ON THE OPTIMUM CHOICE OF CENTRAL VARIABLES IN PROBLEMS OF STELLAR STRUCTURE*

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ABSTRACT

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Mathematical difficulties in calculations of the central regions of a stellar model are shown to be due to a wrong choice of variables. New variables are derived which permit outward integration to commence at the center of a model. They appear to lead to the highest attainable central accuracy for a given integration process in a digital computer.

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I. INTRODUCTION

In many investigations of stellar structure it has been found necessary to commence outward integrations at a position somewhat removed from the center of the model. The values of the dependent variables at this position have been determined by series expansions to the desired order of accuracy. Such expansions are tedious to derive beyond the first terms if complex physical laws are used.

It is the purpose of this paper to show that the technique described above is not necessary. We demonstrate that there exists a natural expansion parameter (the square of the radial distance) at the center of a star. Bearing this in mind, one can then construct a set of variables in terms of which the integrations can begin at the center. Explicit knowledge of the expansions is not required. The new variables are also optimal for accuracy when used in an integration process of a given order.

Throughout the discussion, we bear in mind the application of a digital computer to the calculations, and in particular to the calculation in the central regions of the model.

II. THE USUAL DIFFICULTY

For spherically symmetric models in quasi-static equilibrium, the basic equations of stellar structure take the following form:

$$\frac{dM}{dR} = 4\pi R^2 \rho \tag{1}$$

$$\frac{dL}{dR} = 4\pi R^2 \rho \epsilon \tag{2}$$

$$\frac{dP}{dR} = -\frac{GM}{R^2}\rho \tag{3}$$

$$\frac{dT}{dR} = \frac{-3\kappa\rho L}{16\pi ac\, R^2\, T^3} \qquad (4a)$$

or

$$\frac{dT}{dR} = \frac{\Gamma - 1}{\Gamma}\,\frac{T}{P}\,\frac{dP}{dR} \qquad (4b)$$

The alternative equations (4a) and (4b) correspond to the case of radiative or convective equilibrium respectively. At each point the right-hand sides are computed, the least in modulus being taken for the true temperature gradient in the usual manner.

The notation in equations (1) to (4) is essentially that of Schwarzschild (1958, §12), with the exceptions of R, the radial coordinate, and Γ, the second adiabatic coefficient of Chandrasekhar (1939).

Schwarzschild remarks (Schwarzschild 1958, §14) that a numerical step-by-step integration procedure cannot start at the center because vanishing denominators occur in the basic differential equations (1)-(4). This is not the case. It is for quite the reverse reason that difficulty is experienced. Equations (1) and (2) show that both M and L are of order R^3 near the center. Thus the limiting values of dP/dR and dT/dR at the center are zero. The reason it is pointless to commence at the center with the equations (1) to (4) is that all four differential coefficients are zero. Nor is the situation improved by taking M as independent variable, a natural choice for evolutionary purposes. The limiting values of dP/dM and dT/dM then behave as 1/R. The zero denominator presents a genuine difficulty in this case.

III. THE CENTRAL EXPANSION

The correct approach is in fact suggested by the expansions at the center,

which lead to

$$\left.\begin{aligned} M &= O(R^3) \\ L &= O(R^3) \\ P &= P_c - O(R^2) \\ T &= T_c - O(R^2) \end{aligned}\right\} \qquad (5)$$

From the third and fourth of relations (5) we see that differentials of P and T will be proportional to the differential of R^2 near the center. The relations (5) suggest that one should choose variables which are linearly dependent on R^2 in this region. Thus $r = R^2$, $m = M^{2/3}$ and $l = L^{2/3}$ are indicated. This choice also has desirable computational features. If the variables are almost linearly related near the center, a finite integration process will be more accurate. The integration scheme can be started with precisely evaluated central gradients. (That the latter may have to be specially "set" is not a particularly awkward feature -- it is often the case with automatic integration schemes that one must first define the initial derivatives.) The central point can thus be included in the whole integration scheme whatever its nature, as a matter of course.

Before accepting the above conclusions however, one must check to make sure that an automatic integration will not produce spurious results. Let us suppose for a moment that the expansion for M contained a term in R^4. Then in terms of the new independent variable r ($= R^2$), we might have m ($= M^{2/3}$) given by

$$m = Ar + Br^{3/2} + Cr^2 + \cdots \qquad (6)$$

The expression calculated for dm/dr would yield

$$\frac{dm}{dr} = A + \frac{3}{2} Br^{\frac{1}{2}} + 2Cr + \cdots \qquad (7)$$

Now suppose the usual second order integration process is used (Schwarzschild 1958, §14) to integrate equation (7) away from the origin. One obtains

$$m = Ar + \frac{3}{4} Br^{3/2} + Cr^2 \cdots \qquad (8)$$

One quarter of the second term is lost. The reason is of course clear -- d^2m/dr^2 is unbounded in the initial interval and the Taylor expansion, upon which the integration scheme depends, is invalid. A small point perhaps, but one which might be overlooked with inadequate investigation.

Does this invalidate the use of the variables suggested above? Fortunately, no. The suggested variables will clearly be satisfactory if the physical variables have the form of the relevant leading term multiplied by an expansion in even powers of R. That this is indeed the case will now be shown by two quite independent arguments, the first the application of the concept of parity, the second from the equations themselves.

(a) Take Cartesian axes with origin at the center of the star. Let scalar quantities (e.g., pressure, density, element concentration, etc.) at positions on the x-axis be considered as functions of x and let vector quantities (e.g., gravity, energy flux) be taken as positive when pointing in the $+x$ direction. As the scalars do not change sign under reflection, while the vectors must, the former are even functions of x, the latter odd. Gravity and energy flux are proportional to M/R^2 and L/R^2 respectively. Thus our required property is established.

The argument is nothing more than an application of parity. It is so general that any physical processes (including dynamical changes) can be included in the equations without affecting the conclusion.

(b) The second method is to argue directly from the equations (1) to (4), and cannot be so general as the above since specific physical relations

must be used. There is, however, one point of special interest. We illustrate this by supposing, as is frequently the case, that the energy generation is a function of composition parameters, density and temperature, and also that the pressure is a sum of perfect gas and radiation pressures. We shall now sketch the proof.

Relations (5) are quite general. However, a difficulty arises when trying to obtain the next term in equation (1). It would appear that we cannot know whether the next term is of order R^4 or R^5 since this depends upon the expansion for ρ. The latter, although a function of P and T, depends also on μ, the mean molecular weight, which itself involves the composition parameters. We have no a priori reason for assuming that the composition parameters possess an expansion in even powers of R.

However, consider the initial homogeneous state. Then μ is constant and the whole proof follows quite easily. Now consider subsequent times. Either the center is convecting, so that the result is again true, or it is radiative. If the latter, the first move away from homogeneity involves the rates of nuclear transmutation, which possess the required expansions. Thus the initial homogeneous model generates subsequent models with the required expansions. Within the confines of this argument, it seems necessary to appeal to evolution to prove the result obtained so easily in (a).

The property established above is well-known for polytropes (Chandrasekhar 1939) although it was derived only by expanding the equations. We are not aware of it having been previously established for all physical variables in any stage of a star's evolution. Given this property, we may now usefully express the equations of stellar structure in terms of our new variables.

IV. THE EQUATIONS IN TERMS OF THE NEW VARIABLES

As P and T vary by many orders of magnitude over the whole star, it has been found convenient to use their logarithms as integration variables. Thus the variables we adopt are the set

$$\left.\begin{aligned} m &= M^{2/3} \\ r &= R^2 \\ \ell &= L^{2/3} \\ &\ell n\ P \\ &\ell n\ T \end{aligned}\right\} \qquad (9)$$

For convenience in evolutionary calculations, we take m to be the independent variable, although the form of the variables is such that any one of the five may be so chosen. Let $\alpha = 4\pi/3$. Throughout the body of the star we have

$$\frac{dr}{dm} = \frac{1}{\alpha\rho}\left(\frac{m}{r}\right)^{\frac{1}{2}} \qquad (10)$$

$$\frac{d\ell}{dm} = \epsilon\left(\frac{m}{\ell}\right)^{\frac{1}{2}} \qquad (11)$$

$$\frac{d\ \ell n\ P}{dm} = -\frac{G}{2\alpha P}\left(\frac{m}{r}\right)^2 \qquad (12)$$

$$\frac{d\ \ell n\ T}{dm} = -\frac{1}{8\alpha^2}\ \frac{\kappa}{ac}\ \frac{1}{T^4}\left(\frac{\ell}{r}\right)^{3/2}\left(\frac{m}{r}\right)^{\frac{1}{2}} \qquad (13a)$$

or

$$\frac{d\ \ell n\ T}{dm} = \frac{\Gamma - 1}{\Gamma}\ \frac{d\ \ell n\ P}{dm} \qquad (13b)$$

At the center, the following limiting forms hold,

$$\left(\frac{dr}{dm}\right)_c = \frac{1}{(\alpha\rho_c)^{2/3}} \qquad (14)$$

$$\left(\frac{d\ell}{dm}\right)_c = \epsilon_c^{2/3} \tag{15}$$

$$\left(\frac{d\ \ell n\ P}{dm}\right)_c = -\frac{G}{2}\ \frac{\rho_c^2}{P_c}\ \frac{\alpha}{(\alpha\rho_c)^{2/3}} \tag{16}$$

$$\left(\frac{d\ \ell n\ T}{dm}\right)_c = -\frac{1}{8}\ \frac{\kappa}{ac}\ \frac{\rho_c^2\ \epsilon_c}{T_c^4}\ \frac{1}{(\alpha\rho_c)^{2/3}} \tag{17a}$$

or

$$\left(\frac{d\ \ell n\ T}{dm}\right)_c = \frac{\Gamma - 1}{\Gamma}\ \left(\frac{d\ \ell n\ P}{dm}\right)_c \tag{17b}$$

In situations where L may be temporarily negative (for example, during the helium flash), it may be more convenient to use $RL^{1/3}$ in place of $L^{2/3}$. The appropriate modifications to the above equations are easily made.

V. MAXIMAL CENTRAL ACCURACY

We have already shown that quite general considerations lead to the natural expansion parameter $r = R^2$. Practical considerations require that advantage be taken of this fact. The second order integration process referred to previously gives accurate results to order r^2, i.e., R^4. Similarly, for each increase in order of the scheme used, the new variables attain accuracy to two more powers of R. In a computer, each increase of the integration order requires an extra entry to a generally lengthy subroutine which calculates the derivatives. The latter calculation takes most of the time in integrations. Integration of the equations (1) to (4) as they stand requires twice the time needed to integrate equations (10) to (13) to the same order of accuracy.

We therefore consider our variables to represent an optimum choice in the central regions of a star. In their recent discussion, Larson and Demarque (1964) recognized that the central expansions contained different leading

powers of R. For example, a luminosity variable $q = L\,(1+a/R^2)$ with a a small number was constructed. Thus $q \propto R$ near the center. However, as the integrations were effectively in terms of R, they were not quite as accurate as they might be. We feel our variables would be very useful in the difference equation approaches to stellar structure. Such equations, normally accurate to second order in R, would then give accuracy to 4th order in R in a region where accuracy is highly desirable.

We emphasize that our discussion has dealt entirely with the central variables, and is valid whatever the physical laws might be. For the outer regions the variables can be tailored in the manner of Larson and Demarque to fit the conditions there. Such choices however, seem to depend upon simple expressions for the opacity which may not approach the accurate values as closely as one might wish.

VI. CONCLUSIONS

It has been shown that there exists a natural expansion parameter (the square of the radial distance) at the center of a star. Variables which take advantage of this lead to maximum central accuracy for an integration process of a given order. Integration can be started at the center in terms of these variables. The mathematical difficulties encountered by previous authors are simply due to the wrong choice of variables. The present variables show the central regions for what they are -- both physically and mathematically the best behaved regions of a star.

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