On some inequalities and their application to the Cauchy problem

by

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1. Two inequalities

Let A be a bounded operator in a complex Banach space X, and denote by $\sigma(A)$ the spectrum of A and by $R(\lambda;A)$ its resolvent $(\lambda I - A)^{-1}$. If f is analytic on $\sigma(A)$, i.e., in some neighborhood W of $\sigma(A)$, then f(A) is defined by [1;p.568]

(1.1) $$f(A) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda;A)d\lambda$$

where Γ is a contour lying in $W\setminus\sigma(A)$. If f(z) has a Taylor series expansion $\Sigma\, a_m z^m$ which converges in W then $f(A) = \Sigma\, a_m A^m$. We denote by $\|A\|$ the norm of A. If A is an $N\times N$ matrix then we consider it as an operator in the complex N-dimensional euclidean space.

First inequality. Let A be an $N\times N$ matrix and let f be an analytic function on $\sigma(A)$ having a Taylor series expansion about $z = 0$ which converges in a neighborhood W of $\sigma(A)$. Then

(1.2) $$\|f(A)\| \leq \sum_{j=0}^{N-1} 2^j\|A\|^j \operatorname*{l.u.b.}_{\lambda\in H(A)} |f^{(j)}(\lambda)|,$$

where H(A) is the convex hull of the eigenvalues of A.

This result is due to Gelfand and Shilov [3].

We shall now derive, by a different method, an inequality of the same nature as (1.2), namely: there is a constant C depending only on N, such that, for any δ sufficiently small,

(1.3) $$\|f(A)\| \leq \frac{C}{\delta^{N-1}}(1+\|A\|)^{N-1} \operatorname*{l.u.b.}_{\lambda\in\sigma_\delta(A)} |f(\lambda)|$$

where $\sigma_\delta(A) = \{\lambda;\text{dist.}(\lambda,\sigma(A)) < \delta\}$; δ is restricted only by the require-

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ments that $\sigma_\delta(A) \subset W$ and that $\delta \leq 1$.

Proof of (1.3). We employ (1.1) and shrink Γ to a contour B which, after cancelling out integrals on the same arcs but in reverse orientations, has length $\leq C'\delta$ (C' depending only on N) and is such that $|\lambda - \lambda_j| = \delta$ for any eigenvalue λ_j of A and for all λ in the uncancelled part B' of B. Then $|\det(\lambda I - A)| = |(\lambda-\lambda_1)^{\alpha_1}\dots(\lambda-\lambda_k)^{\alpha_k}| \geq \delta^{-N}$ on B'. Noting that $|\lambda| \leq \|A\| + \delta \leq \|A\| + 1$ on B', we get $\|R(\lambda;A)\| \leq C''(1+\|A\|)^{N-1}\delta^{-N}$ on B' (C'' depending only on N), and (1.3) follows.

Second inequality. Consider a polynomial equation

$$\lambda^N + P_1(s)\lambda^{N-1} + \dots + P_N(s) = 0 \tag{1.4}$$

where $P_j(s)$ are polynomials of degree p_j in the n-dimensional complex variable $s = (s_1,\dots,s_n)$, and set $p_o = \max_{1\leq j\leq N} \frac{p_j}{j}$. Denote by

$$\Lambda(s) = \max_{1\leq j\leq N} \mathrm{Re}\{\lambda_j(s)\}, \qquad \Lambda(r) = \max_{|s|\leq r} \Lambda(s),$$

$$M(s) = \max_{1\leq j\leq N} |\lambda_j(s)|, \qquad M(r) = \max_{|s|\leq r} M(s).$$

Then,

$$\begin{aligned} \Lambda(r) &= \alpha r^{p_o} + O(r^q) \qquad (\alpha > 0,\ q < p_o), \\ M(r) &= \beta r^{p_o} + O(r^q) \qquad (\beta > 0). \end{aligned} \tag{1.5}$$

A slightly weaker result, namely, $\Gamma(r) = O(r^{p_o})$, $\Gamma(r_m) \geq \gamma r_m^{p_o}$ for some $\gamma > 0$, $r_m \to \infty$ and $\Gamma = \Lambda, M$ was proved by Gelfand and Shilov [3], but there is some gap in their proof; this is fixed up in [2], where also the more general version (1.5) is given.

2. The Cauchy problem

Consider the problem of finding a function u satisfying

$$\frac{\partial u}{\partial t} = P\left(\sqrt{-1}\,\frac{\partial}{\partial x}\right)u \qquad (0 < t \leq T,\ x \in R^n), \tag{2.1}$$

$$u(x,0) = u_o(x) \qquad (x \in R^n) \tag{2.2}$$

(u is to be continuous for $0 \leq t \leq T$, $x \in R^n$) where R^n is the real n-dimensional euclidean space, $u = (u_1,\dots,u_N)$, $u_o = (u_{o1},\dots,u_{oN})$, $P(s)$ is an $N \times N$ matrix whose elements are polynomials of degree $\leq p$ in $s = (s_1,\dots,s_n)$, and $\frac{\partial}{\partial x} = (\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n})$. For simplicity we take P to be independent of t, but all the results of this work extend to the case where $P = P(t,\sqrt{-1}\,\frac{\partial}{\partial x})$. The system (2.1),(2.2) is called a Cauchy system. To solve it we first take, formally, the Fourier transform, and get

$$\frac{\partial v}{\partial t} = P(\sigma)v, \tag{2.3}$$

$$v(\sigma,0) = v_o(\sigma), \tag{2.4}$$

whose formal solution is given by $e^{tP(\sigma)}v_o(\sigma)$ $(\sigma \in R^n)$, and then we have to analyze the inverse transform, which should yield a solution of (2.1), (2.2). Actually, this procedure is too crude and a more sophisticated procedure is needed, which employs certain topological spaces and their conjugate spaces; for details the reader is referred to [2],[3].

One concludes that uniqueness holds under the assumption that

$$|u(x,t)| \leq B \exp(\beta |x|^q) \quad \text{for } 0 \leq t \leq T, \tag{2.5}$$

where B,β are positive constants, $\frac{1}{q} + \frac{1}{p_o} = 1$ and p_o is defined as in §1, where (1.4) is the characteristic equation for $P(s)$.

In proving this result one has to show that $e^{tP^*(\sigma)}v_o(\sigma)$ is a solution of (2.3),(2.4) with P replaced by P^* (P^* = transpose of P) in some "W space" of entire functions. This proof is based upon the bound

$$\| e^{tP(s)} \| \leq C(1 + |s|)^{(N-1)p} \exp(ct|s|^{p_o}) \tag{2.6}$$

where C,c are positive constants. Thus the uniqueness proof employs in a substantial manner the inequality (2.6), which in turn follows from the two inequalities of §1.

To prove existence one first reduces the problem (up to some routine estimates of integrals) to the problem of studying the inverse Fourier transform of $e^{tP(\sigma)}$ (i.e., Green's function); see [2]. Next, a better inequality than (2.6) is needed, but only for $s = \sigma$ real. One assumes that

$$\Lambda(\sigma) \leq \gamma|\sigma|^h + \delta \tag{2.7}$$

and, depending on γ, h one obtains different bounds on $e^{tP(\sigma)}$ and on its derivatives, and thus different structures for Green's function. If $\gamma < 0$, $0 < h \leq p_o$ then one can prove that there exists a classical solution of (2.1),(2.2), and that it satisfies (2.5), provided u_o and some of its derivatives have at most an exponential growth; if $h = p_o$ it suffices to assume that $u_o(x)$ is continuous and is $O[\exp(\beta|x|^q)]$ where $0 < \beta < \beta_o T^{-1/(p_o-1)}$, β_o depending only on P. If $\gamma \geq 0$, then more restrictive assumptions are made on u_o (see [2],[3]).

The Goursat problem

$$\frac{\partial^\nu u}{\partial t_1 \ldots \partial t_\nu} = P(i\frac{\partial}{\partial x})u,$$

$$u(x,t)\Big|_{t_i=0} = u_{oi}(x,t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_\nu) \quad (i = 1,\ldots,\nu) \tag{2.8}$$

can be handled along the same lines (see [2]). Instead of $e^{tP(s)}$ we now have to deal with $\sum_{m=0}^{\infty} (P(s))^m/(m!)^\nu$. Uniqueness holds under the assumption (2.5) where $\frac{1}{q} + \frac{\nu}{p_o} = 1$. Existence theorems can also be derived, but there is a remarkable difference between the case $\nu = 2$ where solutions exist under "reasonable" conditions on u_o (i.e., a finite number of derivatives of u_o are assumed to exist and to be bounded by $O(|x|^\gamma)$ for some γ) and on the eigenvalues of $P(\sigma)$, and the case $\nu > 2$ where very restrictive assumptions on u_o are required.

3. Additional inequalities

We shall consider some generalizations of the first inequality of §1 to general bounded operators A. The last result of this section will be substantially used in §4.

Proposition 1. If $\|A\| < r$ and f is analytic in $|\lambda| \leq r$, then

$$\|f(A)\| \leq \frac{r}{r-\|A\|} \operatorname*{l.u.b.}_{|\lambda|\leq r} |f(\lambda)|. \tag{3.1}$$

This follows from (1.1) upon using Neumann's series for $R(\lambda;A)$. A better result holds in Hilbert spaces (but is false in Banach spaces!):

Proposition 2. Let A be a bounded operator in a Hilbert space X and let f be an analytic function in $|\lambda| \leq \|A\|$. Then

$$\|f(A)\| \leq \operatorname*{l.u.b.}_{|\lambda|\leq\|A\|} |f(\lambda)|. \tag{3.2}$$

This result is due to von Neumann [5]; a simpler proof was given by E. Heinz [4] (see also [6]). It is also proved in [5] (and in [4]) that if

$$\operatorname{Re}(A\varphi,\varphi) \geq 0 \quad \text{for all } \varphi \in X \tag{3.3}$$

then $(A-I)(A+I)^{-1}$ exists and has a norm ≤ 1. Employing Proposition 2, one gets:

Proposition 3. Let A be a bounded operator in a Hilbert space X and assume that it satisfies (3.3). If f is an entire function then

$$\|f(A)\| \leq \operatorname*{l.u.b.}_{\operatorname{Re}\lambda\geq 0} |f(\lambda)|. \tag{3.4}$$

4. The Cauchy problem for infinite systems

We shall extend the results of §2 to an infinite system of equations, i.e.,

$$\frac{\partial u_i}{\partial t} = \sum_{j=1}^{\infty} p_{ij}\left(\sqrt{-1}\,\frac{\partial}{\partial x}\right)u_j \qquad (i = 1,2,\ldots), \tag{4.1}$$

$$u_i(x,0) = u_{oi}(x) \qquad (i = 1,2,\ldots). \tag{4.2}$$

At this point we have to introduce the space $W^{p,b}_{p,a}$ which occurs in the case of finite systems. $W^{p,b}_{p,a}$ is a Fréchet space whose elements are those entire functions $f(z)$ $(z = (z_1,\ldots,z_n))$ satisfying

$$|f(z)| \leq C' \exp[-\frac{a'}{p}|x|^p + \frac{b'}{p}|y|^p] \qquad (z = x + iy)$$

for all $a' < a$, $b' > b$ where C' is a constant depending on a', b', f. The metric is given by a sequence of norms $\|f\|_j = \sup M_j(z)|\varphi(z)|$ $(j = 1,2,\ldots)$ where $M_j(z) = a(1 - \frac{1}{j})\frac{|x|^p}{p} - b(1 + \frac{1}{j})\frac{|y|^p}{p}$; i.e.,

$$d(f,g) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\|f - g\|_j}{1 + \|f - g\|_j} .$$

In the case of §2 the component u_j is considered as a functional over W_j, where $W_j = W^{p,b}_{p,a}$ for all j. In the present case we introduce the direct product $\prod_{j=1}^{\infty} W_j$, and the metric

$$(4.3) \qquad \hat{d}(\varphi,\psi) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\|\varphi - \psi\|_j}{1 + \|\varphi - \psi\|_j} \quad \text{where } \|\varphi\|_j = \sum_{j=1}^{\infty} \|\varphi_j\|_j ;$$

here $\varphi = (\varphi_1, \varphi_2, \ldots)$, $\psi = (\psi_1, \psi_2, \ldots)$. The elements φ with $\|\varphi\|_j < \infty$ for $j = 1,2,\ldots$ form a Fréchet space $\hat{W}^{p,b}_{p,a}$.

Assume now that

$$(4.4) \qquad |p_{ij}(s)| \leq \pi_{ij}(1 + |s|^p), \ \pi_{ij} \text{ constants, } \sup_i \sum_{j=1}^{\infty} \pi_{ij} \leq \gamma < \infty .$$

In order to prove uniqueness for (4.1),(4.2), we proceed as in the case of finite systems and thus reduce the problem to showing that

$$\psi(\sigma,t) = e^{tP^*(\sigma)}\psi_o(\sigma) \qquad (\psi_o \in \hat{W}^{p,a}_{p,b},\ P^* = \text{transpose of } P)$$

is a solution in $\hat{W}^{p,b+c}_{p,a-c}$ (for some $c < a$) of

$$(4.5) \qquad \frac{\partial\psi}{\partial t} = P^*(\sigma)\psi, \qquad \psi(\sigma,0) = \psi_o .$$

(It is actually enough to consider ψ_o with all but one component equal to zero.)

Since

$$\sum_{i=1}^{\infty} |(P^*(s)\psi(s,t))_i| \leq (1 + |s|^p) \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \pi_{ji}|\psi_j(s)| \leq \gamma(1 + |s|^p) \sum_{j=1}^{\infty} |\psi_j(s)|,$$

it follows that P^* is a bounded operator in $\hat{W}^{p,b}_{p,a}$. Next,

$$\sum_{i=1}^{\infty} |(e^{tP^*(s)}\psi(s,t))_i| \le \sum_{i=1}^{\infty}\sum_{m=0}^{\infty}\left|\left(\frac{t^m(P^*(s))^m}{m!}\psi(s,t)\right)_i\right|$$

$$\le \sum_{m=0}^{\infty}\frac{\gamma^m t^m(1+|s|)^{pm}}{m!}\sum_{i=1}^{\infty}|\psi_i(s)| = e^{\gamma t(1+|s|)^p}\sum_{i=1}^{\infty}|\psi_i(s)|,$$

and we thus find that $\psi(s,t)$ is in $\hat{W}^{p,b+c}_{p,a-c}$ for some $c < a$ (depending on γ, T). The proof that $\psi(\sigma,t)$ satisfies (4.5) follows without difficulty (compare [2]). We thus obtain the following uniqueness theorem:

Theorem 1. *If (4.4) holds then there exists at most one classical solution of (4.1),(4.2) satisfying*

$$|u_i(x,t)| \le C\, e^{\beta|x|^q} \qquad \left(\frac{1}{q}+\frac{1}{p}=1\right) \tag{4.6}$$

for some $C > 0$, $\beta > 0$ *and* $x \in R^n$, $0 \le t \le T$, $i = 1,2,\ldots.$

As in [2],[3] the condition (4.6) can be replaced by

$$\int_{R^n}|u_i(x,t)|\,e^{-\beta|x|^q}dx \le C. \tag{4.7}$$

Consider now the question of existence. As in the case of a finite system, a "generalized solution" always exists, and we wish to prove that it is also a solution in the classical sense. For this we need to make some differentiability and boundedness assumptions on $u_0(x)$ and also put some conditions on $P(\sigma)$. In the finite case we impose conditions on the eigenvalues $\lambda_i(\sigma)$ of $P(\sigma)$. In the present infinite case we give a different kind of condition on $P(\sigma)$ which will turn out to have the same effect as in the finite case.

We wish to consider $e^{tP(\sigma)}$ as a bounded operator in ℓ^2. We thus need to know that $P(\sigma)$ is a bounded operator in ℓ^2. For this it suffices to assume that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|p_{ij}(\sigma)|^2 < \infty. \tag{4.8}$$

We now impose the following condition: For any $\sigma \in R^n$,

(4.9) $\quad \operatorname{Re}(P(\sigma)v,v) \leq (-C|\sigma|^h + C_1)(v,v) \qquad (C > 0,\ 0 < h \leq p)$

where $(v,w) = \Sigma\, v_i\bar{w}_i$. If $h = p$, $C_1 = 0$ and P is equal to its principal part, then this condition is known as the strong ellipticity condition for P.

Using (4.9) and applying Proposition 3, we deduce:

(4.10) $\quad \|e^{tP(\sigma)}\| \leq C_o e^{-tC|\sigma|^h} \qquad (C_o = e^{T|C_1|})$

where $e^{tP(\sigma)}$ is considered as an operator in l^2. Thus, in particular,

(4.11) $\quad \sum_{j=1}^{\infty} |(e^{tP(\sigma)})_{ij}|^2 \leq C_o e^{-tC|\sigma|^h}.$

Using this bound one can now analyze Green's function $G(x,t)$ (i.e., the inverse Fourier transform of $e^{tP(\sigma)}$) and then the abstract convolution $G(x,t) * u_o(x)$. This is done along the same lines as for finite systems, and we obtain analogous existence theorems. The cases where $C = 0$, $C < 0$ can be treated in a similar manner. We list below just two results which are thus obtained (one could easily write down all the other existence theorems, by following the arguments for finite systems):

Theorem 2. *If $h = p$ in (4.9) then for any continuous function $u_o(x)$ whose components satisfy*

(4.12) $\quad |u_{oj}(x)| \leq C_j e^{\gamma|x|^q}, \quad \sum_{j=1}^{\infty} C_j^{\,2} < \infty, \quad 0 < \gamma \leq \gamma_o T^{-1/(p-1)}$

(γ_o depending only on P), there exists a classical solution of (4.1), (4.2) satisfying (4.6) for some constants C, β.

Theorem 3. *If $C = 0$ in (4.9) and if for some $\nu \geq 0$, $\gamma > 0$ the first $p+\nu+n$ derivatives of $u_o(x)$ are continuous functions satisfying*

(4.13) $\quad \left|\frac{\partial^i}{\partial x^i} u_{oj}(x)\right| \leq C_j(1+|x|)^{\gamma}, \quad \sum_{j=1}^{\infty} C_j^{\,2} < \infty \quad (0 \leq i \leq p+n+\nu)$

<u>where</u> $\gamma + n + 1 \leq \gamma/\mu$ ($\mu > 0$ <u>depending only on</u> P) <u>then there exists a classical solution of</u> (4.1),(4.2) <u>satisfying</u>

$$\left|\frac{\partial^i}{\partial x^i} u_j(x,t)\right| \leq C(1+|x|)^\gamma \qquad (0 \leq i \leq p). \tag{4.14}$$

If $p = 1$ and $C = 0$ in (4.9) then we have the same situation as in the finite hyperbolic case, where Green's function has a compact support.

The norms and metric in (4.3) were chosen quite arbitrarily; other definitions can be made and we then obtain variants of the previous results. Thus if we modify the definition (4.3) by setting $\|\varphi\|_j = \sup_i \|\varphi_i\|_j$ and then modify (4.4) by replacing the last condition by

$$\sup_j \sum_{i=1}^{\infty} \pi_{ij} \leq \gamma,$$

then Theorem 1 remains true if in (4.6) C is replaced by C_i and $\Sigma\, C_i < \infty$.

We finally wish to observe that our results do not yield anything new in the case of finite systems. In fact, the condition (4.9) implies the condition (2.7) with $\gamma = -C$. This follows from the obvious inequality

$$e^{t\Lambda(\sigma)} \leq \|e^{tP(\sigma)}\| \tag{4.15}$$

(as $e^{t\lambda_i(\sigma)}$ are the eigenvalues of $e^{tP(\sigma)}$) and (4.10). From (4.10), (4.15) we also get:

<u>Corollary</u>. <u>If</u> P <u>is strongly elliptic then</u> $\partial u/\partial t = Pu$ <u>is parabolic in the sense of Petrowski</u> (i.e., $h = p_0 = p$; see [2]).

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