California Inst. Tech.

NON-RESONANT NUCLEAR REACTIONS AT STELLAR TEMPERATURES*

The purpose of this note is to describe a systematic and accurate procedure for calculating the rates of non-resonant nuclear reactions at stellar temperatures and to indicate the approximations that are involved, and the corrections that are necessary, when using the currently fashionable formulae (Caughlan and Fowler 1962; Parker, Bahcall, and Fowler 1964).

The number of nuclear reactions, P, occurring per unit of time per unit of volume between nuclei of type one and type two is

$$P = n_1 n_2 (1 + \delta_{12})^{-1} \langle \sigma v \rangle \quad , \tag{1}$$

where n_1 and n_2 are the number densities of nuclei of type one and two and $\langle \sigma v \rangle$ is the interaction cross section times the relative velocity averaged over a Maxwell-Boltzmann distribution. In analyzing non-resonant reactions at stellar temperatures, it is conventional and convenient to represent the cross section (in the center of mass frame) by (Burbidge, Burbidge, Fowler, and Hoyle 1957; Cameron 1963; Fowler and Vogl 1964)

$$\sigma(E) = E^{-1} S(E) \exp - (b E^{-1/2}) \quad , \tag{2a}$$

where the exponential factor represents the Gamow penetration factor, and

$$b = 2 \pi \alpha Z_1 Z_2 (m A c^2/2)^{1/2} \quad . \tag{2b}$$

*Supported in part by the Office of Naval Research [Nonr-220(47)], and the National Aeronautics and Space Administration [NGR-05-002-028].

Here, mA is the reduced mass of the colliding nuclei. The reaction rate P is therefore proportional to an integral of the form

$$I = \int_0^\infty dE\ S(E)\ \exp - (E/kT + bE^{-1/2}) \quad . \tag{3}$$

The usual evaluation (Caughlan and Fowler 1962; Fowler and Vogl 1964) of I is accomplished by representing S in Eq. (3) by a two-term power series about E equals zero and replacing the exponential by a Gaussian centered at an energy E_o chosen such that the exponent, $E/kT + b/\sqrt{E}$, is an extremum. Instead of making approximations with I in the form given by Eq. (3), we shall first rewrite I in a form that permits a power series expansion in a parameter, τ^{-1}, that is small for all stellar problems encountered so far.

In the usual notation (Parker, Bahcall, and Fowler 1964; Fowler and Vogl 1964),

$$P = S_{eff} \times 7.20 \times 10^{-19}\ n(1)\ n(2)\ f_{1,2}\ \tau^2\ e^{-\tau}\ (1 + \delta_{1,2})^{-1}$$

$$(A\ Z_1\ Z_2)^{-1} \text{ reactions } \ \mathrm{cm}^{-3}\ \mathrm{sec}^{-1} \quad , \tag{4a}$$

where $f_{1,2}$ is an electron screening factor and

$$\tau \equiv 3\ E_o/kT$$

$$\approx 42.48\ (Z_1^2\ Z_2^2\ A/T_6)^{1/3} \tag{4b}$$

with $$E_o \equiv (b\ k\ T/2)^{2/3}$$

$$= \left[(\pi\ \alpha\ Z_1\ Z_2\ k\ T)^2\ (m_n\ A\ c^2/2)\right]^{1/3} \quad . \tag{4c}$$

Here, S_{eff} is the correction factor which arises from the variation of S with energy and the departure of the exponential in Eq. (3) from a Gaussian shape. More explicitly, one can show that

$$S_{eff} = \left[(\tau/\pi)^{+1/2} (2 E_o)^{-1} e^{+\tau}\right] \int_0^\infty \exp - (E/kT + bE^{-1/2}) \, S(E) \, dE \quad , \qquad (5)$$

which has, of course, the form indicated in Eq. (3). The bracketed factors have been chosen for simplicity in the final answer.

The quantity τ, defined by Eq. (4b), is large (typically 15 to 40) for all non-resonant nuclear reactions of interest (p-p chain, CNO bi-cycle, etc.). We therefore express S_{eff} as a power series in τ^{-1}. This may be accomplished by the substitution (motivated by Salpeter's treatment of the p-p reaction, Salpeter 1952):

$$E = E_o (1 + \tau^{-1/2} u)^2 \quad . \qquad (6)$$

Inserting the above definition of u in Eq. (5), one finds

$$S_{eff} = (\pi \tau)^{-1/2} \int_{-\tau^{1/2}}^{\infty} du \, (\exp - u^2)(\exp + \left[(2u^3/3)(\tau^{1/2} + u)^{-1}\right]$$

$$(\tau^{1/2} + u) \quad S(E_o (1 + \tau^{-1/2} u)^2) \, du \quad . \qquad (7)$$

Since $e^{-\tau} \ll \tau^{-1}$, S_{eff} can be readily evaluated as a power series in τ^{-1} by replacing the lower limit of the integral in Eq. (7) by infinity and expanding $\exp\left[2u^3/3 \, (\tau^{1/2} + u)\right]$ and $S(E_o (1 + \tau^{-1/2} u)^2)$ as a Taylor series in u about u equals zero. The successive terms can be grouped in descending integral powers of τ (since e^{-u^2} is an even function of u).

One finds in this way that

$$S_{eff} = S(E_o)\left[1 + \tau^{-1}\left(\frac{5}{12} + \frac{5}{2}\,\frac{S'(E_o)\,E_o}{S(E_o)} + \frac{S''(E_o)\,E_o^{\,2}}{S(E_o)}\right) + \theta(\tau^{-2})\right] \quad (8)$$

where $$S'(E_o) \equiv \left.\frac{dS(x)}{dx}\right)_{x=E_o}$$

and $$S''(E_o) \equiv \left.\frac{d^2S(x)}{dx^2}\right)_{x=E_o} .$$

Note that up to order τ^{-2}, only first and second derivatives of S enter Eq. (8).

In order to relate the above expression for S_{eff} to the usual formula (Caughlan and Fowler 1962; Parker, Bahcall, and Fowler 1964), one must express the relevant quantities in terms of their values at E equals zero (not E_o). If one does so, there is no a priori assurance that, to order τ^{-1}, only first and second derivatives of S are important. However, this is a plausible assumption and is necessary in order to obtain a not too complicated formula. We find (neglecting all derivatives higher than the second derivative):

$$S_{eff} \cong S(0)\left[1 + \frac{S'(0)\,E_o + S''(0)\,E_o^{\,2}/2}{S(0)} + \frac{1}{\tau}\left(\frac{5}{12} + \frac{\left(\frac{35}{12}\right)E_o\,S'(0) + \left(\frac{89}{24}\right)S''(0)\,E_o^{\,2}}{S(0)}\right)\right] \quad (9a)$$

or

$$S_{eff} \cong S(0)\left[1 + \frac{5}{12\tau} + \frac{S'(0)}{S(0)}\left(E_o + \frac{35}{36}\ kT\right) + \frac{S''(0)\ E_o}{S(0)}\left(E_o/2 + \frac{89}{72}\ kT\right)\right] . \tag{9b}$$

If the last term in Eq. (9b) is set equal to zero, then the above expression reduces to the usually quoted formula.

June 21, 1965

JOHN N. BAHCALL

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA

REFERENCES

Burbidge, E. M., Burbidge, G. R., Fowler, W. A., and Hoyle, F. 1957, Rev. Mod. Phys. 29, 547.

Cameron, A. G. W. 1963, lectures given at Yale University 1962-63 (unpublished).

Caughlan, G. R. and Fowler, W. A. 1962, Ap. J. 136, 453.

Fowler, W. A. and Vogl, J. 1964, Lectures in Theoretical Physics, Volume VI (University of Colorado Press, Boulder, Colorado).

Parker, P. D., Bahcall, J. N., and Fowler, W. A. 1964, Ap. J. 139, 602.

Salpeter, E. E. 1952, Phys. Rev. 88, 547.