On The Eigenvalues of a Singular Nonself-Adjoint Differential Operator of Second Order

by

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The nonself-adjoint operator $\ell y = y'' + q(x)y$, where $q(x) = q_1(x) + iq_2(x)$, $q_1(x)$ and $q_2(x)$ are real valued, $\lim_{x\to a} q_2(x) = \delta$, $\lim_{x\to b} q_2(x) = \gamma$, was considered over an interval (a,b) in [1]. From ℓ a nonself-adjoint operator L was defined in $L^2(a,b)$. The spectrum and adjoint of L were found, and a "spectral resolution" was derived.

If r is an arbitrary point in (a,b), it was shown that when $\lambda = \mu + i\nu$, $\nu \neq \gamma$, $\ell y = \lambda y$ has a solution $\psi(x,\lambda)$ in $L^2(r,b)$, and when $\nu \neq \delta$, $\ell y = \lambda y$ has a solution $\eta(x,\lambda)$ in $L^2(a,r)$. λ is an eigenvalue of L if and only if the Wronskian of $\psi(x,\lambda)$ and $\eta(x,\lambda)$ is zero. ~~The problem~~ This paper wishes to ~~consider is to~~ characterize these eigenvalues, the zeros of $W[\psi,\eta]$.

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In what follows every expression is a function of the complex variable λ. In the interest of notational clarity, it has been suppressed.

Let $\theta(x)$ and $\phi(x)$ be the solutions of $\ell y = \lambda y$ satisfying $\theta(r) = 1$, $\theta'(r) = 0$, $\phi(r) = 0$, $\phi'(r) = -1$. We choose s in $[r,b)$ such that $|q_2(x)-\gamma|<|\nu-\gamma|/2$ for all x in $[s,b]$ when γ is finite, or such that $|q_2(x)-\nu| > \epsilon$ for all x in $[s,b]$ for some $\epsilon > 0$ when γ is infinite.

We then define $y_{1b}(x) = \theta'(s)\phi(x) - \phi'(s)\theta(x)$ and $y_{2b} = \theta(s)\phi(x) - \phi(s)\theta(x)$.

In $[s,b)$ a sequence of nested circles $C(\beta)$ is found all containing a limit circle or limit point given by

$$M = \lim_{\beta\to b} - \frac{y_{1b}(\beta)z_b + y_{1b}'(\beta)}{y_{2b}(\beta)z_b + y_{2b}'(\beta)},$$

where z_b is any real number. $\psi(x)$ is then given by

$$\psi(x) = \theta(x)[-\phi'(s) + M\phi(s)] + \phi(x)[\theta'(s) + M\theta(s)].$$

(See [1] and [2].)

Similarly we choose t in (a,r) such that $|q_2(x)-\delta| < |\nu-\delta|/2$ for all x in $(a,t]$ when δ is finite, or such that $|q_2(x)-\nu| > \epsilon$ for all x in $(a,t]$ for some $\epsilon > 0$ when δ is infinite.

We define $y_{1a}(x) = \theta'(t)\phi(x) - \phi'(t)\theta(x)$ and $y_{2a}(x) = -\theta(t)\phi(x) + \phi(t)\theta(x)$.

In $(a,t]$ a sequence of nested circles $C(\alpha)$ is found, all containing a limit circle or limit point given by

$$-m = \lim_{\alpha \to a} - \frac{y_{1a}(\alpha) z_a + y_{1a}'(\alpha)}{y_{2a}(\alpha) z_a + y_{2a}'(\alpha)},$$

where z_a is any real number. $\eta(x)$ is then given by

$$\eta(x) = -\theta(x)[\phi'(t) + m\phi(t)] + \phi(x)[\theta'(t) + m\theta(t)].$$

(Again see [1] and [2].)

<u>Lemma</u>. <u>Let</u> $D_b(\beta) = y_{2b}(\beta) z_b + y_{2b}'(\beta)$, $D_a(\alpha) = y_{2a}(\alpha) z_a + y_{2a}'(\alpha)$. <u>Then</u> $W[\psi, \eta] =$

$$\lim_{\substack{\alpha \to a \\ \beta \to b}} \{[\phi(\beta) z_b + \phi'(\beta)][\theta(\alpha) z_a + \theta'(\alpha)]$$

$$- [\theta(\beta) z_b + \theta'(\beta)][\phi(\alpha) z_a + \phi'(\alpha)]\} / D_a(\alpha) D_b(\beta).$$

Proof. We observe that $W[\psi, \eta]$ is independent of x. Computing $W[\psi, \eta]$ at $x = r$ we see

$$W[\psi, \eta] = [\phi'(s) + M\phi(s)][\theta'(t) + m\theta(t)] -$$

$$- [\theta'(s) + M\theta(s)][\phi'(t) + m\phi(t)].$$

Inserting the expressions for m and M and taking limits completes the proof.

<u>Theorem 1</u>. <u>If</u> $D_a(\alpha)$ <u>and</u> $D_b(\beta)$ <u>approach finite limits as</u> $\alpha \to a$ <u>and</u> $\beta \to b$, then $W[\psi, \eta] = 0$ <u>if and only if</u>

$$\lim_{\substack{\alpha \to a \\ \beta \to b}} \{[\phi(\beta) z_b + \phi'(\beta)][\theta(\alpha) z_a + \theta'(\alpha)]$$

$$- [\theta(\beta) z_b + \theta'(\beta)][\phi(\alpha) z_a + \phi'(\alpha)]\} = 0.$$

Note that this last equation in no way depends upon the points $x = s$ or $s = t$.

In the limit point cases the value of M (or m) is independent of the choice of z_b (or z_a). Only in the limit circle cases does M (or m) vary with the choice of z_b (or z_a).

Theorem 2. If the limit point case holds at a and b, then under the conditions of theorem 1

$W[\psi, \eta] = 0$ if and only if

$$\lim_{\substack{\alpha\to a\\ \beta\to b}} [\phi(\beta)\theta(\alpha) - \theta(\beta)\phi(\alpha)] = 0,$$

$$\lim_{\substack{\alpha\to a\\ \beta\to b}} [\phi(\beta)\theta'(\alpha) - \theta(\beta)\phi'(\alpha)] = 0,$$

$$\lim_{\substack{\alpha\to a\\ \beta\to b}} [\phi'(\beta)\theta(\alpha) - \theta'(\beta)\phi(\alpha)] = 0,$$

$$\lim_{\substack{\alpha\to a\\ \beta\to b}} [\phi'(\beta)\theta'(\alpha) - \theta'(\beta)\phi'(\alpha)] = 0.$$

Proof. Multiply the expression in theorem one out and collect the coefficients of $z_a z_b$, z_a and z_b together. Since z_a and z_b are arbitrary the result follows.

Theorem 3. If the limit point case holds at b, and the limit circle case holds at a, then under the conditions of theorem 1, $W[\psi, \eta] = 0$ if and only if

$$\lim_{\substack{\alpha\to a\\ \beta\to b}} \{z_a[\phi(\beta)\theta(\alpha) - \theta(\beta)\phi(\alpha)] + [\phi(\beta)\theta'(\alpha) - \theta(\beta)\phi'(\alpha)]\} = 0,$$

$$\lim_{\substack{\alpha\to a\\ \beta\to b}} \{z_a[\phi'(\beta)\theta(\alpha) - \theta'(\beta)\phi(\alpha)] + [\phi'(\beta)\theta'(\alpha) - \theta'(\beta)\phi'(\alpha)]\} = 0.$$

A similar statement is valid if the limit circle case holds at b, and the limit point cast holds at a.

These results are valid only in nonself-adjoint problems. They are not applicable in the self-adjoint case, since, in that instance, all eigenvalues lie on the real axis. There the existence of ψ and η cannot be established in general.

References

[1] Allan M. Krall, "On the expansion problem for nonself-adjoint ordinary differential operators of second order," submitted for publication.

[2] ____________________, "On nonself-adjoint ordinary differential operators of second order," Doklady Akademii Nauk, to appear.