approaching the Moon. In the area of midcourse guidance, we cite the work of Pfeiffer[13] who has considered such problems by using a penalty function which is equivalent to a quadratic form of the final state vector. His solution is not optimum in the sense of meeting specified terminal constraints. The recent work of Mitchell[14] treats the problem by linearizing along a predetermined optimum trajectory and uses a "method of adjoints." It is not the same as the concept of second variation in the calculus of variations which is the technique used in this report. The difference lies in the loss criterion which comes naturally in the use of second variation.

1.5 Notation

A. Unless otherwise stated, capital letters A, B, ... denote matrices and small letters a, b,... denote vectors. Small letters with subscripts a_{ij},

b_{ij} (or a_i, b_i) denote elements of the matrices A, B (or vector a, b). The transpose of the matrix A (or vector a) is denoted by A' (or a') and $||a||^2 = a'a$.

B. The references cited in each chapter are listed at the end of that chapter.

C. Equations and figures in each chapter are identified as follows: Equation (I, J) (or Figure (I.J)) implies the J's equation (or figure) in the I section. There are no cross references among chapters.

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2. MIDCOURSE GUIDANCE FOR HIGH-THRUST INTERPLANETARY TRANSFER

2.1 Introduction

Consider a space vehicle which is in free fall following injection. The only external controllable force acting on the vehicle is the thrust during short bursts when corrections to the trajectory are executed. Because the injection conditions are not perfect, the vehicle will depart from its desired (or nominal) trajectory and it is the function of the guidance system to: (1) perform measurements (whether on board or not) from which the actual trajectory can be estimated, and (2) apply trajectory corrections to insure the arrival of the vehicle in the close vicinity of the planet.

Lawden[1*] and Breakwell[2] have found solutions to the timing of corrective impulses for the special case where each correction is a full correction which nulls out the estimated miss-distance, and where the error in estimating miss-distance is due entirely to an error in estimating the instantaneous velocity vector. Their solution consists essentially of an early correction to compensate for the initial errors and further corrections, each two-thirds of the remaining distance to the target. Battin[3] has proposed a criterion for the timing of the corrective action based on the ratio of the required velocity correction to the uncertainty in estimated miss-distance, again assuming that each correction is a full correction. Battin's solution, although valid for arbitrary information rates, does not minimize the total velocity corrections. The solution obtained by Pfeiffer[4] (which is concerned with minimizing the terminal miss distance) shows that the timing of the corrective thrusts depends on the estimated miss-distance. It also calls for full corrections.

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*References referred to in this chapter are listed at the end of this chapter.

Since information prior to a correction is not perfect and further corrections will, in general, be required, it may be more economical to under-correct. This report, therefore, re-examines this problem of optimum guidance (i.e., the problem of minimizing the total average velocity correction for specified rms terminal accuracies in the presence of initial injection errors, state measurement errors and control mechanization errors) by formulating it as a problem in stochastic optimal control and applying control theory methods for its solution. It results in a theory which is applicable to arbitrary information rates and in the same time minimizes the required average total velocity correction consistent with a specified reasonable terminal accuracy. We shall concern ourselves to cases involving errors only in the plane of the transfer orbit.

The original work following this approach was done by Breakwell and Striebel[5] in a paper entitled "Minimum Effort Control in Interplanetary Guidance." It was assumed that the magnitude of the control acceleration is linearly related to the predicted miss-distance and points in the direction of maximum effectiveness. The paper showed that, in the absence of mechanization errors, the optimum corrections are continuous instead of discrete. The solution is easily computable in terms of general information rates. We shall call this the Basic Minimum Effort Theory and, for the sake of completeness, include a brief review of the theory in this report. The remaining parts of this report are devoted to various extensions of this basic minimum effort theory.

The assumptions necessary for the development of the basic minimum effort theory were:

- The rms value of only one component of the position at terminal time is specified.

- The magnitude of the control acceleration is linearly related to the predicted miss-distance.
- The information rate histories are specified in advance.
- There is negligible engine mechanization error.

This basic minimum effort theory is applicable to the problem of variable time-of-arrival guidance assuming that all errors lie in the transfer plane. For example, consider the following problem: Suppose that a vehicle travels along a heliocentric ellipse which meets with a planet (whose gravity field is ignored) moving in the same plane as the vehicle. Suppose further that the starting position (Earth) is known, the initial velocity vector is imperfectly known, and that it is desired to control the distance of closest approach but not (directly) the time of closest passage. Then, if the x-axis is chosen perpendicular to the relative velocity of approach to the planet (see Figure 1.1), it is desired to control x(T) but not y(T), where T is the nominal time of arrival.

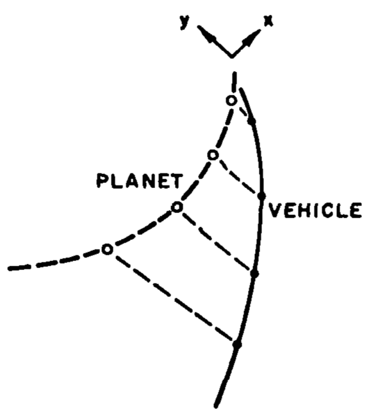

Figure 1.1

Various extensions of the basic minimum effort theory were developed removing one or more of the many assumptions described above. The various extensions undertaken are listed as follows:

- Extension of the theory to the case when the rms values of more than one terminal component are specified.
- Extension of the theory to include the case when engine mechanization errors are taken into account.
- Extension of the theory to include the optimization of the information rate histories (i.e., the rate of measurement and the type of observations).
- Extension of the theory to the case of allowing nonlinear control (i.e., remove the assumption of linear feedback).

No attempt is made in combining these various extensions to form a unified general theory since the computation involved in getting a solution for the general theory is prohibitively complicated.

This chapter is divided into ten sections. Section 2.1 in the Introduction. The mathematical statement of the optimum guidance problem and the equations for estimating its trajectories based on noisy observations are given in Sections 2.2 and 2.3. Section 2.4 introduces the concept of linear control. The various extensions of the basic minimum theory listed above as well as a review of the basic theory are given in Sections 2.5 to 2.9. Each of these sections is, more or less, self-contained. In general, we give in each section the solution of the problem we have proposed, its method of solution and usually a simple example illustrating the results derived in that section. The example used for illustration in all cases is a simple one-dimensional model analogous to the approach to a planet where the only available observation is the vehicle-planet direction. In Section 2.10, we report some related work in connection with the application of the basic minimum effort theory that **is** not specifically required by the contract. The related work consists of the development of a large digital computer program which applies the basic minimum effort theory to the study of guidance problems in typical interplanetary trips. A brief description of this program and the computer results giving the velocity requirements for two typical transfers (Earth-Mars and Earth-Venus-Mars swingby) are given.

2.2 Mathematical Statement of the Problem and the Separation of Estimation and Control

A. The Statement of the Problem

We shall assume that we have a precomputed nominal trajectory and that the departures of the velocities and positions from this nominal trajectory are sufficiently small so that a linearized model evaluated along this nominal path may be used to describe the dynamics of the vehicle. The measurement information will be idealized as continuous and it also suffices to represent the measurement by the deviation of the actual observations from its nominal value.

Given:

(1) The linearized equations of motion describing the dynamics of the vehicle in the neighborhood of the nominal trajectory,*

$$\frac{dx}{dt} = F(t)\ x(t) + G(t)\ u(t) \tag{2.1}$$

(2) The idealized continuous observations,

$$y(t) = M(t)\ x(t) + \epsilon(t) \tag{2.2}$$

where $x(t)$ = a state n-vector ($n \leq 6$) representing the difference between the actual trajectory and the nominal trajectory.

$y(t)$ = an observable r-vector representing the difference between the actual observation and the nominal observation.

$u(t)$ = control m-vector ($m \leq 3$).

$\epsilon(t)$ = a random r-vector accounting for the additive measurement error.

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*Equations given in this chapter are numbered as follows: Eq. (k, J) means the J equation in Section 2.k.

The elements of the matrices $F(t)$, $G(t)$ and $M(t)$ are essentially the partial derivatives evaluated along the nominal trajectory. The random disturbances are assumed to be normally distributed with zero mean and covariances.

$$\mathrm{cov}\ (\epsilon(t),\ \epsilon(s)) = R(t)\ \delta(t-s) \tag{2.3}$$

where $\delta(\cdot)$ is the dirac delta function. The elements of the rxr matrix $R(t)$ are functions of the accuracies and the rate of measurements. We shall assume $R(t)$ is positive definite.

(3) The initial uncertainty $x(0)$ is a zero mean, normally distributed random vector independent of $\epsilon(t)$ with covariance

$$\mathrm{cov}\ (x(0)) = V(0) \tag{2.4}$$

(4) A pxn matrix $H(p < n)$, a nominal arrival time T and a loss function

$$\text{Loss} = \int_0^T \sqrt{E\,||u(t)||^2}\ d\,t \tag{2.5}$$

where $E(\cdot)$ indicates the averaging operator.

Problem. Find the control $u(t)$ as a function of all the past observations $y(s)\ 0 \le s \le t$ which minimizes (2.5) for specified values of $\mathrm{cov}(Hx(T))_{ii}$; $i = 1,2...p$. In other words, the rms values of p of the n states at the nominal arrival time are to be independently controlled.*

Remark. Equation (2.5) is not the same as the total average velocity correction which is given by

$$E \int_0^T \sqrt{||\mu(t)||^2}\ dt \tag{2.6}$$

*With the exception of Section 2.5, all the work to be reported is concerned with the case where $p = 1$.

A slightly modified criterion is used here because the expected amount of total velocity correction given by (2.6) can be expressed only in the form of an infinite series when $p > 1$. This modified criterion (2.5), which is the integral of the square root of the variance of the command accelerations, is, we feel, a most reasonable replacement for the criterion (2.6). It has the properties that

(1) it reduces to the exact amount of total velocity requirements in the absences of random disturbances,

(2) it reduces (except for an unimportant factor of $\sqrt{\frac{2}{\pi}}$) to the same criterion given by (2.6) in the case when $p=1$, and

(3) it sets an upper bound to the expected total velocity correction, i.e.,

$$\int_0^T \sqrt{E \, \| u(t) \|^2} \; dt \quad \geq \quad E \int_0^T \sqrt{\| u(t) \|^2} \; dt \qquad (2.7)$$

which can be easily verified by application of Schwarz's inequality.

B. Separation of Estimation and Control

What we have just stated is a combined optimization problem in estimation and control. In other words, we have, at time t, all the measurements up to this time. The problem is how to make use of this set of data to devise a trajectory correction schedule which meets the optimization criterion. We shall assume that this combined problem in estimation and control can be treated separately in terms of a problem in optimum estimation and a problem in optimum control. We shall first obtain an estimate of the actual miss of those components whose terminal accuracies are to be controlled and then design the controller depending only on these estimated miss components. This is an assumption since the control which is allowed to be a function of all the past observations cannot, in general, be replaced by one which is a function of only those estimated miss-components.

2.3 Optimum Estimation

Since we have assumed that the linear model is sufficient to describe the dynamics of the vehicle as well as the measurement histories in the vicinity of the nominal path, the technique developed for the optimum linear estimation can be used. The method is based on the work of Kalman and Bucy[6,7] where the estimates are updated at each observation time by using the best predicted estimate at this time and the new set of data just received. Only the results will be given here. The derivations may be found in References 6 or 7.

Let $\hat{x}(t)$ be the optimum estimate of $x(t)$ defined by

$$\hat{x}(t) = E \left(x(t) \Big/ y(s) \,,\; 0 \le s \le t \text{ and the past controls} \right) \tag{3.1}$$

and let $V(t)$ be the covariance of the estimation error

$$V(t) = \operatorname{cov}\,(x(t) - \hat{x}(t)) \tag{3.2}$$

Then $V(t)$ satisfies the matrix differential (Riccatti) equation

$$\frac{dV}{dt} = F\,V + V\,F' - V\,M'\,R^{-1}\,M\,V \tag{3.3}$$

and the best estimate $x(t)$ satisfies the differential equation

$$\frac{d\hat{x}(t)}{dt} = F(t)\,\hat{x}(t) + G(t)\,u(t) + K(t)\,(y(t) - M\hat{x}(t)) \tag{3.4}$$

where

$$K(t) = V(t)\,M'(t)\,R^{-1}(t) \tag{3.5}$$

Moreover,

$$\operatorname{cov}\,(\hat{x}(t),\, x(t) - \hat{x}(t)) = 0 \tag{3.6}$$

Initially, $\hat{x}(0) = 0$ and $V(0)$ is given by the apriori information concerning the uncertainty of $x(0)$.

Note $\hat{x}(t)$ is the estimate of the state at time t based on all the data up to time t. This can be used to compute the estimate of the actual miss (i.e., the p components whose final values are to be controlled). To do this, we introduce the nxn transition matrix

$$\Phi(T, t) = \frac{\partial x(T)}{\partial x(t)}$$

which satisfies the matrix differential equation

$$\frac{d\Phi(T,t)}{dt} = -\Phi(T, t) F(t) \quad ; \quad \Phi(T, T) = I \tag{3.7}$$

where I is the identity matrix. Define

$$x(T, t) = \Phi(T, t) x(t) \tag{3.8}$$

and

$$\hat{x}(T, t) = E\left[x(T, t) \Big/ \quad y(s) \quad 0 \leq s \leq t \quad \text{and all the control} < t\right]$$

$$= \Phi(T, t) \hat{x}(t) \tag{3.9}$$

Physically, $\hat{x}(T, t)$ is the predicted miss of the state at the final time based on all the data up to time t and under the assumption that no additional control is applied over the interval (t, T). It follows that the components of $H\hat{x}(T, t)$ are the predicted misses whose terminal rms values are to be controlled.

Our assumption of the separation of control and estimation means that the control acceleration, which executes the trajectory correction, is only a function of $H\hat{x}(T, t)$. Using (3.4) and (3.7), it is seen that $H\hat{x}(T, t)$ satisfies the differential equation

$$\frac{d\, H\hat{x}(T, t)}{dt} = H\Phi(T, t) G(t) u(t) + H\Phi(T, t) \eta(t) \tag{3.10}$$

where

$$\eta(t) = V(t) M'(t) R^{-1}(t) (y(t) - M(t)\hat{x}(t)) \tag{3.11}$$

which may be considered as a white noise with covariance matrix

$$\text{cov}\ (\eta(t),\ \eta(s)\ ) = V(t)\ \overset{\circ}{I}(t)\ V(t)\ \delta(t-s) \tag{3.12}$$

where

$$\overset{\circ}{I}(t) = M'(t)\ R^{-1}\ (t)\ M(t) \tag{3.13}$$

and can be physically interpreted as the information rate matrix.

In summary, the procedure may be broken down as follows:

(a) Obtain an estimate of the actual miss by integrating the differential equation (3.10) where $y(t)$ is the deviation of the actual observation from its nominal value.

(b) Decide the size of the trajectory correction required at this time. The dependence is only going to be a function of $H\ \hat{x}(T,\ t)$. It is interesting to point out that all the strategies to be discussed are not full corrections. A full correction is one which totally nullifies the estimated predicted-miss.

2.4 Linear Control Law

One way of solving the stochastic optimal control problem stated in the previous section is to define some meaningful average quantities and solving an equivalent deterministic problem using these average quantities as the states. It turns out this technique can be fruitfully used if we confine ourselves to linear control laws, i.e., the case where the control depends only linearly on the predicted miss $H \hat{x}(T, t)$. It will now be shown that this assumption on linear control allows us to formulate the given stochastic optimization problem as an equivalent deterministic optimization problem using the elements of the covariance matrix of $H \hat{x}(T, t)$ as the states.

Assume linear control and we may, without loss of generality, represent the optimal linear control law by

$$u(t) = -S(t) H \hat{x}(T, t) \tag{4.1}$$

where $S(t)$ is a mxp matrix whose elements are to be determined such that (2.5) is minimized for specified values of

$$\text{cov } (H \hat{x}(T))_{ii} = \text{cov } (H \hat{x}(T, T))_{ii} , \; i = 1, 2 \ldots p$$

Define

$$P(t) = E \left[H \hat{x}(T, t) \hat{x}'(T, t) H' \right] \tag{4.2}$$

which is equivalent to cov $(H \hat{x}(T, t))$ since $\hat{x}(T, t)$ is a zero mean process. Let $W(t)$ be the covariance of the error in the estimate miss $\hat{x}(T, t)$. Then

$$W(t) = \text{cov } (x(T, t) - \hat{x}(T, t)) = \Phi (T, t) V(t) \Phi' (T, t) \tag{4.3}$$

Using (3.2), (3.10-3.12) and (4.1-4.3), it is seen that

$$\frac{dP}{dt} = -H \Phi (T, t) G(t) S(t) P(t) - P(t) S'(t) G'(t) \Phi'(T, t) H' + H W \Phi' (t, T) \varphi(t) \Phi (t, T) W H' \tag{4.4}$$

$$\frac{dW}{dt} = -W \Phi' (t, T) \overset{o}{I}(t) \Phi (t, T) W \tag{4.5}$$

$$\text{cov}(H\, x(T)) = P(T) + H\, W\, (T)\, H' \tag{4.6}$$

and

$$E\, \|u(t)\|^2 = E\, \|-S(t)\, H\, \hat{x}(T, t)\|^2 = \text{tr}\, P(t)\, S'(t)\, S(t) \tag{4.7}$$

where tr(·) denotes the trace operator.

Now that the last term of (4.6) is independent of the control, it follows that specification of cov $(H\, x(T))_{ii}$ is the same as specifying $P_{ii}(T)$ and the determination of S(t) is equivalent to solving the following deterministic optimization problem.

Given: The dynamic system (4.4) with P(0) = 0, find S(t) which minimizes

$$\int_0^T \sqrt{\text{tr}\, P(t)\, S'(t)\, S(t)} \quad dt \tag{4.8}$$

for specified values of $P_{ii}(t)$, i = 1,2,..., p.

Inspection of (4.4) and (4.8) shows that both are linear in S insofar as the magnitude is concerned. This is a "degenerate" (or singular) problem in the calculus of variations and special techniques are usually necessary for the method of solution. In general, the optimum solution will consist of different subarcs connected at a finite number of points, called the corner points, and the problem is essentially of finding various arcs, the corner points, and the proper arcs to follow between corner points.

The basic minimum effort theory as developed by Breakwell and Striebel[5] is concerned with the case of p = 1. For the case p = 1 (i.e., controlling only one terminal miss), this problem can be solved by application of Green's Theorem.[8] For the case p >1, Green's Theorem cannot be applied. It turns out that, for this particular problem, the solution can be obtained by the use of the maximum principle. Clearly, the case p = 1 can also be obtained using maximum principle.

We now give the solution of the optimal control law stated above. Since the work in this report represents various extensions of the basic minimum effort theory and since the solution by Green's Theorem does provide a different and in fact more elegant way of solving the problem for $p = 1$, we also include, for the sake of completeness, in this report (Section 2.6) a review of the method of solution as was originally derived by Breakwell and Striebel.

2.5 Optimal Control Law for Controlling Several Terminal Components (Maximum Principle)

This section obtains the solution of the optimal feedback gain matrix S(t) by direct application of the maximum principle. The presentation is divided into three parts. Section 2.5.1 gives the necessary conditions for the optimal linear control and the characteristic of the optimal feedback coefficients. It is shown that, in general, the optimum linear corrective strategy consists of an initial period of no control while the information catches up. This is followed by a period of continuous control and finally a period of no control and possibly an impulse at the end. A computation procedure is outlined for obtaining the optimal feedback gains. Section 2.5.2 specializes the result to the case of $p = 1$. It is included here for the purpose of establishing an equivalence between the results in this section and that obtained originally by Breakwell and Striebel using Green's Theorem. (A review of the basic minimum effort theory ($p = 1$) using Green's Theorem is given in Section 2.6.) Finally, in Section 2.5.3 we illustrate the results by giving two examples.

2.5.1 Equations for Optimality and Computation Procedure

To put in evidence the "singular" nature of the problem stated in the previous section, we define*

$$\phi(t) = \sqrt{\operatorname{tr} P S' S} \qquad ; \qquad \phi(t) \geq 0 \tag{5.1}$$

and let the matrix of feedback gains be written as+

$$S = \phi(t)\, B \tag{5.2}$$

*For convenience, we shall hereafter omit the argument t.

+This substitution essentially converts a control problem potentially singular in mxp variables into a problem which is singular in only one variable.

where B is an undetermined mxp matrix (undefined when $\phi = 0$) such that

$$\mathrm{tr}\; P\, B'\, B = 1 \tag{5.3}$$

Substituting (5.2) into (4.4) shows

$$\frac{dP}{dt} = -\phi(t) \left(H\, \Phi\, G\, B\, P + P\, B'\, G'\, \Phi'\, H' \right) + Q \tag{5.4}$$

where

$$Q = H\, \Phi\, V\, \hat{I}\, V\, \Phi'\, H' \tag{5.5}$$

is a known function of time. The problem now reduces to that of finding B and $\phi(t) \geq 0$, which minimizes $\int_0^T \phi\, dt$ subject to (5.3) and specified values of $P_{ii}(T)$.

Let the Hamiltonian be given by

$$\phi(t) + \mathrm{tr}\, \Lambda\, \dot{P}/2 \tag{5.6}$$

where the elements of the pxp symmetric matrix Λ are the adjoint variables. For a given $\phi(t) \neq 0$, minimizing this Hamiltonian with respect to B subject to the constraint (5.3), is a simple, nondegenerate problem in calculus of variation. The necessary equations for optimality are

$$B = \frac{G'\, \Phi'\, H'\, \Lambda}{\sqrt{\mathrm{tr}\; P \Lambda Z \Lambda}} \tag{5.7}$$

$$\frac{dP}{dt} = -\frac{\phi(t)\, (Z\, \Lambda\, P + P\, \Lambda\, Z)}{\sqrt{\mathrm{tr}\; P\, \Lambda\, Z\, \Lambda}} + Q \tag{5.8}$$

and

$$\frac{d\Lambda}{dt} = \phi(t) \frac{\Lambda\, Z\, \Lambda}{\sqrt{\mathrm{tr}\; P\, \Lambda\, Z\, \Lambda}} \tag{5.9}$$

where $Z = H\, \Phi\, G\, G'\, \Phi'\, H'$ and is a given function of time.

The transversality conditions are $\lambda_{ij}(T) = 0$, $i \neq j$; $\lambda_{ii}(T) = c_i$, $i = 1,2...,$ p, where c_i are to be adjusted such that $P_{ii}(T)$ meet the prescribed values.

The Hamiltonian now becomes linear in $\phi(t)$ and can be written as

$$\phi(t) \quad (1 - \sqrt{\mathrm{tr}\ P \Lambda Z \Lambda}\,) \quad + \quad \mathrm{tr}\ \Lambda\, Q/2 \tag{5.10}$$

It only remains to minimize this Hamiltonian with respect to $\phi(t)$. Since $\phi(t) \geq 0$, it follows that $\phi = 0$ if $\mathrm{tr}(P \Lambda Z \Lambda) < 1$, is undetermined if $\mathrm{tr}(P \Lambda Z \Lambda) = 1$, and is infinite if $\mathrm{tr}(P \Lambda Z \Lambda) > 1$. The last case cannot occur over any finite interval since otherwise $\int_0^T \phi(t)\, dt$ will diverge. Now $\mathrm{tr}(P \Lambda Z \Lambda) = 0$ at $t = 0$ and can be shown to be continuous for any $\phi(t) \geq 0$ including impulses (i.e., $\phi(t)$ are Dirac delta functions). Hence, the case $\mathrm{tr}(P \Lambda Z \Lambda) > 1$ cannot occur and we are left with either $\phi = 0$ (when $\mathrm{tr}(P \Lambda Z \Lambda) < 1$), or $\phi \neq 0$, in which case $\mathrm{tr}(P \Lambda Z \Lambda) = 1$.

It turns out that the optimal gain S consists of (in general, but not always) three portions; an initial period of no control where $S = 0$, followed by a period of continuous control, and finally a period of no control and possibly an impulse at the end. Let us now consider the two cases.

(1) $\phi(t) = 0$. Equations (5.8) and (5.9) reduce to

$$\frac{d\Lambda}{dt} = 0 \tag{5.11}$$

and

$$\frac{dP}{dt} = Q \tag{5.12}$$

which show that the adjoint variables remain unchanged during this period.

(2) $\phi(t) \neq 0$. Then

$$\mathrm{tr}\ P \Lambda Z \Lambda = 1 \tag{5.13}$$

This defines a surface which must contain the solution whenever $\phi \neq 0$. We now note that in order to integrate the set of equations (5.8) and (5.9) along this surface, it is necessary to express $\phi(t)$ in terms of P and Λ. This is done by twice differentiating (5.13). It is of interest to note that along this surface $\phi(t)$ is also given by

$$\phi(t) = \operatorname{tr} P \dot{\Lambda} \tag{5.14}$$

which can be verified by combining (5.9) and (5.13). It is a measure of the average "acceleration" and vanishes only when S = 0, or equivalently, Λ = constant.

Differentiating (5.13) once, using (5.8), (5.9), and the commutative properties of the trace operations, we find

$$\operatorname{tr} (P \Lambda \dot{Z} \Lambda + Q \Lambda Z \Lambda) = 0 \tag{5.15}$$

Differentiating (5.15) once more yields a relation between P, Λ, and $\phi(t)$ which, after suitable reduction, can be written as

$$\phi(t) = \frac{-\operatorname{tr} (2Q \Lambda \dot{Z} \Lambda + \dot{Q} \Lambda Z \Lambda + P \Lambda \ddot{Z} \Lambda)}{2 \operatorname{tr} (Q \Lambda Z \Lambda Z \Lambda)} \tag{5.16}$$

We now have the necessary conditions, namely (5.8), (5.9), (5.13), (5.15) and (5.16), for computing the optimal feedback gains. It is noted that the denominator in (5.16) is the trace of the product of two positive semi-definite symmetric matrices and hence is always $\geqq 0$. It will be assumed to be > 0 in this paper. In other words, the matrix $Q \Lambda Z \Lambda Z \Lambda$ is not identically zero.

Since P(0) = 0, it follows that (5.13) cannot be satisfied at t=0. Hence, $\phi(o) = 0$ and there will be an initial period of no control. The time by which the control is first turned on depends on (1) the information rate which is imbedded in Q, and (2) the initial values of Λ. Mathematically, the exact time of turning on is determined by simultaneously satisfying (5.13) and (5.15). It should be noted that satisfaction of (5.15) determines the time. The common multiplicative constant of the adjoint variables is determined by the normalizing equation (5.13).

Computation starts by guessing an initial set of $\Lambda(0)$ and integrating the dynamic equation (5.12) forward until (5.15) is satisfied. This determines t_{on}. Use is then made of (5.13) to compute the normalizing constant which determines the adjoint variables at the time of turning on. We are now on the surface such that $\phi \neq 0$. To proceed along this surface, we use (5.16) to find $\phi(t)$. This is then used in (5.8) and (5.9) to integrate the equations for P and Λ forward. The optimal feedback gain can be obtained by using ϕ and (5.7). Assume that the control is turned off at some time t, say $t_{off} \geq t_{on}$. Then $S(t) = 0$ for $t > t_{off}$. The total average velocity correction required is given by

$$\int_{t_{on}}^{t_{off}} \phi(t)\, dt \tag{5.17}$$

and

$$\Lambda(T) = \Lambda(t_{off}) \tag{5.18}$$

$$P(T) = P(t_{off}) + \int_{t_{off}}^{T} Q(t)\, dt \tag{5.19}$$

The computational procedure we have proposed gives a parametric study of $p(p + 1)/2$ elements consisting of the ratio of the initial adjoint variables and t_{off} as functions of the $p(p + 1)/2$ elements of $P(T)$. Let

$$A(t) = P(t) + H\, \Phi(T, t)\, V(t)\, \Phi'(T, t)\, H' \tag{5.20}$$

Then $A(t)$ is the covariance of the actual terminal miss when the control is turned off at t. Hence, without loss of generality, we may consider that the parametric study is between the $p(p + 1)/2$ elements consisting of the ratio of the initial adjoint variables and t_{off} and the $p(p + 1)/2$ elements of

$A(t_{off})$. If the diagonal elements of $A(t_{off})$ for all t_{off} $\epsilon(t_{on}, T)$ do not meet the specified values, the computation is repeated again with an improved estimate of $\Lambda(0)$.

It should be noted that the computation procedure we have outlined assumes that the computed $\phi(t) > 0$. In the event that $\phi(t)$ becomes negative for some $t \ \epsilon(t_{on}, t_{off})$, then there exists periods of no control in the interval (t_{on}, t_{off}). Physically, this implies that it is not possible to follow the critical surface defined by (5.13). Assume t_1 is the first time that $\phi(t_1) < 0$; then the control must be turned off at some time t before t_1. The problem here is to determine the exact times of leaving the surface and intercepting the surface again. This can be done by using the criterion that the adjoint variables must remain constant during the time that the control is off. It is equivalent to the searching of a normalization constant which must remain the same at the two points. An iterative scheme taking care of this can be easily implemented on the digital computer. This is illustrated in one of the numerical examples given in the next section.

So far we have avoided the possibilities of impulsive corrections, i.e., S or $\phi(t)$ are impulses. Impulsive corrections give rise to discontinuities in P and Λ. Let de be the incremental effort. Then

$$de = \phi(t)\ dt \tag{5.21}$$

so that the effort due to this impulsive correction is

$$\Delta e = \int_{t^-}^{t^+} \phi(t)\ dt \tag{5.22}$$

Using the effort as the independent variable, (5.8) and (5.9) can be written as

$$\frac{d\Lambda}{de} = \Lambda Z \Lambda \tag{5.23}$$

and

$$\frac{dP}{de} = -(Z\Lambda P + P\Lambda Z) \tag{5.24}$$

The relation between the amount of the impulsive effort and the jump (or drop) in Λ (or P) can therefore be obtained by directly integrating (5.23) and (5.24) with respect to the effort. Using (5.23) and (5.24), we find

$$\frac{d\ \mathrm{tr}\ (P\Lambda Z\Lambda)}{de} = 0 \tag{5.25}$$

which implies that impulsive corrections leave tr $(P\Lambda Z\Lambda)$ invariant. In fact, (5.25) is true for any initial values of tr $(P\Lambda Z\Lambda)$. This, incidentally, is necessary for establishing the fact that tr $(P\Lambda Z\Lambda)$ is continuous. We shall now show that impulsive corrections can be applied at t_o if and only if $Q(t)$ is discontinuous at t_o.

Assume that an impulse is applied at t_o and $Q(t)$ is continuous at t_o. The time derivative of tr $(P\Lambda Z\Lambda)$ is tr $(P\Lambda\dot{Z}\Lambda + Q\Lambda Z\Lambda)$ which, immediately after the impulse of area E, is given by

$$\mathrm{tr}\ (P\Lambda\dot{Z}\Lambda + Q\Lambda Z\Lambda)\Big|_{t_o^-} + \int_o^E \frac{d\ \mathrm{tr}\ (P\Lambda\dot{Z}\Lambda + Q\Lambda Z\Lambda)}{de}\ de \tag{5.26}$$

Now, the first term in (5.26) is zero since we were on the singular surface at t_o. Using (5.23) and (5.24) we see that the second term in (5.26) can be written as

$$2\int_o^E \mathrm{tr}\ (Q\Lambda Z\Lambda Z\Lambda)\ de \tag{5.27}$$

which is greater than 0 in view of our assumption that $Q\Lambda Z\Lambda Z\Lambda$ is not identically zero. This implies that tr $(P\Lambda Z\Lambda)$ will be greater than 1 for $t > t_o$, which is not permissible. Hence, impulsive corrections cannot be applied at any time when $Q(t)$ is continuous. (This is the same as requiring that the Hamiltonian be continuous). On the other hand, assume $Q(t)$ is discontinuous at t_o. Inspection of (5.15) shows that it can be satisfied only if P and Λ are discontinuous at t_o.

Hence, impulsive corrections are allowed to occur when $Q(t)$ is discontinuous or at the final time since our argument does not apply there.

Remark 1. In most cases, the optimal corrective strategy consists of an initial period of no control, followed by a period of continuous control, and finally a period of no control and possibly an impulse at the end. Corresponding to that $\Lambda(0)$, the possibility of periods of no control between t_{on} and t_{off} when $\phi(t) > 0$ for all $t \in (t_{on}, t_{off})$ can be established easily by computing the quantity $(\mathrm{tr}\, P(t')\Lambda(t)Z(t')\Lambda(t) - 1)$ for all $t' > t$, $t \in (t_{on}, t_{off})$. If it differs from zero, then it can be concluded that there do not exist periods of no control between t_{on} and t_{off}.

Remark 2. It is not clear whether or not there exists different initial values of the adjoint variable which will give rise to the same terminal conditions. This is the problem involving uniqueness of our solution and as such has not been solved.

Remark 3. It will be shown in Section 2.5.2 that in the case of controlling only one terminal component, the solution we have obtained is unique and that there exist no periods of no control between t_{on} and t_{off} if $\phi(t)$ is positive over this period.

2.5.2 Special Case of Controlling Only One Terminal Miss

This section specializes the results to the case where the rms values of only one of the states at the terminal time is specified. It is included here to establish an equivalence between the solution by maximum principle (as we have done) and that by using Green's Theorem (as was done by Breakwell and Striebel[5] (see Section 2.6 also). Without loss of generality, it will be assumed that the particular terminal miss we wish to control is the final uncertainty in the position, say $p_{11}(T)$. In other words, H is a n-vector consisting of all zero elements except $h_{11} = 1$.

Let the scalar $z_{11} = H\Phi GG'\Phi'H'$ be denoted by D^2 where D is the sensitivity of the miss distance to a change of velocity in the direction of the correction. From (5.8) we see that $p_{11}(t)$ satisfies the scalar differential equation

$$\frac{dp_{11}}{dt} = -2\phi(t)\, D(t) \sqrt{p_{11}} + q_{11} \tag{5.28}$$

On the other hand, Equations (5.13) and (5.15) become

$$p_{11}\, \lambda_{11}{}^2\, D^2 = 1 \tag{5.29}$$

and

$$\lambda_{11}{}^2\, (2D\, \dot{D}\, p_{11} + q_{11}\, D^2) = 0 \tag{5.30}$$

respectively. It follows from (5.30) that t_{on} is determined by the equation

$$\int_0^{t_{on}} q_{11}(t)\, dt = p_{11}{}^* \, (t_{on}) \tag{5.31}$$

where*

$$p_{11}{}^* \, (t) = -\frac{D\, q_{11}}{2\dot{D}} \tag{5.32}$$

Also, for a given $p_{11}(T)$, t_{off} is determined by

*This is called the critical p curve by Breakwell and Striebel.[5]

$$p_{11}^*(t_{off}) + \int_{t_{off}}^{T} q_{11}(t)\, dt = p_{11}(T) \tag{5.33}$$

Moreover, the optimal solution must follow the critical curve defined by (5.29) if

$$\phi(t) = p_{11}^* \dot{\lambda}_{11} > 0 \qquad t \epsilon (t_{on} , t_{off}) \tag{5.34}$$

This can be seen as follows. Assume (5.34) is true. It can be easily verified that this implies

$$\dot{p}_{11}^* < q_{11} \quad , \quad t \epsilon (t_{on} , t_{off}) \tag{5.35}$$

Suppose for some t' where $t_{on} < t' < t_{off}$, we leave the critical curve. Then the control must be turned off and for $t > t'$

$$p_{11}(t) = p_{11}^*(t) + \int_{t'}^{t} q_{11}(s)\, ds \tag{5.36}$$

which by (5.35) is greater than $p_{11}^*(t)$. Hence, the given terminal $p_{11}(T)$ cannot be satisfied. In other words, we cannot come back to the critical curve after leaving it. This establishes our assertion.

Suppose (5.34) is not satisfied. Then there exists an interval within (t_{on} , t_{off}) such that the control must be turned off. This corresponds to the case of an unusual increase in the information rate. Let t_a and t_b be the times of turning off and on, respectively. Since the adjoint variable must remain constant during the time that the control is off, we see from (5.29) that

$$p_{11}^* (t_a) D^2 (t_a) = p_{11}^* (t_b) D^2 (t_b) \tag{5.37}$$

Moreover,

$$a_{11} (t_a) = a_{11} (t_b) \tag{5.38}$$

which is obvious since $a_{11}(t)$ is the actual terminal miss if no control is applied after t. Equations (5.37) and (5.38) provide sufficient conditions for determining the times t_{on} and t_{off}. In other words, optimum transition corresponds to double points in the $a_{11} - D\sqrt{p_{11}}$ plane. It is of interest to note that in the case of the control of only the terminal velocity, the optimum solution, according to our theory, is an impulse at the final time. This solution is certainly correct since the effort necessary to nullify the velocity error remains constant in (0, T) and hence the optimal solution is the one in which all the information is collected before applying the control.

2.5.3 Two Simple Examples and the Computer Results

A. Controlling the Position and the Velocity of a One-Dimensional Model*

Consider a space ship which is "homing" with constant velocity v_f on a massless planet interrupted by velocity impulses perpendicular to the nominal straight line approach to the target or else by a continuous acceleration u in this perpendicular direction. Let x_1 and x_2 be the transverse position and velocity deviations from the nominal orbit. The equation of motion (see Figure 5.1) is therefore

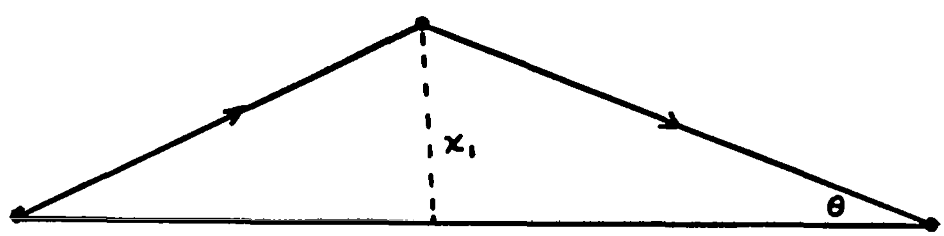


Figure 5.1 A Straight Line Model

*This same straight line model will be used for all the other numerical work in this report.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u \tag{5.39}$$

The 2x2 transition matrix for this example becomes

$$\Phi\,(T, t) = \begin{bmatrix} 1 & T-t \\ 0 & t \end{bmatrix} \tag{5.40}$$

The initial error is to be only in velocity (i.e., $v_{11}(0) = v_{12}(0) = 0$). It is assumed that the information rate is purely positional and that the estimates of the transverse position are obtained by angle measurements at frequent intervals Δt with constant accuracy σ_ϵ. Hence,

$$y_1 = \theta = \frac{x_1}{v_f\,(T-t)} + \epsilon_1(t) \tag{5.41}$$

and the information rate matrix becomes

$$\overset{\circ}{I}\,(t) = \begin{bmatrix} \dfrac{1}{v_f^2\,\sigma_\epsilon^2\,\Delta t\,(T-t)^2} & 0 \\ 0 & 0 \end{bmatrix} \tag{5.42}$$

The product $v_f^2\,\sigma_\epsilon^2\,\Delta t$ may be related to a dimensionless <u>information rate parameter k</u> defined for this example by

$$k = \frac{10^{-4}\,v_{22}(0)\,T}{v_f^2\,\Delta t\,\sigma_\epsilon^2} \tag{5.43}$$

This parameter compares the incoming information with the a priori information $(v_{22}(0))^{-1}$ about the initial velocity error. We shall assume that the variances of both the position x_1 and the velocity x_2 are specified at T.

Using (4.5) and the information rate matrix given by (5.42), it is found that an analytical expression may be obtained for the covariance matrix W(t). It can be easily verified that

$$\dot{w}_{11}(t) = - \frac{t^2}{v_f^2 \sigma_\epsilon^2 \Delta t \, (T-t)^2} w_{11}^2 \tag{5.44}$$

$$w_{12}(t) = \frac{1}{T} w_{11} \tag{5.45}$$

$$w_{22}(t) = \frac{1}{T^2} w_{11} \tag{5.46}$$

with initial condition $w_{11}(0) = T^2 v_{22}(0)$. Solving (5.44), we find

$$w_{11}^{-1}(t_2) = w_{11}^{-1}(t_1) + \frac{1}{v_f^2 \sigma_\epsilon^2 \Delta t \, T^2} \Bigg[(t_2 - t_1) + 2T \log\left(\frac{T-t_2}{T-t_1}\right) + \frac{T^2}{T-t_2} - \frac{T^2}{T-t_1} \Bigg] \tag{5.47}$$

Moreover,

$$Z(t) = \begin{bmatrix} (T-t)^2 & (T-t) \\ (T-t) & 1 \end{bmatrix} \tag{5.48}$$

For the numerical values, we let $(v_{22}(0))^{1/2} = 100$ m/sec, $T = 10^6$ sec, and $k = 1$. Realistic values of k would be much higher and lead to earlier reduction of the predicted miss. For example, if $v_f = 3$ km/sec and $\Delta t = 1$ hr, then $k = 1$ implies $\sigma_\epsilon = 0.32$ degree.

Computation Procedure and the Numerical Results

It can be shown that the adjoint variables are monotonically increasing functions of time if $\lambda_{12} > 0$. (λ_{11} and λ_{22} are always positive.) Since $\lambda_{12}(T) = 0$, we must let $\lambda_{12}(0) < 0$ so that λ_{12} is negative at the time of turning on the control. Moreover, the control must be turned off at the time when λ_{12} reaches zero and not turned on again until possibly at the terminal time. It was shown in the previous section that an impulse may be applied at the final time if (5.13) is satisfied. In our case, this implies

$$\lambda_{22}^2 (T)\, p_{22} (T) = 1 \tag{5.49}$$

It should be noted that an impulse at T brings down $p_{22}(T)$ and cannot change the values of λ_{12} , λ_{11} , and p_{11} . Using (5.23-5.24) we find at time T

$$\frac{dp_{22}}{de} = - 2p_{22}\, \lambda_{22} \tag{5.50}$$

and

$$\frac{d\lambda_{22}}{de} = \lambda_{22}^2 \tag{5.51}$$

where de is the incremental effort due to the impulse. Using (5.50-5.51) and the fact that (5.49) must be satisfied before and after application of the impulse, we find

$$\text{effort due to the impulse} = \frac{1}{\lambda_{22}^-} \left(1 - \sqrt{\frac{p_{22}^+}{p_{22}^-}} \right) \tag{5.52}$$

where p_{22}^- and p_{22}^+ denote the values of $E(x_2^2\, (T))$ immediately before and after the impulse, respectively. Hence, if $p_{22}^+ = 0$ (corresponding to perfect velocity control), then the additional effort required is $\sqrt{p_{22}^- (T)}$. We shall assume that the desired $p_{22}\,(T) = 0$.

The actual computation proceeds as follows:

1. Let $\lambda_{12}(0) = -1$ and guess $\lambda_{11}(0)$ and $\lambda_{22}(0)$.
2. Integrate (5.12) until (5.15) is satisfied. This determines t_{on}.
3. Use (5.13) to determine the value of Λ at t.
4. Integrate along the surface by using (5.8-5.9) and (5.16) until $\lambda_{12} = 0$.

5. Turn off the control until T. This determines P(T) and is a possible solution. But $p_{22}(T)$, in general, will not be zero. Note that $\Lambda(T)$ remains the same as at the time that the control was turned off.
6. If (5.49) is satisfied, an impulse is applied at T to bring $p_{22}(T)$ to zero. The additional velocity required is $\sqrt{p_{22}(T)}$.
7. If (5.49) is not satisfied, we repeat the procedure again with a different guess of $\lambda_{11}(0)$ and $\lambda_{22}(0)$.

The results are given in Figures 5.2-5.4 with the corresponding curves identified by the symbol ME2. Figure 5.2 gives the plot of $\sqrt{p_{11}(T)}$ (which is the same as $\sqrt{a_{11}(T)}$ since $V(T) = 0$) versus the total effort. It is seen that most of the expended effort appears near the beginning of the trip and near the end of the trip when very high terminal accuracy is required. A typical plot of the history of $\sqrt{a_{22}(t)}$ versus time to go is given in Figure 5.3 for the case where $\sqrt{a_{11}(T)}$ = 1530 km. Note the period of no control and the impulse at the end. The corresponding total velocity required as a function of the time to go is shown in Figure 5.4. The jump at T is due to the impulsive correction.

In order to get a "feeling" for these numbers, we include, in the same graph, some typical values obtained from other solutions. The two solutions we have used are the Quadratic Loss (to be denoted by QL) and the Minimum Effort for controlling only the final position (to be denoted by ME1).

<u>QL</u>: This is the problem of minimizing

$$E \int_0^T ||u(t)||^2 \, dt = \int_0^T \mathrm{tr}\, P\, S'\, S \, dt$$

for a specified P(T). The solution of this problem is well known.[9] Let the solution be denoted by *. Then

$$S^* = G' \Phi' \Lambda^*$$

$$\dot{\Lambda}^* = \Lambda^* Z \Lambda^*$$

$$\dot{P}^* = -(Z \Lambda^* P^* + P^* \Lambda^* Z) + Q$$

With the exception of $\emptyset(t)$, we see that this set of equations is the same as that given by (5.7-5.9). However, here the problem is not singular. The solution can be obtained easily by integrating the adjoint equations backwards with an estimated value of $\Lambda^*(T)$. The off diagonal elements of $\Lambda^*(T)$ and zero and the diagonal elements of $\Lambda^*(T)$ are to be adjusted so that the prescribed values of $P_{ii}(T)$ are satisfied. To obtain the solution corresponding to the case that $p_{22}^*(T) = 0$, we let $\lambda_{22}^*(T) = \infty$. The results are also plotted in Figures 5.2-5.4. The numerical values indicate that the difference between this solution and the optimal solution developed in this paper in the total velocity requirement is about ten percent.

<u>ME1</u>: This is the problem of minimizing the effort when only $p_{11}(T)$ is specified. It corresponds to the case of letting $\lambda_{12}(0) = \lambda_{22}(0) = 0$. In other words, we control the position to the specified rms value and turn off the control until T. An impulse is then added to bring $p_{22}(T)$ down to zero. In Figure 5.2 we plot the results of $\sqrt{a_{11}(T)}$ versus the total effort with or without the final impulse. The amount of the additional velocity correction due to the impulse is, of course, $\sqrt{p_{22}^-(T)}$. Similar plots are given in Figures 5.3 and 5.4. As expected, for the same terminal rms values, this design requires a little more effort than that obtained by controlling both components starting at t = 0.

B. <u>Controlling the Two Positions of Two One-Dimensional Models</u>

This example considers the terminal phase of an interplanetary trip where both the in-plane and out-of-plane terminal position components are to be independently controlled. The perturbed motions are assumed to be decoupled and that each one moves in a uniform motion. We shall use the same information rate matrix as that used in the previous example and it will be further assumed that the information rate with respect to the two positions are independent.

The differential equations governing the adjoint variables are:

$$\dot{\lambda}_{11}(t) = (T-t)^2 \lambda_{11}^2 (t) \phi(t) \tag{5.53}$$

$$\dot{\lambda}_{22}(t) = (T-t)^2 \lambda_{22}^2 (t) \phi(t) \tag{5.54}$$

and

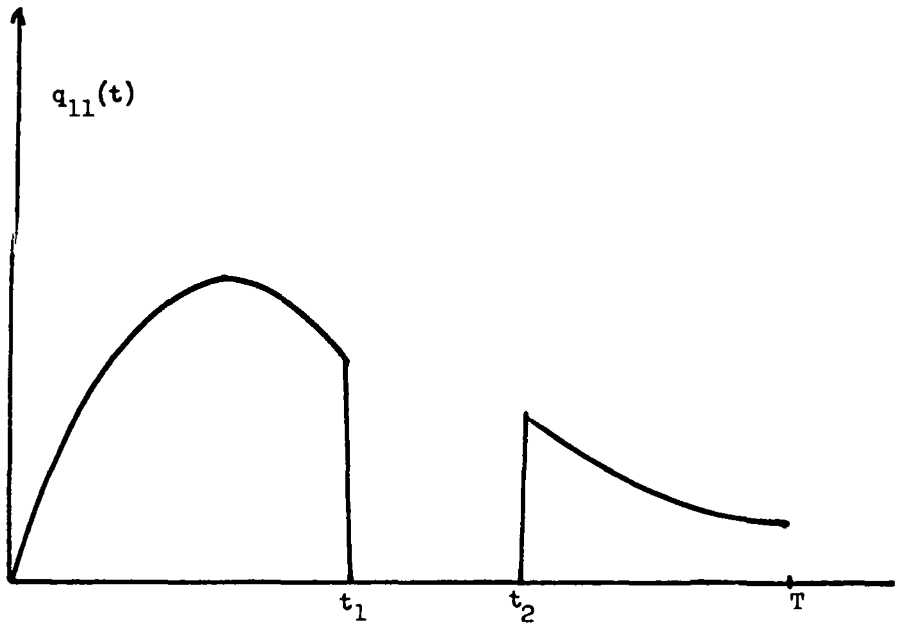
$$\lambda_{12}(t) = p_{12}(t) = 0 \tag{5.55}$$

Equations (5.53-5.54) do not imply that the equations are decoupled. The coupling is introduced by the function $\phi(t)$. By letting $\lambda_{22}(0) = 1$, a family of solutions can be obtained for different values of $\lambda_{11}(0)$. A typical one corresponding to $\lambda_{11}(0) = 1.01$ is given in Figure 5.5. It shows the plot of the history of $\sqrt{a_{11}(T)}$, $\sqrt{a_{22}(T)}$ and the effort versus the time to go. It is seen that the solution consists of an initial period of no control, followed by a period of continuous control and finally a period of no control at the end. The last statement is true since the control may be turned off when sufficient terminal accuracies have been obtained.

<u>Case Involving a Gap in Information Rate</u>

We know that in the event $\phi(t) < 0$, there will exist intervals within (t_{on} , t_{off}) such that the control is turned off. This occurs, for instance, when the information rate suddenly increases. A computation procedure was described in the previous section by which the intervals of no control can be found. For purposes of illustration, we assume that the information vanishes over the interval (t_1 , t_2) and suddenly increases at t_2. In particular, we choose $t_1 = 0.27T$ and $t_2 = 0.45T$.

Now the elements of $Q(q_{11}$ and $q_{22})$ are equal and have the general shape



It is clear that the control cannot follow the sharp rise of q_{11} at t_2; i.e., $\phi(t_2) < 0$. Therefore, the control must be turned off before or immediately after t_1. Since Q is discontinuous at t_1 , it follows from the reasoning given in the previous section that an impulse may be applied at t_1. This is indeed the case. The amount of the impulse (which is not a full correction) is determined by the condition that the adjoint variables after the correction must be the same as the time when the control is turned on again. The amount of the drop or jump in P or Λ can be determined by integrating with respect to the effort at t_1 using (5.23-5.24) which in our case can be written as

$$\frac{dp_{ii}}{de} = -2(T-t_1)^2 \lambda_{ii} \; p_{ii} \tag{5.56}$$

$$\frac{d\lambda_{ii}}{de} = (T-t_1)^2 \lambda_{ii}^{\;2} \quad ; \quad i = 1, 2 \tag{5.57}$$

Let the superscripts - and + denote the times immediately before and after the impulse, respectively. Direct integration of (5.57) yields

$$\text{effort due to the impulse} = \frac{1}{(T-t_1)^2}\left(\frac{1}{\lambda_{ii}^-} - \frac{1}{\lambda_{ii}^+}\right) \tag{5.58}$$

Dividing (5.56-5.57) shows

$$\frac{dp_{ii}}{2p_{ii}} = - \frac{d\lambda_{ii}}{\lambda_{ii}}$$

which can be integrated to give

$$(p_{ii} \lambda_{ii}^2)^- = (p_{ii} \lambda_{ii}^{\;2})^+ \tag{5.59}$$

Equation (5.59) shows, as expected, that (5.13) is satisfied during the impulse.

Prior to t_1, the computation remains the same as before. At t_1, we proceed as follows: Let $d = \frac{\lambda_{11}}{\lambda_{22}}$.

1. Assume an effort due to the impulse and compute λ_{11}^+ , λ_{22}^+ , and d from (5.58).
2. Use (5.59) to determine p_{11}^+ and p_{22}^+ .
3. Integrate the equations for p_{ii} with S = 0 until (5.15) is satisfied. This determines t_1'. Use is then made of (5.13) to determine λ_{ii} and d at t_1' .

4. If $d(t_1^+) \neq d(t_1')$, we repeat the procedure again by assuming a different effort.

The results for the case $\lambda_{11}(0) = 1.01$ are shown in Figure 5.6. The discontinuities at t_1 correspond to the impulsive correction. It is seen that t_1' is greater than t_2 which agrees with the intuitive reasoning that it is necessary to let the information catch up after an interval of no observation.

It is of interest to note that the quadratic loss solution corresponding to this particular example is completely decoupled. In other words, specification of the variance of the terminal in-plane position does not effect the solution of the out-of-plane component and vice versa. The coupling, in our case, is introduced by the loss function.

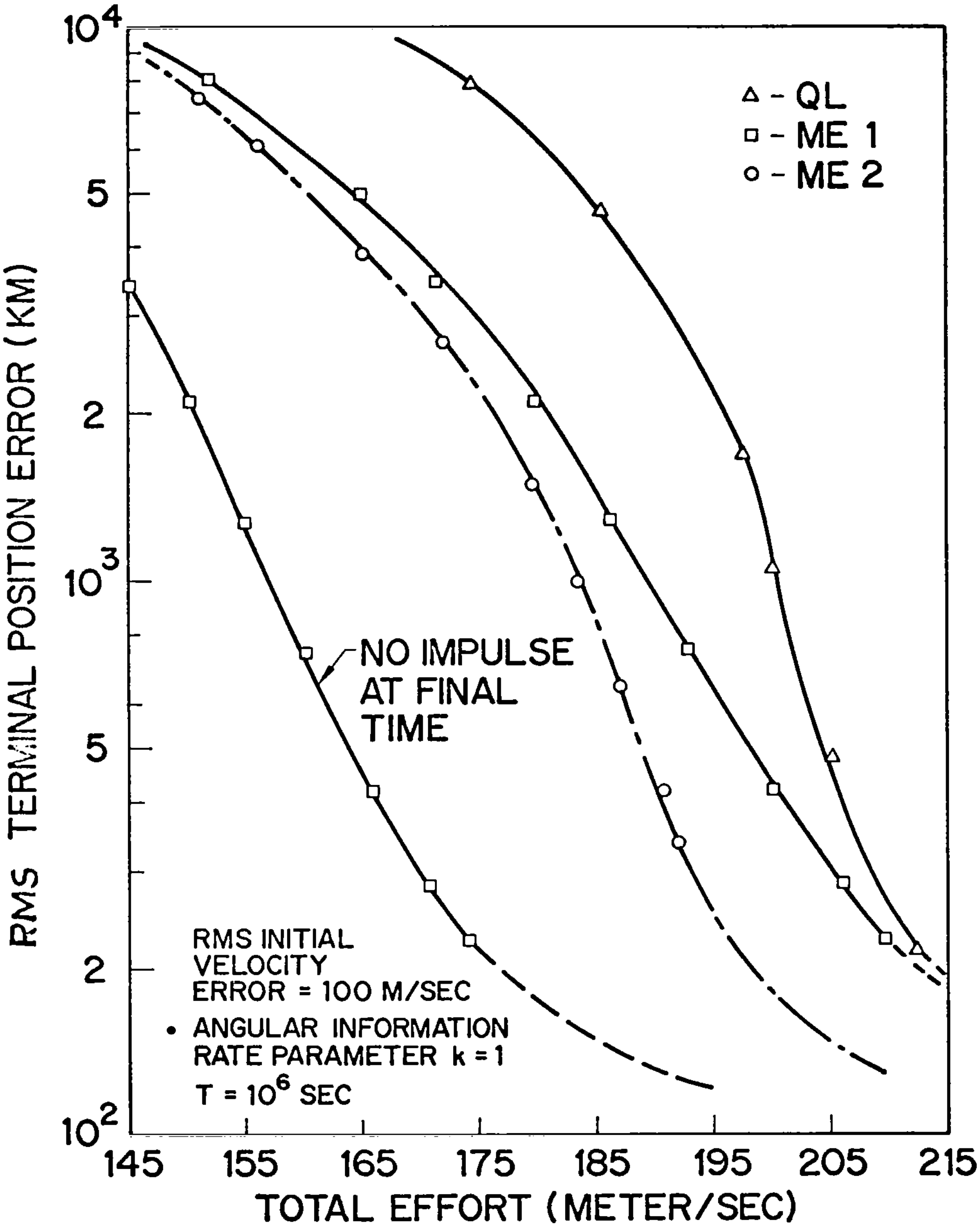


FIG. 5.2 RMS TERMINAL POSITION ERROR VS. TOTAL EFFORT

POSITION AND VELOCITY CONTROL

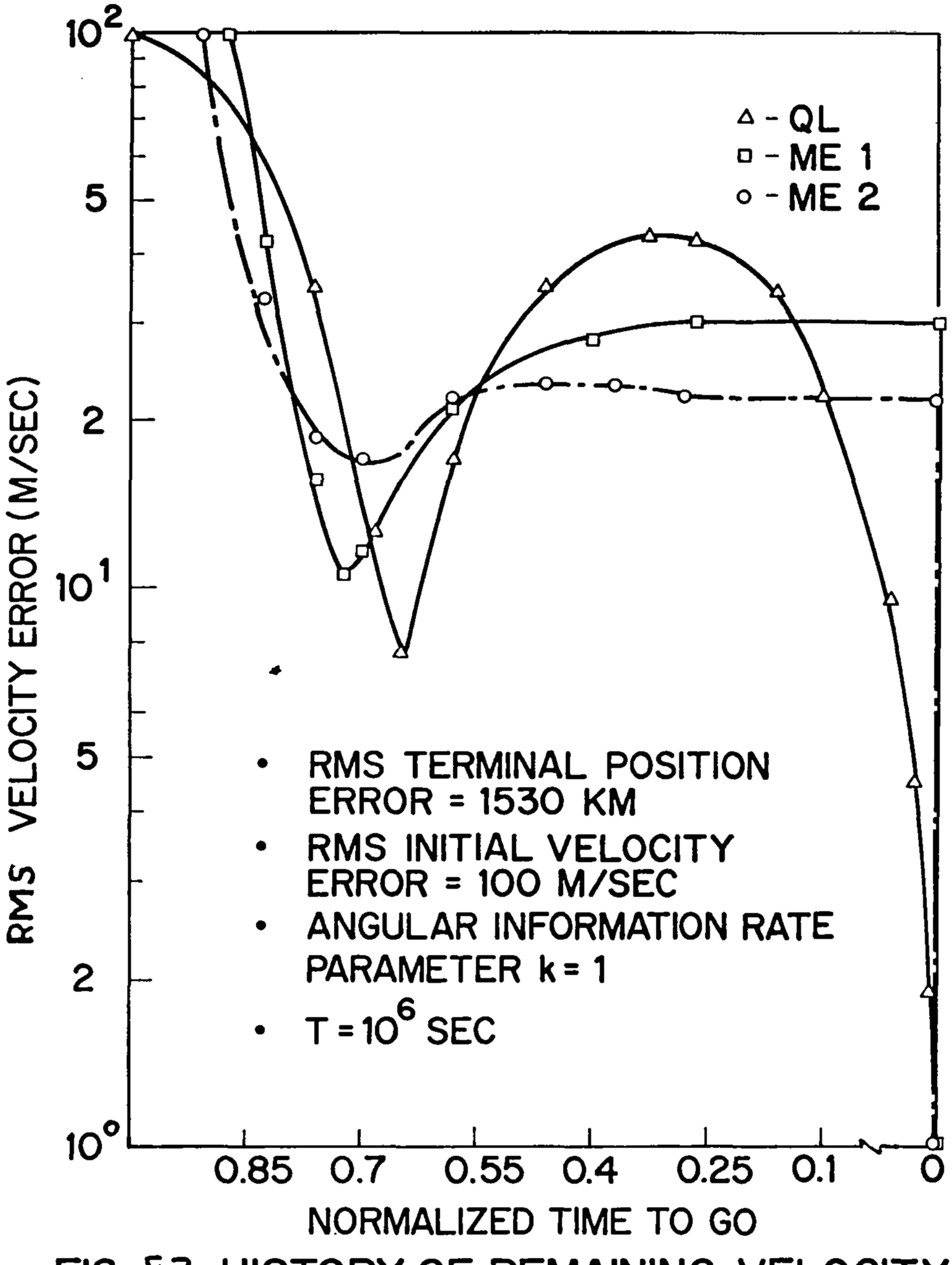


FIG. 5.3 HISTORY OF REMAINING VELOCITY ERROR VS. TIME TO GO

POSITION AND VELOCITY CONTROL

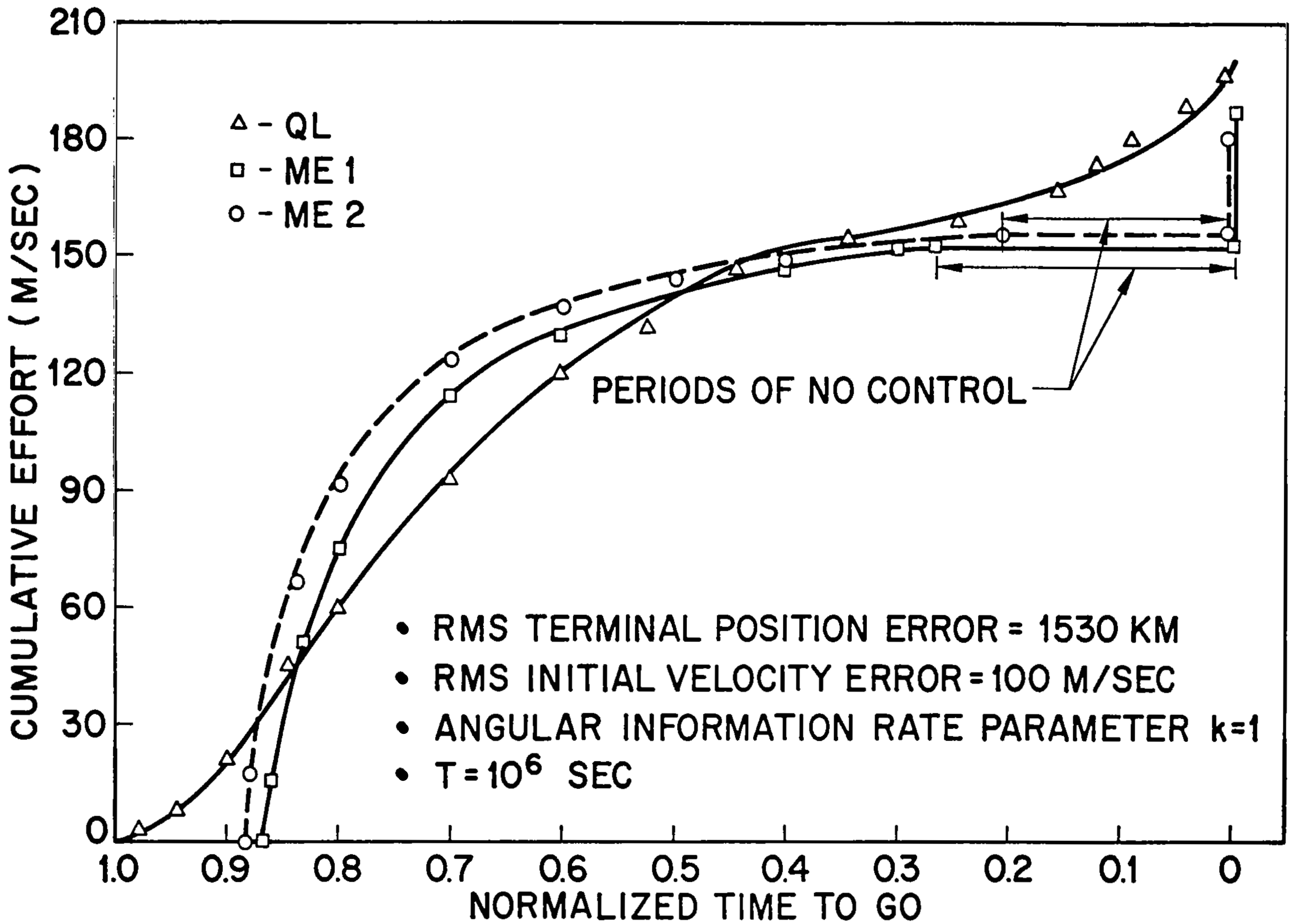


FIG. 5.4 CUMULATIVE EFFORT VS. NORMALIZED TIME TO GO
POSITION AND VELOCITY CONTROL

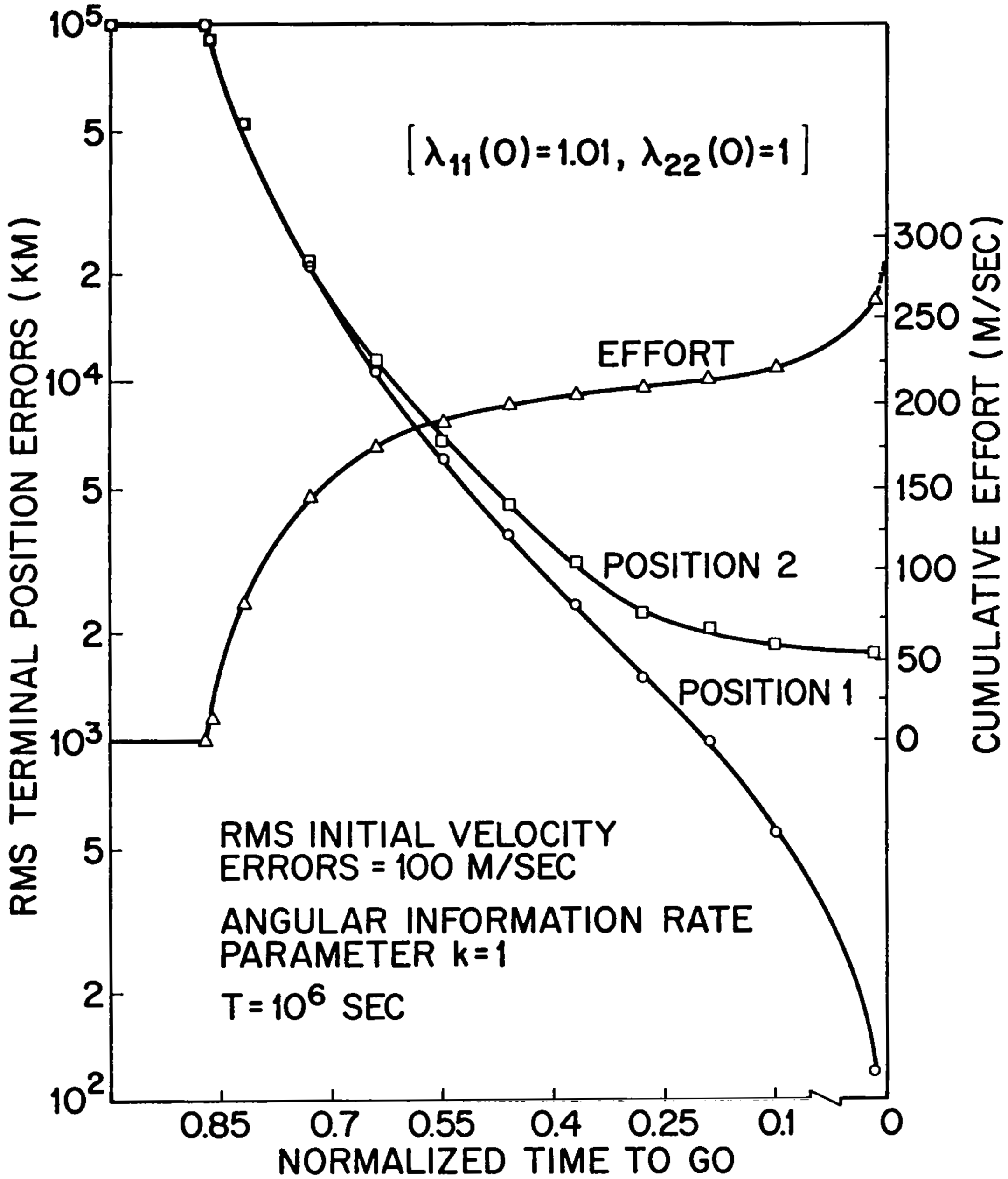


FIG. 5.5 HISTORY OF REMAINING POSITION ERRORS AND CUMULATIVE EFFORT VS. TIME TO GO

TWO POSITION CONTROL

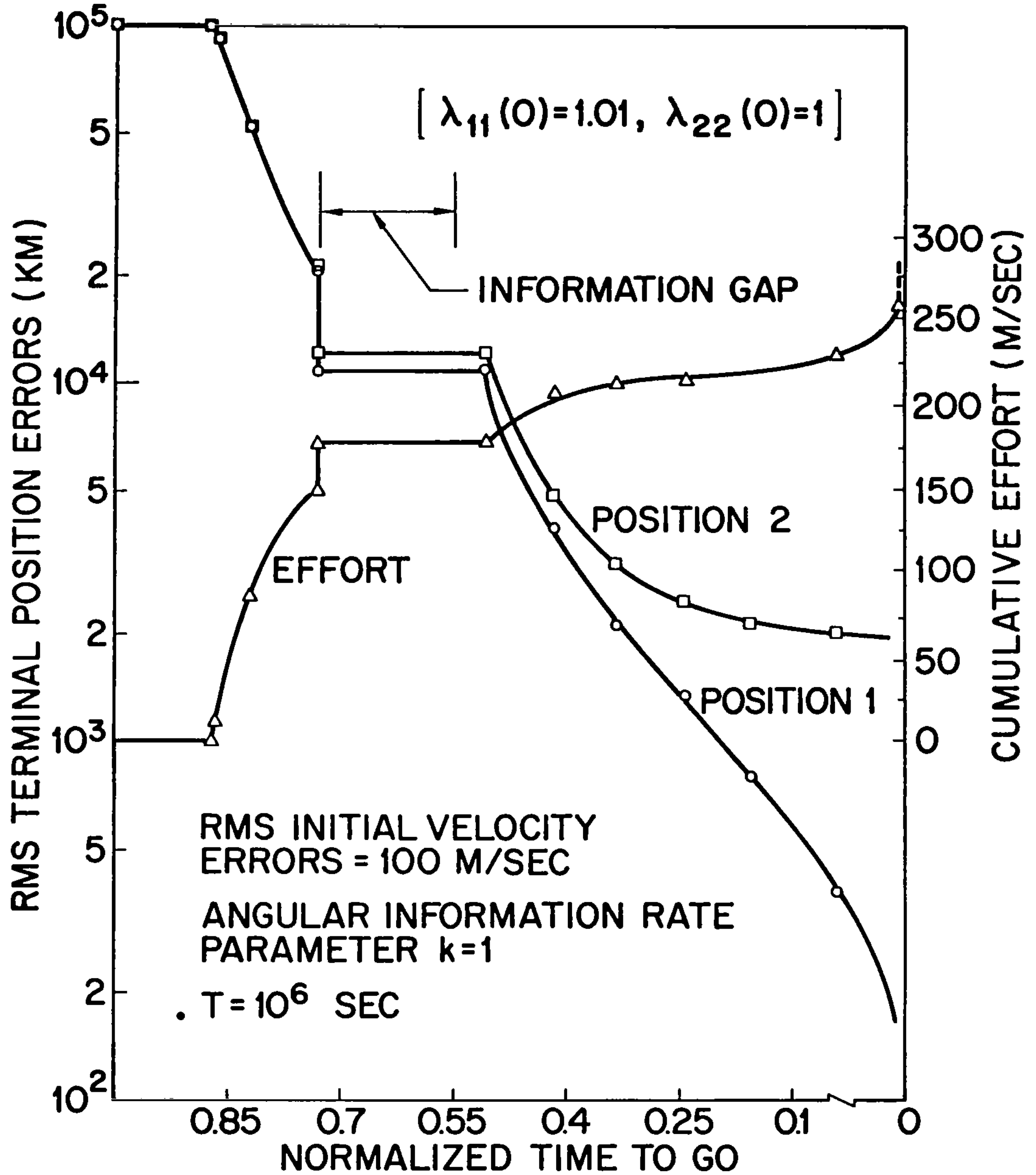


FIG. 5.6 HISTORY OF REMAINING POSITION ERRORS AND CUMULATIVE EFFORT VS. TIME TO GO

TWO POSITION CONTROL

2.6 Solution by Green's Theorem (Review)

This section outlines an alternate method (the one used by Breakwell and Striebel) for solving the optimization problem of controlling only one terminal component. Without loss of generality, it will again be assumed that the particular terminal miss we wish to control is the uncertainty in the position x_1.* The method is based on an ingenious application of Green's Theorem.

From Eq. (4.4), we find

$$\frac{dp_{11}}{dt} = -2H \Phi G S p_{11} + q_{11} \quad ; \quad p_{11}(0) = 0 \tag{6.1}$$

where H is a row vector consisting of all zero elements except $h_1 = 1$. Let the quantity to be minimized be given by

$$\int_0^T \sqrt{S'(t)\ s(t)} \cdot \sqrt{p_{11}(t)} \quad dt \tag{6.2}$$

The problem is to find the elements of the mxl feedback gain matrix S(t) which minimizes (6.2) subject to the differential constraint (6.1) and a prescribed $p_{11}(T)$.

Now, for a prescribed instantaneous $p_{11}(t)$ and $S'(t)\ S(t)$, the negative term in (6.1) is most negative if we choose the m-vector S(t) parallel to the m-vector $H\Phi G$. In other words, we apply the control in the direction of maximum effectiveness. Let

$$S(t) = \frac{g(t)\ G'\ \Phi'\ H'}{D(t)} \tag{6.3}$$

*The work in Sections 2.7-2.9 are all concerned with the problem of controlling only one terminal component.

where

$$D(t) = \sqrt{H \Phi G G' \Phi' H'} \tag{6.4}$$

is the maximum velocity effectiveness and $g(t) \geq 0^*$ is a scalar gain to be determined. Substituting (6.3) into (6.1)-(6.2), we find

$$\frac{dp_{11}}{dt} = -2D(t)\ g(t)\ p_{11}(t) + q_{11}(t) \tag{6.5}$$

while the integrated total effort becomes

$$\int_0^T g(t) \sqrt{p_{11}(t)}\ dt \tag{6.6}$$

The problem now becomes that of finding a scalar gain $g(t) \geq 0$ which minimizes (6.6) subject to the differential constraint (6.5) with specified $p_{11}(T)$.

The problem stated in the previous paragraph is equivalent, by elimination of $g(t)$ between (6.5) and (6.6), to the minimization of the following time integral in the t-p plane

$$e(c) = \int_{(0,0)}^{(T,\ p_{11}(T))} \frac{q_{11}(t)\ dt - dp_{11}}{2D(t) \sqrt{p_{11}}} \tag{6.7}$$

where at each point of the curve C joining (0,0) to $(T,\ p_{11}(T))$

$$-\infty \leq \frac{dp_{11}}{dt} \leq q_{11}(t) \tag{6.8}$$

Now the difference in cost $e(c_1)$ and $e(c_2)$ associated with two different strategies $g_1(t)$ and $g_2(t)$ leading to the specified $p_{11}(T)$ can be expressed as a line integral around a closed curve in the t-p_{11} plane which, according

*We assume $a_{11}(T) < a_{11}(0)$ so that a negative $g(t)$ will not be helpful.

to Green's Theorem, is

$$e(c_1)-e(c_2) = \oint_{c_{12}} \frac{q_{11}(t)\,dt - dp_{11}}{2D(t)\sqrt{p_{11}}} \tag{6.9}$$

$$= \iint_{A_{12}} \left[\frac{\partial}{\partial t}\left(\frac{-1}{2D(t)\sqrt{p_{11}}}\right) - \frac{\partial}{\partial p_{11}}\left(\frac{q_{11}(t)}{2D(t)\sqrt{p_{11}}}\right)\right] dt\,dp_{11}$$

where $\oint_{c_{12}}$ denotes the line integral around the closed curve obtained by following C_1 forward from (0,0) to $(T, p_{11}(T))$ and then C_2 back to (0.0). The area A_{12} is counted as positive if enclosed in a counter-clockwise direction. Evaluating the integral in the double integral in (6.9), we obtain

$$e(c_1)-e(c_2) = \iint_{A_{12}} \frac{2\,p_{11}\,\dot{D}(t) + q_{11}\,D(t)}{4\,p_{11}^{3/2}\,D^2(t)}\,dtdp_{11} \tag{6.10}$$

Assuming that, typically, D(t) is a decreasing function of t, the integral in (6.10) is positive or negative as the point in question lies below or above a critical curve C^* given by

$$p_{11}(t) = p_{11}^*(t) = \frac{q_{11}(t)\,D(t)}{-2\dot{D}(t)} \tag{6.11}$$

which separates (0,0) from $(T, p_{11}(T))$, since $D(T) = 0$ and $q_{11}(0)\,D(0) > 0$. Thus, a possible curve from (0,0) to $(T, p_{11}(T))$ must cross the critical curve C^* an odd number of times. Figure 6.1 illustrates the situation when this number is 3.

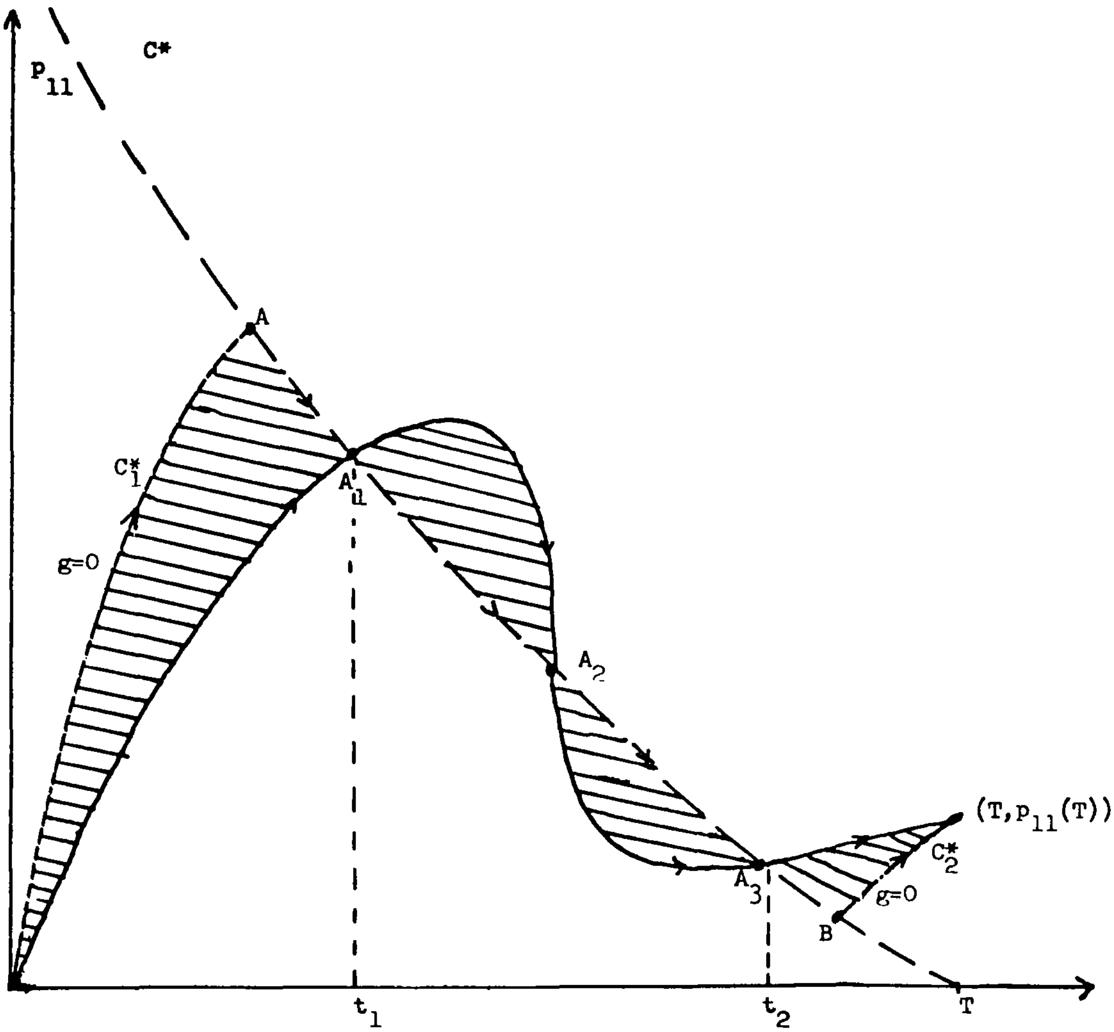


Figure 6.1 Possible p_{11} Histories

Let the crossing points be A_1, A_2, ... Furthermore, let A be a point on C^* obtained by proceeding from (0.0) along a curve $C_1{}^*$ with a maximum slope $q_{11}(t)$ permitted by (6.8) until the critical curve is reached and let B be obtained similarly by proceeding backwards from (T, p_{11}(T)) along a curve $C_2{}^*$ with slope $q_{11}(t)$ until C^* is reached. Then (6.10), together with the plus sign of the integral below C^* shows that the contribution to C from that part of the curve C between (0,0) and A is greater than that obtained by following C_1^* from (0,0) to A and then C^* from A to A_1. Likewise, the contribution of the arc A_2A_3 of C is greater than that of the corresponding arc of C^*. Similarly, because of the minus sign of the integral of (6.10) above C^*, the contribution to C of the arc A_2A_3 of C is greater than that of the corresponding arc of C^*, and the contribution to e of the arc of C between A_3 and $(T, p_{11}(T))$ is greater than that obtained by following C^* from A_3 to B and then C_2^* from B to $(T, p_{11}(T))$.

Putting all this together, we have proved that the optimum curve C is made up of C_1^* , C^* , C_2^* , so that the optimum g(t) is 0 until the time t, at which C_1^* meets C^* and is again 0 after the time t_2 at which C^* meets C_2^*. Between t_1 and t_2 , the optimum g(t) is such as to yield (6.11).

In summary, then, the optimal strategy, in general, consists of a period of no control while $p_{11}(t)$ rise from 0 to the critical curve. This is followed by a period of continuous (non-impulsive) control as long as $q_{11}(t)$ is continuous, and provided that $p_{11}^*(t)$ does not exceed $q_{11}(t)$, and finally followed by a period of no control just before arriving near the planet.

As we have mentioned already, the above solution is not applicable if the critical curve C^* has anywhere a positive slope greater than the maximum allowable $q_{11}(t)$. Such is the case, for example, when a sharp increase in information rate is encountered. Suppose, for example, that there is a sharp rise in the information rate at some time t_c. This implies that $p_{11}^*(t)$ also has a sharp rise at t_c. In this case, the optimum allowable C must leave C^* at some time prior to t_c, proceed at the maximum allowable slope $q_{11}(t)$, and rejoin C^* at some time later than t_c , in such a way as to minimize the sum of the double integrals evaluated over the two shaded areas indicated in Figure 6.2.

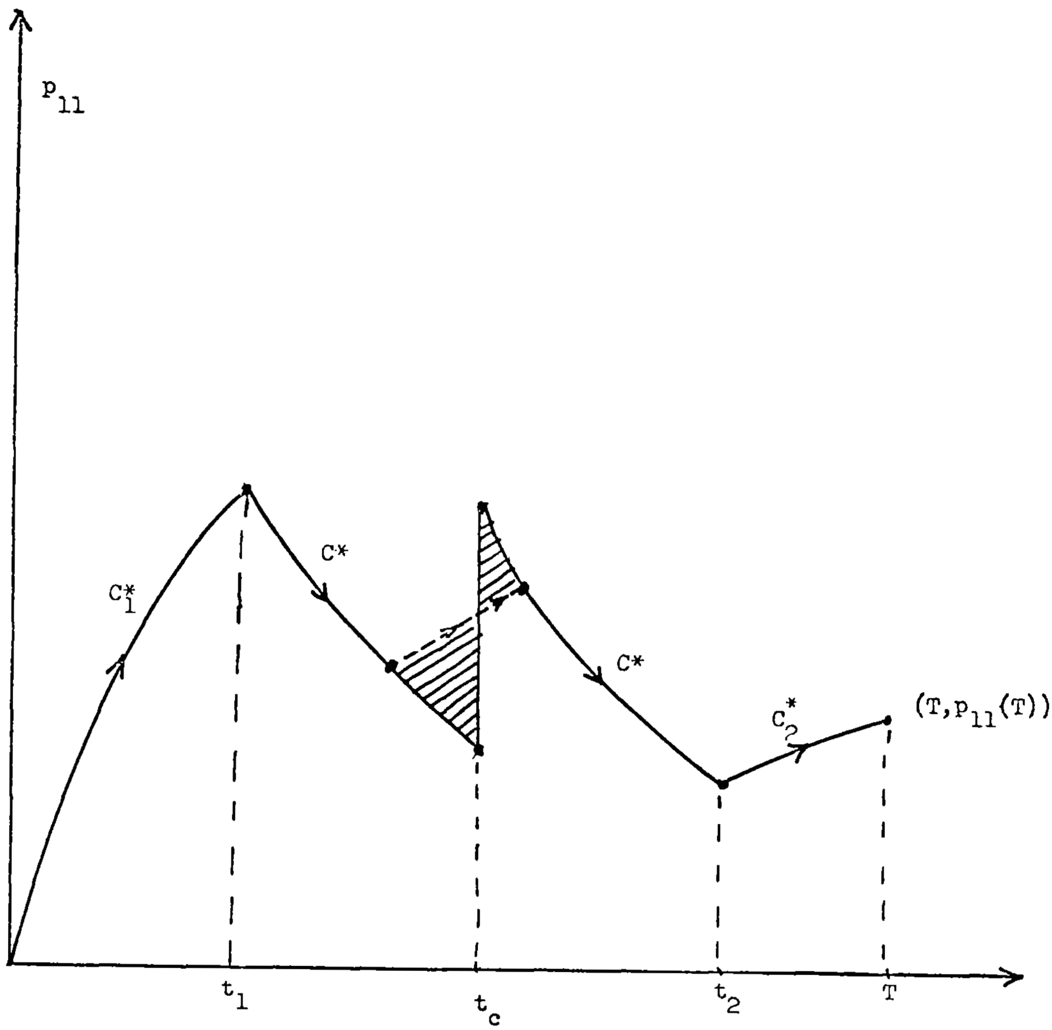


Figure 6.2 Optimum p_{11} History with Jump in Information Rate

This situation also arises if there is a finite interval of time, say t_B to t_C, over which the information rate vanishes. In this case $p_{11}{}^*(t)$

vanishes between t_B and t_C so that the curve C^* drops down to the t-axis between t_B and t_C and rises sharply again at t_C. Again an allowable C cannot follow the sharp rise. The optimum C must, therefore, follow the sharp drop at t_B only part of the way down to $p_{11} = 0$, proceed at slope $q_{11}(t)$, which is 0 between t_B and t_C, and rejoin C^* at some time later than t_C, in such a way as to minimize the sum of the contributions of the two shaded areas in Figure 6.3 The drop part way toward $p_{11} = 0$ at t_B corresponds, of course, to an impulsive correction less than the full correction indicated by the estimated miss just prior to t_B.

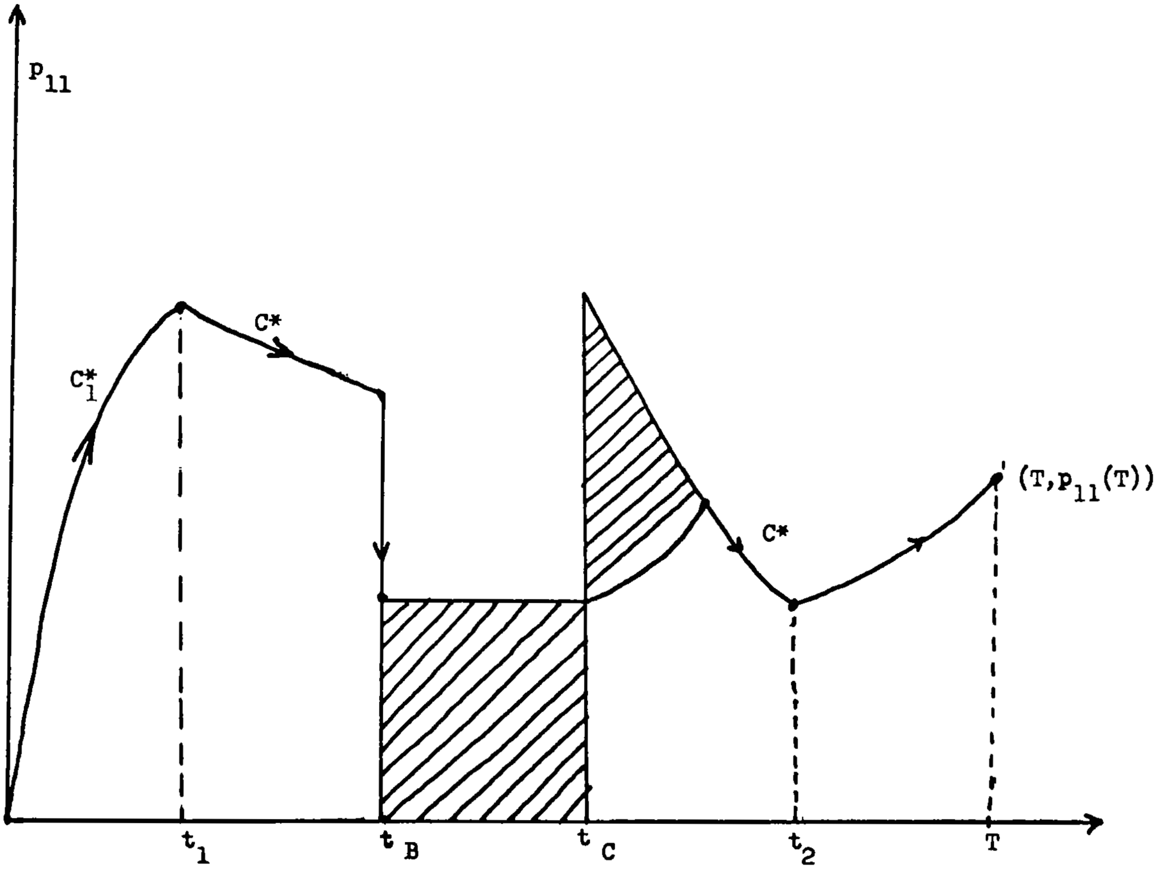


Figure 6.3 Optimum p_{11} History with Break in Information Rate

The search for the optimum transitions in Figures 6.2 and 6.3 is not tedious. It was shown in the previous section (Section 5.2) that optimum transitions correspond to double points in the a_{11} - D $\sqrt{p_{11}}$ plane where, to recapitulate,

$$a_{11}(t) = p_{11}(t) + H \Phi V \Phi' H' \tag{6.12}$$

which is the mean square value of the actual terminal miss when the control is turned off at t.

It should be noted that $p_{11}(t)$ and $a_{11}(t)$ are mean-squared quantities whose optimum histories correspond to an optimal choice of g(t). A typical history of the random process $| \hat{x}_1(T, t) |$ is not necessarily monotonic prior to control turn-on and the value at turn-on is not necessarily at some pre-assigned critical level. Neither is its final value specified.

2.7 Simultaneous Optimization of Control, Measurement Rate and the Type of Measurements

The average total velocity correction computed in the previous sections depends partly on launching accuracy and partly on the information rate history. The latter is especially true near arrival at a planet. Now, whether onboard measurements of the planet against a star background are made photographically by astronauts or by powered star and planet trackers, there are good reasons for reducing the total number of measurements to a number very much smaller than the number possible by measuring throughout at a maximum rate even though the average total velocity correction would thereby be somewhat increased. We are led, thus, to formulate the following problem: Optimize the variable observation rate as well as the correction schedule so as to achieve a desired terminal accuracy with a minimum value of a specified linear combination of total number of observations and average total velocity correction. Again, we will be only concerned with the case of $p = 1$ and controlling the terminal position x_1.

The results of this investigation of simultaneous optimization of control and measurement rate seem to indicate that, in general, the optimal policy consists of periods of measuring separated by periods without measurement or corrective action. Each measurement period starts at a maximum rate with a sub-period without correction action. This is followed by a sub-period of gradual (continuous) correction, and ends with an impulsive partial correction of the miss. The measuring prior to the impulse may be either at maximum rate or at a lower critical rate. If, in addition, a choice is available at any time between various measurements, the optimization procedure automatically selects a most advantageous measurement.

For mathematical simplicity, we shall confine our attention in Sections 2.7.1-2.7.4 to an essentially one-dimensional control problem the same as the example considered in Section 2.5. Section 2.7.5 shows how the solution can be extended to the two-dimensional case including, in addition, a choice between several kinds of observations.

2.7.1 Formulation of the One-Dimensional Model

Consider the one-dimensional problem analogous to the approach to a planet where the only available observation is the vehicle-planet direction (described in Section 2.5.3). The velocity effectiveness of this straight line model is simply the time to go; i.e., $D(t) = T-t$, which will be denoted by τ .

Let $r(t)$ be the variable measurement rate, $0 \le r(t) \le R$; R being the maximum observation rate. The spectrum of the additive measurement noise can be written as

$$\frac{\sigma_\epsilon^2}{r(t)} \tag{7.1}$$

Let $p_{11}(t)$ and $w_{11}(t)$ be the variances of the predicted miss-distance $\hat{x}_1(T,t)$ and its error $x_1(T,t) - \hat{x}_1(T,t)$, respectively, and let $a_{11}(t)$ be the variance of the actual miss. Then

$$a_{11}(t) = p_{11}(t) + w_{11}(t)$$

since the error in $\hat{x}_1(T,t)$ is known to be independent of $\hat{x}_1(T,t)$. It can be readily shown, using the results of the previous sections, that

$$\dot{p}_{11}(t) = -2\tau\, g(t)\, p_{11}(t) + r(t)\, h(t)\, w_{11}^2(t) \tag{7.2}$$

$$\dot{a}_{11}(t) = -2\tau\, g(t)\, p_{11}(t) \tag{7.3}$$

and

$$\dot{w}_{11}(t) = -r(t)\, h(t)\, w_{11}^2(t) \tag{7.4}$$

where $h(t)$ is a measure of the geometrical effectiveness of the measurements and increases markedly as $t \to T$ in our case of angular measurements of the target's instantaneous direction

$$h(t) = \frac{t^2}{v_f^2 \tau^2 \sigma_\epsilon^2} \tag{7.5}$$

Note $p_{11}(0) = 0$, while $a_{11}(0) = w_{11}(0) = T^2 \text{ cov } (x_2(0))$. Now the total number of observations may be represented by

$$\int_0^T r(t)\, dt \tag{7.6}$$

while the average total velocity correction is

$$\int_0^T g(t) \sqrt{p_{11}(t)}\, dt \tag{7.7}$$

Let the cost be given by

$$\text{cost} = \int_0^T 2\, g(t) \sqrt{p_{11}(t)}\, dt + k \int_0^T r(t)\, dt \tag{7.8}$$

where k is a specified constant. The problem to be solved in this section can be stated as follows. Determine the control variables $r(t)$ and $g(t)$, subject to the inequalities

$$0 \le r(t) \le R \quad \text{(the maximum observation rate)} \tag{7.9}$$

$$0 \le g(t) \le \infty \tag{7.10}$$

which minimize the cost (7.8) for a given sum $a_{11}(T)$ of the final values of the "states" p_{11} and w_{11} where known initial values are 0 and T^2 cov $(x_2(0))$ and which satisfy the differential constraints (7.2) and (7.4) where $h(t)$ is a known function. We assume that $a_{11}(T) < a_{11}(0)$ so that a negative $g(t)$ would not be helpful.

Note that this is a doubly singular problem in that both control variables occur only linearly in the appropriate Hamiltonian. A computation procedure for solving this problem is given in the next two sections.

2.7.2 Necessary Conditions for Optimality

To investigate the computation of the solution, we first derive the necessary conditions under which the optimal solution must satisfy. Let $\lambda_{11}(t)$ and $\alpha_{11}(t)$ be the adjoint variables corresponding to $p_{11}(t)$ and $w_{11}(t)$, respectively. The Hamiltonian to be minimized is thus

$$H = \lambda_{11}\,(r\,h\,w_{11}^2 - 2\,\tau\,g\,p_{11}) - \alpha_{11}\,r\,h\,w_{11}^2 + 2\,g\sqrt{p_{11}} + k\,r$$

$$= 2\,g\sqrt{p_{11}}\,(1 - \tau\,\lambda_{11}\sqrt{p_{11}}) + r\,(k - (\alpha_{11} - \lambda_{11})\,h\,w_{11}^2) \tag{7.11}$$

The rates of change of the adjoint variables are:

$$\dot{\lambda}_{11} = -\frac{\partial H}{\partial p_{11}} = 2\,\lambda_{11}\,\tau\,g - \frac{g}{\sqrt{p_{11}}} \tag{7.12}$$

$$\dot{\alpha}_{11} = -\frac{\partial H}{\partial w_{11}} = 2\,(\alpha_{11} - \lambda_{11})\,r\,h\,w_{11} \tag{7.13}$$

The terminal constraint on $p_{11} + w_{11}$ requires that λ_{11} and α_{11} satisfy the end constraint

$$\lambda_{11}(T) = \alpha_{11}(T) \tag{7.14}$$

Note that in this doubly singular problem $\frac{\partial H}{\partial g}$ and $\frac{\partial H}{\partial r}$ are independent of g and r. The minimization of this Hamiltonian with respect to the controls g and r is simple provided that $\frac{\partial H}{\partial g} > 0$ and $\frac{\partial H}{\partial r} \neq 0$. It turns out that $\frac{\partial H}{\partial g}$ is never negative (i.e., $\tau\lambda_{11}\sqrt{p_{11}} \leq 1$ as proven in Section 2.5.1). The procedure when either $\frac{\partial H}{\partial g}$ or $\frac{\partial H}{\partial r}$ vanishes is less direct. The state and control history throughout an interval of time during which either one vanishes is called a "singular arc," or in case they both vanish, a "doubly singular arc" (D-S-arc). Thus, the minimization of H with respect to g shows that

$$\lambda_{11}\,\tau\sqrt{p_{11}} = 1 \tag{7.15}$$

during any control period ($g(t) > 0$) and it can be shown that an impulsive correction preserves the product $\lambda_{11}\sqrt{p_{11}}$, the instantaneous drop in p_{11} being matched by a rise in λ_{11}. Differentiation of (7.15) with respect to t, together with (7.2), (7.12) and (7.15) yields the equation for the critical p^* described in the previous section, namely

$$p(t) \;=\; p^*(t) \;=\; \tfrac{1}{2}\, r\, h\, w_{11}^{\;2}\, \tau \tag{7.16}$$

The minimization of H with respect to r shows that

$$r \;=\; \begin{cases} R & \text{if} \quad F < 0 \\ 0 & \text{if} \quad F > 0 \end{cases} \tag{7.17}$$

where the "switching function" F is given by

$$F = k - (\alpha_{11} - \lambda_{11})\, h\, w_{11}^{\;2} \tag{7.18}$$

To proceed along an arc on which $F = 0$, we need an equation for computing $\dot{r}$ in terms of r, the adjoint variables and the states. This is obtained as follows. From (7.4), (7.12) and (7.14), we find

$$\frac{d}{dt} \left[(\alpha_{11} - \lambda_{11})\, w_{11}^{\;2} \right] = -\frac{g}{\sqrt{p_{11}}}\; w_{11}^{\;2} \tag{7.19}$$

which shows that

$$\frac{dF}{dt} = \frac{hg}{\sqrt{p_{11}}}\; w_{11}^{\;2} \;-\; (\alpha_{11} - \lambda_{11})\, w_{11}^{\;2}\, \frac{dh}{dt} \tag{7.20}$$

It follows easily from (7.18) and (7.20) that F cannot remain zero if $g = 0$. Thus, any interval during which F remains zero constitutes a D-S arc.

On the other hand, substituting (7.16) into (7.2) gives

$$\frac{1}{2}\, \frac{d}{dt}\, (r\, h\, w_{11}^{\;2}\, \tau) \;=\; -g\, r\, h\, w_{11}^{\;2}\, \tau^2 \;+\; r\, h\, w_{11}^{\;2} \tag{7.21}$$

which provides an expression for $g(t)$ in terms of w_{11} , r and $\dot{r}$. Eliminating $g(t)$, we find

$$\frac{dF}{dt} = \frac{1}{\sqrt{p_{11}^*}} \left[- \left(\frac{h\, w_{11}^2}{r\tau} \right) \dot{r} - \frac{1}{2\tau} \frac{d}{dt} (h\, w_{11}^2) + 3/2 \; \frac{h\, w_{11}^2}{\tau^2} \right] - (\alpha_{11} - \lambda_{11})\, w_{11}^2 \; \frac{dh}{dt} \tag{7.22}$$

Since dF/dt must be identically zero along the D-S arc, we find, after setting the right-hand side of (7.22) to zero, a relation for $\dot{r}$ in terms of r, the states p and w_{11}, and the adjoint variables α_{11} and λ_{11}. This allows us to compute the intermediate r-history.

We observe that an impulsive drop in p_{11}, and consequently an impulsive rise in λ_{11} and F, can only occur at the end of a period of observation, i.e., at time when $F = 0$ preceded by times when $F \le 0$. This is equivalent to allowing only those discontinuities in p_{11} and λ_{11} which preserve the continuity not only of $\lambda_{11}\sqrt{p_{11}}$ but also of the Hamiltonian. In particular, we cannot jump onto a D-S arc where F has to remain zero.

Furthermore, the control gain $g(t)$ is always zero at the beginning of any observation period. In particular, we cannot start on a D-S arc at the beginning of an observation period since the simultaneous vanishing to (7.18) and (7.20) is consistent with an increasing $h(t)$.

There remains the possibility of starting on a D-S arc at a time t_1 when $F(t_1) = 0$ preceded by times when $F \le 0$. If however, $\dot{F}(t_1^-) > 0$, starting at t_1 on a D-S arc would require a negative jump in $\dot{F}(t)$. But according to (7.22), $\dot{F}(t)$ is a decreasing function of $\dot{r}$ so that a negative jump in $\dot{F}$ requires a positive jump in $\dot{r}$ from the value $\dot{r}(t_1^-) = 0$ (since $F < 0 \rightarrow r = R$). But this positive jump in $\dot{r}$ is not consistent with $r \le R$. Finally, then, we conclude that we can only start on a D-S arc at time t_1 such that $F(t_1^-) = 0$ and $\dot{F}(t_1^-) = 0$, preceded by times $F < 0$.

The next section outlines a computation procedure for obtaining the optimal r and g histories. It turns out that the optimal policy, in general, consists of periods of measuring separated by periods without measurement or corrective action. Each measurement period starts at maximum rate $r(t) = R$, with a subperiod without corrective action $g(t) = 0$. This is followed by a subperiod of gradual (continuous) correction and ends with an impulsive partial correction of the miss. The measuring prior to the impulse may be either at maximum rate or at a lower rate. The latter case constitutes a "doubly singular" segment of the control history.

2.7.3 Computation Procedure

From the equations derived in Section 2.7.2, we see that (1) if $r = 0$ then $g = 0$, while α_{11} , λ_{11} , p_{11} and w_{11} remain constant; (2) $F(0) > 0$. It follows that $r = 0$ initially. The computation proceeds as follows:

1. Guess an initial positive value for $(\alpha_{11} - \lambda_{11})$.
2. Keep $r = 0$ and w_{11} constant until t_1 when $F(t_1) = 0$; t_1 is specified by the equation

$$h(t_1) = \frac{k}{\left\{\alpha_{11}(0) - \lambda_{11}(0)\right\} \quad w_{11}{}^2(0)} \tag{7.23}$$

3. Keep $r = R$, $g(t) = 0$ and compute $w_{11}(t)$ and $p_{11}(t) = a_{11}(0) - w_{11}(t)$ until time t_2 when $p_{11}(t)$ reaches the critical curve $p_{11}{}^*(t_2) = \frac{1}{2}\, \tau_2\, Rh(t_2)w_{11}{}^2(t_2)$. Computation of $w_{11}(t)$ is done here analytically by using the explicit solution (see Eq. 5.47)

$$w_{11}{}^{-1}(t_2) = w_{11}{}^{-1}(t_1) + \frac{R}{v_f{}^2\, \sigma_\epsilon{}^2\, T^2} \left\{ (t_2 - t_1) + 2T\log\left(\frac{T-t_2}{T-t_1}\right) + \frac{T^2}{T-t_2} - \frac{T^2}{T-t_1} \right\} \tag{7.24}$$

4. Compute $\lambda_{11}(t_2)$,

$$\lambda_{11}(t_2) = \frac{1}{\tau_2 \sqrt{p_{11}*(t_2)}} \tag{7.25}$$

and $\alpha_{11}(t_2)$, which can be obtained by the equation

$$\left\{\alpha_{11}(t_2) - \lambda_{11}(t_2)\right\} w_{11}{}^2(t_2) = \left\{\alpha_{11}(0) - \lambda_{11}(0)\right\} w_{11}{}^2(0) \tag{7.26}$$

since $g(t) = 0$ for $t \epsilon (0,t_2)$.

5. Keep $p_{11}(t) = p^*(t) = \frac{1}{2} R \tau h w_{11}^2$. Compute $\lambda_{11}(t)$ by

$$\lambda_{11}(t) = \frac{1}{\tau \sqrt{p_{11}*(t)}} \tag{7.27}$$

$w_{11}(t)$ by (7.24) and numerically integrate $\alpha_{11}(t)$ until t_3 where $F(t_3) = 0$. If $F(t)$ remains negative for all $t \epsilon (t_2, T)$, the computation is repeated again from procedure 1 with a different initial guess for $(\alpha_{11}(0) - \lambda_{11}(0))$.

6. Consider t_3 as a time of final observation cutoff by applying an impulse whose magnitude is such that

$$\lambda_{11}(t_3^+) = \alpha_{11}(t_3) \tag{7.28}$$

where t_3^+ is the instant immediately after the impulse. Note that $\lambda_{11}(T) = \alpha_{11}(T)$ since $r = g = 0$ for $t \geq t_3^+$, and $\lambda_{11}\sqrt{p_{11}}$ remains unchanged during the impulse. The negative jump in $p_{11}(t_3)$ is determined by the relation

$$\lambda_{11}(t_3^-) \sqrt{p_{11}(t_3^-)} = \lambda_{11}(t_3^+) \sqrt{p_{11}(t_3^+)} \tag{7.29}$$

and the additional average velocity correction due to this impulse is given by

$$\frac{1}{\tau_3} \left(\sqrt{p_{11}(t_3^-)} - \sqrt{p_{11}(t_3^+)}\right) \tag{7.30}$$

7. Consider t_3 as a possible time of temporary but not final observation cutoff. Apply a negative jump in p_{11}, whose amount is to be determined by

iteration, along with a positive jump in λ_{11} determined by (7.29). As a result the switching function at t_3^+ again becomes positive,

$$F(t_3^+) = F(t_3^-) + \left\{\alpha_{11}(t_3) - \lambda_{11}(t_3^-)\right\} h(t_3)\, w_{11}{}^2(t_3) > 0 \quad (7.31)$$

Run through procedures 2, 3 and 4 to obtain t_4 and t_5 where t_4 is the time when F reaches zero and t_5 is the time when $p_{11}(t)$ again reaches the critical curve, i.e.,

$$p_{11}(t_5) = p_{11}{}^*(t_5) = \tfrac{1}{2} Rh(t_5)\, w_{11}{}^2(t_5)\, \tau_5$$

$$= p_{11}(t_3) + \int_{t_4}^{t_5} Rhw_{11}{}^2\, dt \qquad (7.32)$$

Now determine $\lambda_{11}(t_5)$. Since $g(t) = 0$ for $t \epsilon (t_3 , t_5)$, it follows from (7.12) that λ_{11} must remain constant during this interval. An iterative search is therefore used here to determine the size of the impulse at t_3 such that $\lambda_{11}(t_3^+) = \lambda_{11}(t_5)$. If the search is successful, the computation then follows procedure 5 until time t_6 when F again reaches zero. We are now in the same situation as the beginning of procedure 6. The same step is therefore repeated i.e., an impulse is applied whose amount is to be determined by an iterative loop, etc., until t reaches T. Note that each of the times t_3, t_6, ... may be considered as the time of final observation cutoff with mean square terminal miss $a_{11}(t_3^+)$, $a_{11}(t_6^+)$, ... This computation gives a finite number of terminal variances a(T) for every guess of the single quantity $(\alpha_{11}(0) - \lambda_{11}(0))$. The solution in this case has a form where r(t) is either at its maximum or zero, i.e., the observation rate is bang-bang.

8. If the above iterative search is not successful, we look for the possibility of a D-S arc. If in step 5, t_3 is such that $\dot{F}(t_3^-) > 0$, t_3 cannot be the beginning of a D-S arc. This then completes the computation cycle for the particular initial guess of $(\alpha_{11}(0) - \lambda_{11}(0))$.

9. If $\dot{F}(t_3^-) = 0$, t_3 is taken as the beginning of a D-S arc. Note that only a particular value of $(\alpha_{11}(0) - \lambda_{11}(0))$ will lead to this condition. To proceed, we compute $\dot{r}(t)$ from (7.22) by setting $\dot{F}(t) = 0$, use (7.16) to compute $p_{11}(t)$, (7.15) to compute $\lambda_{11}(t)$ and numerically integrate α_{11} and w_{11}. This computation continues until time t_m when r again reaches its maximum value (R). Now every point $t \epsilon (t_3, t_m)$ is a possible time of final observation cutoff by simply applying an impulse which makes $\lambda_{11}(t^+) = \alpha_{11}(t)$, $t \epsilon (t_3, t_m)$. Moreover, every point $t \epsilon (t_3, t_m)$ is also a possible time of temporary observation cutoff; in which case, the computation proceeds through an iterative loop described in procedure 7, and if successful, continues on to a time t_6* (t_6 in Figure 7.1) when F again reaches zero. Note that t_6^* is again a possible time of final or temporary observation cutoff and, if $\dot{F}(t_6^*) = 0$, t_6^* is also a possible starting point for a second D-S arc and we may then proceed along this second D-S arc and repeat the procedure again. Typical F(t), r(t) and $p_{11}(t)$ histories are shown in Figure 7.1 with two observation periods, each one ending with a D-S arc.

2.7.4 Results of Numerical Work

For purposes of illustration, we present in this section some of the numerical results we have obtained in applying the computation procedure outlined in the previous section. The following parameters are used defining this approach guidance: $T = 10^6$ sec (~10 days), $v_f = 3$ km/sec, $\sigma_\epsilon = 1$ milli-rad, $R = 270 \times 10^6$ (sec^{-1}) (approximately once/hour). The initial uncertainty is taken to be $w_{11}(0) = 3000$ km (corresponding to initial velocity uncertainty of 3m/sec). We suppose that k = 0.09.

For various values of the initial guess parameter $(\alpha_{11}(0) - \lambda_{11}(0))$, a finite number of rms terminal misses $\sqrt{a_{11}(T)}$ was obtained together with the associated costs. For an exceptional value of $(\alpha_{11}(0) - \lambda_{11}(0))$, we were

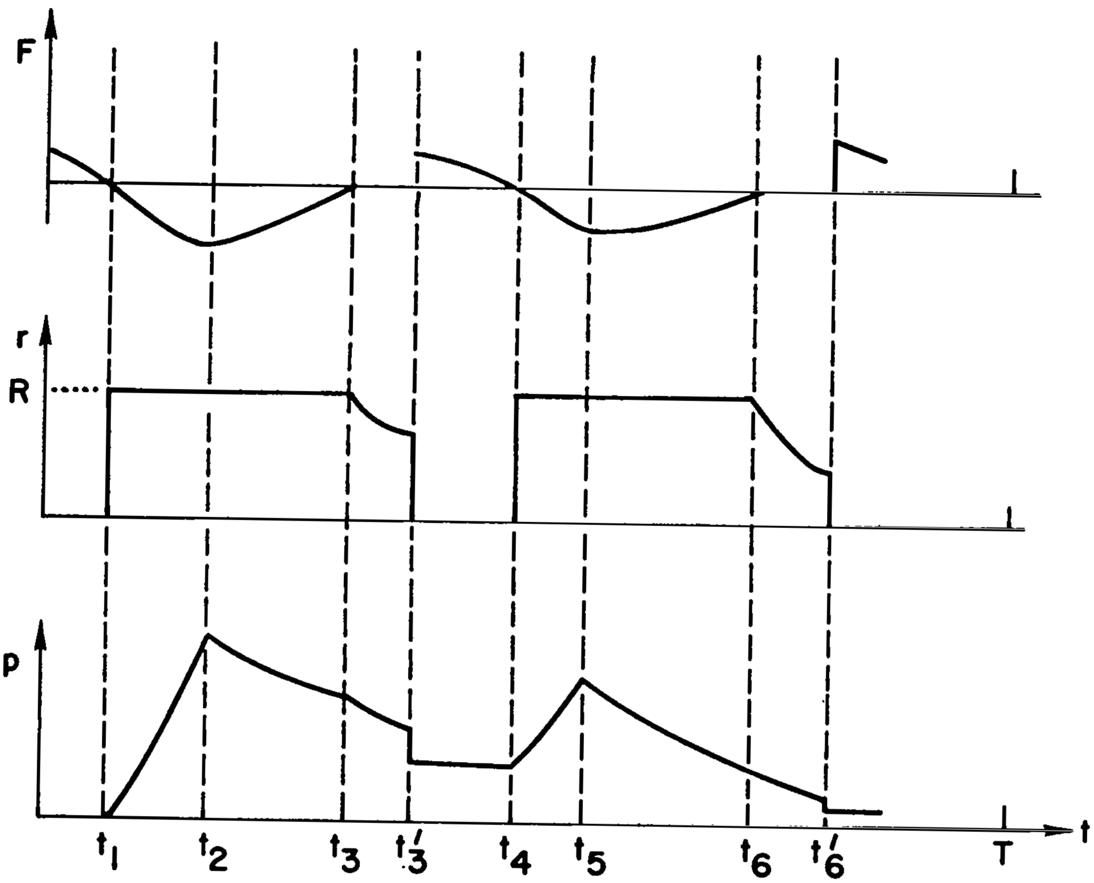


Fig.7.1 Typical F,r and p Histories.

led to a second guess parameter, namely, the time of leaving the first D-S arc. For an exceptional value of the second guess parameter, we were led to a third guess parameter, the time of leaving the second D-S arc. Four different types of observation rate histories were obtained, and these are shown in Figures 7.2-7.5 together with the corresponding projected miss histories. Here, $a_{11}(T)$ and a normalized measuring rate $r(t)/R$ are plotted versus the normalized time t/T. All together, in this way, the minimum cost for any terminal miss was obtained and is shown in Figure 7.6.

Figure 7.2 shows Case 1 where there is only one period of measurement at the maximum rate. In addition to the projected rms miss, we have also included in the same figure the rms predicted miss $\sqrt{p_{11}(t)}$. It is seen that the control is turned on after having measured at a maximum rate for some time and the control always ends with an impulse at the time of final observation cutoff. This is reflected by a downward jump in the rms miss and the predicted miss at that time. Note that $w_{11}(t) = a_{11}(t) - p_{11}(t)$ remains unchanged during the impulse. Case 1 applies to rms misses from 3000 km down to approximately 750 km and is obtained by varying the initial guess of $(\alpha_{11}(0) - \lambda_{11}(0))$ from nearly zero up to a critical value where cases 2, 3 and 4 start.

Figure 7.3 shows the typical solution for Case 2 where there is one period of measurement ending with a subperiod of observation at less than maximum rate. The latter corresponds to the D-S arc. Note that each point on the D-S arc is a possible time of final observation cutoff. A typical one is shown by the dotted line. Case 2 applies to rms misses from approximately 750 km down to 200 km.

By taking each point on the D-S arc as a point of temporary observation cutoff, we obtain type 3 which is plotted in Figure 7.4 It consists of two periods of observation where the second period is always at maximum rate. There are two impulsive controls in this case; the first one is applied on leaving the D-S arc and the second one is applied at the end of the second interval of observation. Case 3 applies to rms misses from 200 km to 140 km and also from 45 km down to zero.

The gap between 45 km to 140 km is filled up by Case 4, shown in Figure 7.5, involving two intervals of observation, each one ending with a D-S arc. Any point on the second D-S arc is a possible time of final observation cutoff as shown by the dotted line.

In Figure 7.6, where we show the plot of rms terminal miss versus cost, we have indicated the range of rms miss corresponding to the various cases. Note that in this example, no matter what the desired terminal accuracy is, not more than two observation periods are required. In a modified example, where the cost of observation k was somewhat lower, we required only one relatively long observation period. We may expect that much higher costs of observation will require several observation periods in order to achieve a reasonably low terminal miss. Each observation period, of course, terminates with an impulsive correction. It is interesting to compare this correction strategy with the purely discrete strategy in References (1), (2) and (3).

2.7.5 A Two-Dimensional Extension of the Problem

The result derived in the previous section is now extended, at least in principle, to a planar transfer problem. A description of such a planar transfer problem is given in Section 2.1. In this case, the variance of the error in the predicted miss, $x_1(T, t) - \hat{x}_1(T, t)$, is now the 1-1 element of a 2x2 position-prediction error covariance matrix $E(t)$, which satisfies the matrix Riccati equation:

$$\dot{E}(t) = -r(t)\, E(t)\, N(t)\, E(t) \tag{7.33}$$

where $N(t)$ is a positive semi-definite matrix measuring measurement effectiveness relative to both terminal position components. The variance $a_{11}(t)$ of $x_1(T, t)$ satisfies in place of (7.5),

$$\dot{a}_{11}(t) = -2\, D(t)\, g(t)\, p_{11}(t) \tag{7.34}$$

where $D(t)$ is the maximum effectiveness of the velocity correction on the predicted miss. Hence, in place of (7.2), we have

$$\dot{p}_{11}(t) = -2D(t)\, g(t)\, p_{11}(t) + r(t)\, \mathrm{tr}\left[\Gamma E(t)\, N(t)\, E(t)\right] \tag{7.35}$$

where Γ is a 2x2 matrix given by

$$\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{7.36}$$

Note that $\mathrm{tr}\left\{\Gamma E(t)\, N(t)\, E(t)\right\}$ essentially picks out the 1-1 element of the 2x2 matrix $E(t)\, N(t)\, E(t)'$.

In place of the scalar adjoint $\alpha_{11}(t)$, we must now introduce a 2x2 symmetric matrix $M(t)$ and minimize a Hamiltonian:

$$\begin{aligned} H = {} & 2g(t)\sqrt{p_{11}(t)} + kr(t) + \lambda_{11}(t)\, \dot{p}_{11}(t) \\ & + \mathrm{tr}\left\{M(t)\, \dot{E}(t)\right\} \end{aligned} \tag{7.37}$$

Since the terminal constraint is only on $a_{11}(T) = p_{11}(T) + w_{11}(T)$, $\lambda_{11}(t)$ and $M(t)$ must satisfy the end constraints,

$$\lambda_{11}(T) = m_{11}(T) \;;\; m_{12}(T) = m_{22}(T) = 0 \tag{7.38}$$

Instead of guessing a single initial quantity $\left\{\lambda_{11}(0) - \alpha_{11}(0)\right\}$, we now have to guess three initial quantities: $\left\{\lambda_{11}(0) - m_{11}(0)\right\}$, $m_{12}(0)$, $m_{22}(0)$, and strive to meet the three end conditions (7.38). Note that $F(t) = 0$ preceded by times when $F \leq 0$ yields a possible final cutoff time only if m_{12} and m_{22} also vanish at that time. We may expect in this way to arrive eventually a one-parameter family of solutions corresponding to various $a_{11}(T)$.

This two-dimensional problem can have an interesting feature which is missing from the one-dimensional problem. Suppose that we not only wish to economize on total fuel and total number of observations but that we have, in addition, a choice between several kinds of observations. For example, we may have onboard capability for measuring angle, range and range-rate from either Earth or the destination planet. In economizing on an appropriately weighted total number of observations we would like to know what proportions of the observations at any time should be allotted to the various kinds. We shall suppose that there is at any time a maximum (appropriately weighted) total measurement rate. Formally, then, we seek to minimize

$$\int_{o}^{T} \left[2g(t) \sqrt{p_{11}(t)} + k \sum_{j} r_j(t) \right] dt \tag{7.39}$$

where

$$\dot{E}(t) = - \sum_{j} r_j(t)\, E(t)\, N_j(t)\, E(t) \;,\; r_j(t) \geq 0 \tag{7.40}$$

and where the (normalized) total rate does not exceed unity:

$$\sum_{j} r_j(t) \leq 1 \tag{7.42}$$

The Hamiltonian may now be written as

$$H = 2g(t) \sqrt{p_{11}(t)} - 2\, \lambda_{11}(t)\, g(t)\, D(t)\, p_{11}(t) + \sum_{j} r_j(t)\, F_j(t) \tag{7.43}$$

$$F_j(t) = k - \text{tr} \quad (M(t) - \lambda_{11}\Gamma\ )\ E(t)\ N_j(t)\ E(t) \tag{7.44}$$

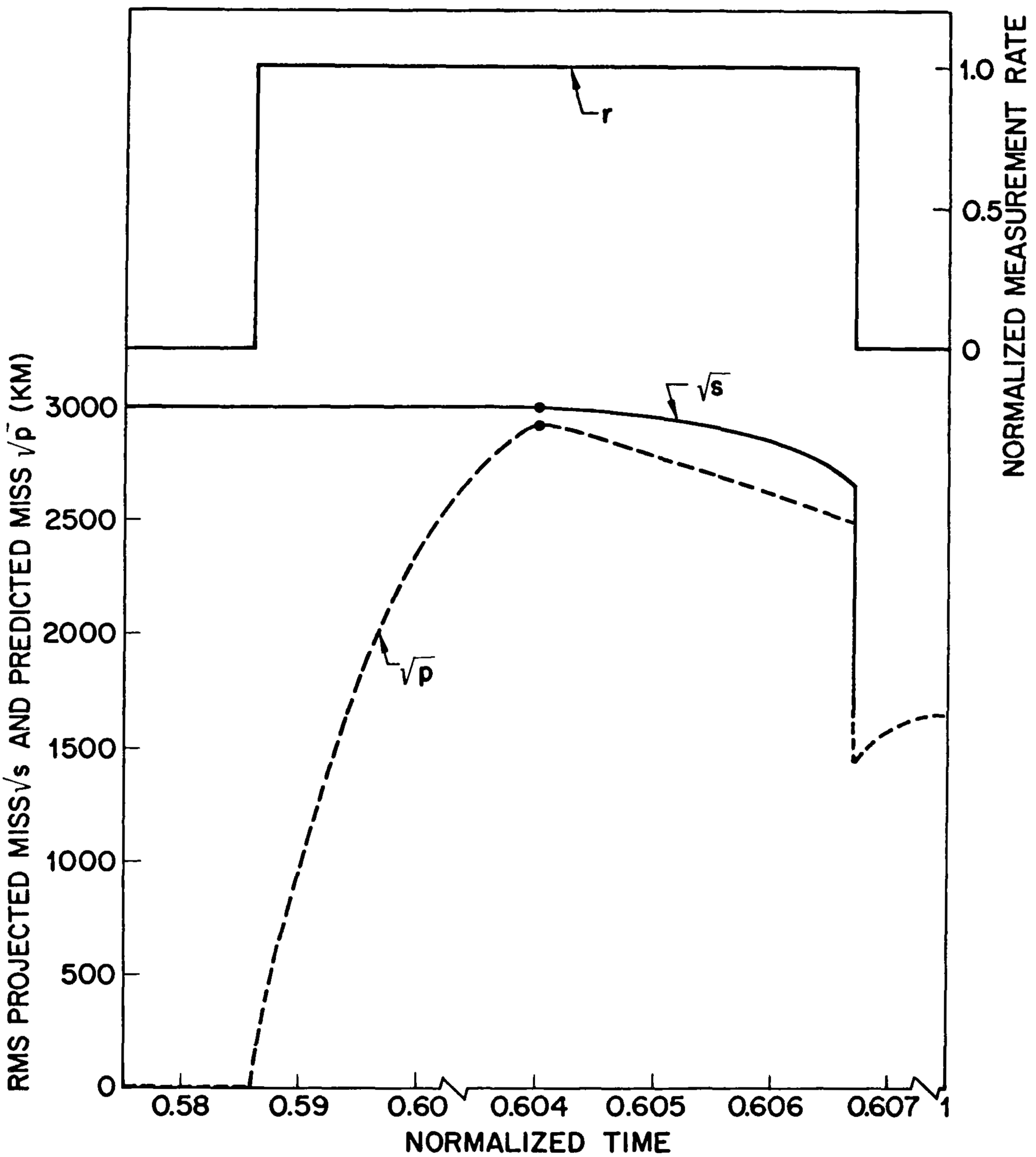
The minimization of H with respect to the r_j's subject to (7.41) and (7.42) leads immediately to:

all $r_j = 0$ at any time when all $F_j > 0$

if, at some time, $\underset{j}{\text{Min}}\ (F_j) < 0$, then $r_{j^*} = 1$ for the j^* which minimizes F_j and other $r_j = 0$

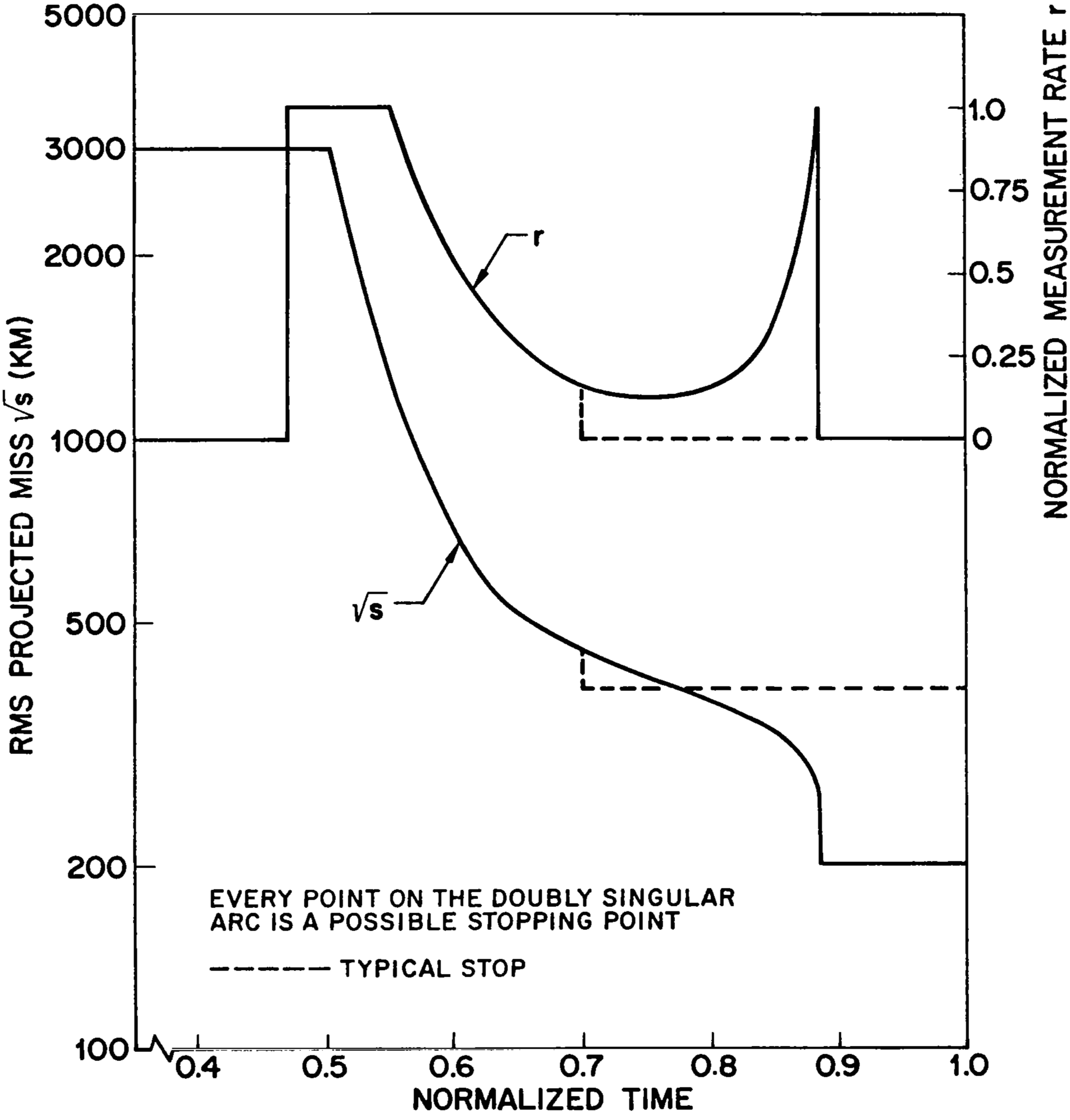
The times at which more than one value of j minimizes F_j are assumed here to be only momentary. If $\underset{j}{\text{Min}}\ (F_j) = 0$, we have the possibility of D-S arc as before.

The reason that this choice is trivial in a one-dimensional problem is that $\underset{j}{\text{Min}}\ (F_j) < 0$ reduces to $(\alpha_{11} - \lambda_{11})\ w_{11}^{\ 2}\ \underset{j}{\text{Max}}\ (a_j) > k\ (> 0)$ so that we always minimize a_j itself (after the r_j's have been normalized). This tells us, for example, when to switch from angular measurement of the vehicle-Earth direction to angular measurement of the vehicle-planet direction. In two dimensions, however, because of the general dynamic coupling between terminal position determination improvement in the x and y directions, it is no longer clear at any time which kind of information is going to be most effective in the long run. The rule for selecting j^*, nevertheless, is easily included in the two-dimensional computation scheme for an optimal policy.



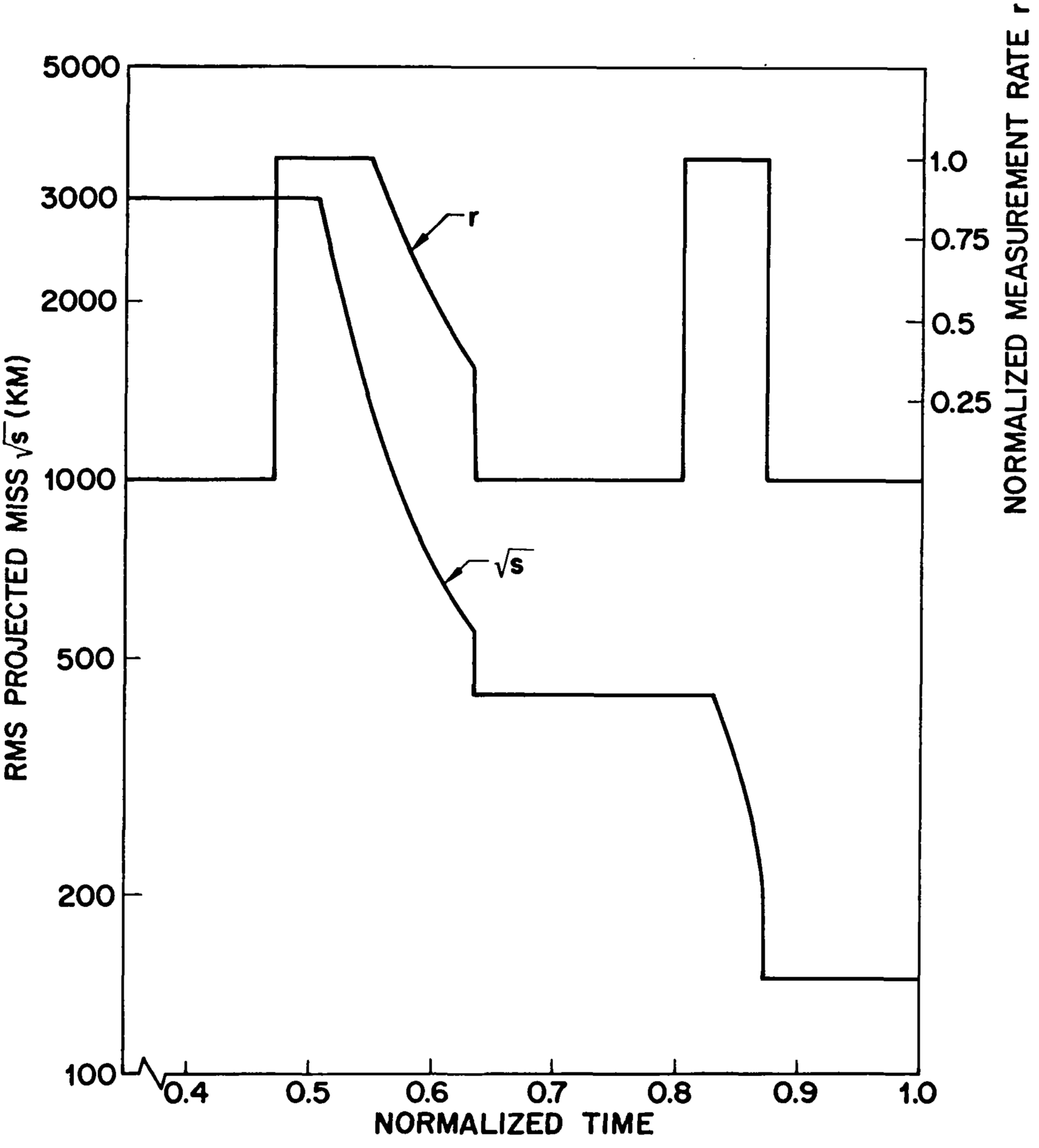
CASE 1 ONE OBSERVATION PERIOD AT MAXIMUM RATE

FIG 7.2



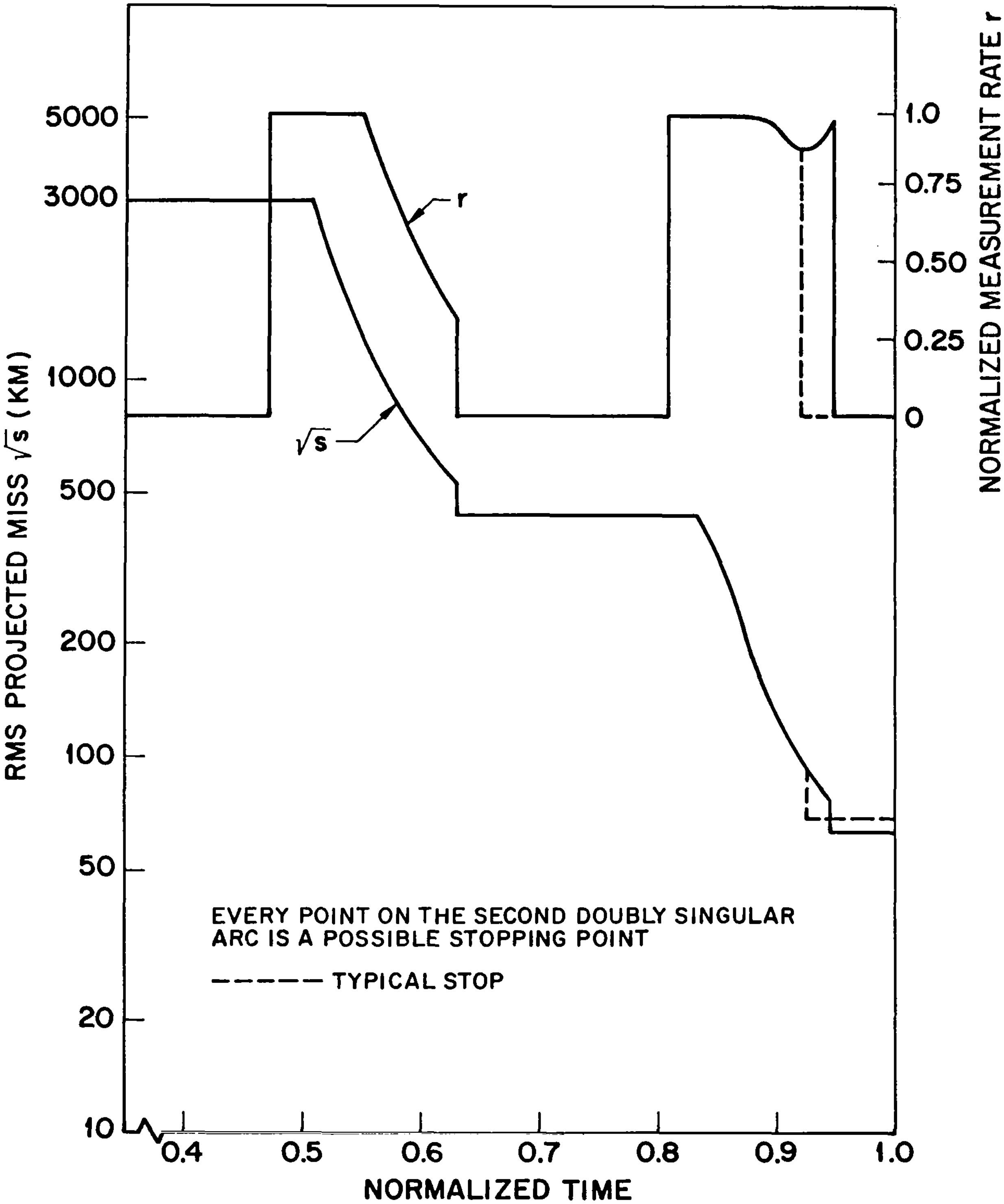
CASE 2 ONE OBSERVATION PERIOD INCLUDING INTERMEDIATE RATE

FIG 7.3



CASE 3 TWO OBSERVATION PERIODS INCLUDING ONE INTERMEDIATE RATE SUBPERIOD

FIG 7.4



CASE 4 TWO OBSERVATION PERIODS EACH INCLUDING AN INTERMEDIATE RATE SUBPERIOD

FIG 7.5

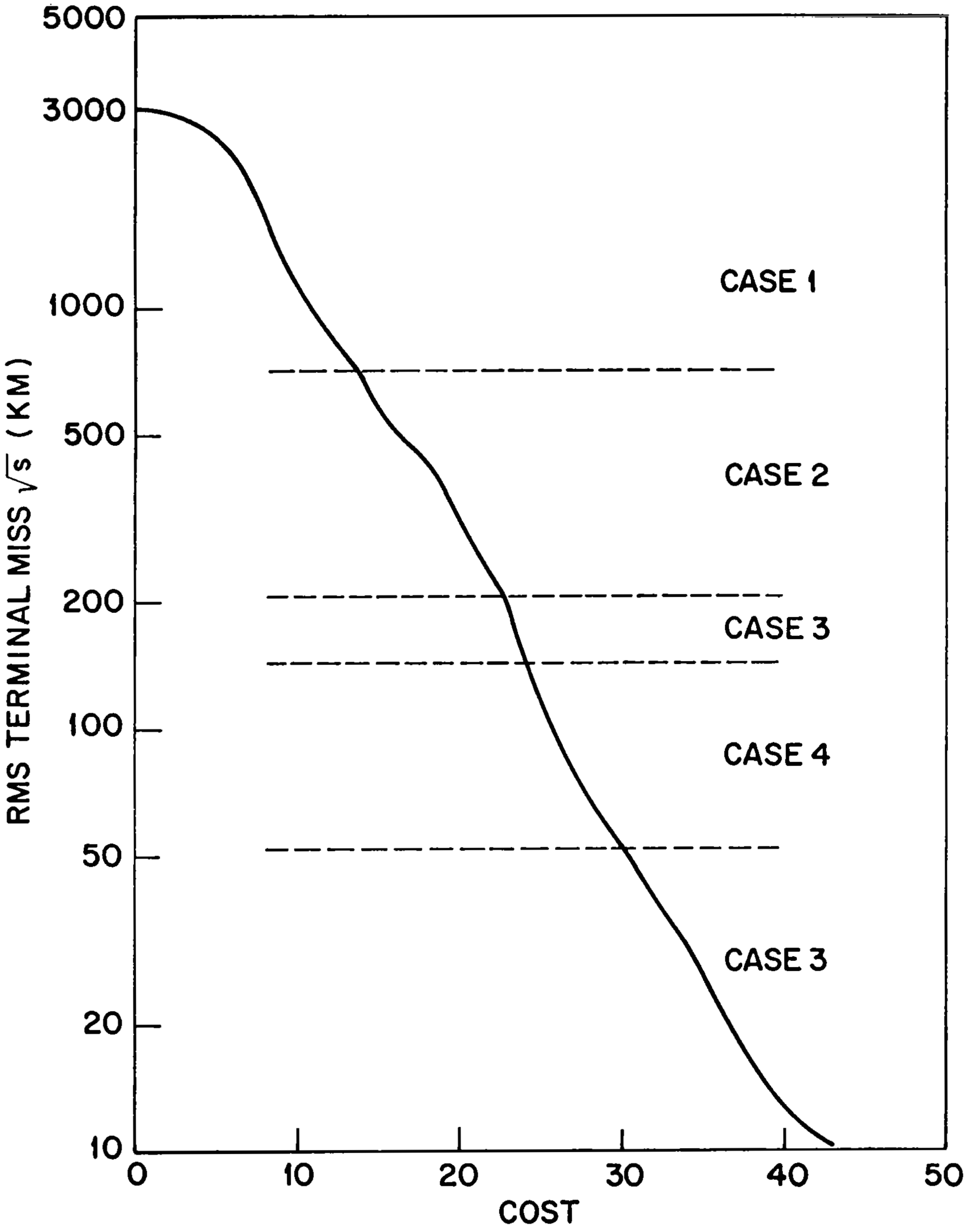

5000
3000
1000
500
200
100
50
20
10
RMS TERMINAL MISS $\sqrt{s}$ (KM)
CASE 1
CASE 2
CASE 3
CASE 4
CASE 3
0
10
20
30
40
50
COST

FIG 7.6

2.8 Optimum Discrete Linear Control Strategy Including Engine Mechanization Errors

Minimum effort theory as developed by Breakwell and Striebel[5] shows that the optimum linear strategy is continuous. However, practical implementation often requires that corrections be carried out at discrete times. A discrete strategy corresponds to the case in which trajectory corrections are executed by discrete impulsive velocity corrections (i.e., the control acceleration $u(t)$ is replaced by several impulses).

This section investigates the solution of the optimum discrete strategy for controlling only one terminal miss (x_1) again assuming linear control. In other words, we assume that the discrete corrective velocity increments are proportional to the instantaneous predicted miss distance. The mathematical problem is essentially that of finding the areas as well as the spacings of these multiple corrections which will steer the vehicle to meet the desired accuracy with a minimum expected amount of total velocity correction.

The results of this investigation seem to indicate that:

- three to four corrections are very close to optimum and that almost no advantages can be obtained by incorporating additional corrections, and
- the optimum corrective strategy is discrete when engine mechanization errors are included and that there is an optimum number of corrections for a given size of engine mechanization error.

The presentation of the material in this section is divided into three parts. Section 2.8.1 outlines a mathematical statement of the discrete optimization problem. Section 2.8.2 shows how the optimum solution (the timings and the amount of each corrective thrust) can be obtained. The technique is based on Dynamic Programming leading to an optimum solution consisting in part analytical and in part computational. Section 2.8.3 gives the results of some numerical work applying the method developed in this study to a simple one-dimensional model.

2.8.1 Formulation of the Discrete Optimization Problem

Let $t, t_2, \ldots, t_N$ be the times by which the corrections are applied. Then the linear feedback gain $g(t)$ can be written as

$$g(t) = \sum_{i=1}^{N} k(t_i)\, \delta(t-t_i) \tag{8.1}$$

where

$$0 \le k(t_i) \le \frac{1}{D(t_i)} \tag{8.2}$$

The required velocity correction at t_i is given by

$$\left| k(t_i)\, \hat{x}_1(T, t_i^-) \right| \tag{8.3}$$

where $\hat{x}_1(T, t_i^-)$ is the predicted miss distance immediately before the correction. Note that equality in (8.2) implies a "full correction" nullifying the predicted miss at t_i. The determination of a N-impulse minimum effort strategy requires the optimization of $k(t_i)$ as well as t_i, the timings of these corrections.*

Let $p_{11}(t_i^-)$ and $p_{11}(t_i^+)$ be the mean square values of the terminal miss distance immediately before and after the correction. Substituting (8.1) into (6.5) and (6.6) shows

$$p_{11}(t_i^+) = (1 - k(t_i)\, D(t_i))^2 \; p_{11}(t_i^-) \tag{8.4}$$

$$p_{11}(t_{i+1}^-) = p_{11}(t_i^+) + b(t_i) \tag{8.5}$$

where

$$b(t_i) = \int_{t_i^-}^{t_{i+1}^+} q_{11}\, dt \tag{8.6}$$

*For the moment, we will assume negligible engine mechanization error. When engine execution errors are included, the total number of corrections, N, also becomes a variable to be optimized.

while the expected total velocity correction becomes

$$\sum_{i=1}^{N} k(t_i) \sqrt{p_{11}(t_i^-)} \tag{8.7}$$

The problem may now be stated as follows:

Given the system (8.4)-(8.6) with $p_{11}(0) = 0$. Find $k(t_i) \geq 0$, t_i $(i=1,2,\ldots,N)$ and N which minimizes (8.7) for a fixed mean square terminal miss $a_{11}(T)$ where

$$a_{11}(T) = E(x_1^2(T)) = p_{11}(t) + w_{11}(t) \quad ; \tag{8.8}$$

t_N being the time of the last correction and $w_{11}(t)$ is the 1-1 element of W(t) which satisfies the matrix differential equation (4.5).

This is a calculus problem with inequality constraints. The solution will be carried out in two steps using Dynamic Programming. Note that the inequality constraints pose no computational difficulty and, in fact, help to eliminate a large number of entries in the tables to be generated.

2.8.2 Solution

We shall first assume that the correction times have been given and proceed with the minimization of $k(t_i)$ conditional on the given set of timings. It is then shown how the spacings between these timings can be obtained. It should be noted that the requirement $k(t_i) \geq 0$ imposes a constraint on the permissible spacings of the correction times. However, the times are not explicitly given. We may, therefore, for the moment assume that all the $k(t_i)$ to be obtained are actually greater than zero. The situation $k(t_i) < 0$ can be eliminated easily by inserting a simple logic in the computational procedure used for optimizing the timings.

Minimization of Gains Given the Correction Times

This will be done by working backwards using Dynamic Programming.[12] For convenience, the constant factor $\sqrt{2/\pi}$ will be dropped. Let

$$L_m = \underset{\substack{k(t_j)\\ j\geq m}}{\text{Min}} \sum_{i=m}^{N} k(t_i)\sqrt{p_{11}(t_i^-)} \qquad (8.9)$$

and meets the specified mean square terminal miss. Application of the Principle of Optimality[12] yields the recursion equation

$$L_i = \underset{k(t_i)}{\text{Min}} \left[k(t_i)\sqrt{p_{11}(t_i^-)} + L_{i+1} \right] \qquad (8.10)$$

Now

$$p_{11}(t_N^+) = a_{11}(T) - w_{11}(t_N^+) = (1 - k(t_N)\,D(t_N))^2\, p_{11}(t_N^-) \qquad (8.11)$$

Since $w_{11}(t_N^+)$ is independent of $k(t_N)$*, it follows that specification of $a_{11}(T)$ completely determines $k(t_N)$. In fact,

$$k(t_N) = \frac{1}{D(t_N)}\left(1 - \sqrt{\frac{a_{11}(T) - w_{11}(t_N^+)}{p_{11}(t_N^-)}}\right) \qquad (8.12)$$

and

$$L_N = \frac{1}{D(t_N)}\sqrt{p_{11}(t_N^-)} + \text{terms which do not involve } p_{11}(t_N^-). \qquad (8.13)$$

Substituting (8.13) into (8.10) and using (8.4)-(8.6), we find

$$\begin{aligned} L_{N-1} = \underset{k(t_{N-1})}{\text{Min}} \Big[& k(t_{N-1})\sqrt{p_{11}(t_{N-1}^-)} \\ & + \frac{1}{D(t_N)}\sqrt{(1-k(t_{N-1})D(t_{N-1}))^2\, p_{11}(t_{N-1}^-) + b(t_{N-1})} \\ & + \text{terms which do not depend on } k(t_{N-1}) \Big] \end{aligned} \qquad (8.14)$$

*This is true because we have, for the moment, assumed negligible engine mechanization error.

Hence, the optimum gain at t_{N-1} is given by

$$k(t_{N-1}) = \frac{1}{D(t_{N-1})} \left(1 - \sqrt{\frac{D^2(t_N)\, b(t_{N-1})}{p_{11}(t_{N-1}^-)\, (D^2(t_{N-1}) - D^2(t_N))}} \right) \tag{8.15}$$

which, when substituted into (8.14) yields

$$L_{N-1} = \frac{1}{D(t_{N-1})} \sqrt{p_{11}(t_{N-1}^-)} \text{ + terms which do not depend on } p_{11}(t_{N-1}^-) \text{ and hence } k(t_{N-2}) \tag{8.16}$$

Equation (8.16) is of the same form as (8.13). It follows that

$$k(t_i) = \frac{1}{D(t_i)} \left(1 - \sqrt{\frac{D^2(t_{i+1})\, b(t_i)}{p_{11}(t_i^-)(D^2(t_i) - D^2(t_{i+1}))}} \right) \tag{8.17}$$

$$i=1,2,\ldots,N-1$$

We now have the equations for computing the gains as a function of the mean square predicted miss. Now $p_{11}(0) = 0$. Using this, Equations (8.4)-(8.6), and the expression for the gains, it is seen that

$$p_{11}(t_i^+) = \frac{D^2(t_{i+1})\, b(t_i)}{D^2(t_i) - D^2(t_{i+1})} \tag{8.18}$$

and

$$p_{11}(t_{i+1}^-) = \frac{D^2(t_i)\, b(t_i)}{D^2(t_i) - D^2\,(t_{i+1})} \tag{8.19}$$

Hence, we may express the gains $k(t_k)$ as functions of only the given correction times by substituting (8.18)-(8.19) into (8.17) and (8.12).

The total average cumulative effort for a given set of correction times is L_o. It can be easily verified and we shall omit the details that

$$L_o = \frac{1}{D(t_1)} \sqrt{b(t_o)} + \sum_{i=1}^{N-1} \frac{1}{D(t_i)\,D(t_{i+1})} \sqrt{(D^2(t_i) - D^2(t_{i+1}))\,b(t_i)}$$

$$- \frac{1}{D(t_N)} \sqrt{(a_{11}(T) - w_{11}(t_N^+))} \qquad (8.20)$$

The only remaining problem now is to find the optimum settings of the correction times which minimize L_o such that $k(t_i) \geq 0$. Unfortunately, this tedious problem does not yield readily to analytical solutions. However, it turns out that the optimum times can be easily computed by using Dynamic Programming. Since only the correction times t_i (which run over a finite interval) need to be quantized and it appears that an adequate solution involves only three to four corrections, the usual difficulty of the storage problem and the accumulation of quantization errors associated with computation by Dynamic Programming disappears. Furthermore, it will be seen that this method of finding the correction times remains essentially unchanged when engine execution error is included.

Computation Procedure for Optimizing the Correction Times

The algorithm for computation involves a procedure based on Dynamic Programming.[12] Inspection of (8.20) shows that L_o can be written as

$$L_o = A_1(t_1\,,\,t_2) + A_2(t_2\,,\,t_3) + \ldots + A_{N-1}(t_{N-1}\,,\,t_N) + A_N(t_N\,,\,a_{11}(T)) \qquad (8.21)$$

where

$$A_1(t_1\,,\,t_2) = \frac{1}{D(t_1)} \sqrt{b(t_o)} + \frac{1}{D(t_1)\,D(t_2)} \sqrt{(D^2(t_1) - D^2(t_2))\,b(t_1)} \qquad (8.22)$$

$$A_i(t_i\,,\,t_{i+1}) = \frac{1}{D(t_i)\,D(t_{i+1})} \sqrt{(D^2(t_i) - D^2(t_{i+1}))\,b(t_i)} \qquad (8.23)$$

$$i=2,3,\ldots,N-1$$

$$A_N(t_N , a_{11}(T)) = - \frac{1}{D(t_N)} \sqrt{a_{11}(T) - w_{11}(t_N^+)} \tag{8.24}$$

Let

$$U_k (t_{k+1}) = \min_{t_1, \ldots t_k / t_{k+1}} (A_1(t_1 , t_2) + \ldots + A_k(t_k , t_{k+1})) \tag{8.25}$$

It follows from the principle of optimality that

$$U_k (t_{k+1}) = \min_{t_k / t_{k+1}} (A_k(t_k , t_{k+1}) + U_{k-1}(t_k)) \tag{8.26}$$

The computation proceeds forward with

$$U_1(t_2) = \min_{t_1 / t_2} (A_1(t_1 , t_2)) \tag{8.27}$$

and involves constructing the tables $U_i(t_{i+1})$. It is to be noted that the total number N need not be fixed in advance and that by the proper selection of the A_k, the optimum k-impulse strategy can be obtained before proceeding to the computation of the k+1-impulse correction strategy. A large number of unnecessary entries in each table can be eliminated by including in the computation procedure the constraints that $t_i \leq t_{i+1}$ and that each $k(t_i)$ must be positive.

Inclusion of the Mechanization Error

The analysis so far has assumed that the engine mechanization errors are negligible. The same analysis, however, can also be used to include the effect of these random errors. The mechanization errors to be considered in this paper are assumed to be of two varieties, both of which are normal, independent with zero mean and constant variances.

a. Engine Execution Error. The random engine execution error is assumed to be in the direction of the correction with a standard deviation σ_β. The effect of this error is to increase the variance of the predicted miss. It should be noted that the errors in the transverse direction effects only the time of arrival since the orientation of the engine has been optimized in the direction of maximum sensitivity miss distance.

b. Accelerometer Reading Error. This is the error in the knowledge of the actual amount of velocity correction used and is assumed to be additive with a standard deviation σ_α. This random error causes a loss in information and increases the variances of the estimation errors. It turns out that this is equivalent to applying impulses at the correction times to the right-hand side of Equation (3.3) (or equivalently Eq. (4.5)). The effect of these impulses is to cause a jump (or discontinuity) in the elements $v_{ij}(t)$, $i,j=3,4$, immediately after the correction.

It can be shown[13] that the uncertainty introduced into the covariance matrix of the estimation error at the correction times is given by

$$V(t_i^+) = V(t_i^-) + B(t_i) \tag{8.28}$$

where $B(t_i)$ is a 4x4 matrix consisting of all zero elements except the four elements in the lower right corner. This submatrix of nonzero elements is given by

$$\frac{\sigma_1^2}{D^2(t_i)} \begin{bmatrix} \phi_{13}^2 (T, t_i) & \phi_{13} (T, t_i)\, \phi_{14} (T, t_i) \\ \phi_{13} (T, t_i)\, \phi_{14} (T, t_i) & \phi_{14}^2 (T, t_i) \end{bmatrix} \tag{8.29}$$

where

$$\sigma_1^2 = \frac{\sigma_\beta^2\ \sigma_\alpha^2}{\sigma_\beta^2 + \sigma_\alpha^2} \tag{8.30}$$

In addition, (8.6) is replaced by

$$b(t_i) = \int_{t_i^+}^{t_{i+1}^-} q_{11}\, dt + \sigma_2^{\,2}\, D^2\,(t_i) \tag{8.31}$$

where

$$\sigma_2^{\,2} = \frac{\sigma_\beta^{\,4}}{\sigma_\alpha^{\,2} + \sigma_\beta^{\,2}} \tag{8.32}$$

The second term in (8.31) shows the effect of the increase of the variance of the predicted miss due to the engine execution error.

Now the computation procedure outlined so far leads to an optimum solution only in the case of $\sigma_\alpha = 0$ (no accelerometer reading error). In general, a more elaborate computation will be required by the addition of the accelerometer reading error. This is because, in this case, (8.21) can no longer be written as a sum of terms, each one being a function of two variables only. In other words, each of the A_i appearing in (8.21) is now an explicit function of all the correction times before t_i as $V(t)$ now depends on the correction times. The method of solution described in this paper clearly leads to an upper bound solution. However, the numerical results of an example studied in the next section indicate that this bound is quite good and, in fact, can be used fruitfully as an initial guess for a gradient procedure if optimum solution is desired. In getting this upper bound solution, the addition of the orbit estimation uncertainty due to the accelerometer reading error is taken care of by storing the elements of $V(t_i^+)$ along with table $U_i(t_{i+1})$.

4.8.3 Application to a One-Dimensional Model and Discussion of Results

The same simple one-dimensional model used in the previous section is considered here. The parameter values assumed defining this approaching guidance is given in Table 8.1. The results are given in Figures 8.1-8.3 and Tables 8.2-8.4.

Table 8.1

Symbol	Description	Values
T	Time from start to impact	10^6 sec~10 days
$\sqrt{v_{22}(0)}$	Standard deviation of initial velocity error	3 m/sec
Δt	Time between measurements	1 hour
σ_ϵ	Standard deviation of measurement error	1 millirad.
v_f	Relative velocity between the vehicle and target planet	3 km/sec
$\sqrt{a_{11}(T)}$	Standard deviation of the desired terminal miss	between 25 km - 2000 km

In Figures 8.1-8.3 the cumulative average velocity corrections are plotted against the rms terminal miss for various combinations of σ_α and σ_β with up to five corrections. Figure 8.1 shows the case of no mechanization error. The result indicates (which seems to be typical in all the other cases) that three to four corrections are very close to optimum and that very little advantage can be gained by adding additional corrections. It is known that the optimum solution for this case requires continuous corrections.[5] A continuous correction strategy based on minimum effort theory developed by Breakwell and Striebel is actually obtained for this particular case. The results fall very close to that obtained by using five corrections and the difference seems to arise from the numerical accuracy of the computation (the grid size used for the Dynamic Programming computation is 10^4 seconds).

The case $\sigma_\alpha = 0$ and $\sigma_\beta = 0.45$ m/sec is shown in Figure 8.2. It differs from that given in Figure 8.1 in that there now exists an optimum number of corrections for a given terminal miss. This can be extracted from our numerical results by noting that the optimum solution with a given rms terminal miss for N corrections requires more average effort than that for N-1 corrections. For example, for an rms terminal miss of 500 km, using four corrections requires less effort than using five corrections. This is, of course, intuitively expected since the effect of the engine execution error is to increase the mean square of the terminal miss at each correction time. Figure 8.3 shows the same plot by including, in addition, the accelerometer reading error. Here we let $\sigma_\alpha = 0.45$ m/sec although realistic values would be much smaller. The curves represent only the upper bound solutions. The effect of this additional noise is very small and seems to shift the solution curves of Figure 8.2 up and to the left. For example, for an rms miss-distance of 500 km, the near-optimum solution given here requires only three corrections.

Tabulated in Table 8.2 are typical values of the corresponding times and the corresponding average effort requirements for an rms distance of 70 km (only the total costs are given for more than four corrections). Results for three separate cases are presented:

1) $\sigma_\alpha = \sigma_\beta = 0$
2) $\sigma_\alpha = 0$, $\sigma_\beta = 0.45$ m/s
3) $\sigma_\alpha = \sigma_\beta = 0.45$ m/s

This table shows, for example, that a total of five corrections is optimum for case (3), whereas the optimum solution for case (2) requires eight corrections. Table 8.3 shows the same comparison for an rms miss-distance of 500 km.

In order to get a "feel" of the difference between our upper bound solution and the optimum solution when the accelerometer reading error is included, a gradient program based on the method of Newton is built using as an initial guess the upper bound solution we have obtained. No provision is made to take care of the inequality constraints in this gradient program since it was anticipated (and verified in this example) that our initial guess is very close to the optimal solution that inequality constraints will not be violated.

Consider a change in the loss function (8.20) due to changes in the correction times t_i , $i=1,2\ldots,N$. This change is expanded to second order in Δt_i so that

$$\Delta L_o = \left(\frac{\partial L_o}{\partial t}\right)' \Delta t + \frac{1}{2} \Delta t' \left(\frac{\partial^2 L_o}{\partial t \partial t'}\right) \Delta t + 0 \ (\Delta t^3) \tag{8.33}$$

where Δt is a N-vector with elements Δt_i. Minimizing (8.33) with respect to Δt by ignoring the third order term yields

$$\Delta t = -\left(\frac{\partial^2 L_o}{\partial t \partial t'}\right)^{-1} \left(\frac{\partial L_o}{\partial t}\right) \tag{8.34}$$

which is then used to obtain the next guess of the correction times. This procedure has proven to be very effective if the initial guess is very close to the optimum and if the number of variables to be optimized are few. It is an improvement over the standard gradient procedure in that it specifies the size and the direction of the next step during each iteration. It is found that the optimal solution can be obtained in three or four iterations. Table 8.4 shows a comparison of the upper bound solution and the true optimum solution for the case of four corrections with an rms terminal miss of 70 km. As expected, the difference is seen to be very small and is only of the order of 0.3%.

Table 8.2

NORMALIZED CORRECTION TIMES AND THE CORRESPONDING AVERAGE EFFORT WITH RMS MISS = 70 KM

Number of Corrections	Case 1 $\sigma_\alpha = \sigma_\beta = 0$ Correction Time (10^6s)	Case 1 Cost $(2/\pi)^{\frac{1}{2}}$m/s	Case 2 $\sigma_\alpha = 0$ $\sigma_\beta = 0.45$ Correction Time	Case 2 Cost	Case 3 $\sigma_\alpha = \sigma_\beta = 0.45$ Correction Time	Case 3 Cost
1	0.94	49.780	0.94	50.070	0.95	59.676
	Total	49.780		50.070		59.676
2	0.65	8.378	0.66	8.615	0.68	9.196
	0.94	5.567	0.94	6.152	0.95	6.921
	Total	13.945		14.767		16.117
3	0.51	5.452	0.51	5.418	0.55	6.136
	0.80	3.108	0.80	3.281	0.85	3.910
	0.94	2.478	0.94	3.158	0.96	3.346
	Total	11.038		11.857		13.292
4	0.44	4.147	0.45	4.330	0.48	4.855
	0.67	2.467	0.70	2.710	0.75	2.892
	0.85	1.935	0.87	2.173	0.90	2.406
	0.94	1.693	0.95	1.882	0.96	2.398
	Total	10.242		11.095		12.551
5	Total	9.914		10.761		12.477
6	Total	9.747		10.627		12.504
7	Total	9.651		10.576		12.617
8	not computed			10.567		not computed
9	not computed			10.581		not computed
10	not computed			10.604		not computed

Table 8.3

COMPARISON OF THE TOTAL AVERAGE EFFORT
WITH RMS MISS = 500 KM

Number of Corrections	Case 1 $\sigma_\alpha = \sigma_\beta = 0$	Case 2 $\sigma_\alpha = 0$ $\sigma_\beta = 0.45$	Case 3 $\sigma_\alpha = \sigma_\beta = 0.45$
	Total Cost $(2/\pi)^{\frac{1}{2}}$m/s)	Total Cost	Total Cost
1	6.797	7.121	7.291
2	5.845	6.074	6.289
3	5.705	5.961	6.273
4	5.658	5.954	6.354
5	5.637	5.973	6.456

Table 8.4

COMPARISON OF THE NORMALIZED CORRECTION TIMES AND THE
CORRESPONDING EFFORT FOR A NEAR-OPTIMUM SOLUTION AND THE
OPTIMUM SOLUTION WITH (RMS MISS = 70 KM; $\sigma_\alpha = \sigma_\beta = 0.45$ M/SEC)

Correction Number	Near Optimum Solution		Optimum Solution	
	Correction Time (10^6s)	Cost $(2/\pi)^{\frac{1}{2}}$m/s	Correction Time	Cost
4	0.48	4.855	0.448	4.386
	0.75	2.892	0.722	3.016
	0.90	2.406	0.897	2.711
	0.96	2.398	0.96	2.408
	Total	12.551		12.521

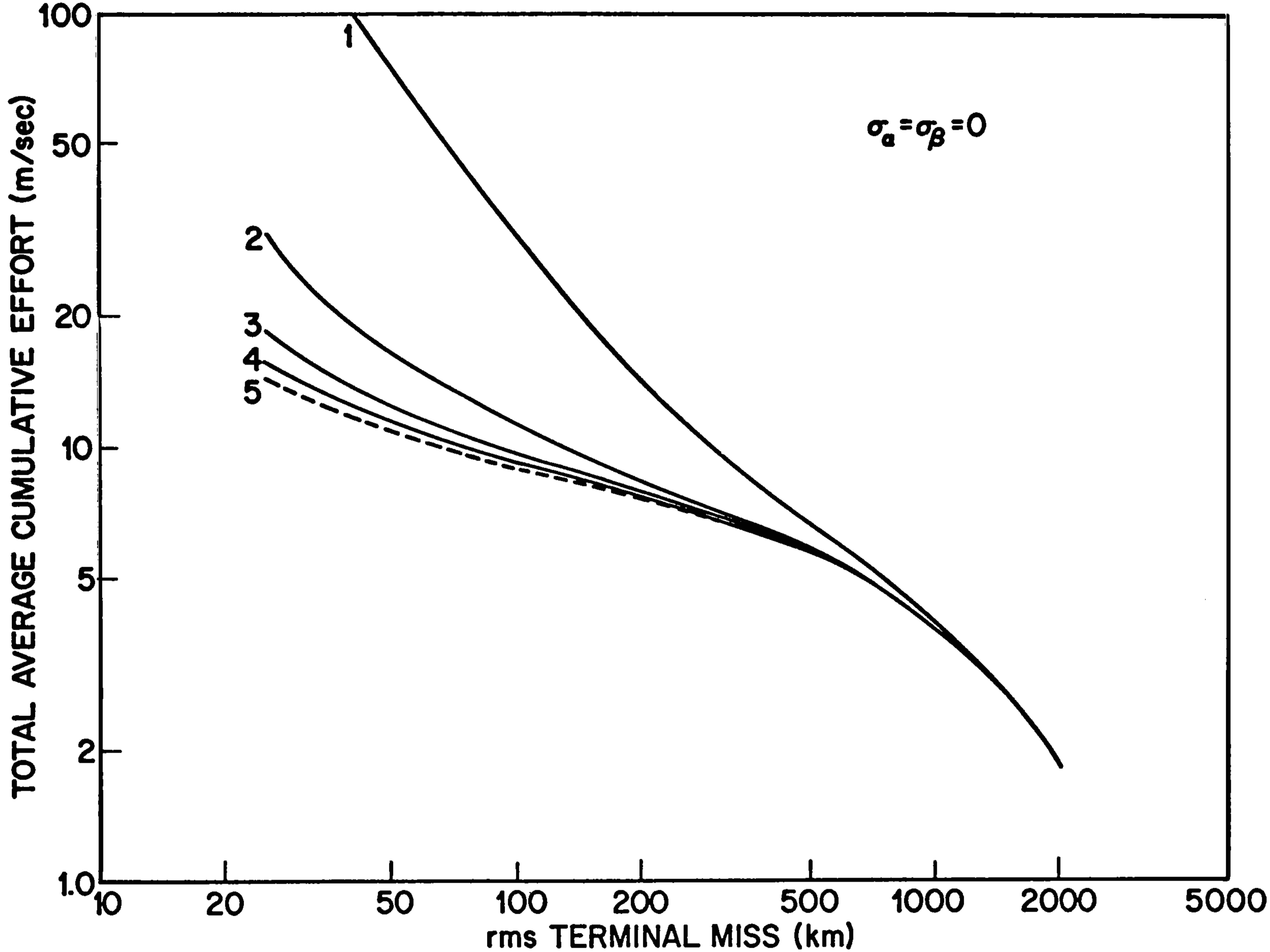


Fig. 8.1 Total average effort vs rms terminal miss for various number of corrections.

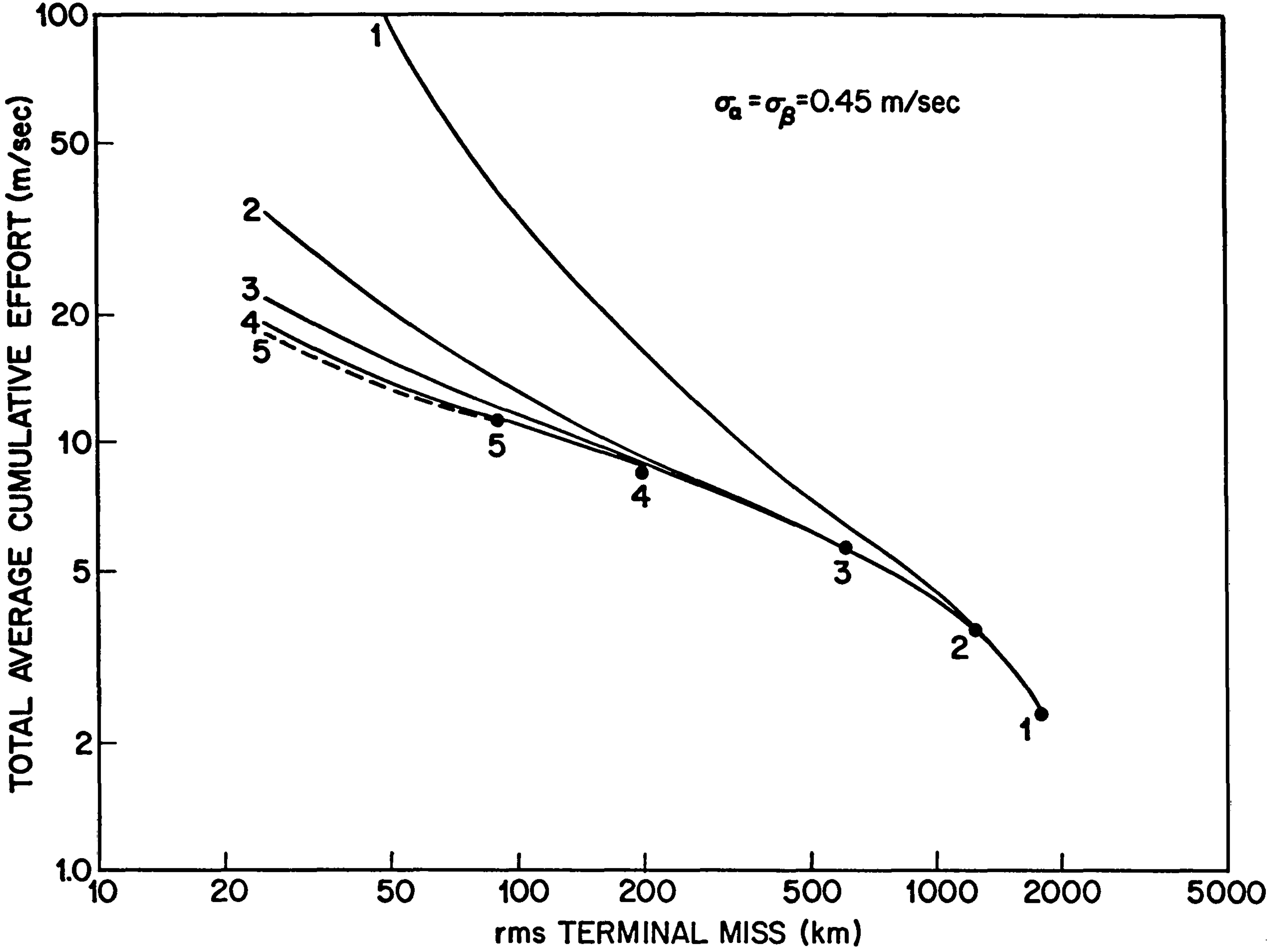


Fig.8.3 Total average effort vs rms terminal miss for various number of corrections.

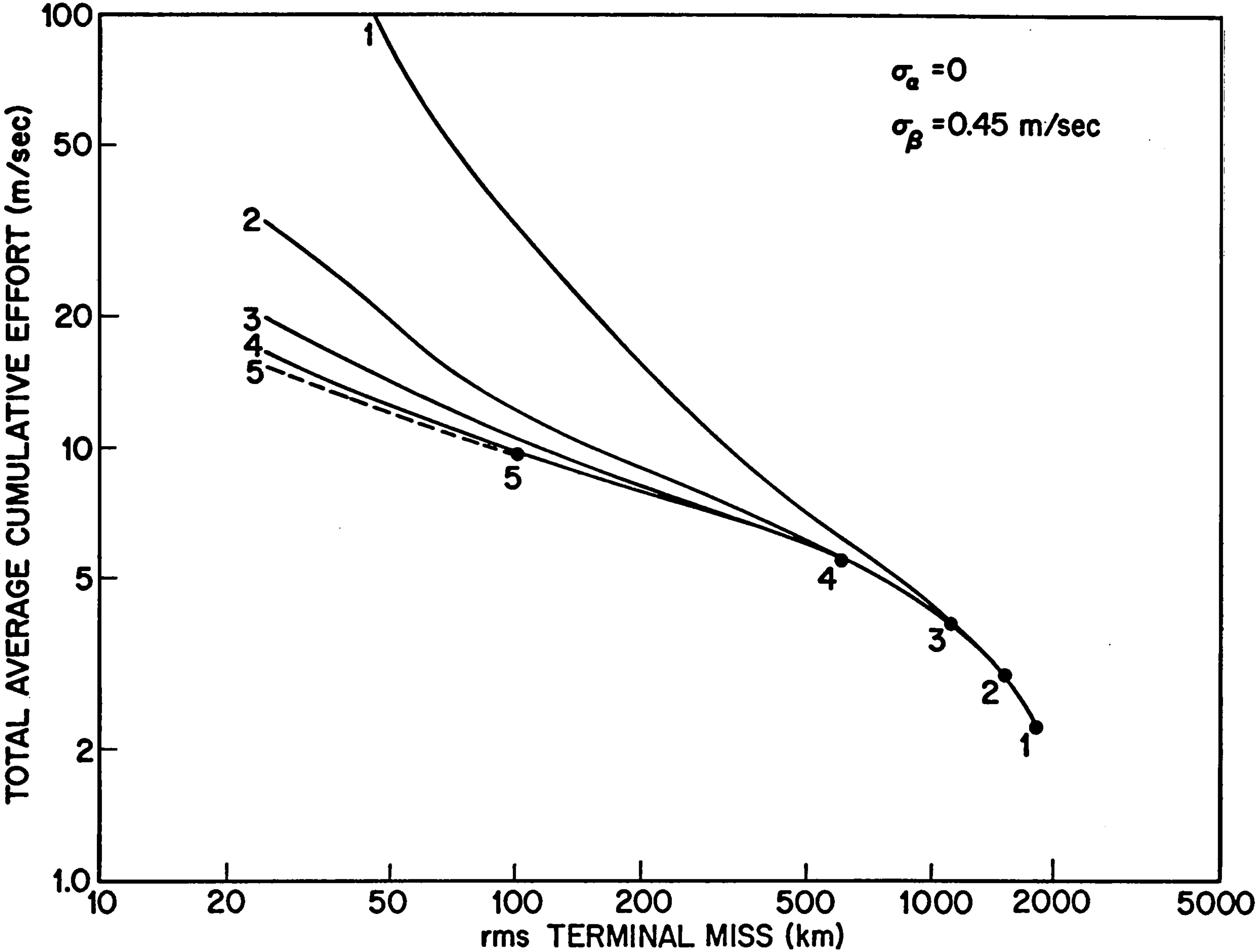


Fig. 8.2 Total average effort vs rms terminal miss for various number of corrections.

2.9 Optimum Nonlinear Control Law

All the results we have presented so far assumed that the magnitude of the control acceleration is proportional to the predicted miss distance $\hat{x}_1(T, t)$. This assumption has allowed us to formulate the guidance problem, originally stochastic in nature in view of the initial injection error and random measurement noises, as an equivalent deterministic problem in optimal control. In this section we shall remove this assumption.

We know that the predicted miss $\hat{x}_1(T, t)$ satisfies the scalar differential equation,

$$\frac{d\hat{x}_1(T,t)}{dt} = \phi_{13}(T, t)\, u_1 + \phi_{14}(T, t)\, u_2 + \eta(t) \tag{9.1}$$

where $\eta(t)$ is a white Gaussian noise with spectrum $q_{11}(t)$. The problem of nonlinear control to be considered in this section is to find $u_1(t)$ and $u_2(t)$ (both of which are assumed to be perfectly executed) as functions of $\hat{x}_1(T, t)$ which minimizes

$$E \int_0^T \sqrt{u_1^2 + u_2^2}\; dt \tag{9.2}$$

for a specified value of

$$p_{11}(T) = E\left[\hat{x}_1^2(T, T)\right] \tag{9.3}$$

The results we have obtained apply only to discrete correction strategies. However, we may let our discrete solution approach the solution of the continuous nonlinear correction strategy by inserting more correction times. The nature of the solution for the discrete model indicates that a correction control is applied at t_i (a given correction time) if, and only if, the predicted miss distance $\hat{x}_1(T, t_i)$ prior to the correction lies above (or below) a certain number, say z_i (or $-z_i$), and that the effect of the control is to bring the predicted miss after the correction to z_i (or $-z_i$).

This section is divided into three parts. The formulation and the solution of the problem are given in Section 2.9.1. In Section 2.9.2, we give the results of some numerical work applying the method to a simple one-dimensional model. The last section gives a brief description of the hard constraint problem. This is a related but different problem in which the total effort is strictly limited while minimizing the terminal miss distance.

2.9.1 Formulation of the Discrete Model and Solution Formulation

First of all we see that the engine must be pointing in the direction of maximum effectiveness. This implies that we may write, without loss of generality,

$$u_1(t) = \frac{\phi_{13}(T, t)}{D(t)} f(t) \tag{9.4}$$

$$u_2(t) = \frac{\phi_{14}(T, t)}{D(t)} f(t) \tag{9.5}$$

where, to recapitulate,

$$D(t) = \sqrt{\phi_{13}^2(T,t) + \phi_{14}^2(T,t)}$$

and $f(t) = f(\widehat{x}_1(T,t))$ is a function to be determined.
Using (9.4)-(9.5), we see that (9.1) and (9.2) can be replaced by

$$\frac{d\widehat{x}_1(T,t)}{dt} = D(t)\, f(t) + \eta(t) \tag{9.6}$$

and

$$E \int_0^T |f(t)|\, dt \tag{9.7}$$

respectively. Let λ be a constant Lagrange multiplier.

The problem is then equivalent to finding $f(t) = f(\hat{x}_1(T,t))$ that minimizes

$$\text{Loss} = E \int_0^T |f(t)|\, dt + \lambda/2 \ E\,(\hat{x}_1^2\,(T, T)) \tag{9.8}$$

with no constraints.

We shall provide a discrete computation procedure for finding the solution of the continuous problem formulated above. Let us assume that the controls are to be executed at a given set of correction times t_i, $i=1,2,\ldots N$ in the form of impulses and let the sizes of the impulsive controls at t_i be denoted by f_i so that

$$f(t) = \sum_{i=1}^{N} f_i\, \delta(t-t_i) \tag{9.9}$$

Let $\hat{x}_i^-$ and $\hat{x}_i^+$ be the values of the estimated miss distance $\hat{x}_1(T, t_i)$ immediately before and after the correction and let $D_i = D(t_i)$. Substituting (9.9) into (9.6) yields the stochastic difference equation

$$\hat{x}_i^+ = \hat{x}_i^- + D_i\, f_i$$

and

$$\hat{x}_{i+1}^- = \hat{x}_i^+ + \epsilon_i \tag{9.10}$$

where ϵ_i are independent normal random variables with zero mean and variances σ_i^2,

$$\sigma_i^2 = b(t_i) = \int_{t_i}^{t_{i+1}} q_{11}(t)\, dt \tag{9.11}$$

Moreover, $\hat{x}_o = 0$ and $\hat{x}_{N+1} = \hat{x}(T,T)$. The loss given by (9.8) becomes

$$\text{Loss} = E \left[\sum_{i=1}^{N} |f_i| + (\lambda/2)\, \hat{x}_{N+1}^2 \right] \tag{9.12}$$

The problem, in this discrete formulation, becomes that of finding f_i as a function of $\hat{x}_i^-$ which minimizes (9.12) subject to the stochastic difference equation constraint (9.10).

Solution

The solution will be obtained using Dynamic Programming.[12] Let

$$U_i(x_i^-) = \underset{f_k(\hat{x}_k^-), k=i, i+1, \ldots, N}{\operatorname{Min}} E\left(\left(\sum_{j=i}^{N} |f_j| + (\lambda/2)\, \hat{x}_{N+1}^2 \right) / \hat{x}_i^- \right) \tag{9.13}$$

where $E(../a)$ indicates a conditional expectation operation given a. It follows from the Principle of Optimality[12] that

$$U_i(\hat{x}_i^-) = \underset{f_i(\hat{x}_i^-)}{\operatorname{Min}} E\left((|f_i| + U_{i+1}(\hat{x}_{i+1}^-)) / \hat{x}_i^- \right) \tag{9.14}$$

From (9.10) we see that $\hat{x}_{i+1}^- / \hat{x}_i^-$ is $N(f_i D_i + \hat{x}_i^-, \sigma_i^2)$ where $N(a,b)$ indicates a Normal random variable with mean a and variance b. Hence,

$$E(U_{i+1}(\hat{x}_{i+1}) / \hat{x}_i^-) = \frac{1}{\sqrt{2\pi}\,\sigma_i}$$

$$\int_{-\infty}^{\infty} U_{i+1}(s) \exp(-(s - f_i D_i - \hat{x}_i^-)^2 / 2\sigma_i^2)\, ds \tag{9.15}$$

Let the optimal control f_i be denoted by f_i° and let

$$z_i = f_i^\circ D_i + \hat{x}_i^- \tag{9.16}$$

It is clear from (9.14) and (9.15) that if $f_i^\circ \neq 0$, then

$$-\operatorname{sgn} f_i^{\circ} = \frac{\partial}{\partial f_i} E\left(U_{i+1}(\hat{x}_{i+1}) / \hat{x}_i^-\right)$$

$$= \frac{D_i}{\sqrt{2\pi}\,\sigma_i} \int_{\infty}^{\infty} K_{i+1}(s) \exp(-(s-z_i)^2 / 2\sigma_i^2)\, ds \qquad (9.17)$$

where

$$K_{i+1}(s) = \frac{\partial U_{i+1}(s)}{\partial s} \qquad (9.18)$$

Let us assume that $K_{i+1}(s)$ is a monotonically non-decreasing odd function of s such that $K_{i+1}(s) = \frac{1}{D_{i+1}} \operatorname{sgn} s$ for s greater than some number z_{i+1}. Inspection of (9.17) and (9.16) shows that there exists a $f_i^{\circ} < 0$ and a $z_i > 0$ such that (9.17) can be identically satisfied.* In other words, the right-hand side of (9.17) is 1. Substituting this z_i into (9.18) and making use of the fact that K_{i+1} is odd lead to the result that the optimal control is given by

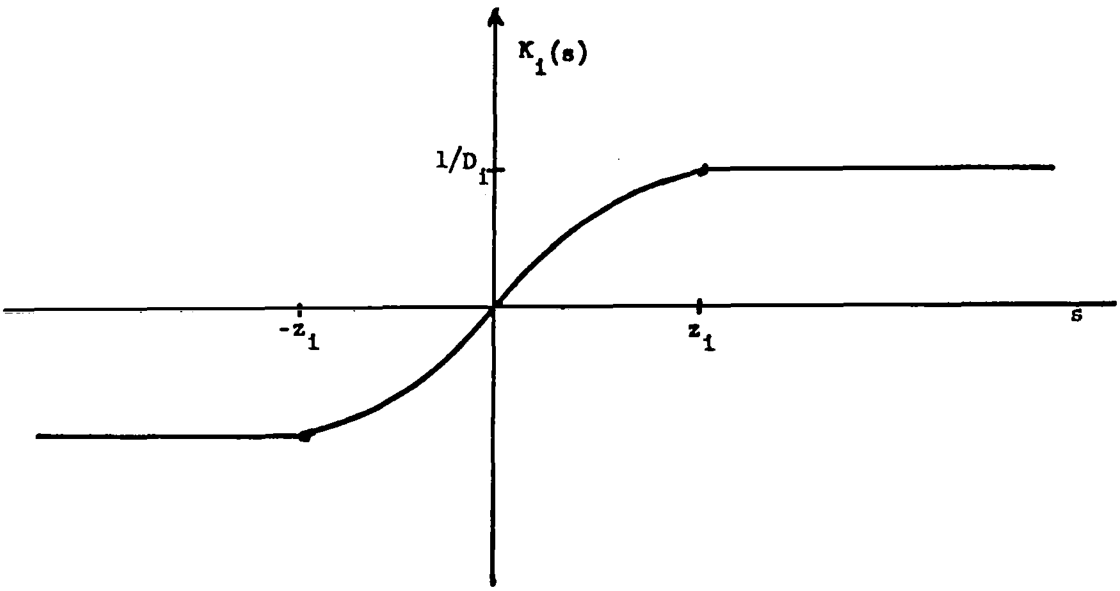
$$f_i^{\circ} = \begin{cases} -\dfrac{\operatorname{sgn} \hat{x}_i^-}{D_i} \left(|\hat{x}_i^-| - z_i \right) & \text{if } |\hat{x}_i^-| \geq z_i \\ 0 & \text{otherwise} \end{cases} \qquad (9.19)$$

Differentiating both sides of (9.14) with respect to $\hat{x}_i^-$ and using (9.15)-(9.16) and (9.19) yields a recursive equation for computing $k_i(s)$

$$K_i(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\,\sigma_i} \displaystyle\int_{-\infty}^{\infty} K_{i+1}(s) \exp(-(s-y)^2/2\sigma_i^2)\, ds & \text{if } |y| \leq z_i \\ \dfrac{1}{D_i} \operatorname{sgn} y & \text{otherwise} \end{cases} \qquad (9.20)$$

*This is because we have assumed that $D_i > D_{i+1}$. It will not have a solution if $D_i < D_{i+1}$. What this means physically is that if the effectiveness of the control is larger at t_{i+1} than t_i , then no control should be applied at t_i.

It is clear from (9.20) that if $K_{i+1}(s)$ is a monotonically non-decreasing odd function of s, so is $K_i(s)$. A typical $K_i(s)$ has the form



It only remains to obtain $K_N(s)$ and to verify that it has the form shown. This can be done by considering the one-stage process. From (9.13) we see

$$U_N(\hat{x}_N^-) = \underset{f_N(\hat{x}_N^-)}{\mathrm{Min}} \quad E\Big((| f_N | + \lambda/2 \; (\hat{x}_N^- + f_N D_N)^2 \; / \; \hat{x}_N^-) + (\lambda/2) \; \sigma_N^{\;2} \tag{9.21}$$

Differentiating the right-hand side of (9.21) with respect to f_N yields

$$\mathrm{sgn}\, f_N^o + \lambda \; (\hat{x}_N^- + f_N^{\,o} D_N \;) \; D_N = 0 \tag{9.22}$$

which implies that the optimal control $f_N^o \neq 0$ is given by

$$f_N^{\;o} = \begin{cases} - \dfrac{\hat{x}_N^-}{D_N} + \dfrac{1}{D_N} \;\; \mathrm{sgn}\, \hat{x}_N^- & \text{if } \; |\hat{x}_N^-| \geq \dfrac{1}{\lambda D_N} \\[2ex] 0 & \text{otherwise} \end{cases} \tag{9.23}$$

Hence, $z_N = \frac{1}{\lambda D_N}$. Substituting (9.23) into (9.21) and differentiating both sides of the resulting equation with respect to $\hat{x}_N^-$, we get

$$\frac{\partial U_N(\hat{x}_N^-)}{\partial \hat{x}_N^-} = K_N(\hat{x}_N^-) = \begin{cases} \lambda \hat{x}_N^- & \text{if } |\hat{x}_N^-| \le 1/(\lambda D_N) \\ \frac{1}{D_N} \operatorname{sgn} \hat{x}_N^- & \text{otherwise} \end{cases} \tag{9.24}$$

which shows that $K_N(s)$ is an odd and monotonically non-decreasing function of s and hence establishes the validity of our solution.

To summarize, we have shown that there exists a set of "threshold" numbers $z_i \ge 0$ such that (see Eq. 9.19) a control is applied at t_i if, and only if, $|\hat{x}_i^-| > z_i$ and the effect of the control is to bring the state after the correction to $\hat{x}_i^+ = z_i \operatorname{sgn} \hat{x}_i^-$. The set of numbers z_i can be computed by (9.16) where $K_{i+1}(s)$ are monotonically non-decreasing odd function of s computable backwards by the recursive equation (9.20). Computation starts at t_N with (9.24).

2.9.2 Results of Numerical Work

Again, the same uniform model is used. The parameters defining this approaching guidance are the same as that given in the previous section. We shall further assume that the last correction is a true correction and that the standard deviation of the desired terminal miss is 1250 km. It can be easily shown that the last correction, in our case, is to be executed at 0.04×10^6 sec. from time of impact. The first correction time is arbitrarily set at 0.64T from time of impact and our numerical work studies the effect of including various numbers of corrections between these two timings. The results are given in Figures 9.1 and 9.2.

Figure 9.1 gives the plot of the threshold levels z_i versus the timing of the corrections with N, the total number of corrections, as a parameter. The computation is done backwards from $t = 0.96T$ using the recursive equations derived in this paper. To save time, the numerical integration at every step (i.e., over the infinite range) is done by Monte Carlo. It was found that, for the case of eleven corrections, the computing time is about four minutes on IBM 7094. It is of interest to note from the results in Figure 9.1 that the boundary lines joining the threshold levels approaches a continuous curve. In Figure 9.2 the total expected velocity correction is plotted against the number of trajectory corrections where the timings of these corrections and the corresponding threshold levels are the same as those given in Figure 9.1. The computation is also done by Monte Carlo. The result indicates, as has been found in other studies using linear control law, that the additional savings obtained with more than four corrections are almost negligible.

2.9.3 Hard Constraint Problem

Within the realm of allowing nonlinear control, there is the so-called hard constraint problem. This is the problem of minimizing the rms values of the terminal miss distance subject to the constraint that the total (not the expected) amount of the velocity capability is limited and specified in advance.

Let c_i^- and c_i^+ be the amount of total velocity correction capability immediately before and after the correction at t_i. Then we have in addition to (9.10) the equations

$$c_i^+ = c_i^- - |f_i| \qquad \text{and}$$

$$c_{i+1}^- = c_i^+$$

Note c_o is the prescribed total velocity capability. The mathematical problem for the hard constraint case is essentially that of finding f_i as a function of two variables c_i^- and $\hat{x}_i^-$ which minimizes

$$E\left[\hat{x}_{N+1}^2\right]$$

subject to the constraint that $c_i^+ \geq 0$ for all i.

This problem, as we have formulated above, has been solved by Rosenbloom[14] and has also been considered by Orford.[15] The optimum solution which can also be obtained by using Dynamic Programming shows that there exists two numbers $z_i^{**}(c_i^-) > z_i^*(c_i^-) > 0$ such that:

(a) no control is applied if $|\hat{x}_i^-| \leq z_i^*(c_i^-)$,

(b) use all the velocity capability left if $|\hat{x}_i^-| \geq z_i^{**}(c_i^-)$, and

(c) apply an intermediate control if $z_i^* < |\hat{x}_i^-| < z_i^{**}$.

Unlike the case we have already considered, the computation involved for obtaining the actual optimum solution is rather unmanageable and involves creating a large number of tables of functions of two variables.

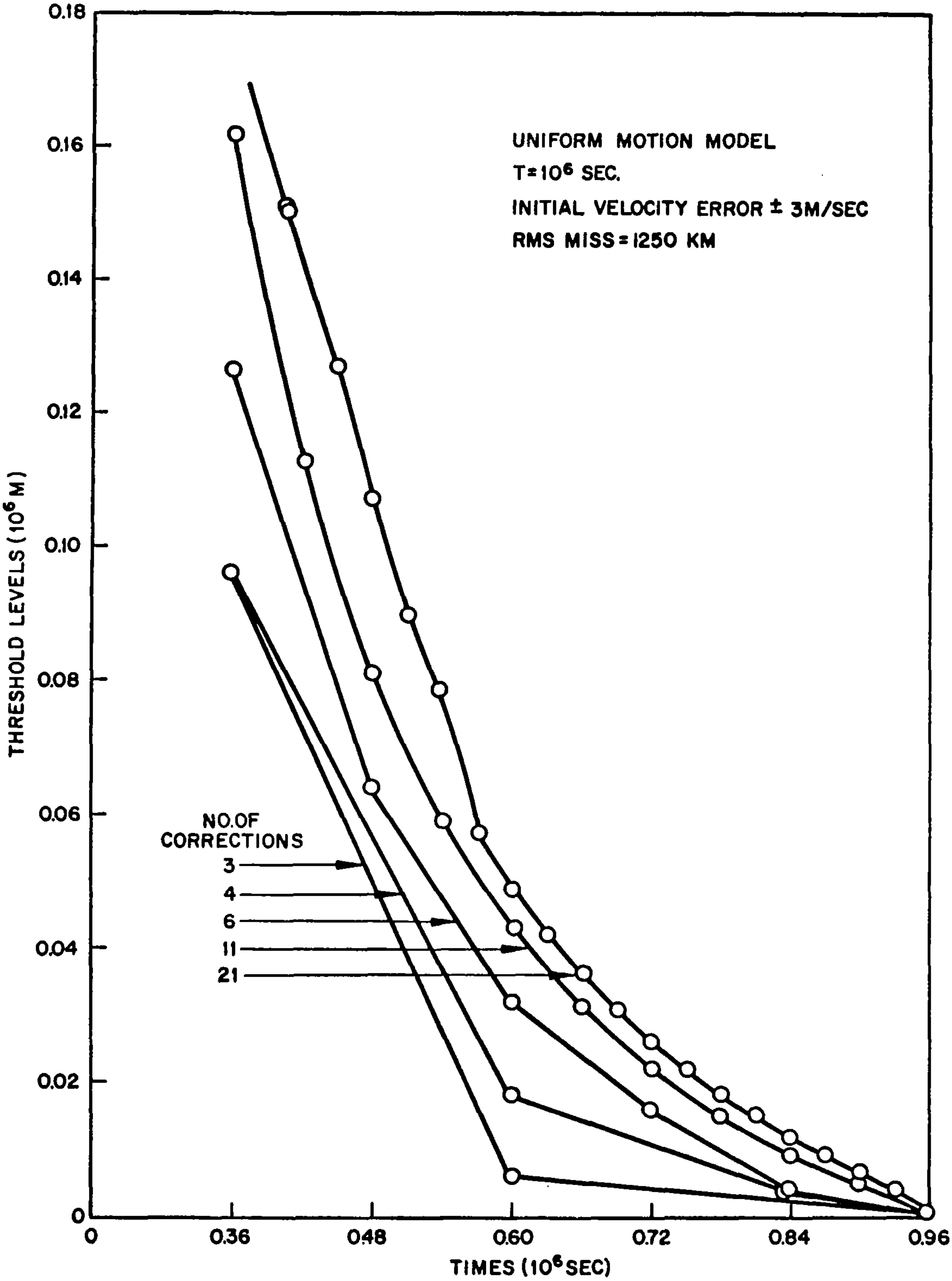


Fig. 9.1 - Threshold Levels vs. Correction Times For Various Number of Corrections.

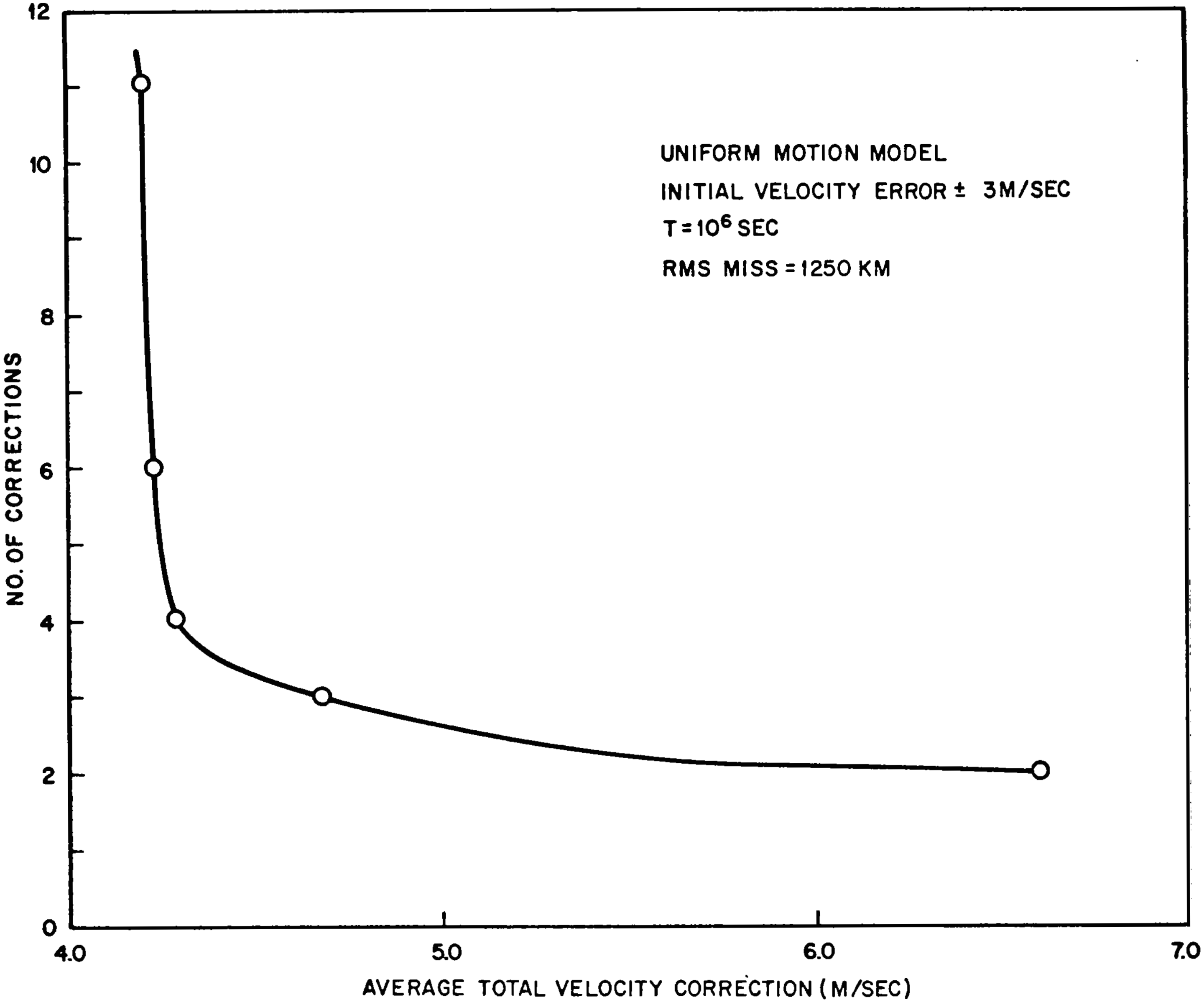


Fig. 9.2 - Average Effort vs. Number of Corrections.

2.10 An Error Analysis Program for Interplanetary Transfer

As indicated in Section 2.1, the theory developed for the basic minimum effort control is directly applicable to the case of variable time of arrival guidance scheme assuming all errors lie in the transfer plane and that correction mechanization errors are negligible. A digital computer program is developed applying this theory to the study of guidance problems in typical interplanetary trips.[16] The program (1) performs a linear error analysis of typical interplanetary trajectories with assumed rms injection errors and measurement histories, and (2) computes a trajectory correction strategy based on the basic minimum effort theory. It includes a near-optimum discrete trajectory correction strategy using impulsive corrections whose spacings are chosen to approximate the ideal continuous strategy. The analysis of these near-optimum discrete strategies extends the study by a Monte Carlo simulation to include the effect of correction mechanization errors as well as the effect of varying the time of the last correction.

We present in this section a brief description of the various subprograms that have been developed as well as the computer results giving the velocity requirements for two typical interplanetary transfers (Earth-Mars and Earth-Venus-Mars swingby).

2.10.1 Description of the Computer Programs

The main program that performs the linear error analysis and includes the continuous minimum effort correction strategy for typical interplanetary trips consists of two versions: (1) a direct transfer program that considers transfers between two massless planets and (2) a swingby transfer program that considers transfers between two massless planets via a third planet whose gravity field is taken into account. Both versions assume that the errors are confined to the transfer plane.

A. Direct Transfer Program

This program assumes that the vehicle is injected from a massless Earth and is transferred to a massless planet. The transition matrices $\Phi(T,t)$, and hence the sensitivities $D(t)$, are obtainable from perturbations of a nominal Keplerian trajectory. The program at present allows four kinds of measurements for orbit determination. These are angular measurements of the direction of the Sun, the target planet, and the Earth, relative to the star background, and range-rate information from an Earth-based radar. The accuracies and frequencies of these measurements are assumed to be constant in time. The program also has the option of turning on as well as turning off the information from the Earth in the middle of the trip, and turning on the angular information from the target planet. It should be noted that the information from the Earth cannot be turned on immediately, as the formulated information rate is initially infinite.

Other inputs to the program are: (1) the eccentricity, semi-major axis, trip time, and the initial true anamoly; these quantities specify the transfer ellipse by Kepler theory; (2) the initial injection errors in the transfer plane which serve as the initial conditions for the Kalman covariance matrix $V(t)$; and (3) the rms accuracy of the measurements and the corresponding intervals of observation; this and the information in (2) allows us to integrate the equations governing $V(t)$.

B. Swingby Transfer Program

This program extends the program for direct transfer to the case where the transfer is via a third planet. The nominal transfer orbit consists essentially of two heliocentric ellipses connected by a planet-centered hyperbola near the flyby planet. The additional inputs for this program are the necessary parameters which specify the second ellipse as well as the ratio of the mass of the flyby planet to that of the sun. A "patched-conic" treatment of the nominal trajectory is used. The trajectory is taken as a planet-centered hyperbola within a "sphere of influence" near the flyby planet such that the ratio of the Sun's "effective" attraction to that of the planet is less than the ratio of the planet's attraction to the (total) attraction of the Sun. The boundary of this "sphere of infuence" is given by

$$\frac{\text{distance from vehicle to planet}}{\text{distance from planet to Sun}} \simeq \left(\frac{\text{mass of planet}}{\text{mass of Sun}}\right)^{2/5}$$

and this is where the two ellipses are patched to the planet-centered hyperbola. The relative velocities of "approach to" and "departure from" the flyby planet must be the same so that the point of closed approach to the flyby planet is halfway along the hyperbolic arc.

C. Discrete Minimum Effort Program

This is an additional program which analyzes near-optimum discrete minimum effort control strategies including the effect of mechanization errors. Given the number and the timings of the corrections, the program computes the optimum gain $k(t_1)$ by using (8.17). A Monte Carlo simulation based on this discrete strategy is performed.

D. Fixed Total Velocity Capability

If we assume a fixed total velocity correction capability somewhat greater than the theoretical expected velocity requirement, the Monte Carlo program has the additional feature of being able to control the time of the last correction so that all remaining fuel is used on this last control, and the estimated miss at this time thereby reduces to zero. In other words, the last correction occurs at the time t_N such that

$$\left[\, \hat{x}_1(T,\ t_N^-) \,\right] / \left[\, D(t_N) \,\right] = \text{velocity capability remaining}$$

t_N being then a random variable. It should be noted that $\hat{x}_i(T,t)$ for $t \geq t_{N-1}$ is a simple Wiener process.

In the (rare) event that the propulsion left aboard is not sufficient to make the correction called for at an earlier time t_i, $i \leq N-1$, then the program assumes, of course, that the correction uses all the fuel available. This run will then give a terminal miss which is the miss immediately after this correction.

2.10.2 Examples and Numerical Results

The programs described in the previous section are used for the study of guidance problems connected with two typical interplanetary trips. The two trips were selected from the results compiled by Lockheed.[17] They are: (1) a 204-day trip from Earth to Mars leaving Earth on December 30, 1966; and (2) a 245-day trip to Mars flying by Venus (seventy-five days to Venus and one hundred and seventy days to Mars). The trip leaves Earth on September 6, 1970.

The initial injection errors are assumed to be in velocity only. These errors can be obtained by propagating the errors at launch along the hyperbolic asymptote predetermined for this trip. Typical values of the errors at launch based on the Atlas-Agena booster were used. It turns out, after a simple computation, that for the 204-day trip to Mars, the 2x2 covariance matrix elements of the initial injection velocity error are of the order of 15-20 m/sec and are given specifically by

$$\begin{bmatrix} 330 & 107 \\ 107 & 130 \end{bmatrix} \left(\text{m/sec}\right)^2$$

The same injection errors were used for the flyby trip.

The following information rates were assumed: (1) angle information from the Sun: $\pm$10 mrad at a rate of 1/hr; (2) angle information from Earth:

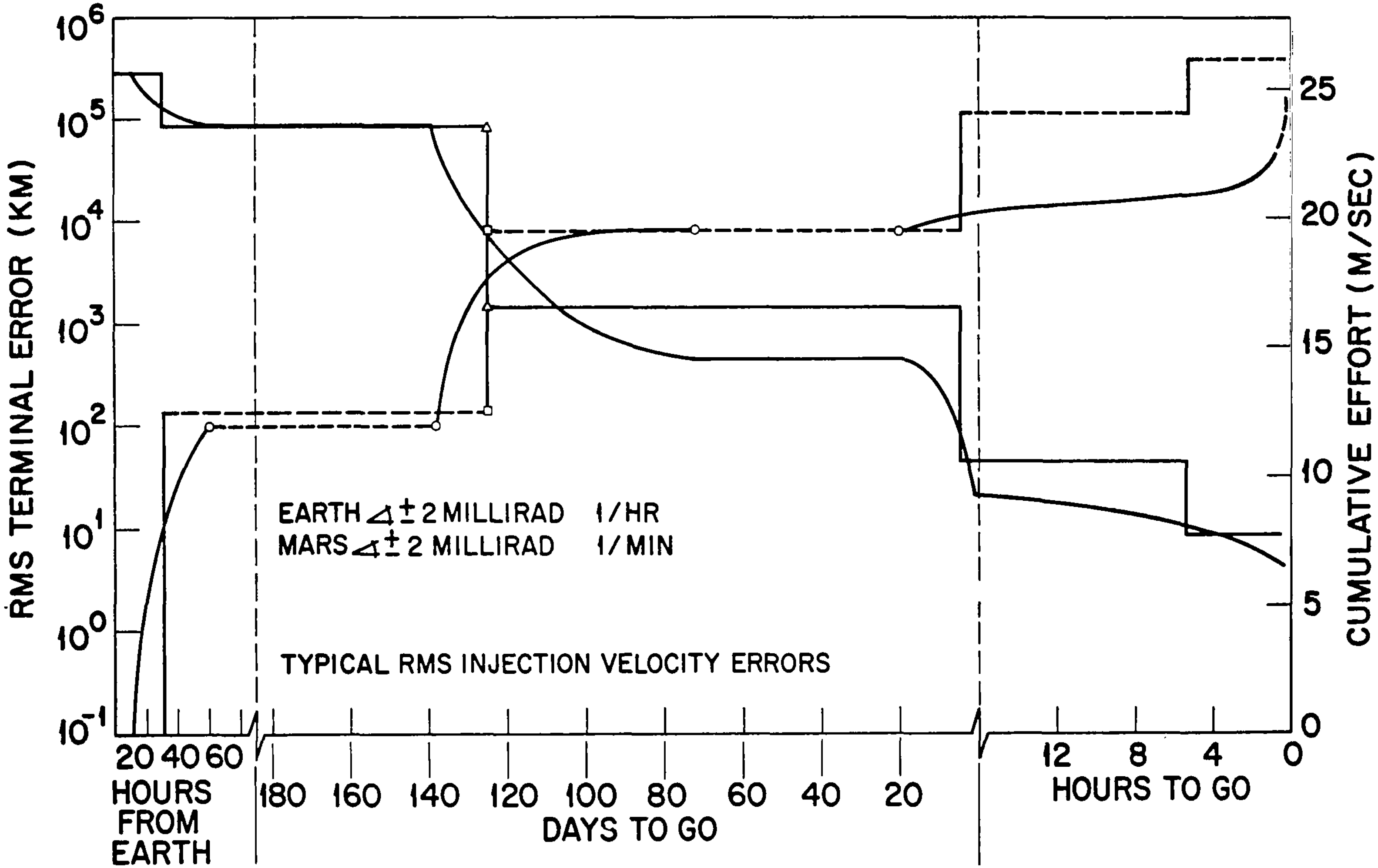
± 2 mrad at a rate of 1/hr; (3) angle information from Mars: ± 2 mrad at a rate of 1/min; (4) Doppler information from Earth: ± 1 m/sec at a rate of 1/min; (5) angle information from Venus (for flyby trip): ± 2 mrad at 1/min on first leg, and ± 2 mrad at 1/hr on the second leg. In all cases, the information from Earth was turned on after a wait of 3.6 hours so as to avoid an infinite information rate at the beginning.

The results of the 204-day trip to Mars are given in Figures 10.1-10.3. In these figures we have plotted the histories of the rms terminal miss and the average cumulative effort vs the time-to-go for different combination of measurement histories. For the purpose of comparison, we have also plotted in Figure 10.1 the corresponding near-optimum discrete strategy using four corrections (near-optimum in the sense that the timings are not optimized). It is seen that the correction effort required by the near-optimum discrete strategy is only about 10 to 15% higher than that required by the corresponding optimum continuous strategy.

The sudden drop in terminal miss in Figure 10.2 is due to the impulsive thrust which is applied just prior to the time when the information from Earth vanishes. The control is turned on again very soon (of the order of a few hours) after the information from Mars is turned on. It is of interest to note that this is only a few percent more costly than the case of continuous observation (see Figure 10.1). The effect of adding Earth range-rate information is shown in Figure 10.3. It shows that most of the errors are corrected out at the beginning; and moreover, in the absence of any angle information from the planet, there exists an uncorrectable terminal miss which reflects our lack of knowledge of the actual error in the orbit estimation. This is of the order of 15 km as can be seen by the leveling off of the curves near the end. It is seen that this uncertainty can be eliminated by supplementing the measurements with angular information from Mars during the last forty days.

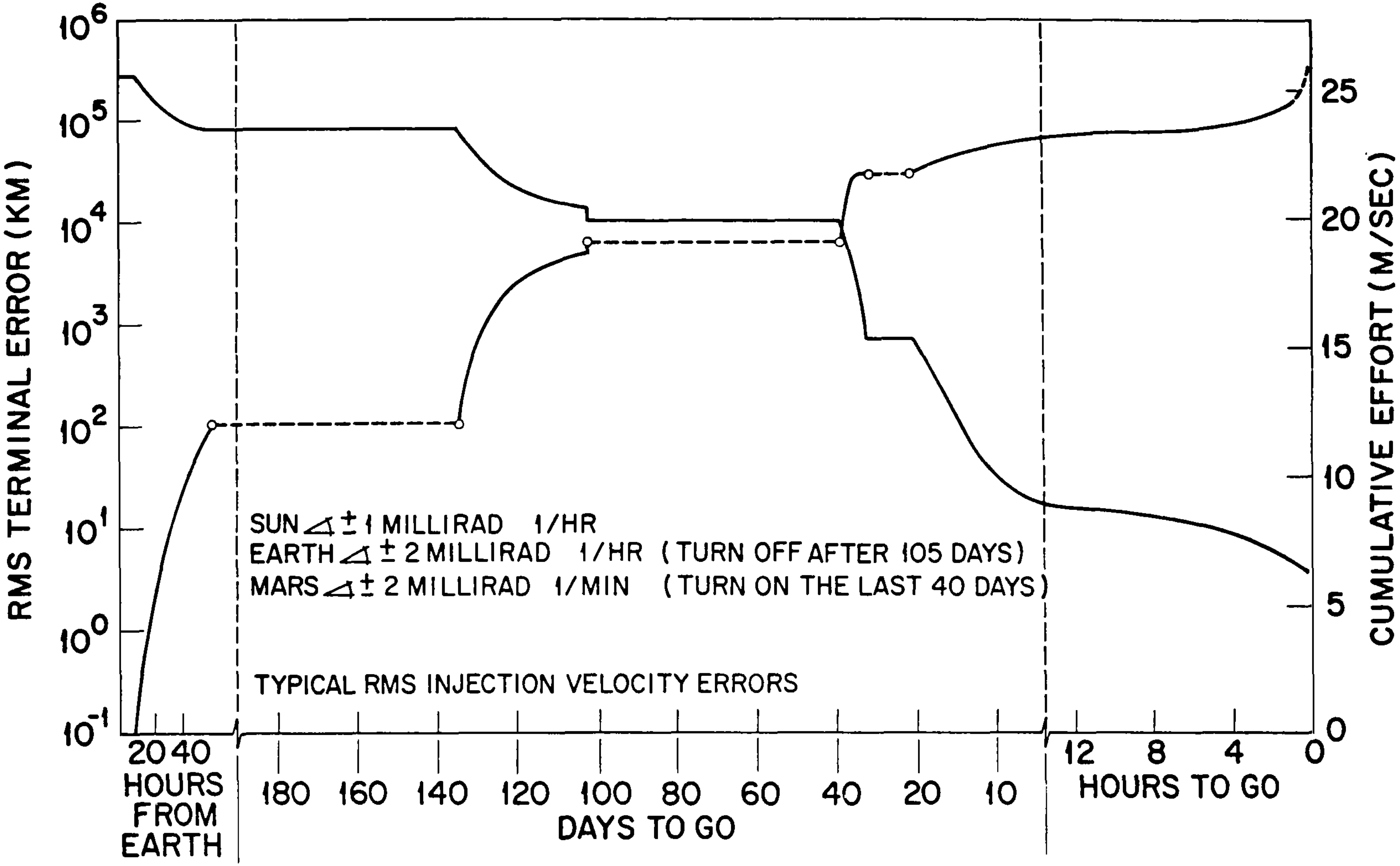
The results of the flyby trip are given in Figures 10.4-10.5. Figure 10.4 shows the histories of the rms terminal miss and the average cumulative effort vs the time to go, for the case where only angle information is used. The Doppler is turned off when the vehicle reaches the end of the hyperbola. It is interesting to note that the corrective effort required for guidance on the flyby trip as far as Mars is not substantially greater than the effort required for the single leg trip to Mars.

Computer results for the Monte Carlo simulation of the discrete strategy shown in Figure 10.1 are given in Figures 10.6-10.8. Figures 10.6 and 10.7 correspond to four fixed correction times, and Figure 10.8 corresponds to three fixed correction times and a randomized last correction. Figure 10.6 gives the empirical probability distribution (a sample size of 100) of the effort used for different rms engine execution errors and different values of rms terminal miss. It is assumed that the accelerometer reading error $\sigma_\alpha = 0$. A scatter diagram of the magnitude of the actual terminal miss and the cumulative effort for the case of $\sigma_\beta = 0.2$ m/sec is given in Figure 10.7. The point marked X indicates the average effort and the average (absolute) terminal error under the assumption of no mechanization error. It is seen that a large number of points fall to the right of this $\boxed{X}$, indicating that the amount of the fuel carried should be considerably in excess of the average amount used with fixed correction times. The results in Figure 10.8 show an improvement in the terminal miss distribution, especially at the low end, as might have been anticipated. The same figure also shows the effect of the loss of information caused by an accelerometer reading error.



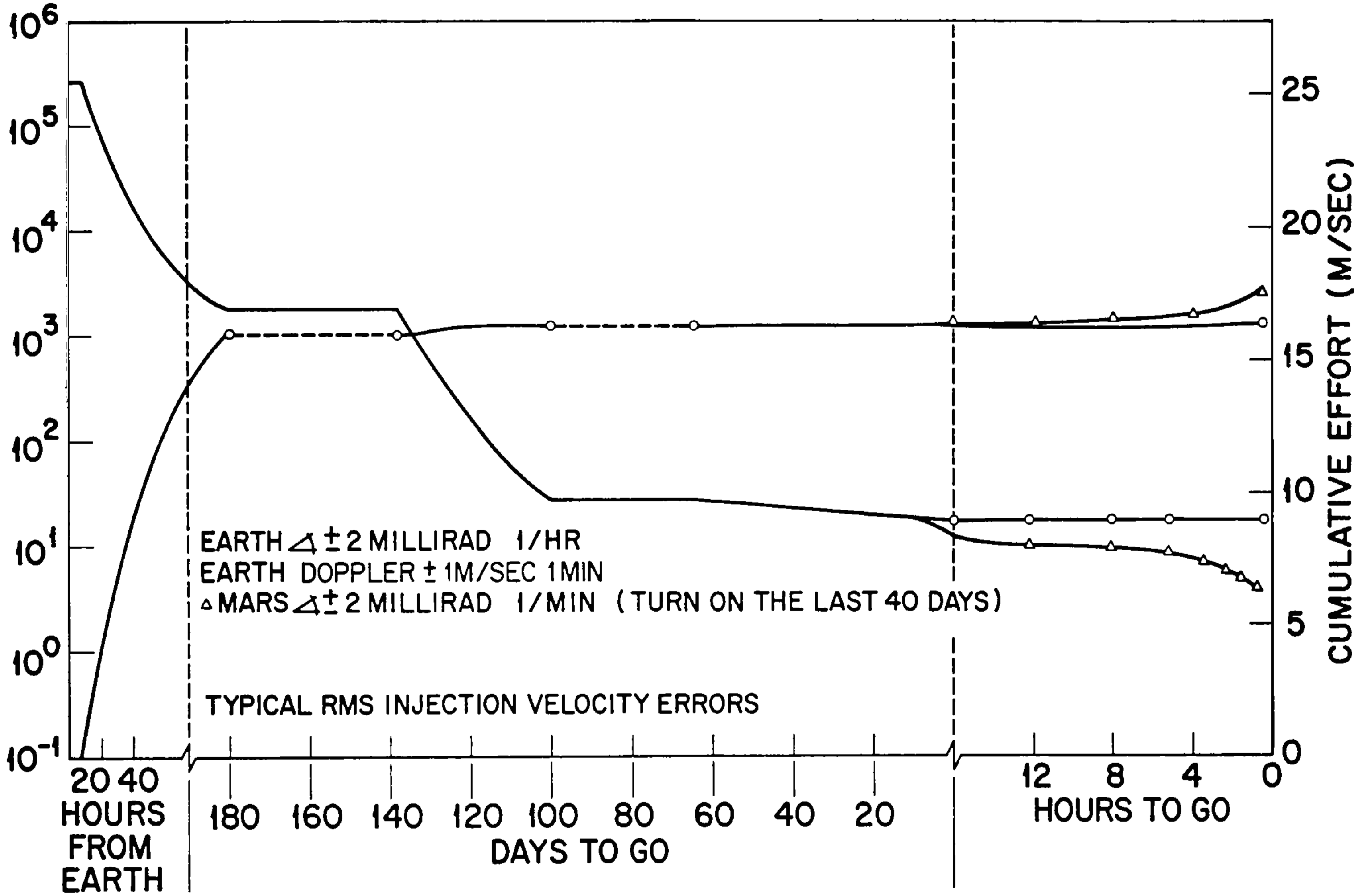
HISTORY OF REMAINING ERROR AND CUMULATIVE EFFORT VS TIME TO GO
204 DAYS TRIP TO MARS

FIG. 10.1



HISTORY OF REMAINING ERROR AND CUMULATIVE EFFORT VS TIME TO GO
204 DAYS TRIP TO MARS

FIG. 10.2



HISTORY OF REMAINING ERROR AND CUMULATIVE EFFORT VS TIME TO GO
204 DAYS TRIP TO MARS

FIG. 10.3

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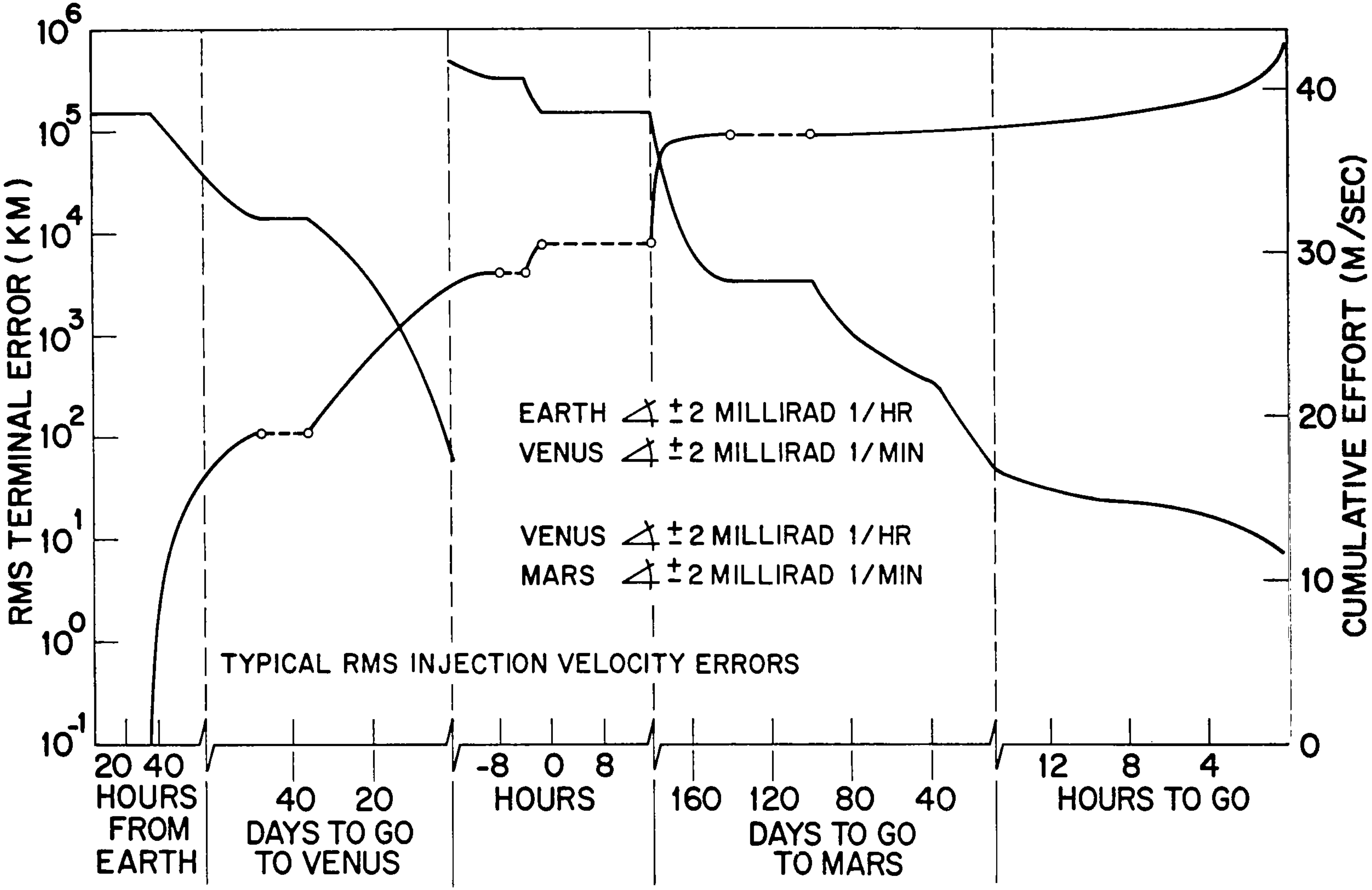
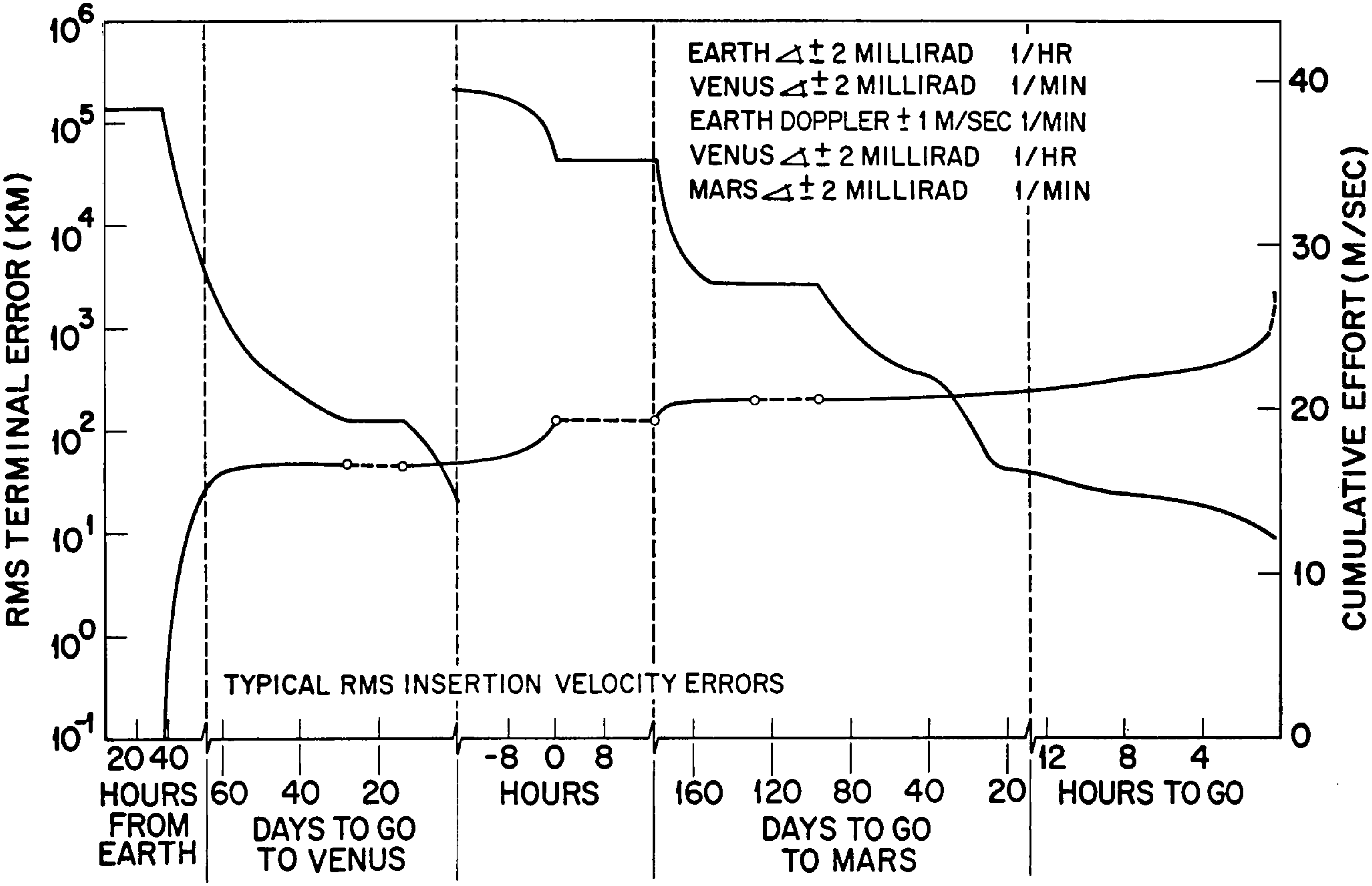


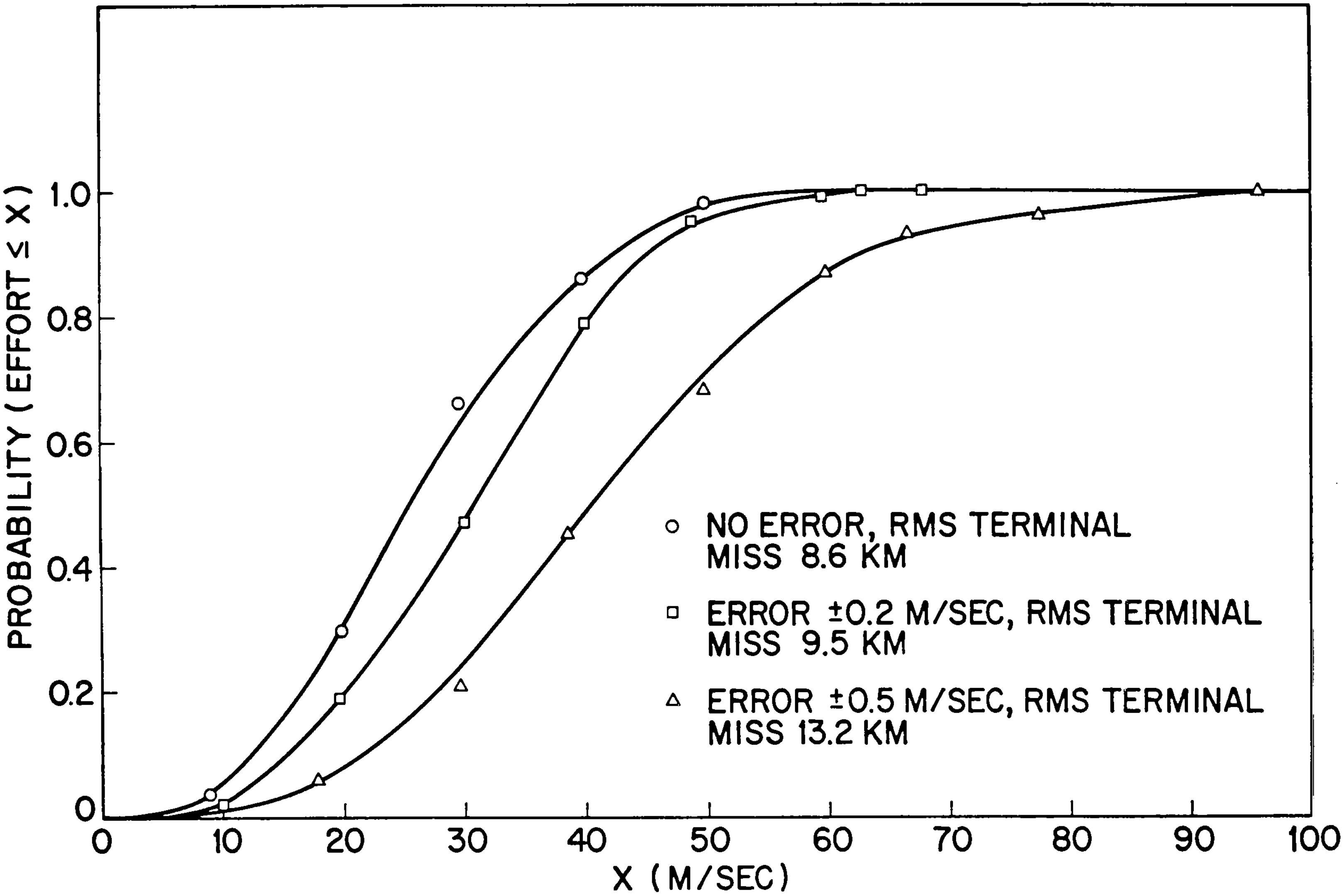
HISTORY OF REMAINING ERROR AND CUMULATIVE EFFORT VS TIME TO GO
FLY-BY TRIP (EARTH - VENUS - MARS)

FIG. 10.4



HISTORY OF REMAINING ERROR AND CUMULATIVE EFFORT VS TIME TO GO
FLY-BY TRIP (EARTH-VENUS-MARS)

FIG. 10.5



PROBABILITY DISTRIBUTION OF THE CUMULATIVE EFFORT

FIG. 10.6

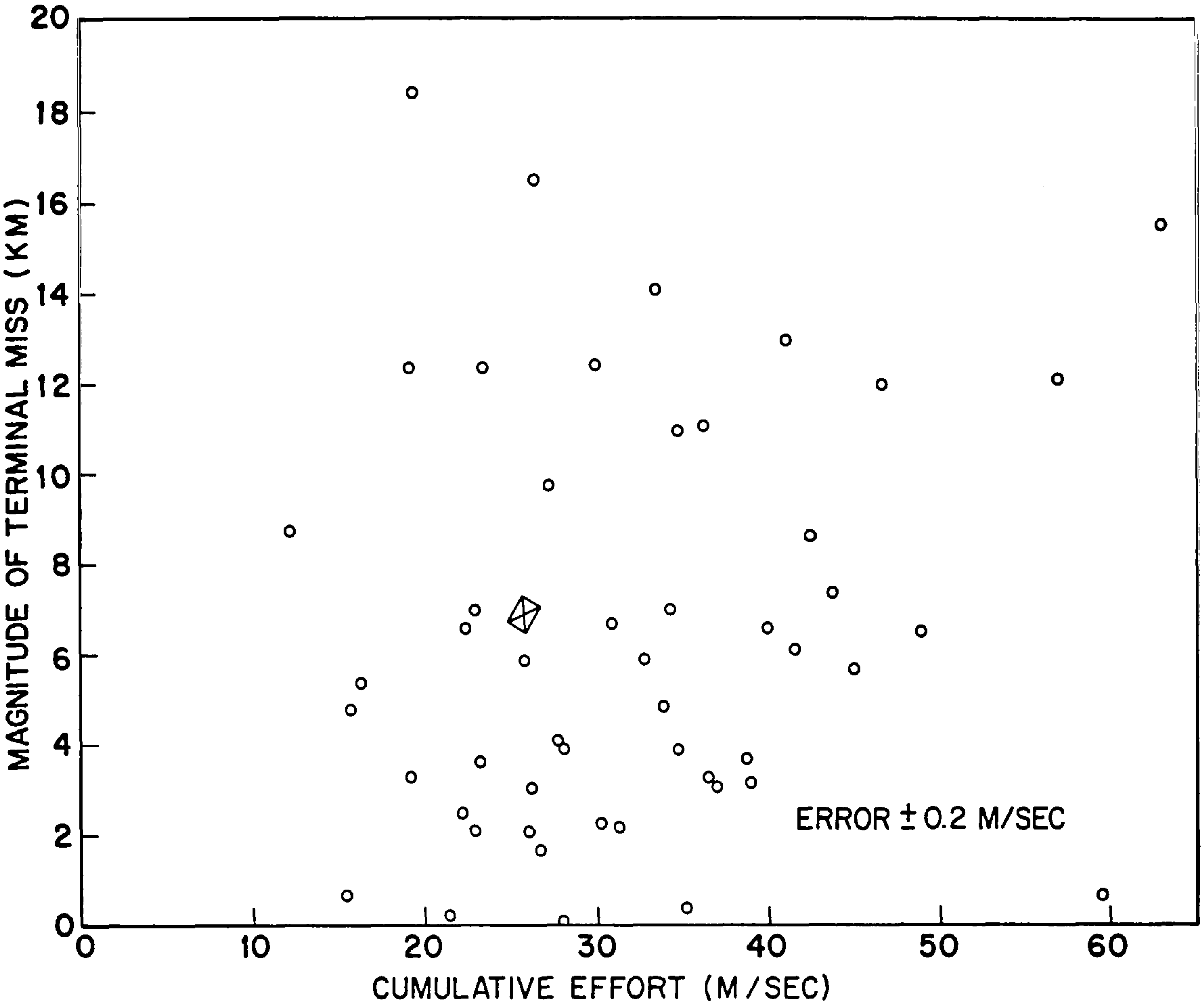
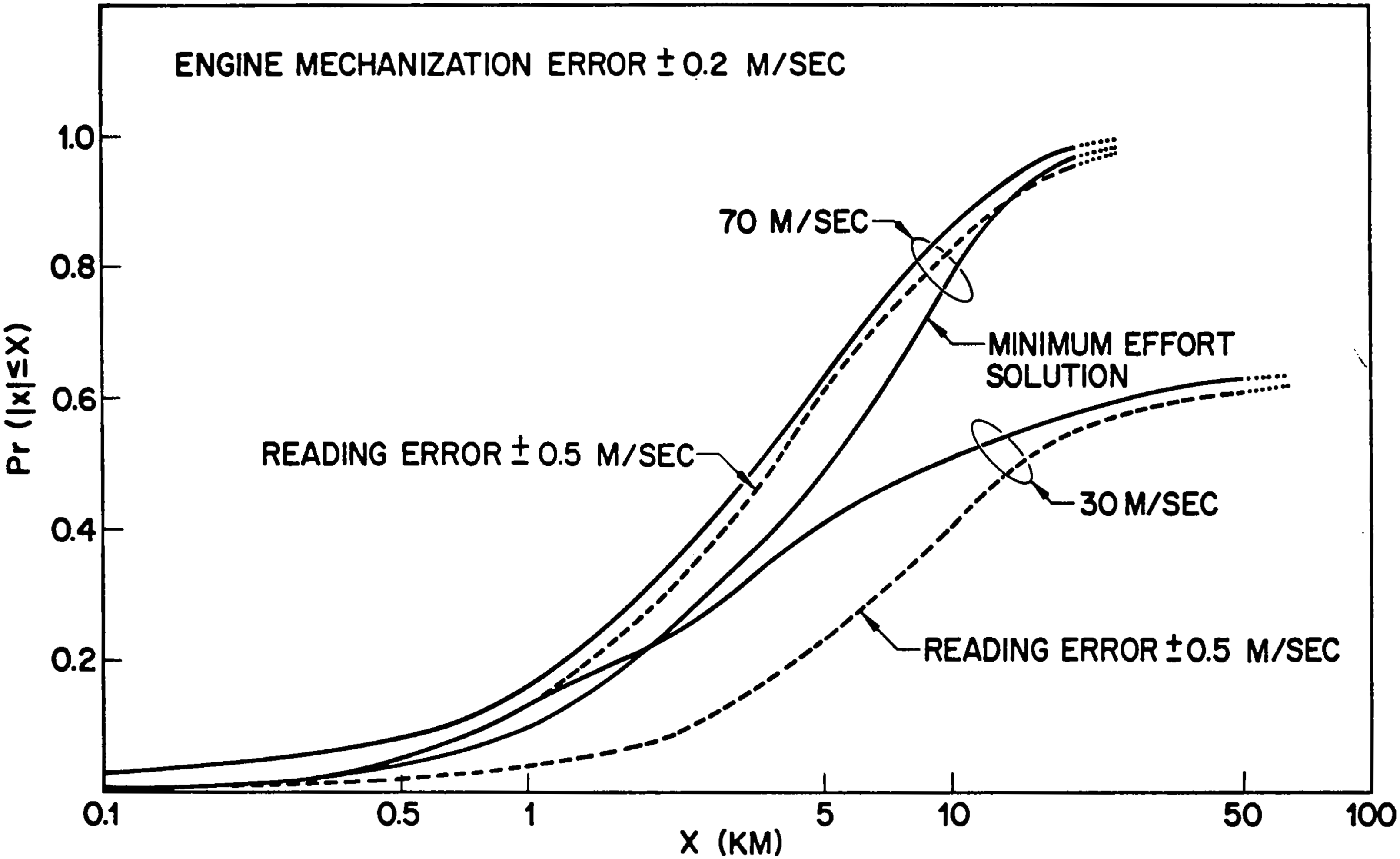


FIG 10.7 SCATTER DIAGRAM



PROBABILITY DISTRIBUTION OF THE MAGNITUDE OF THE TERMINAL MISS FOR FIXED TOTAL VELOCITY CORRECTION (THE TIME OF LAST CORRECTION IS RANDOM)

FIG. 10.8

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3. PLANETOCENTRIC GUIDANCE FOR LOW THRUST INTERPLANETARY TRANSFER

3.1 Introduction

In the low thrust interplanetary mission considered here, it is assumed that the engine is turned off during the central portion of the trajectory so guidance takes place only in the vicinities of the departure planet and the target planet. This means that there is no midcourse guidance and that the midcourse trajectory is determined by the energy and asymptotic direction of the vehicle leaving the vicinity of the departure planet. Accordingly, departure and approach are primarily planetocentric. Furthermore, it is assumed that the low thrust engine has constant specific impulse so that guidance is achieved by changing the direction of thrust rather than the magnitude. Battin[1*] has considered the guidance problem for a variable thrust engine in a lunar mission with characteristics similar to this interplanetary mission. He devised a feasible (non-optimum) guidance scheme. Recently, Mitchell[2] has derived a guidance scheme, also for a variable thrust engine, considering only the heliocentric portion of the transfer. The problem considered here is to find an optimum guidance scheme for a constant thrust vehicle, i.e., a guidance scheme which minimizes the fuel expenditure while attempting to meet the terminal constraints.

For the purpose of simplicity, three additional assumptions are made: (1) all the action takes place in a plane containing the planet and the vehicle acceleration; (2) the vehicle is propelled at constant acceleration; and (3) while the engine is on, the perturbations due to the Sun can be neglected. The first assumption permits the optimization scheme to be worked out as a two-dimensional problem. The extension to three dimensions would

*References referred to in this chapter are listed at the end of this chapter.

be straightforward. Because the change in mass is small, the second assumption should change the original trajectory only slightly, and should have negligible effect on the control scheme. The third assumption is valid in the vicinity of the planet and, once again, should change the original trajectory only slightly and have negligible effect on the control scheme.

The problem of optimizing the original trajectory can be stated as follows: Given a set of initial conditions (position and velocity) and the equations of motion (which include gravity and the operating characteristics of the engine), find the trajectory which meets the specified terminal constraints and minimizes the loss (in this case the mass expenditure). The optimization problem requires the solution to a two-point boundary value problem in the calculus of variations.[3] If deviations from an optimum nominal are small, the application of the minimizing conditions results in a neighboring optimum control scheme. This control scheme is based on a linear perturbation from a nominal optimum path and involves the second variation in the calculus of variations.[4] It results in a linear feedback control law, i.e., the change in the control (in this case the change in the engine orientation) is linear in changes in the trajectory. If the engine is behaving properly and the only disturbances are due to deviations from the original trajectory, any control scheme which satisfies the terminal constraints gives the same loss to first order. In that case, the above guidance scheme is the best one to second order. However, if the engine misbehaves (i.e., the thrust is too high or too low), there is a first order change in the loss.

The analytic approach to the problem is based upon the fact that in the vicinity of the departure planet, the low thrust trajectory can be divided into three regions: (1) the near-planet region which consists of a large number of revolutions while the vehicle gradually spirals away from the planet; (2) the transition region which lasts several revolutions and in which the trajectory spiral straightens out to approach a hyperbola; and (3) the far-planet region in which the vehicle is propelled to a specified energy level while asymptotically approaching a specific angular direction on a near hyperbola. In the approach to the target planet the same three regions are encountered, but in reverse order (that is, (3), (2), (1)).

For both the near-planet and far-planet regions, analytic solutions, including the optimizing condition, have been derived for the optimum and neighboring optimum trajectories. Each solution contains a set of five arbitrary constants (three of which are assumed to be small) that can be used to match initial and final conditions. A numerical integration technique has been used in the transition region to check the analytic solutions and to match the analytic solutions with the numerical solution so as to get one complete solution. Perturbation of the conditions for an optimum trajectory provides a linear feedback relation between changes in the control (thrust direction) and changes in the state (the angular distance and velocity components) for different values of the independent variable (radial distance).

The guidance coefficients are the sensitivities of the optimum control to changes in the state components. The guidance procedure which is envisaged here is that the actual state components of the vehicle are compared with the state components for the original optimum trajectory. If the comparison is made with the same radial distance r for actual and original, the guidance coefficients are called fixed-r guidance coefficients. If the comparison is made with the same energy, they are called fixed-energy guidance coefficients. The same technique is used to determine the guidance coefficients on both the outward spiral when the vehicle is leaving the planet and the inward spiral when the vehicle is approaching the planet. The terminal conditions for the incoming spiral are assumed to be a circular orbit; hence, the incoming guidance law (unlike that for the outgoing spiral) is independent of the instantaneous angular position. It is noted that with initial and final conditions exchanged and no angle constraint, the incoming and outgoing spirals are identical except that the signs of the velocity components are reversed. An additional complication for the inward spiral is to decide when to turn on. The theory of the second variation provides an expression from which the radial distance of turn-on is easily obtained from the known energy and angular momentum on the incoming hyperbola.

If slow fluctuations in engine behavior are anticipated, the effect of such fluctuations must be accounted for in the guidance law. The effect on the optimum trajectory of a constant bias in acceleration is obtained by rescaling. When there are errors in estimating the trajectory, the "best estimates" of the state variables are used along with the same guidance coefficients. The trajectory measurement errors have an effect on both fuel expenditure and terminal accuracy. Note that the terminal errors for the outgoing spiral lead to a non-optimum spiral entry at the target planet and, hence, cause an added fuel expenditure on the incoming spiral.

The general optimization problem is formulated in terms of the calculus of variations in Section 3.2 and the neighboring optimum guidance scheme is outlined. Results for the outward and inward spirals are presented in Sections 3.3 and 3.4, respectively. Section 3.3 also includes an outline of the analytic solution for the far-planet region while the analytic solution for the near-planet region is presented in Section 3.4 with a derivation in

Appendix A. Section 3.5 discusses the consequences of engine fluctuations and presents the effect of measurement errors including a numerical example using a simplified model of the system.

3.2 Optimum and Neighboring Optimum Control

In this section an outline is presented of the calculus of variations approach to the general optimization problem and its application to optimum guidance.[4] In the guidance scheme presented here, it is assumed that the original trajectory is an optimum one and corrections are based on the deviation from that optimum. The guidance does not return the vehicle to the original trajectory, but instead it causes the vehicle to follow a neighboring optimum trajectory.

3.2.1 Optimization Problem

The problem of optimizing the original trajectory can be stated as follows. Given a set of differential equations describing the system

$$\frac{dx}{dr} = g\,(x,\ \alpha,\ r)$$

where x is the state vector with four components

g is a vector valued function with four components

α is the control variable (the thrust direction)

r is the independent variable (the radial distance)

and a set of initial conditions x_o, find the trajectory which maximizes a terminal quantity $\phi(x,\ r)_{r=r_f}$ (where r_f is the terminal value of the independent variable) and which gives the specified values of the terminal quantities

$$\Psi(x,\ r)_{r=r_f} = \Psi_f$$

where Ψ is a vector valued function with q components ($q \le 2$).

The differential equations satisfied by the optimum trajectory are:*

$$\frac{dx}{dr} = g\,(x,\ \alpha,\ r) \tag{2.1}$$

*Equations given in this chapter are numbered as follows: Eq. (k, J) means the J equation in Section 3.k.

$$\frac{d\lambda}{dr} = -\left(\frac{\partial g}{\partial x}\right)^T \lambda \tag{2.2}$$

$$0 = \left(\frac{\partial g}{\partial \alpha}\right)^T \lambda \tag{2.3}$$

where λ is the vector of adjoint variables with four components. $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial \alpha}$ represent 4x4 and 4x1 matrices of partial derivatives, respectively, and superscript T represents the transpose operation.

The boundary conditions for the equations are

$$0 = \Psi(x, r)_{r=r_f} - \Psi_f \tag{2.4}$$

$$0 = (-\lambda^T + \frac{\partial \phi}{\partial x} + \nu^T \frac{\partial \Psi}{\partial x})_{r=r_f} \tag{2.5}$$

$$0 = \frac{d}{dr} (\phi + \nu^T \Psi)_{r=r_f} \tag{2.6}$$

where $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \Psi}{\partial x}$ represent (4x1) and (qx4) matrices, respectively, and ν is a vector of q constant Lagrange multipliers.

The equations (2.1)-(2.3) can also be written in terms of the variational Hamiltonian H which is defined as

$$H = \lambda^T g$$

so that

$$\frac{dx}{dr} = \frac{\partial H}{\partial \lambda} \tag{2.1a}$$

$$\frac{d\lambda}{dr} = -\frac{\partial H}{\partial x} \tag{2.2a}$$

$$0 = \frac{\partial H}{\partial \alpha} \tag{2.3a}$$

3.2.2 Low Thrust Optimization

In the low thrust optimization problem formulated here, the radial distance (r) is the independent variable while the state vector (x) has four components, circumferential velocity (v_θ), radial velocity (v_r), angular position (θ), and time (t). The four components of the adjoint vector (λ) have subscripts one through four corresponding to the respective state variables. The direction (α) of the acceleration is measured positively outward from the local horizontal. The magnitude of the acceleration is a and the gravitational constant of the planet is μ.

For the system under consideration, the differential equations satisfied by the optimum trajectory are as follows.

The equations of motion:

$$\frac{dv_\theta}{dr} = -\frac{v_\theta}{r} + \frac{a\text{Cos}\alpha}{v_r}$$

$$\frac{dv_r}{dr} = \frac{v_\theta^2 - \frac{\mu}{r}}{rv_r} + \frac{a\text{Sin}\alpha}{v_r}$$

$$\frac{d\theta}{dr} = \frac{v_\theta}{rv_r} \tag{2.6}$$

$$\frac{dt}{dr} = \frac{1}{v_r}$$

The adjoint equations:

$$\frac{d\lambda_1}{dr} = \frac{\lambda_1}{r} - \frac{2v_\theta}{rv_r}\lambda_2 - \frac{1}{rv_r}\lambda_3$$

$$\frac{d\lambda_2}{dr} = \frac{a\text{Cos}\alpha}{v_r^2}\lambda_1 + \frac{(ar\text{Sin}\alpha + v_\theta^2 - \frac{\mu}{r})}{rv_r^2}\lambda_2 + \frac{v_\theta}{rv_r^2}\lambda_3 + \frac{1}{v_r^2}\lambda_4$$

$$\frac{d\lambda_3}{dr} = 0$$

$$\frac{d\lambda_4}{dr} = 0 \tag{2.7}$$

The Hamiltonian is defined as

$$H = \lambda_1 \frac{dv_\theta}{dr} + \lambda_2 \frac{dv_r}{dr} + \lambda_3 \frac{d\theta}{dr} + \lambda_4 \frac{dt}{dr}$$

The optimal control equations, found by maximizing the Hamiltonian, are

$$\sin \alpha = \frac{\lambda_2}{(\lambda_1^2 + \lambda_2^2)^{\frac{1}{2}}}$$

$$\cos \alpha = \frac{\lambda_1}{(\lambda_1^2 + \lambda_2^2)^{\frac{1}{2}}} \qquad (2.8)$$

Thus, the control (α) can be eliminated from Equations (2.6) and (2.7) by using the optimizing condition, Equation (2.8).

The terminal constraints for the outward spiral are the final energy (E_f) and the final angular direction of escape ($\theta_{\infty f}$)* where

$$E = \frac{1}{2} v_\infty^2 = \frac{1}{2} v_\theta^2 + \frac{1}{2} v_r^2 - \frac{\mu}{r}$$

$$\theta_\infty = \theta + \Theta$$

$$\Theta = \text{Arc Tan} \left[1/(v_\infty\, v_\theta r) \right] - \text{Arc Tan} \left[(1 - v_\theta^2\, r)/(v_r v_\theta\, r) \right] \qquad (2.9)$$

and the subscript f indicates the value of the quantity at the time of engine cutoff at radial distance r_f. Because the mass flow of the low thrust engine is fixed and positive, minimizing the mass expenditure is the same as minimizing the time the engine is operating. Therefore, the quantity to be maximized is negative time (the same as minimizing positive time). The payoff J (including the terminal constraints) can be written with constant multipliers, ν_1 and ν_2:

$$J = -t + \nu_1 E + \nu_2 (\theta + \Theta) \qquad \text{at } r = r_f \qquad (2.10)$$

*The angular direction (θ_∞) is calculated from the asymptotic properties of a hyperbola.

The terminal conditions for minimizing t subject to the constraints on the final energy and the final angular direction are

$$\lambda_1 = \nu_1 \, v_\theta + \nu_2 \frac{\partial \text{Ⓗ}}{\partial v_\theta}$$

$$\lambda_2 = \nu_1 \, v_r + \nu_2 \frac{\partial \text{Ⓗ}}{\partial v_r}$$

$$\lambda_3 = \nu_2$$

$$\lambda_4 = -1$$

$$0 = -\frac{dt}{dr} + \nu_1 \frac{dE}{dr} + \nu_2 \frac{d}{dr} (\theta + \text{Ⓗ}) \qquad (2.11)$$

For the initial optimum trajectory, the final angle is not constrained because the initial angle for a particular trajectory can be adjusted so as to give any desired final angle. In that case, the adjoint variable, λ_3 , is zero and the conditions in Equation (2.11) reduce to a simpler set of conditions.

$$\lambda_1 = v_\theta \Big/ \frac{dE}{dt}$$

$$\lambda_2 = v_r \Big/ \frac{dE}{dt} \qquad \text{at } r=r_f \qquad (2.12)$$

which further simplify to

$$\frac{\lambda_1}{\lambda_2} = \frac{v_\theta}{v_r}$$

$$\lambda_1^2 + \lambda_2^2 = 1 \qquad \text{at } r=r_f$$

The initial conditions for the outward spiral are circular velocity at some radial distance (r_o).

$$v_\theta = \frac{\mu^{1/2}}{r^{1/2}}$$

$$v_r = 0 \qquad\qquad \text{at } r = r_o$$

The same set of conditions, circular velocity at some radial distance, serves as terminal conditions for the inward spiral.

3.2.3 Control Scheme

The neighboring optimum control scheme presented here results in a linear feedback control law which attempts to meet the terminal conditions while maximizing the payoff. Because the fourth state variable, time (t), is not constrained and does not enter into the equations of motion, it can be neglected in the formulation of the optimum guidance problem (even though it is the quantity to be minimized). Therefore, the changes in the state (δx) and the changes in the adjoint ($\delta\lambda$) are both three-vectors, and the change in the control ($\delta\alpha$) can be written

$$\delta\alpha = c^T \delta x = c_1 \delta v_\theta + c_2 \delta v_r + c_3 \delta\theta$$

where c is a three-vector of guidance coefficients in the linear feedback control law. The guidance coefficients c are chosen so as to satisfy linear perturbations on the conditions for an optimum trajectory.

In particular, from the optimizing condition in Equation (2.8), the change in control is a linear function of the change in the current value of the adjoint variables.

$$\delta\alpha = \frac{-\lambda_2 \delta\lambda_1 + \lambda_1 \delta\lambda_2}{\lambda_1^2 + \lambda_2^2} = \alpha_\lambda \delta\lambda \tag{2.13}$$

The effect of changes in the current variables on the variables at the original final point can be calculated numerically by perturbing the equations of the system (2.6) and (2.7) after using the optimizing condition in Equation (2.8) to eliminate the optimum control (α).

$$\begin{bmatrix} \delta x \\ \delta\lambda \end{bmatrix}_{r=r_f} = \begin{bmatrix} \Phi_{xx} & \Phi_{x\lambda} \\ \Phi_{\lambda x} & \Phi_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta\lambda \end{bmatrix} \qquad (2.14)$$

where the Φ's are 3x3 transition matrices. Linear perturbations on the terminal conditions and the terminal constraints result in three conditions on the values of the variables at the original terminal point which can be written

$$0 = (L_x\ \delta x + L_\lambda\ \delta\lambda)_{r=r_f} \qquad (2.15)$$

where the L's are 3x3 matrices.

For the outward spiral, the explicit form of the above relation can be written from linear perturbations of Equations (2.9) and (2.11) as a set of six equations for nine unknowns (δx, $\delta\lambda$, $\delta\nu_1$, $\delta\nu_2$, δr_f) and hence results in three conditions on the six variables of interest. The six equations are:

$$0 = \delta E = v_\theta\ \delta v_\theta + v_r\ \delta v_r + \frac{dE}{dr}\ \delta r_f$$

$$0 = \delta\theta_\infty = \frac{\partial \Theta}{\partial v_\theta}\ \delta v_\theta + \frac{\partial \Theta}{\partial v_r}\ \delta v_r + \frac{d}{dr}(\theta + \Theta)\ \delta r_f + \delta\theta$$

$$0 = -\delta\lambda_1 + \nu_1\ \delta v_\theta + \delta\nu_1\ v_\theta + \delta\nu_2\ \frac{\partial \Theta}{\partial v_\theta} + \frac{d}{dr}(-\lambda_1 + \nu_1\ v_\theta)\ \delta r_f$$

$$0 = -\delta\lambda_2 + \nu_1\ \delta v_r + \delta\nu_1\ v_r + \delta\nu_2\ \frac{\partial \Theta}{\partial v_r} + \frac{d}{dr}(-\lambda_1 + \nu_1\ v_\theta)\ \delta r_f$$

$$0 = -\delta\lambda_3 + \delta\nu_2$$

$$0 = \frac{\delta v_r}{v_r^2} + \nu_1\ \delta\left(\frac{dE}{dr}\right) + \delta\nu_1\ \frac{dE}{dr} + \delta\nu_2\ \frac{d}{dr}(\theta + \Theta)$$
$$+ \frac{d}{dr}\left(-\frac{1}{v_r} + \nu_1 \frac{dE}{dr}\right)\delta r_f \qquad (2.16)$$

with the nominal values $\nu_2 = \lambda_3 = 0$.

Combining Equations (2.14), (2.15), and (2.16) and solving for the control ($\delta\alpha$) in terms of the state (δx) yields the guidance coefficients

$$\delta\alpha = -\alpha_\lambda (L_x \Phi_{x\lambda} + L_\lambda \Phi_{\lambda\lambda})^{-1} \cdot (L_x \Phi_{xx} + L_\lambda \Phi_{\lambda x}) \delta x \qquad (2.17)$$

For the inward spiral, the terminal conditions (2.15) include the trivial condition $\lambda_3 = 0$ as well as two non-trivial conditions,

$$0 = \delta v_\theta$$

$$0 = \delta v_r \qquad \text{at} \quad r=r_f$$

The same analysis as above holds. Because the third state variable, the angular position (θ), is not constrained and does not enter into the equations of motion, it can be neglected in the formulation of the inward guidance problem. In this case, δx and $\delta\lambda$ are both two vectors instead of three vectors.

3.3 Outward Spiral

In this section the optimum trajectory for the outward spiral and the associated guidance coefficients are presented for a typical set of initial and final conditions and the analytic solution for the far-planet region is outlined. For the purpose of simplifying the presentation, the units of all variables will be normalized so that both the acceleration of the vehicle (a) and the gravitational constant of the planet (μ) are unity. Therefore, when the (normalized) radial distance is unity, the acceleration of the vehicle will be equal to the gravitational attraction of the planet; the circular velocity will be unity; and the period for a circular orbit at that distance will be $t = 2\pi$. Table 1 presents a comparison between normalized and conventional units for the planet Earth using two values for the vehicle acceleration.

Table 1

TWO SETS OF NORMALIZED UNITS FOR EARTH

a	0.1 cm/sec^2	0.4 cm/sec^2
r=1	400,000 mi	200,000 mi
v=1	2600 ft/sec	3700 ft/sec
t=1	9.3 days	3.3 days
v_∞ (for r=10)	11,600 ft/sec	16,500 ft/sec
v_∞ (for r=100)	36,800 ft/sec	52,400 ft/sec

For the remainder of this chapter, these normalized units will be used instead of conventional units. In general, the relation between the variables using conventional units (represented by superscript asterisk, i.e., r^*) and the normalized values is shown below.

$$r^* = a^{-\frac{1}{2}} \mu^{\frac{1}{2}} r$$

$$t^* = a^{-3/4} \mu^{\frac{1}{4}} t$$

$$v_\theta^* = a^{\frac{1}{4}} \mu^{\frac{1}{4}} v_\theta$$

$$v_r^* = a^{\frac{1}{4}} \mu^{\frac{1}{4}} v_r$$

$$E^* = a^{\frac{1}{2}} \mu^{\frac{1}{2}} E$$

The angular position θ is in radians.

3.3.1 Original Trajectory

In Figure 1 the last few revolutions of a typical (optimum) outward spiral are presented. Because the acceleration due to the planet gravity varies inversely as the radial distance (r) while the vehicle acceleration remains constant, the characteristics of the optimum trajectory can best be understood by dividing the trajectory into the three regions discussed in the introduction: (1) the near-planet (or small r) region in which the gravity acceleration is much larger than the vehicle acceleration and the trajectory consists of a large number of revolutions while the vehicle gradually spirals away from the planet; (2) the transition region (perhaps from $r = .3$ to $r = 2$), where the two accelerations are comparable and the trajectory spiral straightens out to approach a hyperbola; and (3) the far-planet (or large r) region, where the gravity acceleration is much less than the vehicle acceleration and the vehicle is propelled to a specified energy level while asymptotically approaching a specific angular direction on a near hyperbola. The actual shape of the trajectory in the transition region is relatively independent of both the initial and final conditions.

The optimum control angle (α) for the same trajectory is presented in Figure 2 (with initial conditions at $r = .02$ and terminal conditions at $r = 10$). The dashed line represents the angle made by the tangent to the motion. At cutoff the thrust direction coincides with the direction of motion. The behavior of the control in the small r region is determined from an approximate analytic solution which is discussed more fully in Section 3.4. The optimum control in the small r region consists of a non-periodic term growing as r squared plus two separate oscillations of period 2π in the angular position. The first oscillation, which is the dominant one shown in Figure 2, grows approximately as r. The second oscillation, which is due to the initial conditions and the dynamics of the motion, decays approximately as the square root of r.

3.3.2 Guidance Coefficients

The optimum guidance coefficients (c_1, c_2, c_3) for the outward spiral, calculated according to Section 3.2, are presented in Figure 3 where the change in control ($\delta\alpha$) is written

$$\delta\alpha = c_1 \, \delta v_\theta + c_2 \, \delta v_r + c_3 \, \delta\theta \qquad (3.1)$$

During the entire trajectory, the guidance is relatively insensitive to changes in the radial velocity (δv_r). These coefficients are called fixed-r guidance coefficients because r is considered to be the independent variable and the changes in both the control and the independent variables are calculated as the difference between the current values and the values on the original optimum trajectory compared for the same radial distance r.

Under certain conditions it might be desirable to use the energy (E) as the independent variable. For instance, one of the conditions for the r solution to be valid is that the variable r be monotonic. For very small r it is possible for oscillatory terms to arise in the radial velocity (v_r) so that the radial velocity changes sign and the radial distance is not monotonic. Using the monotonic variable energy as the independent variable circumvents this difficulty. Fortunately, the fixed-energy guidance coefficients (with energy as the independent variable) can be calculated directly from the fixed-r coefficients without going through a new solution. In order to distinguish between the two sets of variables, the notation convention that will be followed in this chapter is that the changes with the energy (E) as the independent variable will use E as the argument while the changes with r as the independent variable will, in general, still have no argument. Therefore, the optimum fixed-energy guidance coefficients will be written

$$\delta\alpha(E) = c_1(E)\ \delta v_\theta(E) + c_2(E)\ \delta v_r(E) + c_3(E)\ \delta\theta(E) \tag{3.2}$$

The relation between $\delta\alpha$, δx, and $\delta\alpha(E)$, $\delta x(E)$ is

$$\begin{aligned}
\delta\alpha &= \delta\alpha(E) - \frac{d\alpha}{dr}\,\delta r(E) \\
\delta v_\theta &= \delta v_\theta(E) - \frac{dv_\theta}{dr}\,\delta r(E) \\
\delta v_r &= \delta v_r(E) - \frac{dv_r}{dr}\,\delta r(E) \\
\delta\theta &= \delta\theta(E) - \frac{d\theta}{dr}\,\delta r(E)
\end{aligned} \tag{3.3}$$

where $\delta r(E) = -r^2 \left[v_\theta\ \delta v_\theta(E) + v_r\ \delta v_r(E) \right]$.

Substituting Equation (3.3) into Equation (3.1) and comparing terms with Equation (3.2) yields the relation between the two sets of coefficients

$$\begin{aligned}
c_1(E) &= c_1 + kv_\theta \\
c_2(E) &= c_2 + kv_r \\
c_3(E) &= c_3
\end{aligned}$$

where $k = \left(c_1 \frac{dv_\theta}{dr} + c_2 \frac{dv_r}{dr} + c_3 \frac{d\theta}{dr} - \frac{d\alpha}{dr} \right) r^2$ (3.4)

The fixed-energy guidance coefficients for the outward spiral are presented in Figure 4. Notice that in the small r region the large oscillations of the fixed-r guidance coefficient c_1 are not present in the fixed-energy coefficient, $c_1(E)$.

3.3.3 Analytic Solution in the Far-Planet Region

An approximate solution for the state and adjoint variables has been derived which is valid for large values of radial distance (r). The solution contains the first few terms of a power series in $r^{-\frac{1}{2}}$ which satisfies the differential equations of the system (2.6)-(2.8). It contains a set of five arbitrary constants (three of which are assumed to be small) which can be adjusted to meet five of the initial and final conditions. The sixth condition is the current angular position (θ).

In the original optimum trajectory, when the angular position is not constrained, the adjoint variable corresponding to the angle is zero ($\lambda_3 = 0$), and the terminal conditions are satisfied when the three small constants are zero. When the angular position is constrained, one of the small constants is equal to $\delta\lambda_3 = \lambda_3$, while two of the terminal conditions are met by adjusting the other two constants. It turns out that the latter two constants are much smaller than λ_3 so that they can be neglected in the remainder of the analysis.

The approximate analytic solution for the variables is*

$$
\begin{aligned}
v_\theta &= a_o\, r^{-\frac{1}{2}}\,(1 + a_2/r) + \lambda_3 \\
v_r &= 2^{\frac{1}{2}}\, r^{\frac{1}{2}}\,(1 + a_2/r) - a_o\, 2^{\frac{1}{2}}\, \lambda_3\, r^{-1} \\
\theta &= \int_{r_f}^{r} (a_o\, 2^{-\frac{1}{2}}\, r^{-2} + \lambda_3\, 2^{\frac{1}{2}}\, r^{-\frac{1}{2}})dr + \theta_f \\
\lambda_1 &= a_o\, 2^{-\frac{1}{2}}\, r^{-1} + 2^{\frac{1}{2}}\, \lambda_3\, r^{-\frac{1}{2}} \\
\lambda_2 &= 1 - 3/4\, a_o\, r^{-3/2}\, \lambda_3
\end{aligned}
\qquad (3.5)
$$

where a_o and a_2 are arbitrary constants and λ_3 is assumed to be small. The angular distance the vehicle travels after the engine is turned off (Θ) can be approximated by the ratio between the circumferential velocity and the

*It can be shown that Equation (3.5) satisfies the first few terms of the equations of motion and the terminal conditions.

radial velocity at the point the engine is turned off,

$$\Theta_f \cong \left(\frac{v_\theta}{v_r}\right)_f \cong 2^{-\frac{1}{2}} a_o r^{-1} + \lambda_3$$

so the constrained asymptotic angular distance (θ_∞) can be written in terms of the other variables as

$$\theta_\infty = \Theta_f + \theta_f = a_o 2^{-\frac{1}{2}} r^{-1} + 2^{\frac{1}{2}} \lambda_3 r^{-\frac{1}{2}}$$

$$- 2^{-\frac{1}{2}} \lambda_3 r_f^{-\frac{1}{2}} + \theta$$

$$= \lambda_1 \left(1 - \sqrt{\frac{r}{r_f}}\right) + \sqrt{\frac{r}{r_f}} \left(\frac{v_\theta}{v_r}\right) + \theta$$

The control angle (α) is equal to $\tan^{-1}(\lambda_2/\lambda_1)$ so, for α near $\pi/2$ radians, the change in control can be written in terms of the changes in the state variables as

$$\delta\alpha = -\delta\lambda_1 = \frac{\delta\theta + \sqrt{\frac{r}{r_f}}\ \delta\left(\frac{v_\theta}{v_r}\right)}{1 - \sqrt{\frac{r}{r_f}}} \tag{3.6}$$

Thus, the fixed-r guidance coefficients can be determined analytically from Equations (3.5) and (3.6). For the original optimum trajectory in Figure 1, the values of the two arbitrary constants are

$$a_o = 1.28$$

$$a_2 = -.18$$

The fixed-energy guidance coefficients can be derived from the fixed-r ones in Equation (3.6) using the transformation in Equation (3.4). For the large r analytic solution the function k is small so the two sets of analytic coefficients are essentially the same.

$$c_1(E) \cong c_1$$

$$c_2(E) \cong c_2$$

$$c_3(E) \cong c_3$$

The numerical results in Figures 3 and 4 justify this for r greater than about 2. For the trajectories which have been examined, both the analytic solution and the analytic guidance coefficients seem to be adequate for r greater than two or three.

3.4 Inward Spiral

In this section the optimum turn-on point for the inward spiral is derived and the guidance coefficients are presented for a typical set of initial and final conditions. An approximate analytic solution is presented for the small r region and the guidance coefficients are derived in closed form. With the initial and final conditions reversed, the outward spiral and the optimum control in Figures 1 and 2 can be considered as the original optimum trajectory and control for the inward spiral. The only difference between the inward and outward trajectories is that the signs of the velocity components are reversed (v_θ and v_r are negative for the inward spiral).

3.4.1 Engine Turn-On

There is no angular dependency in the inward spiral, so only two independent variables are needed for guidance; for instance, the two components of velocity or, what is more convenient when considering engine turn-on, the radial distance (r) and the angular momentum ($h = rv_\theta$). Before the engine is turned on, the energy (E_o) and the angular momentum (h_o) are both constant. It will be assumed that the energy of the incoming vehicle is the same as the energy on the original optimum; thus, if the angular momentum is the same too, the optimum radial distance to turn on the engine will also be the same. Therefore, the problem is to choose the change in the radial distance of turn-on (δr_o) as a function of the change in the initial angular momentum (δh_o) so as to minimize the additional loss. Furthermore, it would be desirable to make use of the numerical computation scheme which has been set up to calculate the fixed-r guidance coefficients.

The notation introduced here is that $\lambda_h(E)$ and $\lambda_r(E)$ are the adjoint variables (corresponding to h and r) which give the sensitivity of the payoff J to changes in h and r when the energy E is fixed. The first variation of the loss is zero so the expression which must be minimized involves the second variation. The reason the first variation is zero is that the sensitivities $\lambda_h(E_o)$ and $\lambda_r(E_o)$ are zero at the start of the original trajectory because that trajectory optimized the h_o and r_o for fixed E_o. Therefore, the loss due to small changes in h and r is of second order and is given by Equation (4.1).

$$\delta^2 J\ (E_o) = \tfrac{1}{2}\ \delta\lambda_h\ (E_o)\ \delta h_o + \tfrac{1}{2}\ \delta\lambda_r\ (E_o)\ \delta r_o \qquad (4.1)$$

The computation scheme is set up to use r as the independent variable in calculating the effect of small changes in the variables at the initial turn-on point (δv_θ , δv_r , $\delta\lambda_1$, $\delta\lambda_2$) on the variables at the final point as outlined in Equation (2.14). In other words, we do this by perturbing the equations of the system (2.6) and (2.7) after using the optimizing condition in Equation (2.8) to eliminate the control. The terminal constraints are circular velocity at the final radial distance (r_f). Hence, in order to meet the terminal constraints, the changes in the velocity components at the final radial distance must be zero.

$$0 = \Phi_{xx}\ \delta x_o + \Phi_{x\lambda}\ \delta\lambda_o \qquad (4.2)$$

where the Φ's are 2x2 transition matrices. The fixed-r changes are now expressed in terms of the fixed energy changes. From the definition of angular momentum and the relation between the two sets of variables as presented in Equation (3.3), the changes in the variables can be written

$$\begin{bmatrix} \delta v_\theta \\ \delta v_r \end{bmatrix}_{r=r_o} = P \begin{bmatrix} \delta h_o \\ \delta r_o \end{bmatrix}$$

where

$$P = \left[\begin{array}{c:c} 1 + \frac{dv_\theta}{dr} r^2 v_\theta & \frac{dv_\theta}{dr} r^2 v_\theta \\ \hdashline \frac{dv_r}{dr} r^2 v_r & 1+ \frac{dv_r}{dr} r^2 v_r \end{array}\right] \left[\begin{array}{c:c} \frac{1}{r} & - \frac{v_\theta}{r} \\ \hdashline - \frac{v_\theta}{rv_r} & \frac{v_\theta^2 r-1}{r^2 v_r} \end{array}\right]_{r=r_o} \tag{4.3}$$

and where the left 2x2 matrix arises from change of independent variable from E to r. One of the properties of adjoint variables is that in matrix notation they transform as the inverse transpose of the state variables. (This property is due to the fact that the change in the loss must be the same no matter which set of variables is used.) Therefore, the relation between adjoint variables can be written as

$$\begin{bmatrix} \delta\lambda_h \ (E_o) \\ \delta\lambda_r \ (E_o) \end{bmatrix} = P^T \begin{bmatrix} \delta\lambda_1 \\ \delta\lambda_2 \end{bmatrix}_{r=r_o} \tag{4.4}$$

Solving Equations (4.2) - (4.4) for the relation between the two fixed-energy state variables and the two fixed-energy adjoint variables yields

$$\begin{bmatrix} \delta\lambda_h \ (E_o) \\ \delta\lambda_r \ (E_o0) \end{bmatrix} = Q \begin{bmatrix} \delta h_o \\ \delta r_o \end{bmatrix}$$

$$\text{where } Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} = -P^T \left[\Phi_{\lambda x}\right]^{-1} \Phi_{xx} P \tag{4.5}$$

Substituting Equation (4.5) into the expression for the loss in Equation (4.1) gives the change in loss as a quadratic function of the changes in the angular momentum and the radial distance (δh_o and δr_o). The optimum change in the radial distance for turn-on (δr_{opt}) minimizes the quadratic function and both δr_{opt} and the corresponding additional loss can be calculated in terms of δh_o.

$$\delta^2 J\ (E_o) = \tfrac{1}{2}\, q_{11}\ \delta h_o^{\ 2} + \tfrac{1}{2}\ (q_{12} + q_{21})\ \delta h_o\ \delta r_o + \tfrac{1}{2}\, q_{22}\ \delta r_o^{\ 2}$$

$$\delta r_{opt} \quad = \quad - \tfrac{1}{2}\ (q_{12} + q_{21})\ \delta h_o / q_{22}$$

$$\text{Minimum } \delta^2 J\ (E_o) = \tfrac{1}{2} \left[q_{11} - \tfrac{1}{4}\ (q_{12} + q_{21})^2 / q_{22} \right] \delta h_o^{\ 2} \tag{4.6}$$

The optimum radial distance for turn-on (r_{opt}) and the corresponding loss are plotted in Figure 5 as a function of angular momentum for the incoming spiral of Figure 1. For comparison, the total normalized time of the spiral in Figure 1 is 9.6 units.

3.4.2 Guidance Coefficients

For the inward spiral the guidance depends only on the two velocity components

$$\delta\alpha \quad = \quad c_1\ \delta v_\theta + c_2\ \delta v_r$$

The fixed-r guidance coefficients, calculated according to Section 3.2, are presented in Figure 6. Both coefficients are nearly constant in the large r region. In the small r region the coefficient c_1 has absolute value less than unity and oscillates around zero, while the coefficient c_2 is much larger, decreasing to a minimum of one and one-half before rising at the end. The gain increases indefinitely only at the very end of the inward spiral.

The fixed-energy guidance coefficients are presented in Figure 7. The coefficient $c_2(E)$ starts out very large (above 100) because near r=10 about 99 percent of the total energy is in the radial component of velocity. Thus, if one is comparing two trajectories with the same energy and the same circumferential velocity, but different radial velocities, there must be a large difference in r which would call for a large change in control.

3.4.3 Analytic Solution in the Near-Planet Region

The approximate solution in the small r region includes a general solution to the equations of the system (2.6)-(2.8), which is a power series in r that is well behaved near the origin (as r goes to zero), plus a particular solution. The analytic solution which has been obtained is

$$v_\theta = r^{-\frac{1}{2}} \left(1 + \frac{5}{2} r^4 + r^2 \Delta v_\theta\right)$$

$$v_r = 2 r^{3/2} \left(1 - 23 r^4 + \Delta v_r\right)$$

$$\lambda_1 = 1 + \frac{5}{2} r^4 + r^2 \Delta\lambda_1$$

$$\lambda_2 = r^2 \left(1 - 33 r^4 + \Delta\lambda_2\right)$$

where

$$\Delta v_\theta = \sum_{i=1}^{2} B_i r^{-mi} \sin(-\tfrac{1}{4} r^{-2} + \beta_i) \left[1 + \left(\frac{61}{2} - 12 m_i\right) \lambda_3 r^{-3/2}\right] + \lambda_3 r^{-3/2} r^2$$

$$\Delta v_r = -\sum_{i=1}^{2} B_i r^{-mi} \cos(-\tfrac{1}{4} r^{-2} + \beta_i) \left[1 + \left(\frac{61}{2} - 12 mi\right) \lambda_3 r^{-3/2}\right] - 4 \lambda_3 r^{-3/2} r^4$$

$$\Delta\lambda_1 = \sum_{i=1}^{2} (42 - 16 mi) B_i r^{-mi} \sin(-\tfrac{1}{4} r^{-2} + \beta_i) \left[1 + (5 - 2 mi) \lambda_3 r^{-3/2}\right] + \lambda_3 r^{-3/2} (-1 + 26 r^4) r^{-2}$$

$$\Delta\lambda_2 = -\sum_{i=1}^{2} (22 - 8 mi) B_i r^{-mi} \cos(-\tfrac{1}{4} r^{-2} + \beta_i) + \lambda_3 r^{-3/2} (-3 - 44 r^4)$$

$$m_1 = \frac{7 - \sqrt{10}}{4} \cong 0.959$$

$$m_2 = \frac{7 + \sqrt{10}}{4} \cong 2.541 \qquad (4.7)$$

and where B_1, B_2, and λ_3 are small constants and β_1 and β_2 are arbitrary constant angles.

Appendix A contains a derivation of the solution for the case where $\lambda_3 = 0$. Except for phase differences, β_i, the arguments of the sine and cosine terms $(-\frac{1}{4} r^{-2} + \beta_i)$ approximate the angular distance θ, so the oscillatory terms are periodic in θ with period 2π. The r dependence of the amplitude of the oscillatory terms was obtained by the method of averaging (see Appendix A). One of the oscillatory terms grows with r while the other decays. The solution has been checked numerically on the digital computer by integrating from r = .06 to r = .1 (about eight revolutions), and the theoretical and numerical values for the exponents m_1 and m_2 matched to within one percent. For most of the numerical results presented in this paper, numerical integration was used down to r = .1 and the analytic solution for r less than .1. For the original optimum trajectory in Figure 1, $\lambda_3 = 0$ and the values of the other four constants are

$$B_1 = 39 \times 10^{-3}$$

$$B_2 = 66 \times 10^{-6}$$

$$\beta_1 = 27.38 \text{ radians}$$

$$\beta_2 = 23.72 \text{ radians}$$

For the inward guidance the angle is not constrained, $\lambda_3 = 0$, and four constants $(\beta_1, \beta_2, B_1, B_2)$ determine the motion in Equation (4.7). Four conditions on the motion are the two current components of velocity (v_θ and v_r) and the two terminal components of velocity (circular velocity at r_f).

Therefore, the four constants can be calculated in terms of r_f and the current velocity components. Substituting that expression for the four constants into the adjoint variables ($\Delta\lambda_1$ and $\Delta\lambda_2$) in Equation (4.7) yields the latter in terms of the velocity components as shown in Equation (4.8).

$$\lambda_1 = 1 + (14 + 4\sqrt{10})\,(r^{\frac{1}{2}} v_\theta - 1) + 8\sqrt{10}\;\frac{(r^{\frac{1}{2}} v_\theta - 1)\,\rho^{m_2 - m_1} + r^2 \sin\Delta}{1 - \rho^{m_2 - m_1}}$$

$$\lambda_2 = r^2 + (4 + \sqrt{10})\,(r^{\frac{1}{2}} v_r - 2r^2) + 2\sqrt{10}\;\frac{(r^{\frac{1}{2}} v_r - 2r^2)\,\rho^{m_2 - m_1} + r^2 \cos\Delta}{1 - \rho^{m_2 - m_1}}$$

where $\rho = \dfrac{r_f}{r} < 1$

$$\Delta = -\tfrac{1}{4}\,(r_f^{-\frac{1}{4}} - r^{-\frac{1}{4}}) \tag{4.8}$$

Because the adjoint variables are known, both the optimum control and the guidance coefficients can be written in closed form. The guidance coefficients are obtained by differentiating the expression for the inverse tangent

$$\delta\alpha = \frac{\lambda_1\,\delta\lambda_2 - \lambda_2\,\delta\lambda_1}{(\lambda_1^2 + \lambda_2^2)} \tag{4.9}$$

From Equations (4.8) and (4.9) the approximate guidance coefficients can be determined.

$$c_1 = r^{5/2}\left[\,14 + 4\sqrt{10} + \frac{8\sqrt{10}\,\rho^{m_2 - m_1}}{(1 - \rho^{m_2 - m_1})}\,\right]$$

$$c_2 = r^{\frac{1}{2}}\left[\,4 + \sqrt{10} + \frac{2\sqrt{10}\,\rho^{m_2 - m_1}}{(1 - \rho^{m_2 - m_1})}\right] \tag{4.10}$$

For small r the coefficient c_2 is much larger than c_1, i.e., the optimum guidance is dependent primarily on the radial velocity component.

The fixed-energy guidance coefficients can be derived from the fixed-r ones by using the transformation in Equation (3.4) and the result (for small r) is

$$c_1(E) = \tfrac{1}{2} c_1$$

$$c_2(E) = c_2$$

It is interesting to compare the analytic solution for the optimum trajectory in Equation (4.7) with the analytic solution for a family of non-optimum trajectories suggested by Reference (5). Let the control angle of the non-optimum trajectory be given by $\tan^{-1}(sv_r/v_\theta)$ where s is a parameter which can take on values such as zero (for circumferential acceleration) and one (for tangential acceleration). Using the same technique as in Appendix A, it can be shown that the approximate analytic solution for this non-optimum trajectory is

$$v_\theta = r^{-\frac{1}{2}} \left[1 + (3 - s)\, r^4 + r^2\, \Delta v_\theta^* \right]$$

$$v_r = 2r^{3/2} \left[1 + (-27 + 9s - 2s^2)\, r^4 + \Delta v_r^* \right]$$

$$\Delta v_\theta^* = Br^{-m} \sin(-\tfrac{1}{4} r^{-2} + \beta)$$

$$\Delta v_r^* = -Br^{-m} \cos(-\tfrac{1}{4} r^{-2} + \beta)$$

$$m = \frac{11-s}{4}$$

where B is a small constant and β is an arbitrary constant angle.

For the case of tangential acceleration, the non-optimum solution is similar to that of Reference (5) which uses θ as the independent variable. For the case where s is one-half, the root m is 2.5 which is very close to the root $m_2 \cong 2.541$ in Equation (4.7). The non-optimum solution is not used in this report, but it is introduced here to show how the dynamics of the optimum solution compare with a non-optimum solution.

3.5 Engine Misbehavior and Measurement Errors

The guidance coefficients derived in the previous sections are based on a vehicle with constant acceleration when the deviations from the nominal are known exactly. This section shows how the guidance law must be modified to account for both long-term fluctuations in acceleration and errors in measuring the terminal value of the state. A method is presented for calculating the additional loss due to measurement errors, and numerical results are given for a simplified model of the system.

3.5.1 Modified Guidance Law for Engine Misbehavior

Before one can talk about a guidance law accounting for engine fluctuations, it must be assumed that the estimated future acceleration deviations from nominal are proportional to the present deviation, i.e., for $r' \geq r$.

$$\delta a(r') = k(r', r)\, \delta a(r) \qquad (5.1)$$

For example, the estimated acceleration change represented by $k(r', r)$ could be exponentially decaying with time or it could be constant. The modified guidance law will have the form

$$\delta\alpha = c_1\, \delta v_\theta + c_2\, \delta v_r + c_3\, \delta\theta + c_4\, \delta a \qquad (5.2)$$

where the first three coefficients are the same as before and the fourth allows for the expected deviations in acceleration. The derivation of the modified guidance law is based on the original derivation of the neighboring optimum control scheme in Section 3.2 and uses much of the same notation. The effect of small changes in acceleration (δa) on the state and adjoint variables can be calculated by perturbing the equations of the system (2.6) and (2.7) after using the optimizing condition Equation (2.8) to eliminate the control. In particular, the variations of the state and adjoint variables at the original final point (r_f) are given by

$$\begin{bmatrix} \delta x \\ \delta \lambda \end{bmatrix}_{r_f} = \begin{bmatrix} \Phi_{xx} & \Phi_{x\lambda} \\ \Phi_{\lambda x} & \Phi_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta \lambda \end{bmatrix}$$

$$+ \int_{r}^{r_f} \begin{bmatrix} \Phi_{xx} & \Phi_{x\lambda} \\ \Phi_{\lambda x} & \Phi_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} \dfrac{\partial g}{\partial a} \\ \left(\dfrac{\partial^2 g}{\partial x \partial a} \right)^T \lambda \end{bmatrix} \delta a(r')\, dr' \qquad (5.3)$$

where the notation of Equation (2.14) has been used.

When the future deviations in acceleration $\delta a(r')$ are proportional to the current deviations as in Equation (5.1), the last term in Equation (5.3) can be expressed as a linear function of $\delta a(r)$:

$$\int_{r}^{r_f} \begin{bmatrix} \Phi_{xx} & \Phi_{x\lambda} \\ \Phi_{\lambda x} & \Phi_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} \dfrac{\partial g}{\partial a} \\ \left(\dfrac{\partial^2 g}{\partial x \partial a} \right)^T \lambda \end{bmatrix} k(r', r)\, dr'\, \delta a(r)$$

$$= \begin{bmatrix} x_a \\ \lambda_a \end{bmatrix} \delta a(r) \qquad (5.4)$$

The terminal constraints allowing for $\delta a(r_f)$ become

$$0 = (L_x\ \delta x + L_\lambda\ \delta_\lambda + L_a\ \delta a)_{r_f} \qquad (5.5)$$

in the notation of Equation (2.15) where L_a is a 2x1 vector which gives the effect of changes in the final value of acceleration on the terminal conditions. For the inward spiral, L_a is zero because the terminal conditions are not a function of acceleration. The change in control which is needed to meet the terminal constraints is calculated as in Equation (2.17) and it is given by

$$\delta\alpha = -\alpha_\lambda \left(L_x \Phi_{x\lambda} + L_\lambda \Phi_{\lambda\lambda}\right)^{-1} \cdot \left[\left(L_x \Phi_{xx} + L_\lambda \Phi_{\lambda x}\right) \delta x + \left(L_x x_a + L_\lambda \lambda_a + L_a\right) \delta a\right] \tag{5.6}$$

which we note has the form of Equation (5.2).

The modified guidance coefficient c_4 has not been calculated numerically, but, for short term fluctuations in acceleration, a very rough approximation to the coefficient can be obtained relatively simply. If the acceleration deviation lasts for a short time, the expected future change in velocity components ($\delta\hat{v}_\theta$ and $\delta\hat{v}_r$) can be approximated by the integral of the expected acceleration deviation

$$\delta\hat{v}_\theta = \int_r^{r_f} \frac{\cos\alpha}{v_r}\, k\,(r', r)\, dr'\ \delta a(r)$$

$$\delta\hat{v}_r = \int_r^{r_f} \frac{\cos\alpha}{v_r}\, k\,(r', r)\, dr'\ \delta a(r)$$

Thus, one might say that an acceleration deviation $\delta a(r)$ which lasts for a short time has approximately the same effect as deviations in the velocity components equal to $\delta\hat{v}_\theta$ and $\delta\hat{v}_r$. Under this assumption, a rough approximation to the guidance coefficient c_4 would be

$$c_4 = c_1 \int_r^{r_f} \frac{\cos\alpha}{v_r}\, k(r', r)\, dr' + c_2 \int_r^{r_f} \frac{\cos\alpha}{v_r}\, k(r', r)\, dr'$$

For example, if the expected acceleration decayed exponentially with time

$$k(r', r) = e^{-\gamma\,(t'-t)} \tag{5.7}$$

The approximate guidance coefficient c_4 would be

$$c_4 = \frac{c_1}{\gamma} \cos\alpha + \frac{c_2}{\gamma} \sin\alpha \tag{5.8}$$

If the guidance coefficients oscillate, the average values of c_1, c_2 and α should be used in Equation (5.8).

The guidance coefficient c_4 can be calculated for any fluctuation in acceleration which can be predicted. For the special case where the engine has a constant acceleration bias, a different approach may be more appropriate. Normalized variables were explained in Section 3.3 and for most of the results it has been assumed that all variables have been normalized so that the original (constant) acceleration of the vehicle is unity. If the acceleration changes to a new constant value, all the variables can be renormalized to the new value. For instance, assume that it was suddenly discovered that the engine acceleration was eight percent higher than the original value. The new current (normalized) values of the state variables would be

r	4% higher
v_θ	2% lower
v_r	2% lower
E	4% lower

Furthermore, the new (normalized) terminal energy would be four percent lower than on the original optimum trajectory. The change in terminal distance (δr_f) would be given by

$$\delta r_f = \delta E \Big/ \frac{dE}{dr}$$

In the early part of the spiral, the change in the control due to the change in the final terminal position can be calculated in the same way as before; however, near the later part of the spiral, both the original optimum trajectory and the guidance coefficients can be written in closed form, so it is only necessary to change r_f in Equation (3.6) for the outward spiral, or Equation (4.10) for the inward spiral, to obtain the new guidance coefficients.

3.5.2 Modified Guidance with Terminal Errors

For the outward spiral there is an additional source of loss because terminal errors in the outgoing spiral lead to a non-optimum spiral entry at the target planet and, hence, cause an added fuel expenditure on the incoming spiral as explained in Section 3.4. Let h_θ represent the sensitivity of the angular momentum at the arrival planet (h_o) to changes in the asymptotic angular direction (θ_∞) at the departure planet so that

$$\delta h_o = h_\theta \, \delta\theta_\infty$$

From Equation (4.6) the change in the payoff is

$$\delta^2 J = -\tfrac{1}{2} k \, (\delta\theta_\infty)^2 \tag{5.9}$$

where

$$k = \left[q_{11} - \tfrac{1}{4} (q_{12} + q_{21})^2 / q_{22} \right] h_\theta^2$$

It is possible to modify the guidance law of the outgoing spiral to take into account the additional loss in Equation (5.9). The payoff function J in Equation (2.10) will be modified from

$$J = -t + \nu_1 E + \nu_2 \theta_\infty \Big|_{r=r_f}$$

to

$$J = -t + \nu_1 E - \tfrac{1}{2} k \, (\theta_\infty - \theta_{\infty f})^2 \Big|_{r=r_f}$$

where $\theta_{\infty f}$ is the asymptotic angular direction of the original optimum trajectory when the angle is not constrained. The analysis proceeds as before except that instead of $\delta\theta_\infty$ being constrained to zero,

$$\delta\theta_\infty = -\delta\lambda_3 / k \tag{5.10}$$

The analytic expression for the guidance coefficients in the large r region is derived in Equation (3.6) which is

$$\theta_\infty = \lambda_1 (1 - \rho^{-\frac{1}{2}}) + \rho^{-\frac{1}{2}} (v_\theta/v_r) + \theta \qquad (5.11)$$

where $\rho = r_f/r \geq 1$.

Also, from the analytic solution in the large r region, an approximate expression for the adjoint variable λ_3 is

$$\lambda_3 = 2^{\frac{1}{2}} r^{\frac{1}{2}} (\lambda_1 - v_\theta/v_r) \qquad (5.12)$$

From Equations (5.10)-(5.12) the change in control can be written in terms of changes in the state variables as

$$\delta\alpha = -\delta\lambda_1 = \frac{\delta\theta + (\rho^{-\frac{1}{2}} - 2^{\frac{1}{2}} r^{\frac{1}{2}}/k)\ \delta(v_\theta/v_r)}{(1 - \rho^{-\frac{1}{2}} + 2^{\frac{1}{2}} r^{\frac{1}{2}}/k)} \qquad (5.13)$$

where $\rho = r_f/r \geq 1$

For large k, the modified guidance law in Equation (5.13) only deviates from the original one in Equation (3.6) near engine cutoff, and it reduces to the original one as k approaches infinity.

The actual values of both h_θ and k will depend on the particular trip. For instance, for a low energy 180-day trip from Earth to Mars in 1975, excess velocities at departure and arrival are about 25,000 ft/sec and the value of h_θ is about 10^9 $(\text{mi})^2/\text{sec}$. For this trip the normalized value of k for leaving the planet Earth with $a = 0.4$ cm/sec^2 would be about 10^4.

For the inward spiral there is no additional loss due to terminal errors, but it is possible to reformulate the optimization problem so as to specify the covariance of the terminal errors. We know[6] that including additional constant multipliers in the loss function is equivalent to minimizing the loss function subject to constraints on the covariance of the terminal error. In particular, for the inward spiral, the payoff J is modified from

$$J = -t + \nu_1 v_\theta + \nu_2 v_r$$

$$\left.\begin{aligned} v_\theta &= r^{-\frac{1}{2}} \\ v_r &= 0 \\ \lambda_1 &= \nu_1 \\ \lambda_2 &= \nu_2 \end{aligned}\right|_{r=r_f}$$

to

$$\left.\begin{aligned} J &= -t + \tfrac{1}{2} k_1 \, (v_\theta - r^{-\frac{1}{2}})^2 + \tfrac{1}{2} k_2 \, (v_r)^2 \\ \lambda_1 &= k_1 \, (v_\theta - r^{-\frac{1}{2}}) \\ \lambda_2 &= k_2 \, (v_r) \end{aligned}\right|_{r=r_f} \qquad (5.14)$$

where k_1 and k_2 are the two constant multipliers.

The terminal errors (i.e., the deviation from circular velocity at the terminal point) are due to errors in estimating the terminal value of the state as well as the fact that the control may not drive the estimated terminal error to zero. Therefore, the allowed terminal errors must be greater than the covariance of the errors in estimating the terminal state, and they will increase as the two constant multipliers decrease.

In the small r region the optimum control and the guidance coefficients for the modified payoff in Equation (5.14) can be obtained in closed form using the same approach as in Section 3.4. For the case where $k_1 = 4\, k_2$ the new guidance coefficients can be written in a particularly simple form, namely,

$$c_1 = r^{5/2} \, (14 + 4\sqrt{10}) \, k(\rho) + r^{5/2} \, \frac{8\sqrt{10} \, \rho^{m_2-m_1}}{(1 - \rho^{m_2-m_1} + 8\sqrt{10} \, r_f^{\frac{1}{2}}/k_1)}$$

$$c_2 = r^{\frac{1}{2}} \, (4 + \sqrt{10}) \, k(\rho) + r^{\frac{1}{2}} \, \frac{2\sqrt{10} \, \rho^{m_2-m_1}}{(1 - p^{m_2-m_1} + 8\sqrt{10} \, r_f^{\frac{1}{2}}/k_1)}$$

$$k(\rho) = \frac{(1 - \rho^{m_2 - m_1})^2}{(1 - \rho^{m_2-m_1} + 8\sqrt{10}\, r_f^{\frac{1}{2}}/k_1)\,(1 - \rho^{m_2-m_1} + 8\sqrt{10}\, r_f^{\frac{1}{2}}\, \rho^{m_2-m_1} / k_1)}$$

$$\rho = \frac{r}{r_f} \leq 1 \qquad (5.15)$$

For large k_1 the modified guidance coefficients in Equation (5.15) deviate from the original ones in Equation (4.10) only near engine cutoff, and they reduce to the original ones as k_1 approaches infinity. In the limit, as r approaches r_f, the guidance coefficients in Equation (5.15) become

$$c_1 = r_f^2 k_1$$
$$c_2 = \tfrac{1}{4} k_1 = k_2$$

3.5.3 Loss Due to State Measurement Errors

The original trajectory is optimized with respect to the control angle (α), so that, with nominal engine performance, all control schemes which meet the terminal conditions give the same loss to first order. To second order, the change in payoff between the optimum control scheme (α_{opt}) and any other control scheme (α) is equal to the integral of a known function (the second partial derivative of the Hamiltonian with respect to the control) times the square of the difference in control from the optimum.

$$\delta^2 J = \int_{r_o}^{r_f} \frac{1}{2} \frac{\partial^2 H}{\partial \alpha^2} (\alpha - \alpha_{opt})^2 \, dr \qquad (5.16)$$

where $\delta^2 J$ is the second order change in payoff.

When the optimum guidance law derived here is used,

$$\delta\alpha = c_1 \delta v_\theta + c_2 \delta v_r + c_3 \delta\theta$$

the only difference, to second order, between the optimum control and the actual control is due to errors in estimating the state variables used in the guidance law. The expected value of the second order change in payoff is a negative definite quadratic function of the covariance of the errors in estimating the state. The conditional mean is the estimate which minimizes the covariance of the error, so it is the estimate which should be used. The expected value of the change in payoff, $E\left[\delta^2 J\right]$, is

$$E\left[\delta^2 J\right] = \int_{r_o}^{r_f} \tfrac{1}{2} \frac{\partial^2 H}{\partial \alpha^2} \operatorname{cov} (c_1 \delta v_\theta + c_2 \delta v_r + c_3 \delta\theta) \, dr \tag{5.17}$$

where $\frac{\partial^2 H}{\partial \alpha^2} = -(\lambda_1{}^2 + \lambda_2{}^2)^{\frac{1}{2}}/v_r$ and cov means the covariance of the error in estimation.

When there are errors in measuring the independent variable (r), there will be an additional difference between the optimum control and the actual control because of the error in calculating the original optimum control (α_{opt}). In that case, the expected value of the change in payoff in Equation (5.17) is changed to

$$E\left[\delta^2 J\right] = \int_{r_o}^{r_f} \tfrac{1}{2} \frac{\partial^2 H}{\partial \alpha^2} \operatorname{cov} (c_1 \delta v_\theta + c_2 \delta v_r + c_3 \delta\theta - \frac{d\alpha}{dr} \delta r) \; dr \tag{5.18}$$

When there are fluctuations in engine acceleration, the loss due to control is still second order, but there may be a first order change in the loss function which is independent of control. The actual loss is the total mass expenditure, so the loss depends on the mass flow as well as the elapsed time. Because the mass flow ($\dot{m}$) may be a function of acceleration, i.e., $\dot{m} = \dot{m}$ (a, t), changing the acceleration may result in a first order change in the mass expenditure.

$$\delta J = \delta \left[\int_{t_o}^{t_f} \dot{m} \, (a, t) \, dt \right]$$

The second order loss due to control can still be calculated using Equation (5.16), but when there are fluctuations in engine acceleration, the optimum control (α_{opt}) must allow for all the fluctuations. If the future acceleration deviations are proportional to the current deviations as shown in Equation (5.1), and the modified optimum guidance law in Equation (5.2) is used, the expected value of the additional loss includes the effect of estimating the current acceleration deviation, and it is given by

$$E\left[\delta^2 J\right] = \int_{r_o}^{r_f} \frac{1}{2} \frac{\partial^2 H}{\partial \alpha^2} \operatorname{cov}\left(c_1\ \delta v_\theta + c_2\ \delta v_r + c_3\ \delta\theta + c_4\ \delta a\right) dr \tag{5.19}$$

If there are statistical fluctuations in the acceleration which cannot be predicted accurately, these fluctuations introduce a second source of loss. In deriving the modified optimum guidance law, the coefficient c_4 was calculated under the assumption that the estimated future acceleration deviation $\delta a(r')$ was $k(r', r)$ times the current deviation $\delta a(r)$. Because of statistical fluctuations, the difference, $\delta a(r')-k(r', r)\ \delta a(r)$, will have a probability distribution (with zero mean). Therefore, in addition to Equation (5.19), there will be a second source of loss given by

$$E\left[\delta^2\ J\right] = \int_{r}^{r_f} \frac{1}{2} \frac{\partial^2 H}{\partial \alpha^2} \operatorname{cov}\left(\delta\tilde{\alpha}\right) dt$$

where the notation in Equation (5.3) is used, and

$$\delta\tilde{\alpha} = -\alpha_\lambda \left(L_x\ \Phi_{x\lambda} + L_\lambda\ \Phi_{\lambda\lambda}\right)^{-1} \cdot \left[L_x\ \delta\tilde{x} + L_\lambda\ \delta\tilde{\lambda} + L_a\ \delta a(r_f) - L_a\ k(r_f, r)\ \delta a(r)\right]$$

with

$$\begin{bmatrix} \delta\tilde{x} \\ \delta\tilde{\lambda} \end{bmatrix} = \int_r^{r_f} \begin{bmatrix} \Phi_{xx} & \Phi_{x\lambda} \\ \Phi_{\lambda x} & \Phi_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} \dfrac{\partial g}{\partial a} \\ \dfrac{\partial^2 g^T}{\partial x \partial a}\ \lambda \end{bmatrix} \left[\delta a(r') - k(r',r)\ \delta a(r)\right] dr'$$

Notice that the second source of loss can be calculated numerically if $\operatorname{cov}\left[\delta a(r') - k(r', r)\ \delta a(r)\right]$ is a known function.

This covariance depends only on the statistical properties of the engine fluctuations and not on observation errors. Thus, statistical fluctuations in engine acceleration introduce an additional loss which cannot be corrected by making more accurate observations. In the steady state example given in the following section, only losses due to Equation (5.19) will be considered.

3.5.4 Example with Steady State Errors

When there are both measurement errors and stochastic fluctuations in acceleration, it is possible for there to be a balance between the gain in information due to additional measurements and the loss in information due to engine fluctuations. A simplified model of the system will be used here to represent error propagation and to determine the steady state balance. For this simplified model, the additional loss (burning time) due to measurement errors will be calculated numerically using representative numerical values for engine fluctuations and measurement accuracy and rate.

It is assumed that the deviations in acceleration are exponentially correlated so that

$$E\left[\delta a(t')\ \delta a(t)\right] = \frac{q}{2\gamma}\ e^{-\gamma(t'-t)} \tag{5.20}$$

where t' is greater than t and $1/\gamma$ could be considered the correlation time. For the purpose of estimation, time will be the independent variable so the equations of motion in Equation (2.1) can be written

$$\begin{aligned} r\frac{d}{dt}\theta &= v_\theta \\ \frac{d}{dt}(rv_\theta) &= ra\cos\alpha \\ \frac{d}{dt}r &= v_r \\ \frac{d}{dt}v_r &= \frac{v_\theta^2 - 1/r}{r} + a\sin\alpha \end{aligned} \tag{5.21}$$

A three-dimensional version of the estimation problem has been derived for a near-Earth satellite in Reference 7. The fluctuations in acceleration there are due to stochastic fluctuations in atmospheric drag. In that paper there are eight state variables; a set of six mean orbit elements which give position and velocity, as well as two additional state variables which represent the average value of the acceleration and the instantaneous deviation in acceleration (δa). For the estimation problem considered here, the following, much simpler model of the system will be used.

The equations of motion are

$$r \frac{d}{dt} \delta\theta = \delta v_\theta$$

$$\frac{d}{dt} \delta v_\theta = \delta a$$

$$\frac{d}{dt} \delta a = - \gamma \delta a + u \tag{5.22}$$

where u is a white noise process so that

$$E\left[u(t')\ u(t)\right] = q\ \delta^* \ (t'-t)$$

where $\delta^*(t'-t)$ is the Dirac delta function.

Noisy measurements of angular position $\delta\theta$ are made every τ units of normalized time with root-mean-square accuracy σ_M radians. The system in Equation (5.22) represents circumferential motion where $r\delta\theta$, δv_θ, δa are position, velocity, and acceleration, respectively. Furthermore, it is assumed that (1) the radial distance r varies slowly and is known very accurately so that it need not be estimated, and (2) the covariance in radial velocity is assumed to be equal to the covariance in circumferential velocity but uncorrelated with it so that

$$\mathrm{cov}\left[\delta v_r\right] = \mathrm{cov}\left[\delta v_\theta\right]$$

In the small r region the above assumptions are not unreasonable, although the error in estimating radial velocity should probably be somewhat less than that in circumferential velocity. These assumptions are not valid in the large r region, but the obtained estimate of the state covariance will be extended to the large r region anyway.

Let P be the 3x3 matrix which represents the covariance of the errors in estimating the state $r\delta\theta$, δv_θ, δa. The solution to the steady state version of the estimation problem is obtained from the matrix differential equation representing the propagation of errors.[8,9]

$$0 = \frac{dP}{dt} = FP + PF^T - P S^{-1} P + Q \qquad (5.23)$$

where, for the model presented here,

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\gamma \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} s^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$s = r^2 \sigma_M^2 \tau$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q \end{bmatrix}$$

and the discrete measurements have been approximated by an equivalent set of continuous measurements. An approximate solution to Equation (5.23) when $\gamma(s/q)^{1/6}$ is somewhat smaller than 1* is

*The assumption $\gamma(s/q)^{1/6}$ is much smaller than 1 corresponds to assuming that many relatively accurate measurements are taken during the time $1/\gamma$.

$$p_{11} = 2\, s^{5/6}\, q^{1/6} \qquad p_{22} = 3\, s^{\frac{1}{2}}\, q^{\frac{1}{2}}$$
$$p_{12} = 2\, s^{2/3}\, q^{1/3} \qquad p_{23} = 2\, s^{1/3}\, q^{2/3}$$
$$p_{13} = s^{\frac{1}{2}}\, q^{\frac{1}{2}} \qquad p_{33} = 2\, s^{1/6}\, q^{5/6}$$
$$p_{ij} = p_{ji} \tag{5.24}$$

where p_{ij} are the components of the matrix P. Because p_{11} is the covariance of $r\delta\theta$, it must be divided by r^2 to obtain the covariance of $\delta\theta$. Hence, the covariance of the errors in $\delta\theta$, δv_θ, δa are

$$\mathrm{cov}(\delta\theta) = p_{11}/r^2 \qquad \mathrm{cov}(\delta v_\theta) = p_{22}$$
$$\mathrm{cov}(\delta\theta,\ \delta v_\theta) = p_{12}/r \qquad \mathrm{cov}(\delta v_\theta\ \delta a) = p_{23}$$
$$\mathrm{cov}(\delta\theta,\ \delta a) = p_{13}/r \qquad \mathrm{cov}(\delta a) = p_{33} \tag{5.25}$$

To obtain numerical values for the additional loss due to measurement errors, let

$$s = 10^{-9}\, r^2$$
$$q = 10^{-3}$$

The value for s corresponds to measurements with a root-mean-square accuracy of 10^{-3} radians taken once each 10^{-3} normalized units of time. (From Table 1 for the planet Earth with engine acceleration 0.4 cm/sec^2, this is about once each seven minutes.) The value for q corresponds to a steady state root-mean-square deviation in acceleration of about two percent with $1/\gamma$ equal to one normalized unit of time. (From Table 1, this means that it takes about 3.3 days for a particular deviation in acceleration to die out.)

The additional loss will be calculated numerically from Equation (5.19) which is repeated here.

$$E\left[\delta^2 J\right] = \int_r^{r_f} \frac{1}{2}\ \frac{\partial^2 H}{\partial\alpha^2}\ \mathrm{cov}\ (c_1\ \delta v_\theta + c_2\ \delta v_r + c_3\ \delta\theta + c_4\ \delta a)\ dr$$

The guidance coefficients in Figure 3 are used in the outward spiral modified in the large r region by Equation (5.13) which includes an additional loss due to terminal errors. The guidance coefficient c_4 is approximated by Equation (5.8) and in the large r region c_4 rapidly goes to zero. The additional loss due to measurement errors is almost entirely due to errors in estimating the final angle θ_∞ and it is given by

$$E\left[\delta^2 J\right] = .75\ k \cdot 10^{-6}$$

where k is defined in Equation (5.9) and it relates the additional loss at the inward spiral due to terminal errors in the outward spiral.

$$\delta^2 J = \tfrac{1}{2} k \left(\delta\theta_\infty\right)^2$$

For $k = 10^4$ the additional normalized time due to errors in estimating the final angle θ_∞ is 7.5×10^{-3} which, for the case of leaving Earth with $a = 0.4$ cm/sec^2, is only thirty minutes and is extremely small in comparison with a typical burn time of thirty days.

For the inward spiral, the guidance coefficients in Figure 6 are used and in the small r region they are modified by Equation (5.15). The guidance coefficient c_4 is approximated by Equation (5.8). The additional loss due to measurement errors is approximately

$$E\left[\delta^2 J\right] = 7 \cdot 10^{-3} + 4\ k_2 \cdot 10^{-7}$$

where the first term is primarily due to errors in estimating δa in the large r region and the second term (which includes the constant multiplier k_2) is due to errors in estimating the terminal value of v_r in the small r region. Errors in estimating δa are important in the large r region because changing the acceleration a changes the initial point at which the engine should be turned on. Therefore, if the initial value of a is not known because of measurement errors and engine fluctuations, the vehicle is coming in on a non-optimum trajectory in the large r region.

For this example, a maximum value for the constant multiplier k_2 might be about 10^3 because for that value the modified guidance coefficients in Equation (5.15) are essentially the same as the guidance coefficients in Figure 6 until less than one revolution before the engine is turned off. For $k_2 = 10^3$ the additional time on the inward spiral would also be about thirty minutes. Thus, as one might expect, for the model presented here and long term fluctuations in engine acceleration with rms value of about two percent, the additional loss due to estimation errors is quite small on both the outward spiral and the inward spiral.

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Appendix A

DERIVATION OF ANALYTIC SOLUTION FOR NEAR-PLANET REGION

The analytic solution for r will be derived for the case where λ_3 is zero. The differential equations which must be satisfied are presented in Equation (A.1) where λ_3 is zero and λ_4 is minus one.

$$\frac{dv_\theta}{dr} = -\frac{v_\theta}{r} + \frac{\lambda_1}{v_r (\lambda_1^2 + \lambda_2^2)^{\frac{1}{2}}}$$

$$\frac{dv_r}{dr} = \frac{v_\theta^2 - \frac{1}{r}}{rv_r} + \frac{\lambda_2}{v_r (\lambda_1^2 + \lambda_2^2)^{\frac{1}{2}}}$$

$$\frac{d\lambda_1}{dr} = \frac{\lambda_1}{r} - \frac{2v_\theta}{rv_r} \lambda_2$$

$$\frac{d\lambda_2}{dr} = \frac{(\lambda_1^2 + \lambda_2^2)^{\frac{1}{2}} - 1}{v_r^2} + \frac{v_\theta^2 - \frac{1}{r}}{rv_r^2} \tag{A.1}$$

The solution is assumed to be of the form given by Equation (A.2) where the Δ's are to be determined.

$$\begin{aligned} v_\theta &= r^{-\frac{1}{2}} (1 + \tfrac{5}{2} r^4 + r^2 \Delta v_\theta) \\ v_r &= 2 r^{3/2} (1 - 23 r^4 + \Delta v_r) \\ \lambda_1 &= 1 + \tfrac{5}{2} r^4 + r^2 \Delta\lambda_1 \\ \lambda_2 &= r^2 (1 - 33 r^4 + \Delta\lambda_2) \end{aligned} \tag{A.2}$$

When the Δ's are all zero, Equation (A.2) satisfies Equation (A.1) up to

r^4.* Substituting Equation (A.2) into Equation (A.1) and keeping terms of order Δ/r^3 and Δ/r yields Equation (A.3).

$$\begin{bmatrix} \Delta v'_\theta \\ \Delta v'_r \\ \Delta\lambda'_1 \\ \Delta\lambda'_2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2r} & -\frac{1}{2r^3} & 0 & 0 \\ \frac{1}{2r^3} & -\frac{3}{r} & 0 & \frac{1}{4r} \\ -\frac{1}{r} & +\frac{1}{r^3} & -\frac{1}{r} & -\frac{1}{r^3} \\ \frac{1}{2r^3} & -\frac{4}{r} & +\frac{1}{4r^3} & -\frac{1}{2r} \end{bmatrix} \begin{bmatrix} \Delta v_\theta \\ \Delta v_r \\ \Delta\lambda_1 \\ \Delta\lambda_2 \end{bmatrix} \tag{A.3}$$

where the prime indicates the derivative with respect to r.

There are also terms of order $\Delta\cdot\Delta/r^3$ which have been discarded. Because there are terms of order Δ/r^3 in Equation (A.3) and because $\frac{1}{2}/r^3$ approximates $d\theta/dr$, a solution containing sines and cosines suggests itself. In particular, a solution of the form of Equation (A.4), with the coefficients A_1, A_2, C_1 and C_2 (some slowly varying functions of r), satisfies the dominant part (Δ/r^3 terms) of Equation (A.3).

$$\begin{aligned} \Delta v_\theta &= A_1 \sin\theta + A_2 \cos\theta \\ \Delta v_r &= -A_1 \cos\theta + A_2 \sin\theta \\ \Delta\lambda_1 &= C_1 \sin\theta + C_2 \cos\theta \\ \Delta\lambda_2 &= -(A_1 + \frac{C_1}{2}) \cos\theta + (A_2 + \frac{C_2}{2}) \sin\theta \end{aligned} \tag{A.4}$$

More precisely, substituting Equation (A.4) into Equation (A.3) yields Equation (A.5).

$$A'_1 \sin\theta + A'_2 \cos\theta = -\frac{5}{2r} (A_1 \sin\theta + A_2 \cos\theta)$$

*The next terms in the expression for v_θ and λ_1 in Equation (A.2) are $-307\frac{1}{8}r^8$ and $-352\frac{7}{8}r^8$, respectively. Adding those two terms satisfies Equation (A.1) up to r^6 when the Δ's are zero.

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$$C_1 = \sum_{i=1}^{2} (42 - 16\, m_i)\, B_i\, r^{-m_i} \cos \beta_i$$

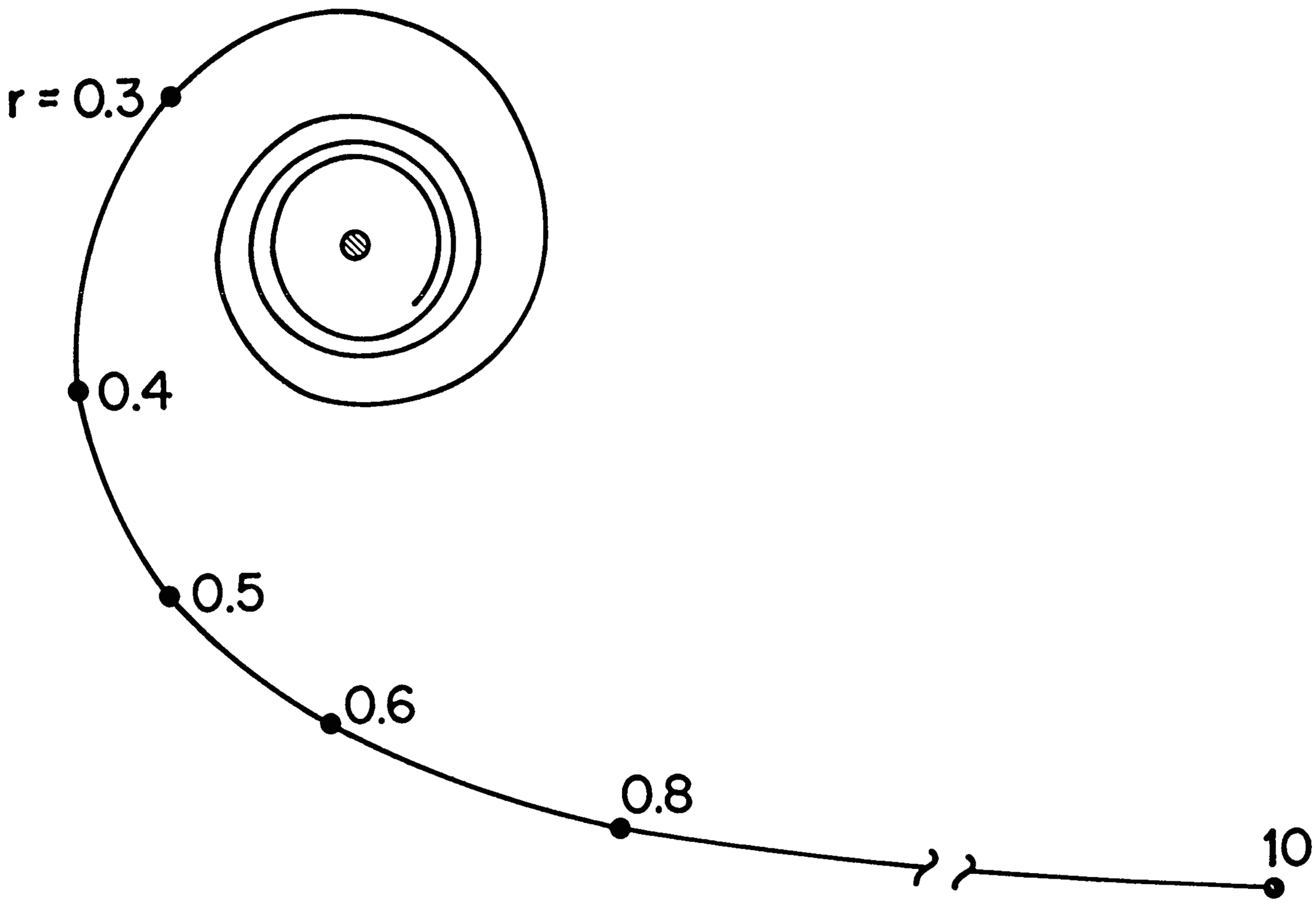
$$C_2 = \sum_{i=1}^{2} (42 - 16\, m_i)\, B_i\, r^{-m_i} \sin \beta_i \tag{A.7}$$

where

$$m_1 = \frac{7 - \sqrt{10}}{4}$$

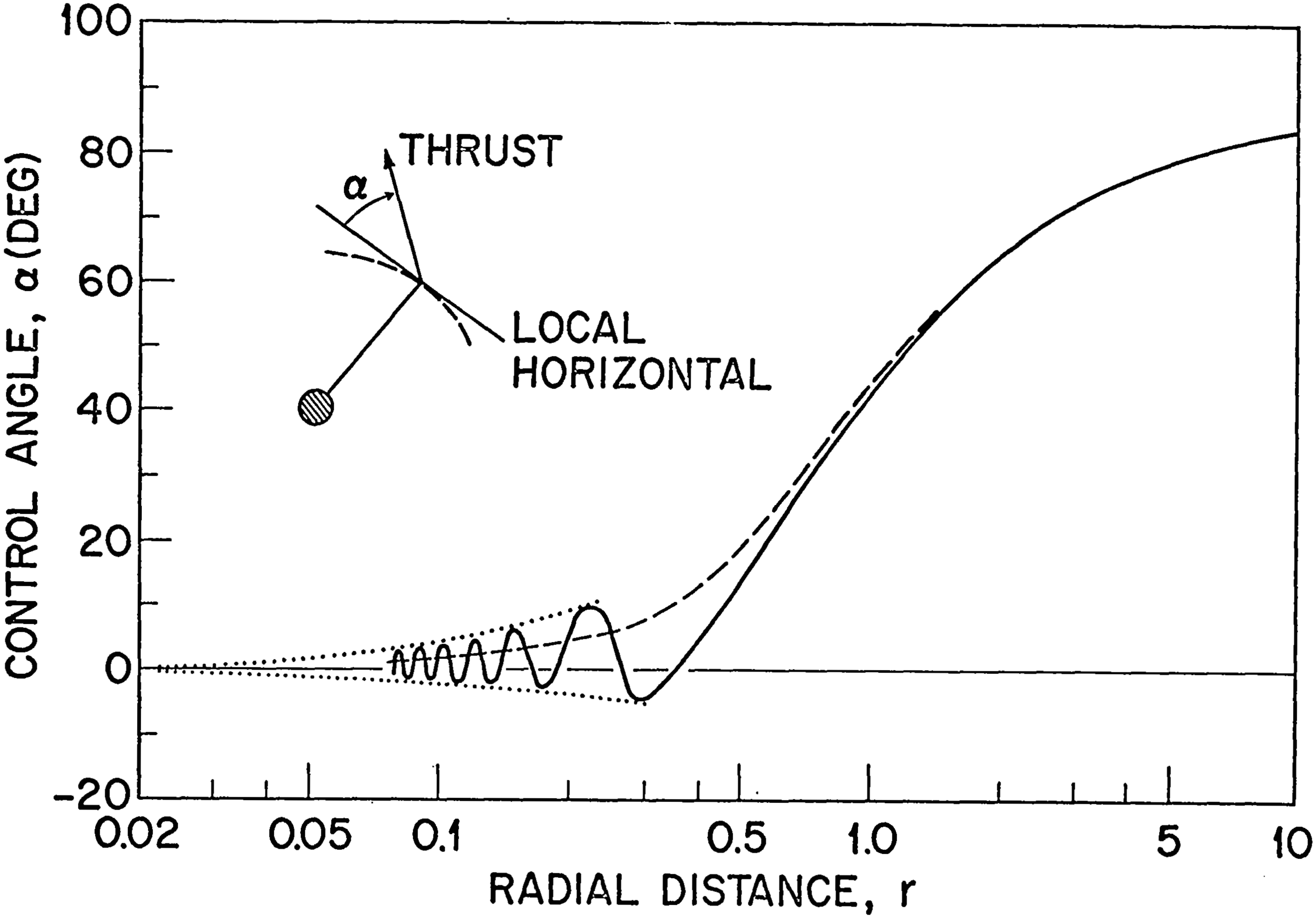
$$m_2 = \frac{7 + \sqrt{10}}{4} \tag{A.8}$$

Substituting Equation (A.7) into Equations (A.2) and (A.4) yields the analytic solution. For small non-zero λ_3, Equation (A.6) and the solution in Equation (A.7) are slightly perturbed.



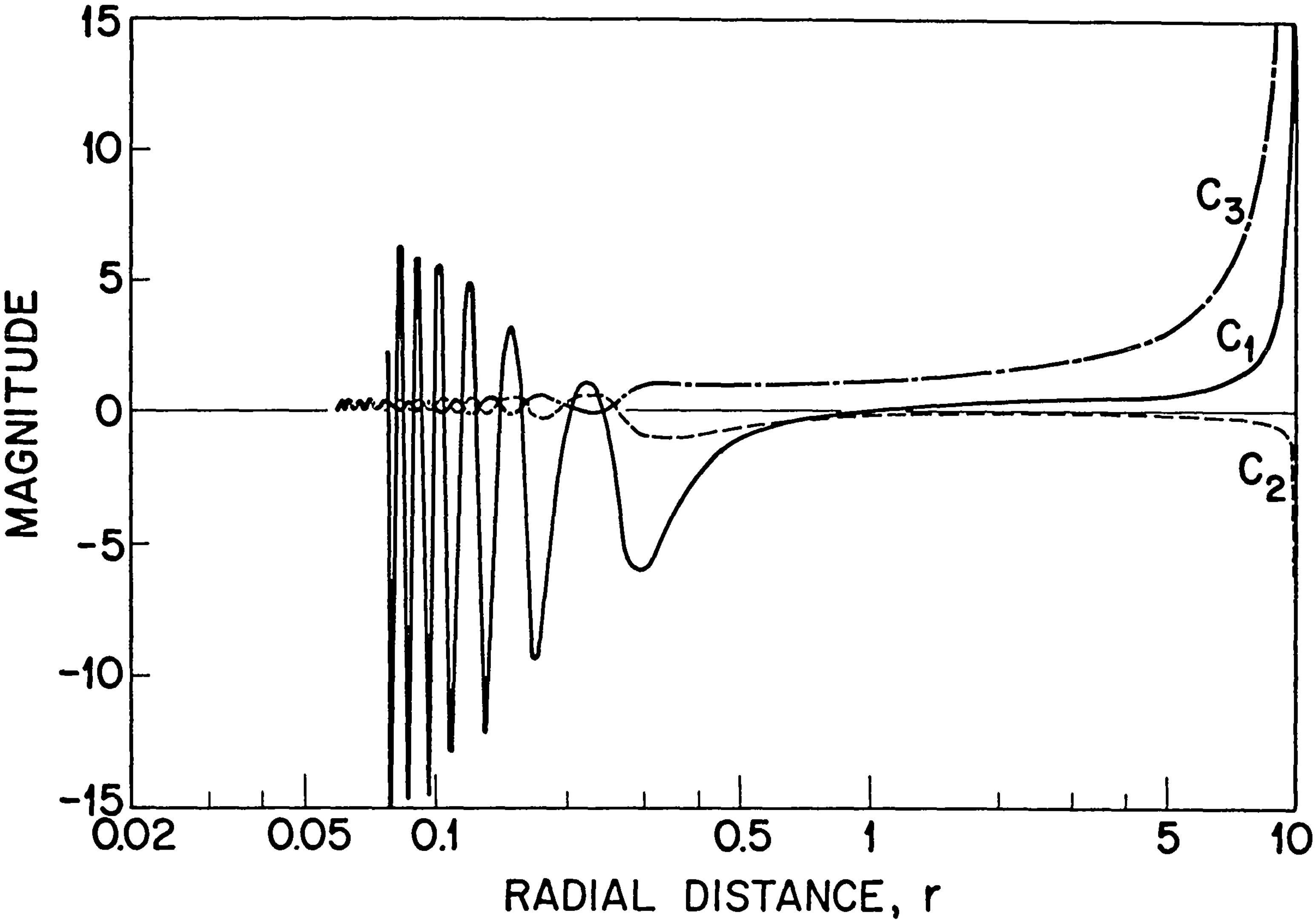
POLAR ORBIT OF OPTIMUM TRAJECTORY

Fig. 1

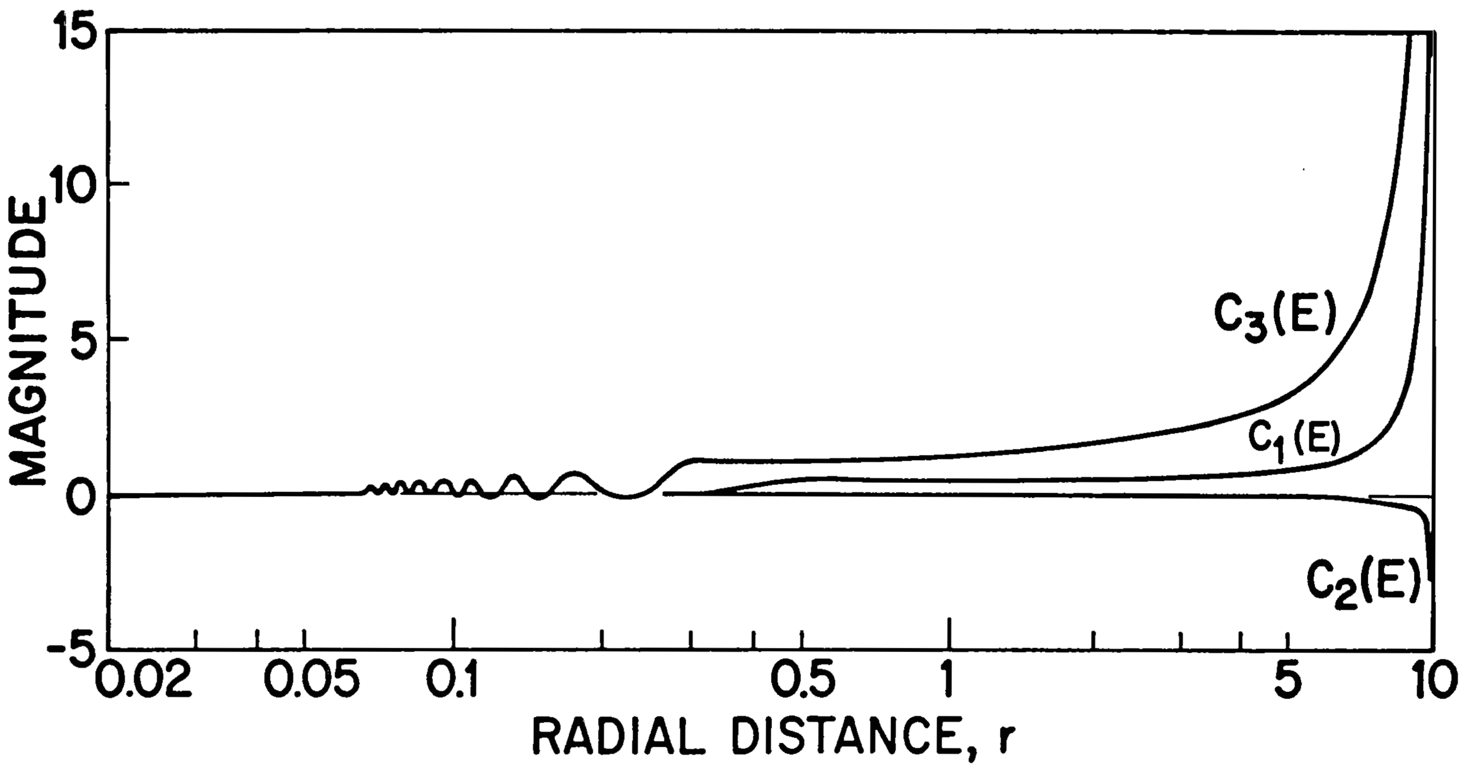


OPTIMUM THRUST ANGLE

Fig. 2

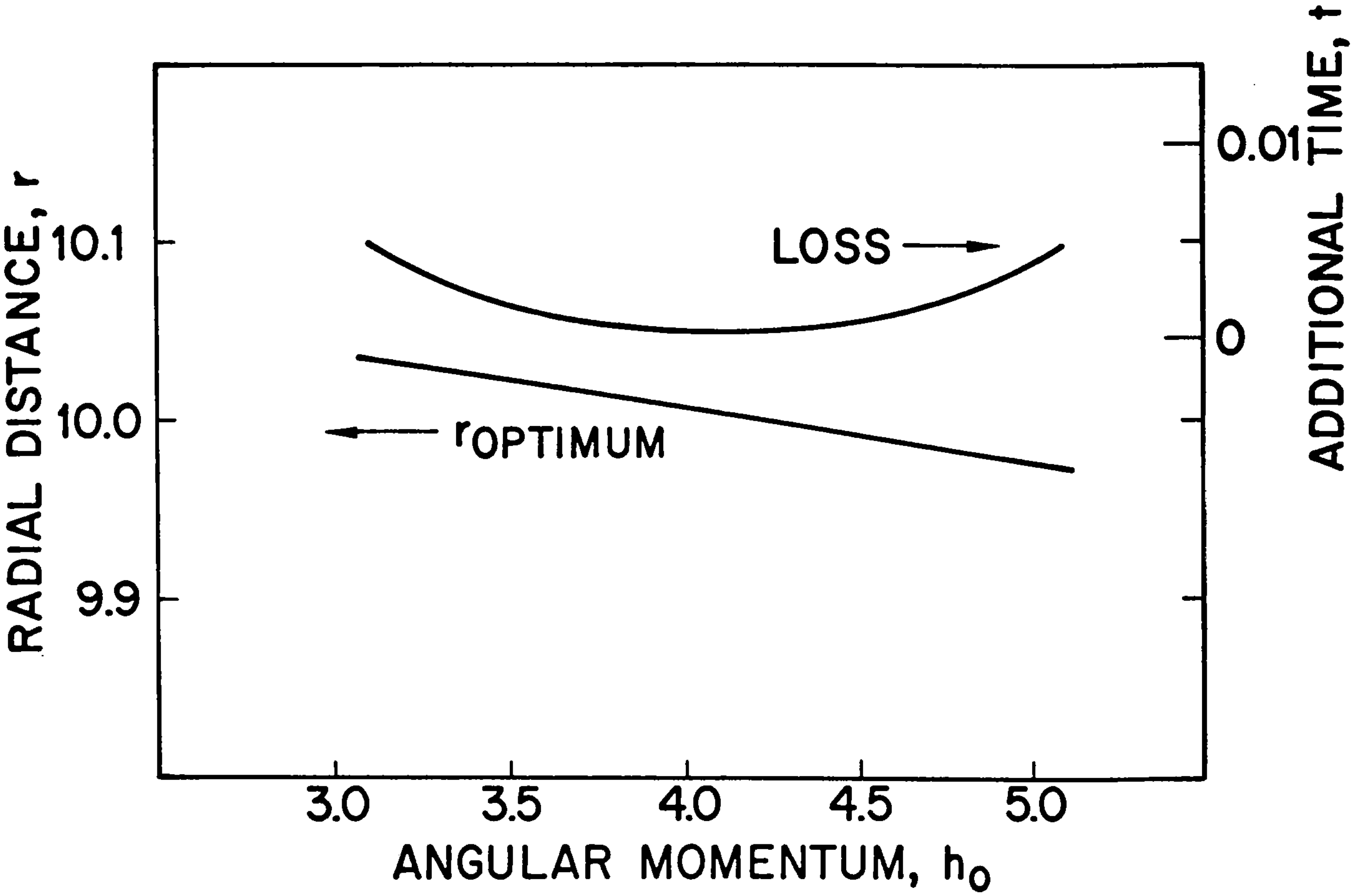


OUTWARD GUIDANCE
FIXED-r COEFFICIENTS



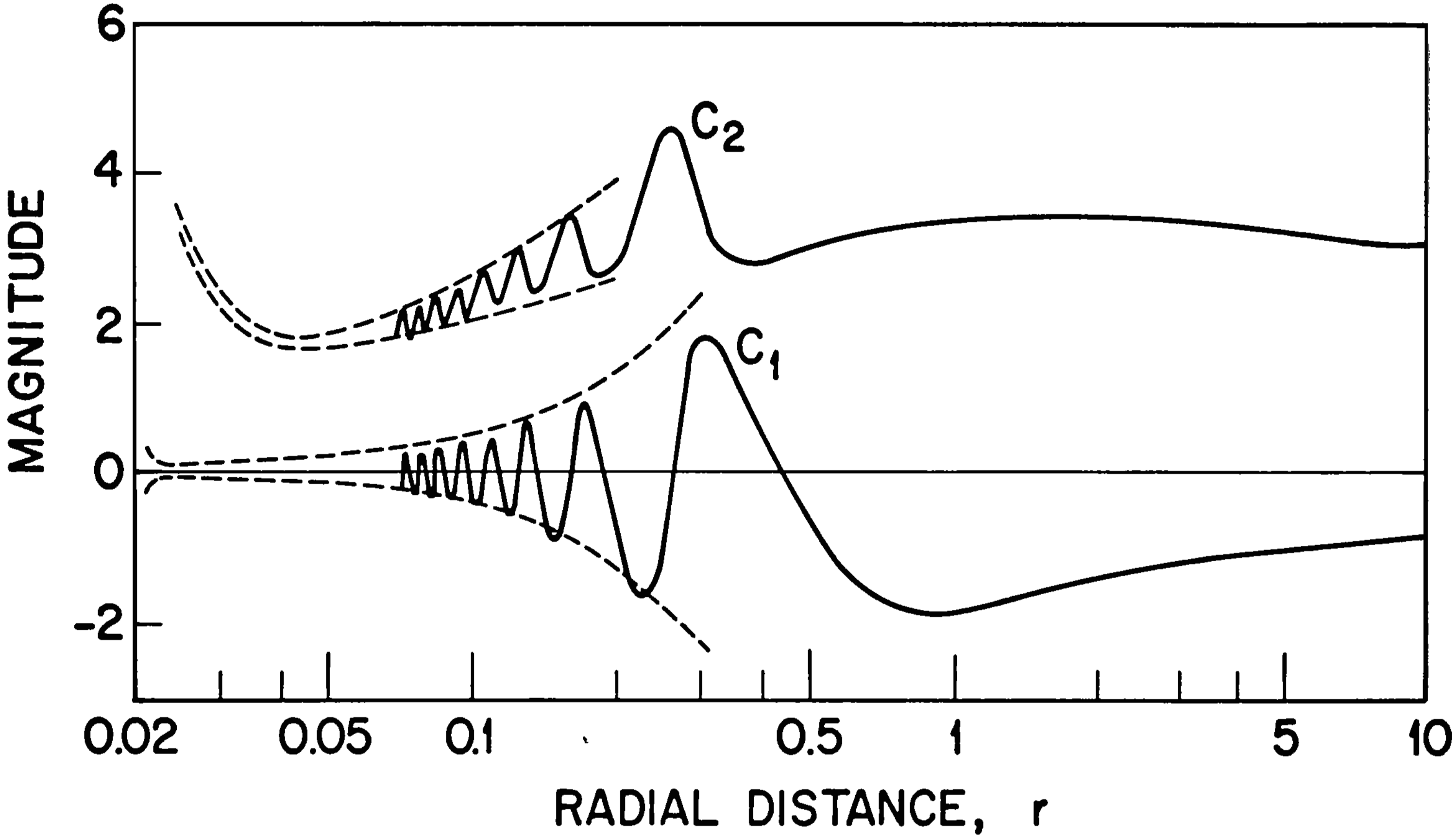
OUTWARD GUIDANCE
FIXED ENERGY COEFFICIENTS

Fig. 4



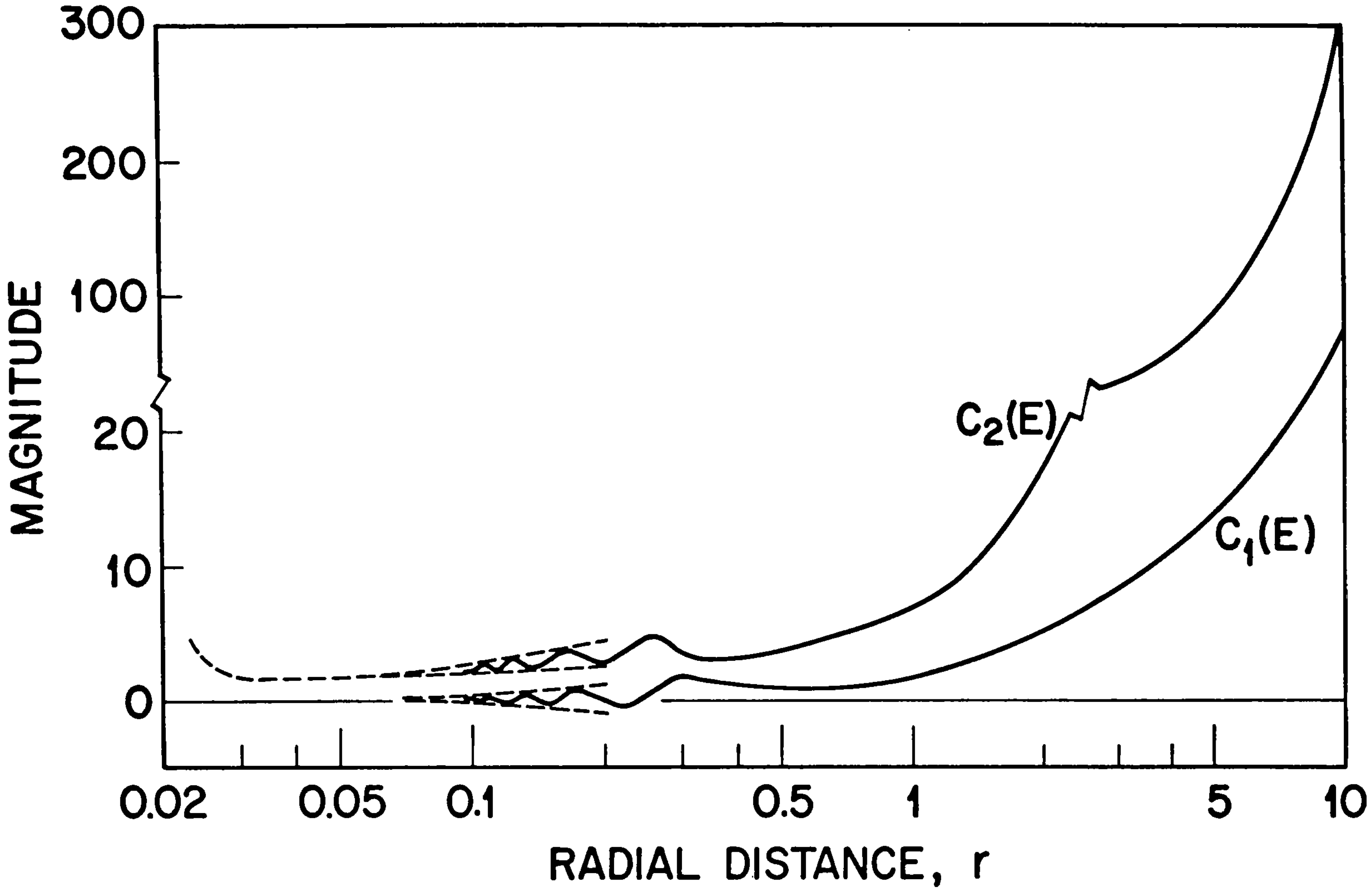
OPTIMUM ENGINE TURN-ON

Fig. 5



INWARD GUIDANCE
FIXED-r COEFFICIENTS

Fig. 6



INWARD GUIDANCE
FIXED ENERGY COEFFICIENTS

Fig. 7

4. SUMMARY

4.1 Summary of Results

4.1.1 Midcourse Guidance Using High-Thrust Engines (Chapter 2)

Our studies in this area have been concerned with extending in various ways the theory of minimum effort control developed by Breakwell and Striebel.[1] The various extensions and their results can be summarized as follows:

A) Control of Several Terminal Components (Section 5)

It is shown that the optimum corrective strategy for controlling independently the rms accuracies of the in-plane and the out-of-plane terminal positions as well as their velocities has essentially the same characteristic as the unidimensional problem of the basic minimum effort theory. The optimal control history consists of an initial period of no control, followed by a period of continuous control, and finally a period of no control with possibly an impulse at the end. The numerical computation requires the proper guessing of the ratio of the initial values of adjoint variables and, in general, an iterative procedure is necessary in order to satisfy the specified terminal accuracies.

B) Optimization of the Control Histories as well as the Observation Rate and the Selection Among a Choice of Observations (Section 7)

The problem considered here is to incorporate into the solution of the basic minimum effort theory the additional feature of selecting a most advantageous measurement among a choice of observations and the corresponding rate of observation subject to the following constraints: (1) a maximum observation rate and (2) a fixed total number of measurements. It is shown that the solution, in general, includes intermediate values of observation rates. The optimal policy consists of periods of measuring separated by periods of no measurement or corrective action. Each measurement period starts at maximum rate with a subperiod without corrective action. This is followed by a subperiod of gradual (continuous) correction and ends with an impulsive partial correction of the terminal miss distance. The measuring prior to the impulse may be either at maximum rate or at an intermediate lower rate. In addition, if a choice is available at any time between various possible measurements, the optimal solution automatically selects the one which is most advantageous.

C. Optimization of the Control Histories Including the Effect of Mechanization Errors (Section 8)

This study extends the basic minumum effort theory to include the case when random engine mechanization errors are taken into account. Included are the engine execution error, which is error in the magnitude of the velocity correction along the direction of the thrust, and the accelerometer reading error, which is an error in the knowledge of the actual amount of velocity correction used. The latter type of error causes a loss of information which increases the uncertainty of the orbit. A computational method is found for obtaining the solution of this problem assuming constant variances for both types of errors. The method is based on Dynamic Programming and leads to a solution which is in part analytical and in part computational. Typical results indicate that:

1) Optimum Corrective strategy is discrete when engine mechanization errors are included; hence, there exists an optimum number of corrections for a given size of mechanization error.

2) The improvement in velocity correction obtained using more than three or four corrections is negligibly small.

D. Optimum Nonlinear Control Strategy (Section 9)

One of the important simplifying assumptions in the development of the basic minimum effort theory is the requirement that the magnitude of the applied control acceleration at any time be linearly related (with a variable gain) to the predicted miss distance at that time. In the present study, this assumption is removed. For the sake of simplicity, only a discrete system has been considered. It is shown that a corrective velocity is applied if, and only if, the predicted miss distance at the time t_i lies above (or below) a certain number, say z_i (or $-z_i$) and that the effect of the control is to bring the predicted miss distance after the correction to z_i (or $-z_i$). A relatively simple computation procedure for obtaining z_i recursively based on the technique of Dynamic Programming is given. In addition, consideration has also been given to the hard constraint problem, in which the total amount of velocity correction capability is limited and specified in advance. The characteristic of the solution is very different. In this case, there exists two numbers, say $z_i^{**} > z_i^{*} > 0$, such that (1) no control is applied if the magnitude of the predicted miss distance is less than z_i^{*}, (2) all the velocity capability available is used if the magnitude of the predicted miss is greater than z_i^{**}, and (3) a certain intermediate control is applied otherwise. The z_i^{*} and z_i^{**} are functions of the available corrective velocity capability and appear to be obtainable only in tabulated form by using Dynamic Programming.

D. Computer Program for Studying Guidance Problems for Typical Interplanetary Trips* (Section 10)

Section 10 of Chapter 2 reported some related work in connection with the study of guidance requirements for typical interplanetary trips. A computer program is developed which (1) performs a linear error analysis of typical interplanetary trajectories with assumed rms injection errors and

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*This was not required by the contract.

measurement histories, and (2) computes a trajectory correction strategy based on the basic minimum effort theory. It includes a near-optimum discrete trajectory correction strategy using impulsive corrections whose spacings are chosen to approximate the ideal continuous strategy. The analysis of these near-optimum discrete strategies extends the study by a Monte Carlo simulation to include the effect of correction mechanization errors as well as the effect of varying the time of the last correction. Orbit determination is assumed to be based on the information obtained from onboard angular measurements as well as Earth-based radar. Computer results for two trips are given for typical injection errors indicating the total velocity correction as a function of the required rms accuracy for various information histories. The two trips are: (1) a 204-day trip to Mars, and (2) a 245-day swingby trip, Earth-Venus-Mars.

4.1.2 Planetocentric Guidance Using Low-Thrust Engines (Chapter 3)

The low-thrust portion of the work is concerned with the guidance of a vehicle with a constant impulse low-thrust engine while spiraling away from one planet, and later, in toward another planet. Guidance is achieved by changing the direction of thrust rather than the magnitude. The optimization problem - to reach a specified escape energy with minimum mass expenditure (minimum time, in this case) - is solved using the calculus of variations; the guidance problem is solved by using a neighboring optimum control scheme which generates a linear feedback control law which minimizes the mass expenditure while attempting to meet a terminal constraint on direction of escape. The terminal constraint for the subsequent arrival spiral is horizontal circular velocity at a desired radial distance. In particular, the results of the study may be summarized as follows:

- The optimum guidance coefficients for both the outward and inward spiral are derived. The guidance law using radial distance as the independent variable is transformed to one using energy as the independent variable.
- In both the near-planet and far-planet regions, approximate analytic solutions are derived for both the optimum and neighboring optimum trajectories and, in some cases, these permit the guidance coefficients to be written in analytic form.
- For the inward spiral, the optimum turn-on time is derived as a function of the radial distance and the angular momentum.
- The guidance law is modified to account for long term (predictable) fluctuations in engine acceleration and the guidance coefficients are adjusted to account for the effect of measurement errors in estimating the terminal quantities. The latter adjustment prevents the coefficients from becoming large without bound as the time of engine cutoff approaches.
- A method is presented for calculating the effect of measurement errors on mass expenditure. This error analysis is extended to include the case of statistical fluctuations in engine thrust.

For a simplified model of the system, numerical results are obtained for the additional mass expenditure. The simplified model illustrates the steady state balance of information, i.e., when the loss of information due to statistical fluctuations in engine acceleration just balances the gain in information due to additional measurements.

4.2 Publications

Most of the investigations presented in this report have been (or will be) published in outside journals. Listed below are the papers which have been supported by this NASA Contract NAS 1-3777 from Langley Research Center.

1) Breakwell, J. V., and Tung, F., "Minimum Effort Control of Several Terminal Components," J. SIAM Control, Ser. A, Vol. 2, No. 3 (1965), pp. 295-316.

2) Tung, F., "An Optimal Discrete Control Strategy for Interplanetary Guidance," IEEE Trans. Automatic Control, AC-10, July (1965).

3) Breakwell, J. V., Tung, F., and Smith, R. R., "Application of the Continuous and Discrete Strategies of Minimum Effort Theory to Interplanetary Guidance," AIAA Journal, Vol. 3, No. 5, (1965), pp. 907-913.

4) Tung, F., and Striebel, C. T., "A Stochastic Optimal Control Problem and its Applications," to appear in the J. of Math. Analysis and Applications.

5) Breakwell, J. V., "A Doubly Singular Problem in Optimal Interplanetary Guidance," to appear in J. SIAM Control, Ser. A (1965).

6) Breakwell, J. V., and Rauch, H. E., "Optimum Guidance for a Low-Thrust Interplanetary Vehicle," presented at the AIAA Guidance and Control Conference, Minneapolis, Minnesota, August 16-18 (1965).

4.3 Recommendations

(i) A three-dimensional version of the program described in Section 2.10 should be written to serve as a realistic guide to an optimal control policy for high-thrust interplanetary trajectories.

(ii) More basic work is needed to investigate the strictly optimal nonlinear strategies in more than one dimension, whether in minimizing average fuel consumption for given terminal accuracy or in minimizing terminal error for fixed amount of fuel available.

(iii) The neighboring-optimum control and error analysis, discussed in Section 3 for low-thrust spiral trajectories, should be applied to combination high and low thrust interplanetary trajectories.

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1) Breakwell, J. V., and Striebel, C. T., "Minimum Effort Control in Interplanetary Guidance," presented at the 1963 IAS Meeting, New York, Preprint 63-80 (1963).

NASA-Langley, 1966 CR-378