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On The Number of L^2 Solutions of $-y'' + q(t)y = \lambda y$

Where $q(t)$ Is Complex Valued.

by

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Let us consider the differential equation $-y'' + q(t)y = \lambda y$ on an interval with the right end at ∞ . It has been shown [1] that when $q(t)$ is real valued; under certain conditions the differential equation does not possess two linearly independent solutions, both square summable toward ∞ . ($\int_{\infty}^{\infty} | \cdot |^2 dt < \infty$). This is known as the classical limit point case of Hermann Weyl [6], its negation - the existence of two square summable solutions - being the limit circle case. The reason for these names is readily apparent from the theory [6].

While the correspondence between the number of square summable solutions and the limit point - circle cases is not quite so precise when $q(t) = q_1(t) + i q_2(t)$, where $q_1(t)$ and $q_2(t)$ are real valued and $q_2(t) \neq 0$, a great many results can be extended. (See [2], [3], [4], and [5]). The purpose of this paper is to show that when $q_2(t) \neq 0$, if $q_1(t)$ satisfies the same conditions stated for $q(t)$ when $q(t)$ is real valued, then the number of square summable solutions toward ∞ can still be restricted in the same way.

Theorem. Suppose there exists a positive, differentiable function
 $M(t)$ satisfying $M(t) \geq M_0 > 0$, $q_1(t) \geq -k_1 M(t)$ for some constant
 k_1 , $|M'(t) M^{-3/2}(t)| \leq k_2$ for some constant k_2 , $\int_{\infty}^{\infty} M^{-1/2}(t) dt = \infty$.

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Then the differential equation $-y'' + (q_1(t) + i q_2(t))y = \lambda y$
does not possess two linearly independent solutions, square summable
toward ∞ .

Proof. It has been shown ([1] or [2]) that if two square summable solutions exist for any value of λ , then two exist for all values of λ . Thus, without loss of generality, we can choose $\lambda = i\nu$ where ν is real.

Let X satisfy $-X'' + (q_1(t) + i q_2(t))X = i\nu X$ be square summable toward ∞ . We then have

$$\int_c^t \frac{X'' \bar{X}}{M} dt = \int_c^t \frac{q_1 |X|^2}{M} dt + i \int_c^t \frac{[q_2 - \nu] |X|^2}{M} dt.$$

Integrating the left side by parts,

$$\int_0^t \frac{|X'|^2}{M} dt = \frac{X' \bar{X}}{M} \Big|_c^t + \int_c^t \frac{X' \bar{X} M'}{M} dt - \int_c^t \frac{q_1 |X|^2}{M} dt +$$

$$- i \int_c^t \frac{[q_2 - \nu] |X|^2}{M} dt.$$

Taking real part of this equation, we have

$$\int_c^t \frac{|X'|^2}{M} dt = \frac{1}{2} \frac{d}{dt} |X|^2 \Big|_c^t + \operatorname{Re} \int_c^t \frac{X' \bar{X} M'}{M} dt - \int_c^t \frac{q_1 |X|^2}{M} dt.$$

Using Cauchy's inequality on the second term of the right side.

$$\int_c^t \frac{|X'|^2}{M} dt \leq \frac{1}{2} \frac{d}{dt} |X|^2 \Big|_c^t + \sup \left| \frac{M'}{M^{3/2}} \right| \left(\int_c^t \frac{|X'|^2}{M} dt \right)^{1/2} \left(\int_c^t |X|^2 dt \right)^{1/2} -$$

$$- \int_c^t \frac{q_1 |X|^2}{M} dt.$$

Letting $H(t) = \int_c^t \frac{|X'|^2}{M} dt$,

$$H(t) - \frac{1}{2} \frac{d}{dt} |X|^2 \Big|_c^t - \sup_{t \geq c} \left| \frac{M'}{M^{3/2}} \right| \left(\int_c^t |X|^2 dt \right)^{1/2} H(t)^{1/2} \\ \leq - \int_c^t \frac{q_1 |X|^2}{M} dt.$$

Choose c sufficiently large so that

$$k_2 \left(\int_c^\infty |X|^2 dt \right)^{1/2} < \frac{1}{2}. \quad \text{Then } H(t) - \frac{1}{2} \frac{d}{dt} |X|^2(t) - \frac{1}{2} H(t)^{1/2} < k_3,$$

for some k_3 . If $\lim_{t \rightarrow \infty} H(t) = \infty$, then eventually $\frac{d}{dt} |X|^2 > H(t)/2$.

Thus $|X|^2(t)$ is increasing, and $X \notin L^2(0, \infty)$. Thus $H(t)$ is bounded.

Now let θ, ϕ be solutions square summable toward ∞ .

Assume $W[\theta, \phi] = \theta \phi' - \phi \theta' = 1$ for all t . Then

$$\frac{\theta \phi'}{M^{1/2}} - \frac{\phi \theta'}{M^{1/2}} = \frac{1}{M^{1/2}}.$$

The left is summable by Schwarz's inequality. The right is not, giving us a contradiction.

We are now in a position to begin to characterize the roles of $q_1(t)$ and $q_2(t)$. [2] shows that if $\lim_{t \rightarrow \infty} q_2(t) = \gamma$ and

$\lim_{t \rightarrow \infty} q_1(t) = \delta$, then with the exception of a countable number of eigenvalues, the lines $\lambda = i\gamma$ and $\lambda = i\delta$ contain the spectrum of the operator L defined by $Ly = -y'' + q(t)y$. On the other hand $q_1(t)$ determines the nature of the spectrum. If two square summable solutions of $-y'' + q(t)y = \lambda y$ exist toward ∞ , the spectrum on $\lambda = i\gamma$ consists of a countable number of eigenvalues.

If only one exists, the spectrum is still unspecified. Similarly if two square summable solutions exist toward $-\infty$, then the spectrum on $\lambda = i\delta$ consists of a countable number of eigenvalues. If only one exists, the spectrum is unspecified. (See [3].)

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