RATIONAL APPROXIMATIONS TO THE GENERALIZED DUFFING EQUATION

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by

Wyman G. Fair
Yudell L. Luke

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PREFACE

This report covers research initiated by Headquarters, National Aeronautics and Space Administration on Contract NASA Hq. R&D 80X0108(64), 10-74-740-124-08-06-11, PR 10-2487, "Nonlinear Dynamics of Thin Shell Structures." The research work upon which this report is based was accomplished at Midwest Research Institute with Mr. Howard Wolko as project monitor.

This report covers work conducted from January 15, 1965, to January 14, 1966.

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Approved for:

MIDWEST RESEARCH INSTITUTE

Sheldon L. Levy, Director
Mathematics and Physics Division

21 January 1966

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SUMMARY AND INTRODUCTION

In this report we use a linear fractional transformation to obtain rational approximations to the response of a physical system which is described by a second order nonlinear differential equation which includes Duffing's equation as a special case.

In Section I the general problem is stated and discussed. In Section II we develop the recurrence relations which define the approximations. Section III contains some examples and applications which exhibit the uses and advantages of the rational approximations. A FORTRAN program and its operating procedures are given in Section IV.

Author

I. THE DAMPED MASS SPRING OSCILLATOR EQUATION

Many physical problems, such as large amplitude vibration and response of curved panels, can be resolved by obtaining the solution to the nonlinear differential equation

$$y'' + ay' + \omega^2 h(y) = Q(t) \ , \ y = y(t) \quad , \tag{1.1}$$

where $h(y)$ is a cubic in y and $Q(t)$ is an arbitrary forcing function. The difficulties involved in computing the solutions to (1.1), to within a desired degree of accuracy, are well known.

Techniques are available for deducing local and asymptotic solutions (e.g., power series, perturbation, Fourier series, etc.) but these schemes are of limited use if reasonably high accuracy is desired. Further, these methods, in general, require much algebraic manipulation which can be tedious and lengthy.

Numerical integration can be used effectively only when one has accurate information on the behavior of the solution. For example, without knowledge as to the location of poles of the solution to (1.1), numerical integration can be disastrous. A Taylor's series expansion has this same drawback. Since the existence of poles of solutions to nonlinear differential equations is the rule, rather than the exception, a method for obtaining, simultaneously, both the solution and information about the location of poles is very desirable.

Now rational approximations are useful for numerical evaluation of the solutions. But more important, they provide a valuable and effective technique for deducing global behavior of the solutions including zeros and poles.

Some work has been done in this area, see [1] and [2]. In this report, we construct rational approximations to a second order nonlinear differential equation which includes as special cases (1.1), Ricatti's equation and Abel's equation.

II. THE GENERALIZED SECOND ORDER RICATTI EQUATION

In the present study we generalize (1.1) by considering the equation

$$(A_o+B_oy)y'' + (C_o+D_oy)y' - 2B_o(y')^2 + E_o + F_oy + G_oy^2 + H_oy^3 = 0 \quad ,$$

$$y(0) = \alpha_o \ , \ y'(0) = \beta_o \ , \ \alpha_o\beta_o \neq 0 \quad , \qquad (2.1)$$

where each of the coefficients is expandable in a Taylor's series about $x = 0$, and $A_o(0) = B_o(0) = C_o(0) = D_o(0) = G_o(0) = H_o(0) = 0$. In particular,

$$A_o = x^2 \sum_{k=0}^{\infty} a_k x^k \ , \qquad B_o = x^2 \sum_{k=0}^{\infty} b_k x^k \quad ,$$

$$C_o = x \sum_{k=0}^{\infty} c_k x^k \quad , \qquad D_o = x \sum_{k=0}^{\infty} d_k x^k \quad ,$$

$$E_o = \sum_{k=0}^{\infty} e_k x^k \ , \ e_o \neq 0 \ , \qquad F_o = \sum_{k=0}^{\infty} f_k x^k \ , \ f_o \neq 0 \quad ,$$

$$G_o = x \sum_{k=0}^{\infty} g_k x^k \quad , \qquad H_o = x \sum_{k=0}^{\infty} h_k x^k \quad . \qquad (2.2)$$

We further assume that the solution of (2.1) has a power series expansion of the form

$$y = \alpha_o + \sum_{k=1}^{\infty} \beta_k x^k \quad . \qquad (2.3)$$

Note that (2.2) and (2.3) together with (2.1) uniquely determine α_0 and β_1. We also require that the coefficients in (2.3) have the property that

$$\Delta_p = \begin{vmatrix} \alpha_0 & \beta_1 & \cdots\cdots & \beta_p \\ \beta_1 & \beta_2 & \cdots\cdots & \beta_{p+1} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \beta_p & \beta_{p+1} & \cdots\cdots & \beta_{2p} \end{vmatrix} \neq 0 \ , \ p = 0,1,2,\ldots \ ,$$

and

$$\Gamma_{2p+1} = \begin{vmatrix} \beta_1 & \beta_2 & \cdots\cdots & \beta_{p+1} \\ \beta_2 & \beta_3 & \cdots\cdots & \beta_{p+2} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \beta_{p+1} & \beta_{p+2} & \cdots\cdots & \beta_{2p+1} \end{vmatrix} \neq 0 \ , \ p = 0,1,2,\ldots \ . \tag{2.4}$$

Then y has a continued fraction expansion of the form

$$y = \cfrac{\alpha_0}{1 + \cfrac{\alpha_1 x}{1 + \cfrac{\alpha_2 x}{1 + \ddots}}} \ . \tag{2.5}$$

For further information on continued fractions, see Wall [3].

A transformation of the type

$$y = \frac{m(x) + n(x)y^*}{p(x) + q(x)y^*} \tag{2.6}$$

where m, n, p and q are polynomials in x may be necessary to bring the differential equation into the required form. We suppose that this has already been done. See [4] for the results of applying transformations of this type to (2.1). We give an example in Section III.

The even approximants of (2.5) are the main diagonal Padé approximations which have the following properties. Let

$$y_n = \frac{P_n}{Q_n} = \frac{\sum_{k=0}^{n} p_{n,k}x^k}{\sum_{k=0}^{n} q_{n,k}x^k} \tag{2.7}$$

be the n^{th} order main diagonal Padé approximant. If Q_n is formally divided into P_n, the resulting power series agrees with the power series solution to (2.1) for the first (2n+1) terms. The polynomials P_n and Q_n both satisfy the relation

$$P_n = \left[1 + (\alpha_{2n-1}+\alpha_{2n})x\right]P_{n-1} - \alpha_{2n-1}\alpha_{2n-2}x^2P_{n-2} ,$$

$$P_o = \alpha_o , \ P_1 = \alpha_o(1+\alpha_2 x) , \ Q_o = 1 \ \text{and} \ Q_1 = 1 + (\alpha_1+\alpha_2)x . \tag{2.8}$$

Thus, rational approximations to the solution of (2.1) are immediately forthcoming if the values $\alpha_1, \alpha_2, \alpha_3, \ldots$ can be computed. We compute these values by utilizing a linear fractional transformation. Let

$$y = y_o , \ y_n = \alpha_n(1+xy_{n+1})^{-1} , \quad n \geq 0 \quad . \tag{2.9}$$

Repeated application of (2.9) to (2.1) and division by $\alpha_n x$ at each step yields

$$(A_{n+1}+B_{n+1}y_{n+1})y''_{n+1} + (C_{n+1}+D_{n+1}y_{n+1})y'_{n+1} - 2B_{n+1}(y'_{n+1})^2 + E_{n+1}$$

$$+ F_{n+1}y_{n+1} + G_{n+1}y^2_{n+1} + H_{n+1}y^3_{n+1} = 0 \quad , \tag{2.10}$$

where

$$A_{n+1} = - A_n - \alpha_n B_n \quad ,$$

$$B_{n+1} = - xA_n \quad ,$$

$$C_{n+1} = - 2x^{-1}(A_n+\alpha_n B_n) - C_n - \alpha_n D_n \quad ,$$

$$D_{n+1} = 2A_n - xC_n \quad ,$$

$$E_{n+1} = x^{-1}(\alpha_n^{-1}E_n+F_n+\alpha_n G_n+\alpha_n^2 H_n) \quad ,$$

$$F_{n+1} = - x^{-1}(C_n+\alpha_n D_n) + 3\alpha_n^{-1}E_n + 2F_n + \alpha_n G_n \quad ,$$

$$G_{n+1} = 2x^{-1}A_n - C_n + 3\alpha_n^{-1}xE_n + xF_n \quad ,$$

and

$$H_{n+1} = \alpha_n^{-1}x^2E_n \quad . \tag{2.11}$$

It is easily shown that

$$A_{n+1}(0) = B_{n+1}(0) = C_{n+1}(0) = D_{n+1}(0) = G_{n+1}(0) = H_{n+1}(0) = 0$$

and

$$E_{n+1}(0) \neq 0 \quad , \quad F_{n+1}(0) \neq 0 \quad . \tag{2.12}$$

It follows that after setting $x = 0$ in (2.10), one gets

$$F_{n+1}(0)y_{n+1}(0) = - E_{n+1}(0) \quad ,$$

or

$$\alpha_{n+1} = - \frac{E_{n+1}(0)}{F_{n+1}(0)} \tag{2.13}$$

and this value is well defined. Computation of the Padé approximations (2.7) by using (2.13) is now easily accomplished.

It is evident that, for realistic computer application, the functions appearing in (2.11) must be polynomials. Accurate polynomial expansions are available for the usual transcendental functions. The validity of the rational approximations to the solution of (2.1) are limited by the range of accuracy of these polynomial approximations. This is not serious for in practice one computes approximations for a restricted range and then obtains approximations over an adjacent range by the method of analytic continuation. We briefly discuss this technique.

Suppose a rational approximation $y_{n,o}(t)$ is obtained to the solution of (2.1) which is valid for $t_o \le t \le t_1$. The transformation $t = \tau + t_1$ is then employed to convert (2.1) into a new initial value problem with the initial conditions $y_{n,o}(t_1)$ and $y'_{n,o}(t_1)$. A new rational approximation, $y_{n,1}$, is obtained which is valid for $t_1 \le t \le t_2$. Repetition of this process yields a sequence of rational approximations $y_{n,j}(t)$, valid over the interval $t_{j-1} \le t \le t_j$. This can be continued until the entire desired range is covered.

Convergence of the rational approximations (2.7), in general, is still an unresolved problem and warrants further investigation. In the important special case of the first order Ricatti equation convergence proofs are available for a number of examples, see [1] and [5].

A very reliable estimate of the error incurred by the n^{th} order approximation is easily obtained by comparing the n^{th} order approximation with the $(n+1)^{st}$ order approximation (see the third example in Section III). In the cases investigated, the magnitude of the error of the n^{th} approximant, y_n, is the same order of magnitude of the difference, $y_{n+1} - y_n$. This method of error analysis is quite common in the stepwise integration of differential equations.

It should be noted that once rational approximations have been constructed for y, like approximations for y' and y'' are easily obtained by differentiation.

III. EXAMPLES AND APPLICATIONS

In this section we exhibit the varied uses of the approximations developed in Section II. In the first two examples we construct rational approximations to Painlevé's first and second transcendents. These approximations are accurate for a surprisingly wide range of the variable. At the same time they effectively predict the poles of the solutions. The third example shows the use of the rational approximations and the idea of analytic continuation to compute functional values of the solution of Duffing's equation.

Painlevé's first and second transcendents are defined by the differential equations

$$u'' - 6u^2 - \lambda x = 0 \text{ , } u(0) = 1 \text{ , } u'(0) = 0 \text{ ,} \tag{3.1}$$

and

$$v'' - 2v^3 - xv - \delta = 0 \text{ , } v(0) = 1 \text{ , } v'(0) = 0 \text{ ,} \tag{3.2}$$

respectively. In what follows, $\lambda = \delta = 1.0$.

To cast (3.1) and (3.2) into the required form of (2.1), we set

$$u = 1 + 3x^2\bar{u}$$

and

$$v = 1 + 1.5x^2\bar{v} \tag{3.3}$$

in which case (3.1) and (3.2) become

$$3x^2\bar{u}'' + 12x\bar{u}' + (6-36x^2)\bar{u} - 54x^4\bar{u}^2 - (6+x) = 0 \text{ , } \bar{u}(0) = 1 \tag{3.4}$$

and

$$3x^2\bar{v}'' + 10x\bar{v} + (5-18x^2-6x^3)\bar{v} - 27x^4\bar{v}^2 - 13.5x^6\bar{v}^3$$
$$- (5+4x) = 0 \text{ , } \bar{v}(0) = 1 \text{ .} \tag{3.5}$$

Now u has a pole of the second order at $x = 1.2067$ and v has a simple pole at $x = 1.1577$. This behavior manifests itself in Tables III.1 and III.2 below where $\bar{u}_6$ and $\bar{v}_6$ are the sixth order main diagonal Padé approximations to $\bar{u}$ and $\bar{v}$ obtained using the algorithm of Section II. We have

$$u_6 = 1 + 3x^2\bar{u}_6$$

and

$$v_6 = 1 + 1.5x^2\bar{v}_6 \quad . \tag{3.6}$$

TABLE III.1

x	u(x)	$u_6(x)$
0.0	1.0000	1.0000
0.1	1.0305	1.0305
0.2	1.1264	1.1264
0.3	1.3015	1.3015
0.4	1.5831	1.5831
0.5	2.0228	2.0228
0.6	2.7212	2.7212
0.7	3.8909	3.8909
0.8	6.0383	6.0383
0.9	10.6226	10.6223
1.0	23.3936	23.3860
1.1	87.7732	87.3769

TABLE III.2

x	v(x)	$v_6(x)$
0.0	1.0000	1.0000
0.1	1.0152	1.0152
0.2	1.0626	1.0626
0.3	1.1464	1.1464
0.4	1.2742	1.2742
0.5	1.4592	1.4592
0.6	1.7254	1.7254
0.7	2.1184	2.1184
0.8	2.7369	2.7369
0.9	3.8344	3.8343
1.0	6.3110	6.3104

The values of $u(x)$ and $v(x)$ were taken from a paper by Simon [6] who used (3.1) and (3.2) as examples in a study of a numerical integration technique for the solution of initial value problems in ordinary differential equations.

The poles of smallest magnitude of u_6 and v_6 are $1.2051 \pm i0.0134$ and 1.1578 , respectively. These values are deduced from the rational approximations which are very accurate near $x = 0.0$. If more accurate estimation of the poles are desired, the method of analytic continuation can be used to obtain approximations in a region closer to the true poles.

For our third example we develop approximations to the solution of Duffing's equation (with constant coefficients)

$$y'' + Ay' + By + Cy^3 = D\cos(\omega t+\varphi) \quad ,$$

$$A > 0 \ , \ y(0) = \alpha_0 \ , \ y'(0) = \beta_0 \ . \qquad (3.7)$$

This equation describes a damped mass-spring system with control proportional to $By + Cy^3$ and driven by the force $D\cos(\omega t+\varphi)$.

If C is large, the usual perturbation scheme is not adequate. The validity of our rational approximations does not depend on the relative magnitude of C , and in our example we purposely choose a large value for C . We also illustrate the method of extending the range of validity of the rational approximations by analytic continuation.

In (3.7) let $A = 0.2$, $B = 5.0$, $C = 10.0$, $D = \alpha_0 = 1.0$, and $\beta_0 = \varphi = 0.0$. The equation becomes

$$y'' + 0.2y' + 5y + 10y^3 = \cos\omega t \ , \ y(0) = 1.0 \ , \ y'(0) = 0.0 \ . \qquad (3.8)$$

To cast this equation in the required form, set

$$y = 1 - 7t^2 v \quad . \qquad (3.9)$$

Then (3.8) becomes

$$7t^2v'' + (28t+1.4t^2)v' + (14+2.8t+245t^2)v - 1740t^4v^2$$

$$+ 3430t^6v^3 - 15 + \cos\omega t = 0 \ , \ v(0) = 1.0 \quad . \qquad (3.10)$$

In (3.10) we replace $\cos\ \omega t$ by a polynomial approximation which is accurate to five decimals for $0 \le \omega t \le 1$. Using our technique to obtain rational approximations to the solution of the resulting equation, we construct rational approximations v_n to v and y_n to y, where

$$y_n = 1 - 7t^2 v_n \quad . \tag{3.11}$$

It is clear that the range of validity of our approximations is limited to the range of validity of the approximation to $\cos \omega t$. For purposes of illustration, we consider two cases, $\omega = 0$ and $\omega = 1$.

In Tables III.3 and III.4 the sixth order approximations to y for $\omega = 0$ and $\omega = 1$ are listed. Also given are values determined by stepwise numerical integration which we call the true values. As is evident, the rational approximations are quite accurate.

Since the accuracy of our approximations decreases as t increases, we employed the analytic continuation technique for the $\omega = 1$ case in order to compute accurate values for $0.4 \le t \le 1.0$. Thus the rational approximations were computed for $0.0 \le t \le 0.4$, and then the transformation $t = \tau + 0.4$ was utilized to convert (3.10) into a new initial value problem. Then rational approximations were computed for $\tau = 0.0(0.04)0.6$, i.e., $t = 0.4(0.04)1.0$. Note that the approximations can be used to tabulate zeros of the solution and hence may be used to obtain an accurate estimate of periods of periodic solutions.

The approximants y_2, y_3, y_4 and y_5 were also computed but, for the sake of brevity, these are not given here.

We do, however, illustrate our remarks in Section II concerning the error involved in these approximations. For the case $\omega = 1$, $t = 0.8$, $y(0.8) = -0.68961$, $y_4(0.8) = -0.69119$ and $y_5(0.8) = -0.68966$. Note that the true error incurred by $y_4(0.8)$ is $y(0.8) - y_4(0.8) = 0.00158$, whereas $y_5(0.8) - y_4(0.8) = 0.00153$, so that $y_{n+1}(t) - y_n(t)$ does indeed give an accurate estimation of the error of the n^{th} approximation.

TABLE III.3

$\omega = 0.0$		
t	y(t)	$y_6(t)$
0.00	1.00000	1.00000
0.04	0.98888	0.98888
0.08	0.95625	0.95625
0.12	0.90399	0.90399
0.16	0.83481	0.83481
0.20	0.75186	0.75186
0.24	0.65838	0.65838
0.28	0.55742	0.55742
0.32	0.45162	0.45162
0.36	0.34315	0.34315
0.40	0.23373	0.23373
0.44	0.12469	0.12469
0.48	0.01708	0.01708
0.52	-0.08822	-0.08821
0.56	-0.19037	-0.19033
0.60	-0.28848	-0.28837
0.64	-0.38151	-0.38127
0.68	-0.46826	-0.46774
0.72	-0.54734	-0.54628
0.76	-0.61722	-0.61516
0.80	-0.67629	-0.65482
0.84	-0.72298	-0.71633
0.88	-0.75596	-0.74481
0.92	-0.77421	-0.75632
0.96	-0.77720	-0.77346
1.00	-0.76495	-0.75084

TABLE III.4

$\omega = 1.0$		
t	y(t)	$y_6(t)$
0.00	1.00000	1.00000
0.04	0.98888	0.98888
0.08	0.95625	0.95625
0.12	0.90398	0.90398
0.16	0.83478	0.83478
0.20	0.75179	0.75179
0.24	0.65825	0.65825
0.28	0.55718	0.55718
0.32	0.45121	0.45121
0.36	0.34251	0.34251
* 0.40	0.23276	0.23276
0.44	0.12328	0.12328
0.48	0.01509	0.01509
0.52	-0.09095	-0.09095
0.56	-0.19402	-0.19402
0.60	-0.29324	-0.29324
0.64	-0.38760	-0.38760
0.68	-0.47589	-0.47589
0.72	-0.55672	-0.55672
0.76	-0.62851	-0.62851
0.80	-0.68961	-0.68961
0.84	-0.73840	-0.73840
0.88	-0.77347	-0.77348
0.92	-0.79375	-0.79379
0.96	-0.79865	-0.79876
1.00	-0.78817	-0.78841

* Analytic continuation begins here.

IV. FORTRAN PROGRAM FOR COMPUTATION OF RATIONAL APPROXIMATIONS

Here we give a listing of a FORTRAN program used to compute the rational approximations developed in Section II. We also give a description of operating procedures, input and output. We assume that the differential equation is already in the desired form, see Eqs. (2.1) - (2.4). We assume also that the coefficients in (2.1) are polynomials.

Since two particular transformations occur frequently in the development of rational approximations to the solution of (2.1), provisions were made in the program for the incorporation of these transformations into the final approximations. We briefly discuss these transformations and the way in which the program accommodates them.

Type I

Suppose k transformations of the type $y_n = \alpha_n(1+xy_{n+1})^{-1}$ are needed to bring (2.1) into the required form. The result of the transformations is a differential equation of type (2.1) in the independent variable y_{k+1}, where $\alpha_0,\alpha_1,\ldots,\alpha_{k-1}$ and α_k are determined. The program accepts the values $\alpha_0,\alpha_1,\ldots,\alpha_k$ and the coefficients of the equation in y_k computes the main diagonal Padé approximations $y_{k,n}$ to y_k and then computes the following approximation $\bar{y}_n$ to y,

$$\bar{y}_n = \cfrac{\alpha_0}{1 + \cfrac{\alpha_1 x}{1 + \cdots \cfrac{}{1 + xy_k}}} \quad . \tag{4.1}$$

Type II

If $y(0) \neq 0$ and $y'(0) = 0$, a transformation of the form $y = a + bx^2v$ is needed (see (3.3)) to bring the equation into the required form. The program accepts $S1 = a = y(0)$, $S2 = b = y''(0)/2$, computes the main diagonal Padé approximations v_n to v and then computes $y_n = a + bx^2v_n$. If this transformation is not needed, no values are entered for S1 and S2 .

In some cases, combinations of the two types of transformations discussed above are needed.

Description of Input in Order

$M = D+1$, D being the degree of highest order polynomial in (2.1).

$N = 2L+1$ where L is desired order of main diagonal Padé approximations.

$K = k$ where k is defined by (4.1).

$S1 = a$, $S2 = b$ where a and b are defined in the Type II transformation. (No entry if no transformation is made.)

$XI = x_o$, $XF = x_n$, $Z = x_{k+1} - x_k = \Delta x$ where the evaluation of the main diagonal Padé approximations are desired for $x = x_o(\Delta x)x_n$.

$A(J)$, $B(J), \ldots, H(J)$ are the coefficients appearing in (2.2), i.e., $A(J) = a_{j-1}$, $j = 1,2,\ldots,M$, etc.

$ALPH(J) = \alpha_{j-1}$, $j = 1,2,\ldots,k$ where k is defined in (4.1).

Description of Output in Order

1. Coefficients of polynomials in (2.1).

2. $\alpha_o, \alpha_1, \ldots, \alpha_{N+k+1}$, where N and k are defined above.

3. Order of Padé , x , $y_n(x)$, and $y_n'(x)$ for $n = 1,2,\ldots,N$ and $x = x_o(\Delta x)x_n$.

We conclude this section with a listing of the FORTRAN program.

```
      DIMENSION A(40),B(40),C(40),D(40),E(40),F(40),G(40),H(40),A1(40),B
     11(40),C1(40),D1(40),E1(40),F1(40),G1(40),H1(40),ALPH(30)
      DIMENSION T(16),T1(16),T2(16),R(16),R1(16),R2(16)
      COMMON A,B,C,D,E,F,G,H,A1,B1,C1,D1,E1,F1,G1,H1,ALPH,M,N,X,XF,Z,T,T
     11,T2,R,R1,R2
8     DO 10 I=1,640
10    H1(I)=0.
      DO 12 I=1,30
12    ALPH(I)=0.
      READ 1,M,N,K,S1,S2,XI,XF,Z
1     FORMAT(3I4,2E19.0/(3E19.0))
      READ 109,(A(I),I=1,M)
      PRINT 101,(A(I),I=1,M)
      READ 109,(B(I),I=1,M)
      PRINT 102,(B(I),I=1,M)
      READ 109,(C(I),I=1,M)
      PRINT 103,(C(I),I=1,M)
      READ 109,(D(I),I=1,M)
      PRINT 104,(D(I),I=1,M)
      READ 109,(E(I),I=1,M)
      PRINT 105,(E(I),I=1,M)
      READ 109,(F(I),I=1,M)
      PRINT 106,(F(I),I=1,M)
      READ 109,(G(I),I=1,M)
      PRINT 107,(G(I),I=1,M)
      READ 109,(H(I),I=1,M)
      PRINT 108,(H(I),I=1,M)
101   FORMAT(5X,14HA COEFFICIENTS/(6F19.9)//)
102   FORMAT(5X,14HB COEFFICIENTS/(6F19.9)//)
103   FORMAT(5X,14HC COEFFICIENTS/(6F19.9)//)
104   FORMAT(5X,14HD COEFFICIENTS/(6F19.9)//)
105   FORMAT(5X,14HE COEFFICIENTS/(6F19.9)//)
106   FORMAT(5X,14HF COEFFICIENTS/(6F19.9)//)
107   FORMAT(5X,14HG COEFFICIENTS/(6F19.9)//)
108   FORMAT(5X,14HH COEFFICIENTS/(6F19.9)//)
109   FORMAT(4E19.0)
      ALPH(1)=-E(1)/F(1)
      DO 50 I=1,N
      ALP=ALPH(I)
      L=M+(I-1)/2+3
      DO 18 J=1,L
      A1(J)=A(J)
      B1(J)=B(J)
      C1(J)=C(J)
      D1(J)=D(J)
      E1(J)=E(J)
      F1(J)=F(J)
      G1(J)=G(J)
18    H1(J)=H(J)
      A(1)=-A1(1)-ALP*B1(1)
      C(1)=-2.*(A1(2)+ALP*B1(2))-C1(1)-ALP*D1(1)
      D(1)=2.*A1(1)
      E(1)=E1(2)/ALP+F1(2)+ALP*G1(2)+ALP*ALP*H1(2)
      F(1)=-(C1(2)+ALP*D1(2))+3.*E1(1)/ALP+2.*F1(1)+ALP*G1(1)
      G(1)=2.*A1(2)-C1(1)
      A(2)=-A1(2)-ALP*B1(2)
      B(2)=-A1(1)
      C(2)=-2.*(A1(3)+ALP*B1(3))-C1(2)-ALP*D1(2)
      D(2)=2.*A1(2)-C1(1)
      E(2)=E1(3)/ALP+F1(3)+ALP*G1(3)+ALP*ALP*H1(3)
      F(2)=-(C1(3)+ALP*D1(3))+3.*E1(2)/ALP+2.*F1(2)+ALP*G1(2)
      G(2)=2.*A1(3)-C1(2)+3.*E1(1)/ALP+F1(1)
```

```
      DO 24 J=3,L
      A(J)=-A1(J)-ALP*B1(J)
      B(J)=-A1(J-1)
      C(J)=-2.*(A1(J+1)+ALP*B1(J+1))-C1(J)-ALP*D1(J)
      D(J)=2.*A1(J)-C1(J-1)
      E(J)=E1(J+1)/ALP+F1(J+1)+ALP*G1(J+1)+ALP*ALP*H1(J+1)
      F(J)=-(C1(J+1)+ALP*D1(J+1))+3.*E1(J)/ALP+2.*F1(J)+ALP*G1(J)
      G(J)=2.*A1(J+1)-C1(J)+3.*E1(J-1)/ALP+F1(J-1)
24    H(J)=E1(J-2)/ALP
50    ALPH(I+1)=-E(1)/F(1)
      N=N+1
      Y=K
      IF(Y)54,54,52
52    DO 53 I=1,N
      L=N+1-I
      J=K+L
53    ALPH(J)=ALPH(L)
      READ 7,(ALPH(I),I=1,K)
7     FORMAT(4E19.0)
54    X=XI
      PRINT 5,ALPH
5     FORMAT(38X,43HALPHAS FOR CONTINUED FRACTION APPROXIMATION//(6E19.1
     11)/)
      PRINT 2
2     FORMAT(5X,13HORDER OF PADE,10X,8HARGUMENT,25X,13HAPPROXIMATION,19X
     121HDERIVATIVE OF APPROX.)
56    P1=ALPH(1)
      P2=ALPH(1)
      Q1=1.+ALPH(2)*X
      Q2=1.0
      P1P=0.
      P2P=0.
      Q1P=ALPH(2)
      Q2P=0.
      J=N+K
      DO 60 I=3,J
      P=P1+ALPH(I)*X*P2
      Q=Q1+ALPH(I)*X*Q2
      PP=P1P+P2P*ALPH(I)*X+ALPH(I)*P2
      QP=Q1P+Q2P*ALPH(I)*X+ALPH(I)*Q2
      KORD=I/2
      IF(2*KORD-I)57,59,59
57    IF(S1)58,55,58
55    APPROX=P/Q
      APXPRI=(Q*PP-P*QP)/(Q*Q)
      GO TO 61
58    APPROX=S1+S2*X*X*P/Q
      APXPRI=2.0*S2*X*P/Q + (S2*X*X*(Q*PP-P*QP))/(Q*Q)
61    PRINT 6,KORD,X,APPROX,APXPRI
6     FORMAT(10X,I2,7X,E19.11,2(19X,E19.11))
59    P2=P1
      P1=P
      P2P=P1P
      P1P=PP
      Q2=Q1
      Q2P=Q1P
      Q1P=QP
60    Q1=Q
      IF(X-XF)62,8,8
62    X=X+Z
      GO TO 56
      END
```

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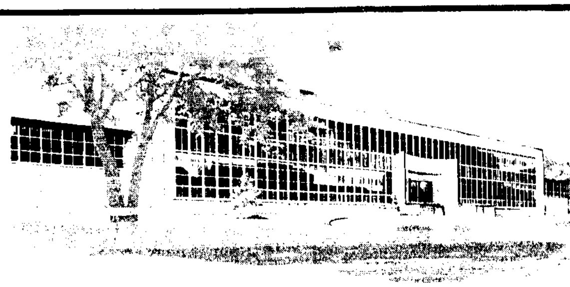

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