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THE EFFECT OF COHERENT RADIATION ON THE STABILITY OF A CROSSED-FIELD ELECTRON BEAM

by

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ABSTRACT

It has been suggested that a toroidal space vehicle might be raised to a very high positive potential relative to the surrounding space by causing a stable crossed-field electron beam to circulate around its outer surface. In this arrangement the electric field exists only between the space vehicle and the electron beam; the magnetic field is imposed by coils within the space vehicle. It has been shown that such a beam can be stabilized against the diocotron (slipping stream) instability by being made sufficiently thick. The stability against coherent radiative perturbations of such a thick beam is studied under the following simplifying assumptions: 1) the geometry is taken as infinite cylindrical; 2) the ratio of all frequencies in the problem to the electron cyclotron frequency is negligible; 3) no perturbation electric fields along the magnetic field lines; 4) on the basis of an analogy, but without direct proof, certain continuous spectra of real eigenvalues occurring in the problem are unimportant; 5) the electrons are cold and 6) the outer boundary of the "space vehicle", i.e. the cylinder, is perfectly conducting. It is concluded that if there is a gap between the inner edge of the beam and the cylinder, one unstable mode is present for each azimuthal mode number $\ell \geq 1$.

Author

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LIST OF SYMBOLS

r, Θ, z	Cylindrical coordinates.*
t	Time.
m	Electronic mass.
e	Absolute value of electronic charge.
ϵ_o, μ_o	Permittivity and permeability of free space.
c_o	Speed of light.
$\underline{E}$	Electric field vector.
E_o	Unperturbed (radial) electric field.
E_r, E_Θ	Complex amplitudes of perturbation components of the electric field.
$\underline{B}$	Magnetic field vector.
$B_o(r)$	Unperturbed (axial) magnetic field.
B	Complex amplitude of perturbation axial magnetic field.
B_∞	$B_o(\infty)$
n_o	Unperturbed electron number density.
$\underline{v}$	Velocity vector.
v_o	Unperturbed (azimuthal) electron velocity.
β	$v_o(b)/c_o$
$\Omega_o(r)$	Angular velocity of unperturbed electron beam.
Ω_o	$\Omega_o(b)$
$V_o(r)$	Unperturbed electrostatic potential. $V_o(\infty) = 0$.

*MKS units are used throughout this paper.

ω_p	Plasma frequency = $(n_o e^2/\epsilon_o m)^{1/2}$
ω_c	Cyclotron frequency = eB_o/m
ω_o	ω_p^2/ω_c. Basic frequency in this paper.
k_o	ω_o/c_o. Free space wave number at frequency ω_o.
a, b, c	Dimensions of electron beam. See Fig. 1.
r_1	Location of jump in ω_o.
ℓ	Azimuthal mode number.
ω	Eigenfrequency (complex in general).
z	ω/ω_o
ζ	$(z^2-1)^{1/2}$
$I_\ell, K_\ell, J_\ell, Y_\ell, H_\ell^{(1)(2)}$	Bessel functions.
s	Dummy variable.
F_1, F_2, F_5, F_6	Functions defined in Section 6.
Q_1, Q_2, Q_3, Q_4	Functions defined in Section 6.
g_1, g_2	Functions defined in (6. 24) and (6. 25); modified definitions given in (7. 32) and (7. 33).
M(z)	Function defined in (6. 26).
P_1, P_2, P_3, P_4	Functions defined in Section 7.
f_1, f_2	Functions defined in (7. 14) and (7. 15).
D	Discriminant of a quadratic; see (7. 8).
$\omega_{D\pm}$	Two diocotron frequencies.
$z_{D\pm}$	$\omega_{D\pm}/\omega_o$; see (7. 9).
$\delta\omega_{D\pm}$	Radiative correction to $\omega_{D\pm}$
$\delta z_{D\pm}$	$\delta\omega_{D\pm}/\omega_o$
$\mathcal{L}$	Linear operator defined in (3. 12).

1. INTRODUCTION

It has been suggested[1] that a toroidal space vehicle might be raised to a very high positive potential relative to the surrounding space by causing a crossed-field electron beam to circulate around its outer surface. The magnetic field would be provided by a field coil running around the torus while the electric field would come from the separation of the negative charge in the electron beam from a positive charge on the surface of th space vehicle.* These two charges being equal in magnitude but opposite in sign, the arrangement would have no net charge. The large mobility of the electrons parallel to the magnetic field will ensure that the magnetic field lines are equipotentials and hence that the magnetic and electric fields are everywhere mutually perpendicular. This scheme is reasonable in the sense that losses due to such causes as collisions of the electrons with neutral gas atoms or with each other can lead to long (~days) containment times.[1] However, certain types of instability to which crossed-field electron beams are subject can, in appropriate circumstances, have rapid growth rates. The crucial question therefore concerns the stability of the equilibrium situation as outlined above.

The most important instability of the crossed-field electron beam is the diocotron (or slipping stream) instability.[2,3,4,5] This instability is of great importance in the theory of the crossed-field microwave magnetron. It can be thought of as arising from the interaction of two surface waves propagating one along each edge of the beam. This interaction gives

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*A magnetron in roughly this geometry was proposed by Buneman.[13]

no amplification for wavelengths much shorter than the beam thickness. If now the beam travels periodically around a closed path, the longest admissible wavelength will be roughly the perimeter of the path. The possibility therefore arises that in periodic geometries, the beam may be made sufficiently thick that it is stable against perturbations of all admissible wavelengths. In a geometry related (but not identical) to the toroidal configuration discussed above, stabilization along these lines, i.e., by making the beam thick enough, is in fact possible.[6]

In this paper we consider a potential instability of a different kind, one furthermore that is apparently not of much interest in magnetron work. We refer to a mechanism whereby a bunching of the electron beam causes the emission of electromagnetic radiation directly into space. The question arises whether the emission of this radiation increases the bunching and thereby causes an instability to grow. In this case, in order to supply the radiated energy, the electron beam would have to fall in towards the space vehicle. Alternatively, the emission of the radiation might cause debunching, and hence lead to stability. In either case, the dynamical reaction of the electromagnetic radiation field back on the electron beam is critical. This type of study, which appears to be new, might be called "flexible antenna theory". Two considerations distinguish this problem from problems studied in connection with the microwave magnetron. In the first place, the radiation takes place directly into space, so that elements such as periodic slow-wave structures designed to excite instabilities and external antennas are absent. In these circumstances the boundary condition applied at large distances from the electron beam is the radiation condition requiring only outgoing waves to be present. In the second place emphasis is placed on the search for configurations in which all modes are stable (i. e. damped) rather than the search (characteristic of magnetron studies) for instabilities exhibiting the largest possible growth rates or propagation constants. In spite of these differences from standard magnetron theory, the form of analysis which we apply to this problem is entirely classical. A perturbation procedure yields a standard boundary value problem leading to an equation for the permitted (complex) frequencies of oscillation. From the roots of this equation we draw conclusions as to the possible presence of unstable modes of oscillation.

In this initial study of the problem, several simplifying assumptions are introduced. These are:

1) the electron cyclotron frequency ω_c is much greater than all the other frequencies in the problem; these are a) the electron plasma frequency ω_p, b) the frequency Ω with which the electrons perform their orbits in the drift direction and, c) the frequency ω of the perturbation studied. It follows from this assumption that all cyclotron resonance phenomena are neglected, and that the electron dynamics are governed by the simple equation

$$\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B} = 0 \tag{1.1}$$

except possibly in the direction of the magnetic field. It does not appear possible at this stage to make a prediction concerning the possibility of an instability associated with electron cyclotron resonance. This topic is therefore left for study in a later paper.

2) the electron temperature is negligible. One might suppose that a cold electron beam is more subject to coherent radiative instabilities

than a warm one. In this sense, the assumption is a pessimistic one from the point of view of overall stability. However, the effects of this assumption definitely require further study.

3) No perturbations involve components of the electric field parallel to the magnetic field. In a geometry infinitely long in the magnetic field direction, a proper accounting for motions of the electrons along the field lines might possibly result in instabilities. But it is also likely that in a geometry of finite length in this direction (or, as in the application to space vehicles, a geometry in which the field lines curve and close on themselves) a certain minimum length is required before such an instability could manifest itself. The quantitative establishment of such a criterion remains to be calculated.

Taken in conjunction with the assumption that the electrons are cold, this assumption allows (1. 1) to be solved for the electron velocity which is:

$$\underset{\sim}{v} = (\underset{\sim}{E} \times \underset{\sim}{B})/B^2 \tag{1.2}$$

This velocity can also be interpreted as the velocity of the magnetic field lines.

4) It is assumed on the basis of an analogy but without direct proof that a certain continuous spectrum of real eigenvalues which arises in the analysis does not lead to instability. This assumption is discussed further in context in Section 4.

5) The surface of the cylinder is assumed to be perfectly conducting. A number of studies have been made[7,8] of the stability of a crossed-field electron beam moving between plane parallel resistive walls. If one wall

is assumed perfectly conducting and one resistive, damping or growth results according as the resistance is present in the cathode or the anode. These analyses have a doubtful application to the present case since the space charge effect is neglected; thus a uniform velocity in the beam is obtained. In our case, there being no cathode, the space charge effect is vital and must always lead to a strongly sheared beam velocity profile. Obviously, a definite answer to this question must await further analysis, however the growth and damping rates resulting from this effect are arbitrarily small for arbitrarily good conductors.

In order to study the physical problem outlined above without undue mathematical complication it is desirable to simplify the geometry. The simplest way of doing this would be to suppose the radius ratio of the torus very large, and treat a segment of it as an infinite, straight, charged, current-carrying wire. In this geometry the magnetic field would be azimuthal, the electric field would be radial, and the E x B drift would be parallel to the wire. The difficulty with this scheme is as follows: we would expect in this geometry to study waves propagating parallel to the wire, that is, in a straight line. But in a frame moving at the phase velocity of such a wave the motion would appear to be steady; it follows that no radiation of power from such a wave is possible. The radiation we seek must arise as a result of the acceleration of a bunch of charge around a curved path. This condition is satisfied in the toroidal geometry, but the mathematical complexity of this geometry appears formidable. For this reason we treat a geometry which is topologically related to the torus problem, but is nevertheless different. The chosen geometry is illustrated in Fig. 1; it is the same arrangement as that in which the diocotron effect was studied by Levy.[6] It consists of a perfectly conducting infinite circular cylinder immersed in a magnetic field everywhere parallel to the axis of the cylinder. This axis is then chosen as the z-axis of a system of cylindrical polar coordinates r, Θ, z. The electron beam circulates in the Θ-direction around the cylinder. The surface of the cylinder carries a positive charge per unit axial length equal in absolute magnitude to the total negative charge per unit axial length in the electron beam. This arrangement gives a radial electric field in the electron beam and overall electrical neutrality. The cylinder stands at a positive potential relative to "space". All quantities are assumed uniform in the axial direction. The circulating electron beam considered as an electric current causes a radial variation in the axial magnetic field strength. In this geometry a charge bunch does move in an accelerated path and will therefore radiate. It is thought that this change in geometry can show the radiative effect we seek while greatly simplifying the mathematics.

The organization of this paper is as follows: In Section 2 we consider the available range of equilibrium configurations. In Section 3 a linearized equation is derived which represents a perturbation of any of the equilibrium configurations of Section 2. In Section 4 the analysis is much simplified by restricting attention to a special class of equilibrium configurations in which the ratio of electron number density to magnetic field strength is piecewise constant. In Section 5 we specialize still further and consider that the electron number density is non-vanishing only in a

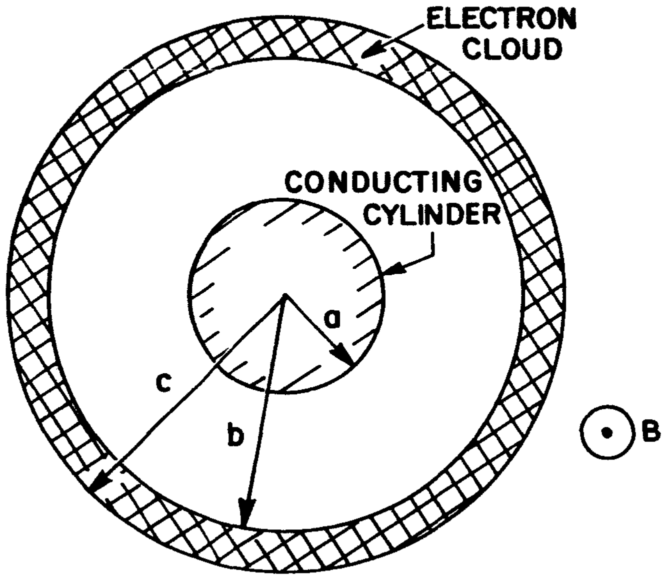

Fig. 1 This is the equilibrium geometry in which the radiative stability of a crossed-field electron beam is studied. The cylinder carries a positive charge, equal in magnitude to the total charge in the electron beam. The configuration is two dimensional, and the regions $a < r < b$ and $r > c$ are empty. The electric field is pure radial. The magnetic field is only axial, but the strength varies across the beam in view of the electric current in the beam. The beam rotates clockwise, the velocity declining to zero at the outer edge.

single region, that is we treat a bounded beam. The range of geometrical and physical parameters available to such a beam is exhibited. In Section 6 we derive the eigenfunctions and dispersion relation for the boundary value problem which is now completed posed. Sections 7 and 8 are devoted to treatment of the dispersion relation by approximate analytical and numerical techniques, respectively. Section 9 summarizes the conclusions reached, and offers some comment of a general nature on the physical meaning of these conclusions.

2. EQUILIBRIUM CONFIGURATIONS

In this section we derive a general relation which the field quantities must satisfy in equilibrium. The specialization from this general class of equilibrium to the particular case of a single bounded electron beam is postponed to Sections 4 and 5.

We denote quantities relevant to the unperturbed state (zeroth order) by the subscript o. Since E_o and B_o are in the radial and axial directions respectively, the unperturbed velocity is in the azimuthal (Θ) direction from (1. 2) it is:

$$v_o(r) = - E_o(r)/B_o(r) \tag{2. 1}$$

This equation is subject to an important limitation. It is clear that (2. 1) can be made to indicate speeds in excess of the speed of light merely by reducing B_o. It is equally clear that such a result would be absurd. However, (1. 1) was written down assuming the electron cyclotron frequency much greater than any other frequency of interest. This condition is inevitably violated if we increase the electron speed towards the speed of light since the cyclotron frequency falls as the electron mass increases. We shall have to recall, as the analysis proceeds, that speeds in excess of the speed of light are not permitted even though this will not be evident from equations derived basically from (2. 1). From the practical point of view the decrease in the cyclotron frequency caused by increasing the electron speed only amounts to a factor of two when the electron speed is 87% of the speed of light; thus speeds up to at least this value (for which the electron energy is 1 MeV) should be attainable within the present framework of assumptions.

Other equations governing the equilibrium configurations are

$$\frac{1}{r}\frac{d}{dr}(rE_o) = -n_o e/\epsilon_o \tag{2. 2}$$

and

$$\frac{dB_o}{dr} = \mu_o n_o e\, v_o \tag{2. 3}$$

Combining these relations yields:

$$\frac{E_o}{r}\frac{d}{dr}(rE_o) - c_o^2 B_o \frac{dB_o}{dr} = 0 \tag{2. 4}$$

c_o refers to the speed of light; the unsubscripted symbol c will be used to denote a length. We introduce a frequency $\omega_o(r)$ defined by

$$\omega_o = \frac{\omega_p^2}{\omega_c} = \frac{n_o e}{\epsilon_o B_o} = -\frac{1}{B_o r}\frac{d}{dr}(rE_o) = -\frac{c_o^2}{E_o}\frac{dB_o}{dr} \tag{2. 5}$$

the last equality following from (2. 4). Further, let $k_o(r)$ be the vacuum wave number of electromagnetic radiation of frequency ω_o. Thus:

$$\omega_o / k_o = c_o \tag{2.6}$$

The statement (2.4) amounts to the statement that the divergence of the Maxwell electromagnetic stress tensor must vanish in electromagnetic equilibrium. For the symmetric conditions considered, (2.4) is the only non-vanishing component of this statement.

At the present stage the equilibrium configuration need not be defined more precisely than has already been done. We do not discuss what methods might be used to establish any given equilibrium configuration satisfying (2.4), since an "inductive ejection" method of producing any desired distribution has been described elsewhere.[1]

3. DERIVATION OF THE LINEAR BOUNDARY VALUE PROBLEM

In this section we carry out a perturbation analysis of the general equilibrium just derived, and obtain a boundary value problem consisting of a linear second order differential equation, and two boundary conditions. The general equilibrium appears in the coefficients of the differential equation.

Let $\underset{\sim}{E}$, $\underset{\sim}{B}$, n and $\underset{\sim}{v}$ denote the total electric and magnetic fields, number density and velocity, respectively. These quantities satisfy the following equations:

$$\text{curl } \underset{\sim}{E} = - \frac{\partial \underset{\sim}{B}}{\partial t} \tag{3.1}$$

$$\text{curl } \underset{\sim}{B} = - \mu_o \, n e \, v + \frac{1}{c_o^2} \frac{\partial \underset{\sim}{E}}{\partial t} \tag{3.2}$$

$$\text{div } \underset{\sim}{E} = - \; ne/\epsilon_o \tag{3.3}$$

$$\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B} = 0 \tag{3.4}$$

(3.4) depends on the assumption that all frequencies of interest are much less than the cyclotron frequency.

(3.2), (3.3) and (3.4) can be combined to yield:

$$- \underset{\sim}{E} \text{ div } \underset{\sim}{E} + \left\{ \frac{\partial \underset{\sim}{E}}{\partial t} - c_o^2 \text{ curl } \underset{\sim}{B} \right\} \times \underset{\sim}{B} = 0 \tag{3.5}$$

This equation together with (3.1) are the basic equations to be discussed henceforward. In a vacuum ($n = 0$) these equations represent the propagation of waves at a speed c_o but there is no way of knowing from the equations that E_o/B_o is not allowed to exceed this same number c_o. Thus no singularity need be anticipated in equations derived from this system as E_o/B_o approaches c_o. This is, of course, an artifact since, in fact, our assumptions break down before this point is reached.

We consider only two dimensional perturbations from the basic equilibrium; that is, only E_r, E_Θ and B_z are considered in first order and are assumed independent of z. This restriction is based on the idea that the mobility of the electrons parallel to the magnetic field lines is extremely high, so that no significant gradients need be expected in the axial direction. We look for oscillations in which the unknown field quantities are each proportional to $\exp i(\ell\Theta - \omega t)$. ℓ is the azimuthal mode number and must be a non-negative integer. ω is the complex frequency associated with the oscillation and will eventually be found as the root of a dispersion relation. We are interested primarily in the sign of the imaginary part of ω. If $\text{Im}\,\omega > 0$, we have a growing (i.e. unstable oscillation). If $\text{Im}\,\omega < 0$ we have a damped oscillation. If ω is pure real the corresponding oscillation is neutral and does not grow.

We modify our notation somewhat and suppose the perturbed radial electric field to be given by the real part of the expression $E_r(r) \exp i(\ell\Theta - \omega t)$ and likewise for E_Θ and B. (In the case of B, the subscript has been dropped.) E_r, E_Θ and B now have the meaning of complex amplitudes. On substitution of the assumed forms for the first order variables, (3.1) yields a significant component only in the axial direction:

$$\frac{1}{r}\frac{d}{dr}(rE_\Theta) - \frac{i\ell}{r}E_r - i\omega B = 0 \qquad (3.6)$$

On linearization and substitution (3.5) yields components in the radial and tangential directions as follows:

$$-E_o\left[\frac{1}{r}\frac{d}{dr}(rE_r) + \frac{i\ell}{r}E_\Theta + \omega_o B\right] + B_o\left[\omega_o E_r - i\omega E_\Theta + c_o^2\frac{dB}{dr}\right] = 0 \qquad (3.7)$$

$$i\omega E_r + \omega_o E_\Theta + i\ell\frac{c_o^2 B}{r} = 0 \qquad (3.8)$$

In deriving (3.7) and (3.8) use was made of the definitions (2.5) of ω_o. (3.6), (3.7) and (3.8) are now three ordinary linear differential equations in r for the complex amplitudes E_r, E_Θ, and B. Algebraic elimination between (3.6) and (3.8) gives:

$$(\ell^2 - r^2\omega^2/c_o^2)\, i\, E_r = \ell \frac{d}{dr}(rE_\Theta) + \frac{r^2 \omega\, \omega_o}{c_o^2} E_\Theta \qquad (3.9)$$

$$(\ell^2 - r^2\omega^2/c_o^2)\, i c_o B = -\frac{r\omega}{c_o}\frac{d}{dr}(rE_\Theta) - \ell \frac{r\omega_o}{c_o} E_\Theta \qquad (3.10)$$

Substitution of (3.9) and (3.10) into (3.7) yields after some reduction:

$$\left[\omega + \frac{\ell}{r}\frac{E_o}{B_o}\right]\mathcal{L}(E_\Theta) + \left[\ell + \frac{r\omega}{c_o^2}\frac{E_o}{B_o}\right]\frac{rE_\Theta}{\ell^2 - r^2\omega^2/c_o^2} \cdot \frac{d\omega_o}{dr} = 0 \qquad (3.11)$$

where $\mathcal{L}$ is the following second-order differential operator:

$$\mathcal{L}(E_\Theta) \equiv \frac{d}{dr}\left\{\frac{r\frac{d}{dr}(rE_\Theta)}{\ell^2 - r^2\omega^2/c_o^2}\right\} - E_\Theta\left\{1 + \frac{(k_o r)^2}{\ell^2 - r^2\omega^2/c_o^2} - \frac{2\ell k_o r\,(r\omega/c_o)}{(\ell^2 - r^2\omega^2/c_o^2)^2}\right\} \qquad (3.12)$$

This is the single linear differential equation for $E_\Theta(r)$ which, together with appropriate boundary conditions, defines our problem. At the surface r = a of the (assumed) perfectly conducting boundary we have

$$E_\Theta(a) = 0 \qquad (3.13)$$

At infinity we require the condition that only outgoing waves be present (the radiation condition).

4. CHOICE OF EQUILIBRIUM CONFIGURATION

In equation (3. 11) any one of E_o, B_o and ω_o can be given arbitrarily as a function of r, and the others then follow from the relations presented in Section 2. The solution of (3. 11) for arbitrary equilibrium profiles is evidently not easily found. We therefore restrict our attention to a special class of equilibrium profiles, whose choice is primarily dictated by their resulting convenience. This special class is still very extensive, so that for purposes of detailed study it will be necessary (in Section 5) to specialize still further the chosen equilibrium configuration. In this section, however, we exhibit the simplification achieved by the first specialization, and give some discussion of certain problems which arise in consequence.

It is immediately clear that (3. 11) will simplify enormously if ω_o is independent of r, for in this case the equation for E_Θ reduces (after cancellation of a factor about which more later) to a form which can be shown to be a variant of the Bessel equation. It is, however, unrealistic to take ω_o as independent of r everywhere since ω_o is proportional to n_o/B_o. Where $n_o = 0$, i. e. outside the beam at any rate $\omega_o = 0$. We therefore assume as our first specialization that ω_o is piecewise constant. This assumption is only weakly restrictive, for any continuous distribution of $\omega_o(r)$ can be approximated to some degree by a piecewise constant function. It is doubtful that any important change in conclusions about stability could be traced to this approximation. Arguing by analogy, a procedure virtually identical to that proposed here is commonly adopted in the mathematically related problem of the stability of plane parallel shear flows. In that case arbitrary unperturbed distributions of vorticity are replaced by distributions which are stepwise constant. Goldstein[9] has analyzed a problem of this type involving no less than five such steps.

Before turning to the simplified forms of (3. 11) we consider the condition to be applied across a point where ω_o is discontinuous. Such a point is best treated by considering $d\omega_o/dr$ to have δ-function behavior and integrating (3. 11) across a vanishing range containing the point in question. Suppose $r = r_1$ is such a point. We observe that E_Θ, the tangential component of the electric field is continuous. E_o and B_o are at least bounded in the vicinity of $r = r_1$. We find:

$$\left[\frac{d}{dr}(rE_\Theta)\right]_{r_{1-}}^{r_{1+}} = -\left\{\frac{\dfrac{r_1\omega}{c_o^2}\dfrac{E_o(r_1)}{B_o(r_1)} + \ell}{\dfrac{\ell}{r_1}\dfrac{E_o(r_1)}{B_o(r_1)} + \omega}\right\} E_\Theta(r_1)\left[\omega_o\right]_{r_{1-}}^{r_{1+}} \tag{4. 1}$$

The corresponding jump in E_r is found from (3. 9) to be:

$$\left[E_r\right]_{r_{1-}}^{r_{1+}} = \frac{i\,E_\Theta(r_1)\left[\omega_o\right]_{r_{1-}}^{r_{1+}}}{\dfrac{\ell}{r_1}\dfrac{E_o(r_1)}{B_o(r_1)} + \omega} \tag{4. 2}$$

The corresponding jump in B is found from (3.10) to be:

$$\left[c_o B\right]_{r_{1-}}^{r_{1+}} = \frac{i\ E_o(r_1)\ E_\Theta(r_1)\ [\omega_o\ ]_{r_{1-}}^{r_{1+}}}{\frac{c_o \ell}{r_1}\ E_o(r_1) + c_o\,\omega\, B_o(r_1)} \tag{4.3}$$

This discontinuity (4.2) in E_r indicates an accumulation of charge at the surface $r = r_1$; the motion of the surface charge constitutes a surface current which gives rise to the discontinuity (4.3) in B. These results can also be obtained[4] by supposing the surface $r = r_1$ to be slightly deformed and expressing the kinematic and electromagnetic relations in terms of the coordinates of this (moving) surface.

For any range in which ω_o is constant, (3.11) becomes:

$$[\,\omega + \frac{\ell}{r}\ \frac{E_o}{B_o}\,]\mathcal{L}(E_\Theta) = 0 \tag{4.4}$$

We shall shortly divide out by the factor $(\,\omega + \ell E_o/rB_o)$ but this step requires careful justification. The argument which, for the diocotron problem, allows this division is given at some length by Levy[6], and depends on an extrapolation of the work of Case,[10, 11] and Dikii.[12] Inclusion of this factor leads to a class of discontinuous eigenfunctions having a continuous spectrum of real eigenvalues. This spectrum corresponds to the range of angular velocities present in the unperturbed beam. In the problem treated in the references cited above a Laplace transform technique shows that the continuous spectrum leads to the decay of perturbations like algebraic powers of the time.

If ω_o is taken as a constant ab initio, the factor presently under discussion could be missed; this comment applies to much diocotron work. The diocotron problem (for $\omega_p \ll \omega_c$) can be obtained from the present problem by letting $c_o \rightarrow \infty$ so that the quasi-static approximation is recovered. In this approximation B_o is unaffected by the circulating electron current and can be considered constant. ω_o piecewise constant therefore amounts to n_o piecewise constant; this is in fact the choice of most authors of pape on the diocotron effect.

The modes corresponding to the continuous spectrum of eigenvalues a) have been shown to decay in the fluid dynamic studies of Case and Dikii and, b) have been ignored in much diocotron work. If, as seems likely, the analysis used in the fluid dynamic case can be carried over to the diocotron work without affecting the results, then the neglect of these modes in much of the literature is of no consequence. This extension of the work of Case and Dikii was explicitly assumed by Levy.[6] Possibly the most doubtful point in this extension is the following: according to Dikii the introduction of a finite fluid viscosity which is subsequently made to tend to zero does, in fact, substantiate the results of the purely inviscid analysis. In the case of the electron beam, some physical mechanism more involved than a simple fluid viscosity must be invoked to remove the physically unacceptable discontinuity in the eigenfunctions. While it is not clear exactly what mechanism is appropriate, it would certainly involve more detailed consideration of the electron dynamics (finite temperature, finite Larmor radius, Landau damping) than has been given up to this point. In connection with the present problem we will simply assume that whatever the nature of the important mechanism, Dikii's result holds and that we can neglect the continuous spectrum of eigenvalues. In making this assumption we have also assumed that by making c_o finite we have not introduced any important modification to results connected with the continuous spectrum.

After division through by the factor discussed, (4. 4) becomes:

$$\mathcal{L}(E_\Theta) = 0 \qquad (4.5)$$

The complete formulation of our problem now involves the differential equation (4. 5), the jump condition (4. 1) together with the continuity of E_Θ at each jump, the boundary condition (3. 12) and the radiation condition at infinity.

5. SPECIAL EQUILIBRIUM CONFIGURATION

At this stage further simplification is necessary not because the differential equation cannot be easily solved, but merely to keep the algebraic manipulation within reasonable bounds. This second simplification restricts the number of points at which ω_o is discontinuous to two (or possibly only one). Further, we shall suppose ω_o (i. e. , n_o) to be zero except between these points. In this section we shall exhibit the range of possibilities still available in spite of these two successive specializations of the original class of equilibrium configurations.

To fix the problem we suppose the electron beam to be confined to the annular region $b \leq r \leq c$ (see Fig. 1). Outside this region, the electron density being zero, $\omega_o = 0$. We can therefore use the symbol ω_o unambiguously to refer to the constant value of ω_o in the beam. Since ω_o is constant (2. 5) amounts to two simultaneous equations for E_o and B_o:

$$E_o + \frac{dc_o B_o}{dk_o r} = 0 \; ; \qquad c_o B_o + \frac{1}{k_o r} \frac{d}{dr}(rE_o) = 0 \tag{5.1}$$

Elimination of E_o between these equations yields the modified Bessel equation of order zero for B_o.*

In all the equilibrium configurations so far studied, an important feature has been the net electrical neutrality of the cylinder and beam combination. In the present context this means that there is no net charge inside the cylinder $r = c$. Since there is no charge at all outside it we find

$$\left. \begin{aligned} E_o(r) &= 0 \\ B_o(r) &= B_\infty, \text{ say} \end{aligned} \right\} \quad r \geq c \tag{5.2}$$

In order to make E_o and B_o continuous at $r = c$ we let:

$$E_o(r) = c_o B_\infty k_o c \left\{ -K_1(k_o c)\; I_1(k_o r) + I_1(k_o c)\; K_1(k_o r) \right\} \tag{5.3}$$

$$B_o(r) = B_\infty k_o c \left\{ K_1(k_o c)\, I_o(k_o r) + I_1(k_o c)\; K_o(k_o r) \right\} \tag{5.4}$$

These formulae are valid in the range $b \leq r \leq c$. Other properties of the electron beam are found as follows: from (2.5):

$$n_o(r) = (\omega_o \epsilon_o / e)\; B_o(r) \tag{5.5}$$

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*All definitions and formulae in this paper involving Bessel functions are from G. N. Watson, "A Treatise on the Theory of Bessel Functions," Cambridge University Press, Second Ed., 1958.

while $v_o(r)$ is found directly from (2.1) to be the quotient of (5.3) and (5.4). Now $v_o(r)/c_o$ must be restricted to values less than unity in order to satisfy the relativistic requirements discussed earlier. Since it has its largest value when r has its smallest value, that is when $r = b$, we define

$$\beta = -\frac{v_o(b)}{c_o} = \frac{-K_1(k_oc)\, I_1(k_ob) + I_1(k_oc)\, K_1(k_ob)}{K_1(k_oc)\, I_o(k_ob) + I_1(k_oc)\, K_o(k_ob)} \tag{5.6}$$

Figure 2 shows β as a function of k_ob and k_oc. The values of k_oc available when k_ob is given are limited by the conditions $\beta \leq 1$. At this stage it is helpful to introduce another quantity having the dimensions of a frequency, and defined as follows:

$$\Omega_o(r) = \frac{v_o(r)}{r} = \frac{v_o(r)}{c_o} \cdot \frac{\omega_o}{k_or} \tag{5.7}$$

$\Omega_o(r)$ is the angular frequency of rotation of the electron beam about the cylinder at any radius r. Since it is positive in the sense of Θ increasing, it is negative with v_o. We define $\Omega_o(b) = \Omega_o$ and find:

$$-\frac{\Omega_o}{\omega_o} = \frac{\beta}{k_ob} \tag{5.8}$$

Figure 3 is a plot of Ω_o/ω_o as a function of k_ob and k_oc. It can be seen that Ω_o/ω_o can take essentially any value, but that large values are only present when $k_ob \ll k_oc \ll 1$. When Ω_o is large, the convective part of the term $m\ dv/dt$ missing from the left hand side of (3.4) becomes more important than the part indicating local time rates of change. There-

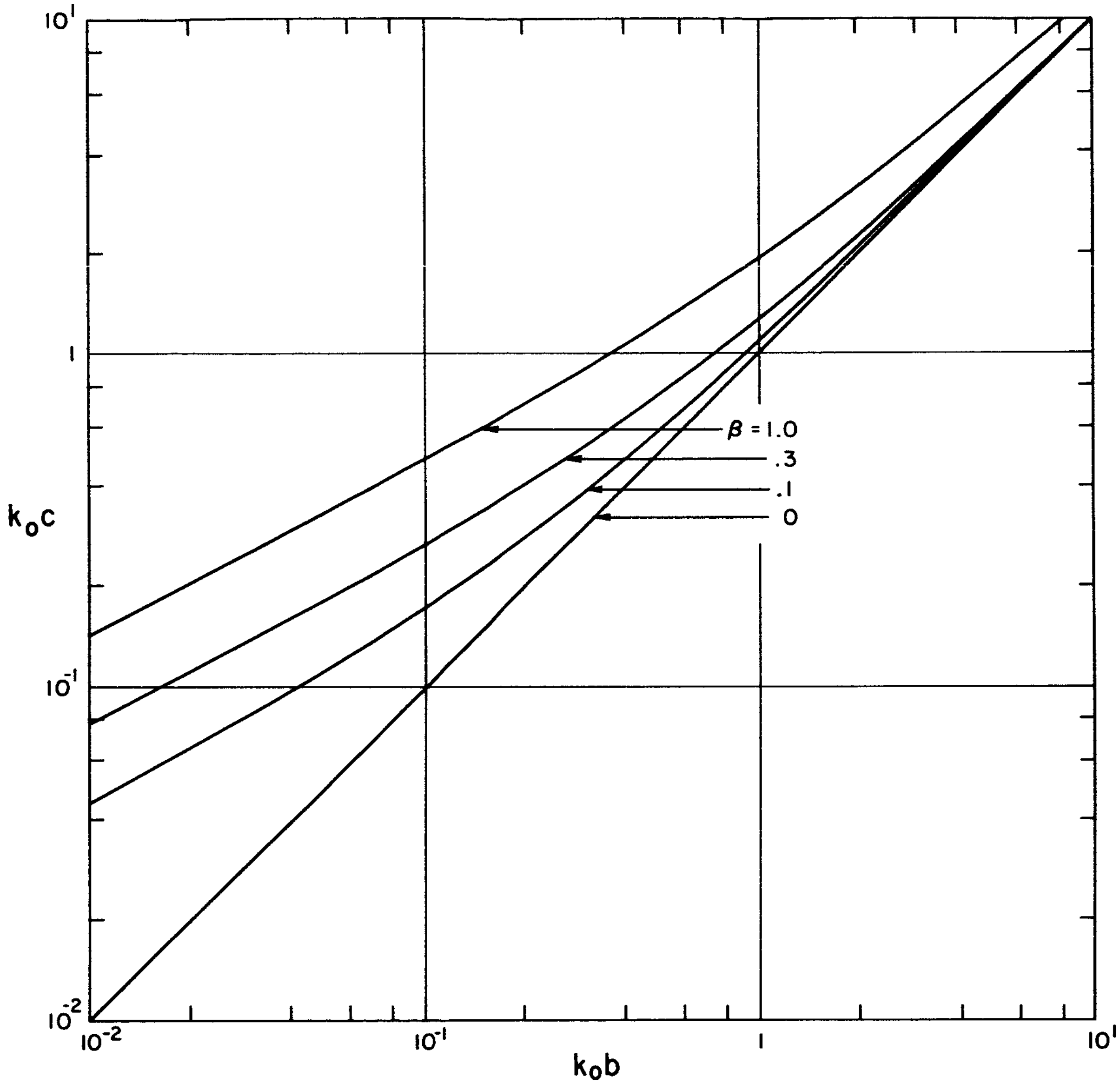

Fig. 2 This figure shows the values of $\beta = - v_0(b)/c_0$ corresponding to different choices of k_0b and k_0c in (5.6). β is restricted by relativistic considerations to values less than unity. For small values of k_0b and k_0c, a large range of ratios c/b is available. For large k_0b and k_0c, the ratio c/b cannot differ by much from unity.

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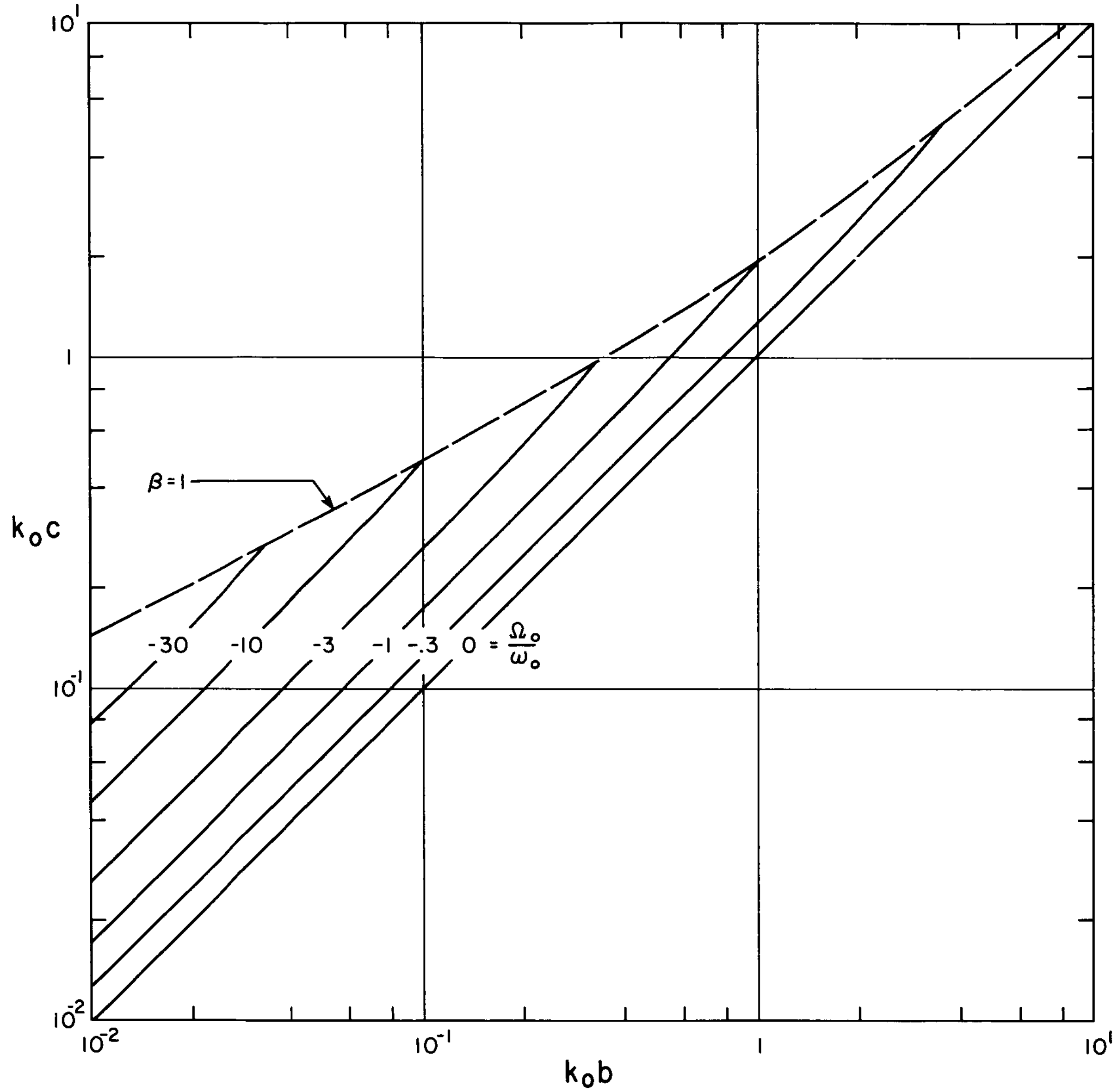

Fig. 3 This figure shows the values of Ω_o/ω_o corresponding to different choices of k_ob and k_oc. Ω_o is the angular velocity of the inner edge of the electron beam. The curves are continued only so far as $\beta = 1$ since values of β larger than this are forbidden.

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fore, m dv/dt will only be negligible compared to the terms in (3.4) if the cyclotron frequency satisfies the condition $\omega_c \gg \mathrm{Max}\,(|\Omega_o|, \omega_o)$. We shall suppose this condition satisfied where necessary. If $k_o b > 1$, Ω_o/ω_o is always less than unity provided $\beta < 1$.

One further quantity relating to the electron beam is of interest, namely the electrostatic potential which we may define by:

$$V_o(r) = \int_r^c E_o(r')\,dr' = (c_o/k_o)\,[\,B_o(r) - B_\infty\,] \tag{5.9}$$

The potential of the outer edge of the beam has been taken to be zero. The potential of the inner edge of the beam is:

$$\frac{V_o(b)}{bE_o(b)} = \frac{1}{\beta\, k_o b}\left[1 - \frac{B_\infty}{B_o(b)}\right] \tag{5.10}$$

Figure 4 is a plot of the potential across the beam, $V_o(b)$, non-dimensionalized as in (5.10), as a function of $k_o b$ and $k_o c$. The choice of $bE_o(b)$ as the reference potential seems reasonable since this quantity is easily shown to be just a constant $(e/2\pi\epsilon_o)$ times the total number of electrons in the beam per unit axial length.

In the empty space between the cylinder $(r = a)$ and the inner edge of the electron beam $(r = b)$ the magnetic field has the constant value

$$B_o(r) = B_o(b) \tag{5.11}$$

The electric field is given by

$$E_o(r) = \frac{b}{r}\,E_o(b) \tag{5.12}$$

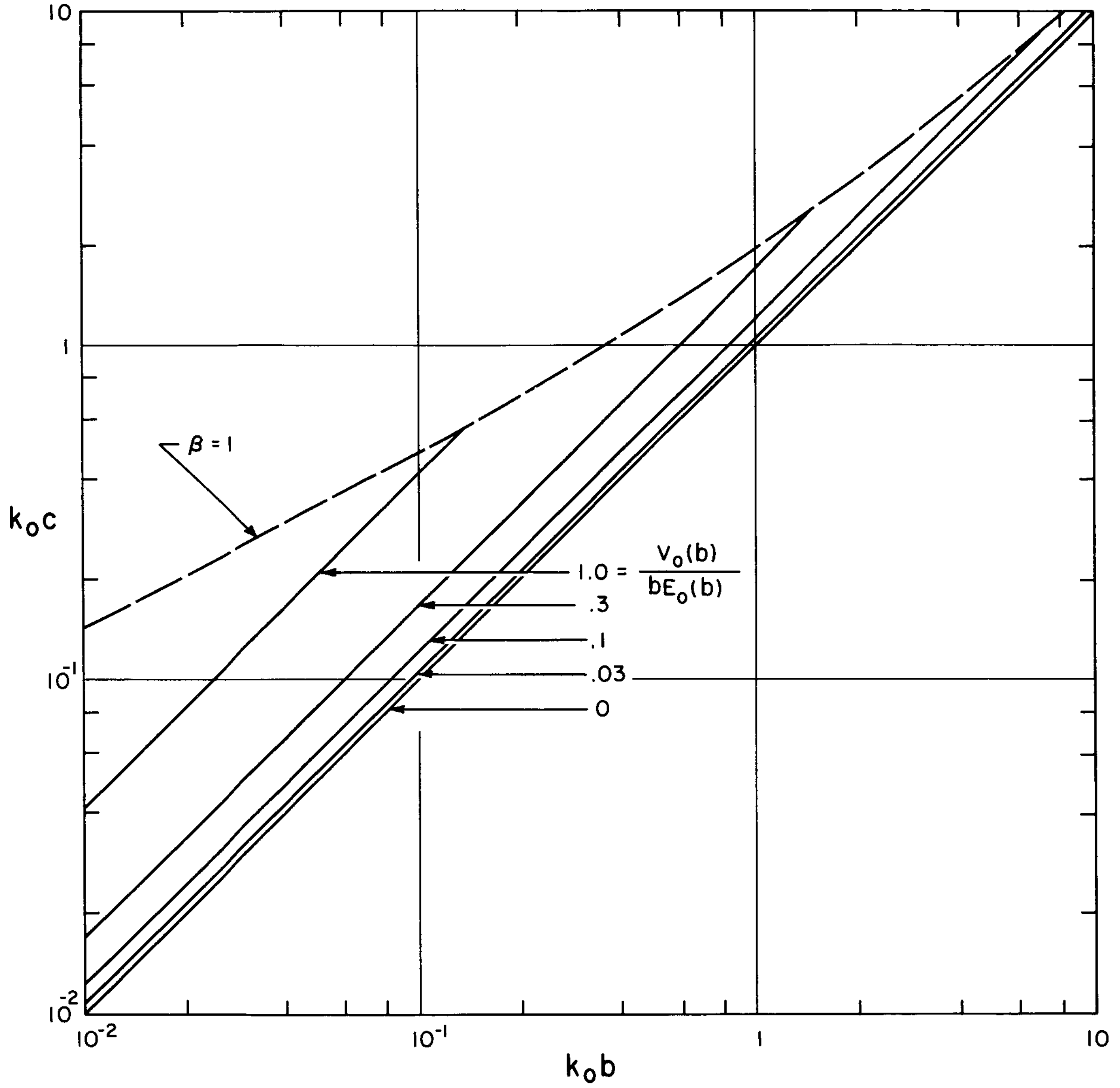

Fig. 4 This figure shows values of the electric potential across the electron beam, $V_o(b)$, normalized to $b\ E_o(b)$. This quantity, $b\ E_o(b)$, is proportional to the total charge in the electron beam. The curves are continued only as far as $\beta = 1$ since values of β larger than this are forbidden.

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These formulae are valid for $a \leq r \leq b$. In this region, there being no matter, the velocity E_o/B_o may exceed c_o. The potential difference between the cylinder and the inner edge of the beam is:

$$\frac{V_o(a) - V_o(b)}{bE_o(b)} = \frac{V_o(a) - V_o(b)}{aE_o(a)} = \ell n \frac{b}{a} \tag{5. 13}$$

The total potential between the cylinder and space is found by adding (5. 1' and (5. 13). We observe that for fixed $k_o b$ and $k_o c$, this potential can be raised to an arbitrary level by reducing a.

Figure 5 is a plot against radius of all the quantities so far discussed for one particular set of values of $k_o b$ and $k_o c$. It is thus a typical profile of the class of equilibrium configurations to which our attention is henceforth restricted. We choose the case $k_o b = .301$, $k_o c = .903$ since this case is discussed in detail numerically in a later section. In this case, $c/b = 3$, $\beta = 1$, $\Omega_o/\omega_o = -3.32$ and $V_o(b)/bE_o(b) = .718$. Choosing a case for study for which $\beta = 1$ exhibits the fact that no singularity appears at this point, and also brings out as strongly as possible the radiative effects which vanish with vanishing β.

6. DERIVATION OF THE DISPERSION RELATION

In this section we derive the dispersion relation appropriate to the class of equilibrium configurations just studied, and describe some of its properties. We introduce the notations:

$$z = \omega/\omega_o \quad ; \quad \zeta = (z^2 - 1)^{1/2} \tag{6. 1}$$

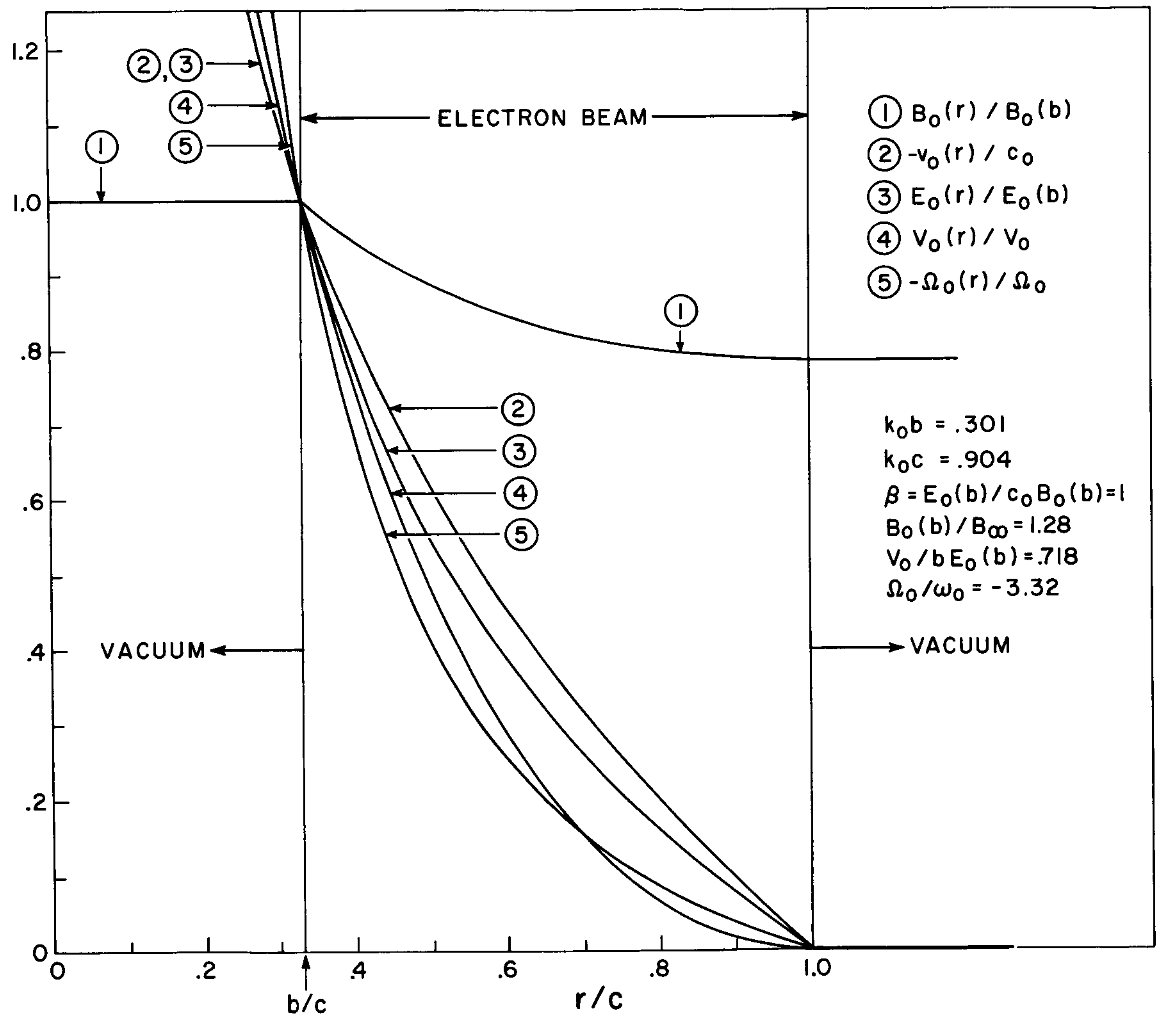

Fig. 5 This figure shows the variation of the field quantities through the electron beam in the special case $k_o b = .301$, $k_o c = .904$. The stability of this case, for which $\beta = 1$, is studied numerically in the text. All the quantities are normalized to their values at $r = b$. Also shown are the values taken by these quantities in the empty regions inside and outside the beam. The inner cylinder may be any radius $a < b$.

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z is thus a non-dimensional complex frequency. A solution to (4.5) is:

$$E_\Theta = F_1(\zeta k_o r) \tag{6.2}$$

where

$$F_1(s) = J'_\ell(s) - (\ell/zs)\, J_\ell(s) \tag{6.3}$$

If we substitute Y for J in (6.3) we obtain a second solution to (4.5) which we define as $F_2(s)$. The most general solution to (4.5) is then a linear combination of $F_1(\zeta\, k_o r)$ and $F_2(\zeta\, k_o r)$. The corresponding solutions for E_r and B are found from (3.9) and (3.10). The solution to the equations for the regions outside the electron beam ($a \le r \le b$, and $r \ge c$) may be found in various ways of which the simplest is to let $\omega_o \to 0$ in the preceding. k_o also $\to 0$, z and $\zeta \to \infty$, and $\zeta/z \to 1$. In the region $r \ge c$ we must choose that combination of solutions that satisfies the radiation condition at infinity. We find:

$$E_\Theta \propto H_\ell^{(1)'}(z\, k_o r) \tag{6.4}$$

This choice of Hankel function requires $-\pi < \arg z < 2\pi$. Thus the upper (unstable) half of the complex frequency plane is covered once, while the lower (stable) half is covered twice. In the region $a \le r \le b$, in order to satisfy the boundary condition (3.13) at $r = a$, we choose

$$E_\Theta \propto F'_5(z\ k_o r) \tag{6.5}$$

where

$$F_5(s) = J_\ell(s)\ Y'_\ell(z\, k_o a) - Y_\ell(s)\, J'_\ell(z\, k_o a) \tag{6.6}$$

We turn next to the jump conditions given in (4. 1), (4. 2) and (4. 3). Of these three conditions, any two can be derived from the third. At $r = r_1 = b$ we use (4. 3) to find:

$$c_o B(b+) - c_o B(b-) = \frac{\beta k_o b \, i \, E_\Theta(b)}{\beta \ell + z k_o b} \tag{6. 7}$$

where (5. 8) was used. We recall that β is a function of $k_o b$ and $k_o c$ given by (5. 6). At $r = r_1 = c$, $E_o(c) = 0$ and (4. 3) shows that B is continuous. The formulae given above now permit the derivation of the dispersion relation. Defining:

$$t_1 = z k_o a; \quad t_2 = z k_o b; \quad t_3 = z k_o c; \quad t_4 = \xi k_o b; \quad t_5 = \xi k_o c \tag{6. 8}$$

$$F_5 = F_5(t_2); \quad F_5' = F_5'(t_2) \tag{6. 9}$$

$$F_6 = F_5 + \frac{\beta k_o b}{\beta \ell + z k_o b} F_5' \tag{6. 10}$$

$$\left.\begin{aligned} Q_1 &= F_1(t_4) Y_\ell(t_5) - F_2(t_4) J_\ell(t_5) \\ Q_2 &= J_\ell(t_4) Y_\ell(t_5) - Y_\ell(t_4) J_\ell(t_5) \\ Q_3 &= F_1(t_4) F_2(t_5) - F_2(t_4) F_1(t_5) \\ Q_4 &= J_\ell(t_4) F_2(t_5) - Y_\ell(t_4) F_1(t_5) \end{aligned}\right\} \tag{6. 11}$$

$$\left.\begin{aligned} g_1 &= \xi F_6 Q_1 - (\xi^2/z) F_5' Q_2 \\ g_2 &= z F_6 Q_3 - \xi F_5' Q_4 \end{aligned}\right\} \tag{6. 12}$$

The dispersion relation is:

$$M(z) = g_1 H_\ell^{(1)'} (t_3) - g_2 H_\ell^{(1)} (t_3) = 0 \tag{6.13}$$

Since ζ is known when z is known, and β is known when $k_o b$ and $k_o c$ are known, $M(z)$ is a function of z, $k_o a$, $k_o b$, $k_o c$ and ℓ; of these we regard z as the variable and the others (which are all real) as parameters. The complexity of the relation (6. 13) renders it hopeless to attempt a solution for general values of all the parameters. Before proceeding to treat (6. 13) by approximate and numerical methods, however, we make certain observations of a general nature.

First, it can be shown that if z is real, g_1 and g_2 are both pure real. It follows that $M(z)$ can only vanish if, for real z,

$$\frac{J_\ell (z k_o c)}{Y_\ell (z k_o c)} = \frac{J_\ell' (z k_o c)}{Y_\ell' (z k_o c)} \tag{6.14}$$

This equality is approximately satisfied if $z k_o c$ is small (an important case). Otherwise, an application of the Wronskian shows that the equality (6. 14) is impossible. We reach the important conclusion that there can be no pure real eigenvalues in the present problem, although for small $z k_o c$ some eigenvalues might appear near the real axis. This conclusion has a simple physical interpretation as follows: the Hankel functions describe radiative transfer of energy to infinity. Such radiation obviously cannot be stationary (as would be implied by a real eigenvalue) without violating the conservation of energy.

Secondly, when $z = \pm 1$, $\zeta = 0$, the solution (6. 2) breaks down in each case and in addition, g_1 and g_2 become indeterminate forms. It can easily be shown, however, that the difficulty is purely mathematical. A singular solution can be found in either case to replace (6. 2), and limits can be found toward which g_1 and g_2 tend smoothly as $z \to \pm 1$. Since they have no intrinsic importance, these limits are not recorded here.

Another value of z that requires special consideration is $z = -\beta\ell/k_o b$, or from (5. 10) $\omega = \ell\,\Omega_o$. This is a resonance at a real frequency which is a simple multiple of the angular frequency of rotation of the unperturbed beam at $r = b$. We see from (6. 10) that F_6 is infinite when z has this value. The difficulty can be traced back to the integration of (3. 11) across a jump in ω_o when the quantity $(\omega + \ell E_o/rB_o)$ vanishes. It can be seen that in this case we must require that E_Θ vanish at the jump. Furthermore, the jump in $d(rE_\Theta)/dr$ is not given by (4. 1) but is arbitrary. That is, the point in question becomes a node of the particular mode of oscillation. (An exception to this analysis arises if the quantity $(\ell + r\omega E_o/c_o^2 B_o)$ also vanishes at the same point. This, however, can only happen if $\beta = 1$ at that point, a condition which cannot be satisfied.) When $r = b$ is a node of the oscillation, and $d(rE_\Theta)/dr$ has an arbitrary jump at this point our problem splits into two, since oscillations in the electron beam and the region outside it $(r > b)$ are uncoupled from oscillations in the vacuum region $r < b$. The problem for the region $a < r < b$ is now just a standing wave problem. The amplitude of the oscillation in this region will be zero or arbitrary depending on whether the parameters of the problem are such that:

$$J'_\ell(\beta\ell)\,Y'_\ell(\beta\ell\,a/b) - Y'_\ell(\beta\ell)\,J'_\ell(\beta\ell\,a/b) = 0 \qquad (6.\ 15)$$

A standing wave of this type appears to be fortuitous and uninteresting; we shall discuss this case no further.

7. REDUCTION TO THE DIOCOTRON CASE

In this section we consider the simplification that results when the electron speeds are everywhere small compared to the speed of light. In the extreme limit when β is negligible we simply recover the diocotron dispersion relation. In the present geometry this dispersion relation has either two real roots, or two complex conjugate roots of which one is necessarily unstable.[6] Since we are interested in stable (or at least nearly stable) configurations we shall concentrate our attention on the former case. We therefore seek corrections to the two real diocotron roots for the case of β small, but not negligible. These are the roots near (but not on) the real axis discussed as an exceptional case in connection with (6. 14). We shall find expressions for the imaginary parts of these corrections (which determine growth or damping) and discuss the variation of these with the geometrical parameters of the beam.

A simple way to attain the desired limit mathematically is to let $c_o \to \infty$ keeping ω_o fixed. Thus $k_o \to 0$ and the arguments of the Bessel functions all become small. Also, from (5. 6) and (5. 8)

$$\frac{\beta}{k_o b} = - \frac{\Omega_o}{\omega_o} \approx \frac{1}{2} \left(\frac{c^2}{b^2} - 1 \right) \tag{7. 1}$$

Since, for small argument, Y_o behaves in a different manner than Y_ℓ ($\ell \geq 1$), it will be convenient to postpone consideration of the case $\ell = 0$. For $\ell \geq 1$, retaining only leading terms, we find:

$$g_1 \approx \frac{-2\, b^2\, P_1(z)}{\pi^2 k_o^{\,2}\, ab\, z^2 \left\{ 2b^2\, z + \ell\, (c^2 - b^2) \right\}} \tag{7.2}$$

$$g_2 \approx \frac{-\, 2b^2\, \ell\, P_2\,(z)}{\pi^2 k_o^{\,3}\, abc\, z^2 \left\{ 2b^2\, z + \ell\, (c^2 - b^2) \right\}} \tag{7.3}$$

where $P_1(z)$ is a quadratic expression in z and P_2 is a linear expression in z; the coefficients in both these expressions depend only on a, b, c and ℓ. Referring now to the dispersion relation (6. 13), we suppose $z\, k_o c$ so small that the only important contribution to $H_\ell^{(1)}$ comes from the term iY_ℓ. This amounts to satisfying the phase relationship indicated in (6. 14) (and hence allowing real eigenvalues to exist) by letting both sides vanish. Alternatively, allowing c_o to tend to infinity amounts to suppressing the radiation of energy. Replacing $H_\ell^{(1)}$ and $H_\ell^{(1)'}$ by their leading terms taken in this way, (6. 13) becomes:

$$P_3(z) = P_1(z) + z\, P_2\,(z) = 0 \tag{7.4}$$

where

$$P_3(z) = \frac{c^\ell}{a^\ell} \left[4z^2 + 2z \left\{ \ell \left(\frac{c^2}{b^2} - 1 \right) + \frac{a^{2\ell}}{b^{2\ell}} - \frac{a^{2\ell}}{c^{2\ell}} \right\} + \ell \left(\frac{c^2}{b^2} - 1 \right) \left(1 - \frac{a^{2\ell}}{c^{2\ell}} \right) - \left(1 - \frac{a^{2\ell}}{b^{2\ell}} \right) \left(1 - \frac{b^{2\ell}}{c^{2\ell}} \right) \right] \tag{7.5}$$

As explained above, this dispersion relation, a quadratic in the unknown frequency $z = \omega/\omega_o$, is identical to that obtained by Levy[6] treating the same geometry as a diocotron problem ($c_o \rightarrow \infty$) ab initio. Since the coefficients in (7. 5) are real, the two roots are either both real or else complex conjugate. In the latter case one of the roots (the one having positive imaginary part) corresponds to instability. Thus the only chance of stability is for (7. 4) to have real roots in which case the stability is of the neutral type--perturbations neither growing nor decaying in amplitude. The condition for stability in this sense is:

$$D^2 = \left[-\ell \left(\frac{c^2}{b^2} - 1 \right) + 2 - \frac{a^{2\ell}}{b^{2\ell}} - \frac{a^{2\ell}}{c^{2\ell}} \right]^2 - 4 \frac{b^{2\ell}}{c^{2\ell}} \left(1 - \frac{a^{2\ell}}{b^{2\ell}} \right)^2 \geq 0 \tag{7. 6}$$

We refer to the two roots of (7. 4) with the symbols $z_{D\pm}$, the subscript D indicating "diocotron". We have

$$z_{D\pm} = \frac{1}{4} \left[\pm D - \ell \left(\frac{c^2}{b^2} - 1 \right) - \frac{a^{2\ell}}{b^{2\ell}} + \frac{a^{2\ell}}{c^{2\ell}} \right] \tag{7. 7}$$

It is easy to show that if the two roots $z_{D\pm}$ are both real, they are both negative.

Having accomplished the reduction of (6. 13) to the known result (7. 4) we now turn (as indicated earlier) to the more interesting task of calculating small complex corrections to $z_{D\pm}$ for the case of β small but not negligible. The method used for this calculation is to separate out from (6. 13) the imaginary part (which gives the diocotron dispersion relation) and a much smaller real part. Now $H_\ell^{(1)}$ satisfies the radiation condition when its argument lies between $-\pi$ and 2π. Therefore since $z_{D\pm}$ are both negative, we suppose their arguments near π. Using the relation

$$H_\ell^{(1)}(z) = -\, e^{-\ell\pi i}\, H_\ell^{(2)}(z\, e^{-\pi i}) \tag{7.8}$$

we rewrite (6. 13) as:

$$\begin{aligned} i(-1)^\ell \Big[Y'_\ell (z\, k_o c\, e^{-\pi i})\, g_1 + Y_\ell (z\, k_o c\, e^{-\pi i})\, g_2 \Big] \\ = (-1)^\ell \Big[J'_\ell (z\, k_o c\, e^{-\pi i})\, g_1 + J_\ell (z\, k_o c\, e^{-\pi i}) g_2 \Big] \end{aligned} \tag{7.9}$$

In this form when z is nearly real and negative, both braces are nearly pure real. It will be convenient to regard the ratios a:b:c as fixed and to treat the parameters in the dispersion relation as z and $k_o c$. Upon supposing $k_o c$ to be small we can expand in a power series and define:

$$(-1)^\ell \Big[Y'_\ell (z\, k_o c\, e^{-i\pi})\, g_1 + Y_\ell (z\, k_o c\, e^{-i\pi})\, g_2 \Big] \approx \frac{f_1(z)}{(k_o c)^{\ell+3}} + 0\, (k_o c)^{-\ell-2} \tag{7.10}$$

$$(-1)^\ell \Big[J'_\ell (z\, k_o c\, e^{-i\pi})\, g_1 + J_\ell (z\, k_o c\, e^{-i\pi})\, g_2 \Big] \approx f_2(z)\, (k_o c)^{\ell-3} + 0\, (k_o c)^{\ell-2} \tag{7.11}$$

We evaluate $f_1(z)$ using (7.2) and (7.3) and find:

$$f_1(z) = \frac{2^{\ell+1} \ \ell! \ bc^2 \ P_3(z)}{\pi^3 \ z^{\ell+3} \ a\left\{2b^2 \ z+\ell\,(c^2-b^2)\right\}} \tag{7.12}$$

The appearance of $P_3(z)$ in (7.12) is not unexpected, since, in calculating the diocotron limit, $H^{(1)}_\ell$ was approximated by iY_ℓ. We note that, as a result:

$$f_1\,(z_{D\pm}) = 0 \tag{7.13}$$

Evaluating f_2 we find:

$$f_2(z) = \frac{z^{\ell-3} \ bc^2 \ P_4(z)}{\pi^2 \ 2^{\ell-1} \ (\ell-1)! \ a\left\{2b^2 \ z+\ell(c^2-b^2)\right\}} \tag{7.14}$$

where

$$P_4(z) = P_1(z) - zP_2(z) \tag{7.15}$$

We now seek a correction to $z_{D\pm}$ to account for a small but finite value of k_oc. Suppose this correction is $\delta z_{D\pm}$. Keeping linear terms in $\delta z_{D\pm}$ the dispersion relation becomes:

$$i\left[\frac{\delta z_{D\pm} \ f_1'\,(z_{D\pm})}{(k_oc)^{\ell+3}} + 0\,(k_oc)^{\ell-2}\right] = \left[\left\{f_2(z_{D\pm}) + \delta z_{D\pm} \ f_2'\,(z_{D\pm})\right\}(k_oc)^{\ell-3} + 0\ (k_oc)^{\ell-2}\right] \tag{7.16}$$

Now all the terms in both braces, except for $\delta z_{D\pm}$, are pure real. Therefore, the most important contribution to the imaginary part of $\delta z_{D\pm}$ comes from the first term on the right hand side. This situation corresponds to the physical idea that the Hankel functions are basically responsible for radiation. We find:

$$\mathrm{Im}\,(\delta z_{D\pm}) = -\frac{(k_o c)^{2\ell} f_2\,(z_{D\pm})}{f_1'\,(z_{D\pm})} \tag{7.17}$$

Using the fact that $P_3(z_{D\pm}) = 0$ we find:

$$\mathrm{Im}\,(\delta z_{D\pm}) = \frac{-\pi}{\ell!(\ell-1)!}\left(\frac{z_{D\pm}\,k_o c}{2}\right)^{2\ell}\frac{P_4(z_{D\pm})}{P_3'\,(z_{D\pm})} \tag{7.18}$$

The denominator of (7.18) is the derivative of a quadratic at a root. It therefore cannot vanish unless the roots coalesce--a limiting case with which we shall not concern ourselves. We find:

$$P_3'\,(z_{D\pm}) = \pm\ 2\ \frac{c^{\ell}}{a^{\ell}}\ D \tag{7.19}$$

giving

$$\mathrm{Im}\,(\delta z_{D\pm}) = \frac{-\pi}{2\ell!\,(\ell-1)!}\left(\frac{z_{D\pm}\,k_o\sqrt{ac}}{2}\right)^{2\ell}\frac{P_4(z_{D\pm})}{\pm\ D} \tag{7.20}$$

The question of stability now can be seen to depend on the sign of $P_4(z_{D\pm})/\pm D$. A certain amount of algebra, here omitted for the sake of brevity, shows that:

$$\frac{P_4(z_{D-})}{-D} \leq 0 \leq \frac{P_4(z_{D+})}{+D} \tag{7.21}$$

We conclude that the diocotron root z_{D-} is in general destabilized by radiation, while the diocotron root z_{D+} is stabilized by the same effect. Further, the case $\ell = 1$ is more important than the higher modes. We illustrate this conclusion by considering the unstable root for the case $\ell = 1$. The real frequency ω_{D-} obtained for $\ell = 1$ from (7.7) is plotted in Fig. 6 as a function of a/c and b/c; only values satisfying the condition $D^2 \geq 0$ are considered interesting. We have chosen to non-dimensionalize ω_{D-} by using Ω_o rather than ω_o to bring out the fact that the mode in question always has the real part of its frequency close to Ω_o. The destabilizing imaginary correction to z_{D-} is for this case:

$$\mathrm{Im}\delta z_{D-} = \frac{\pi}{8} \frac{\beta^2 b^2 z_{D-}^2}{(c^2-b^2)} \cdot$$

$$\left[\left(1-\frac{a^2}{bc}\right)\left\{\frac{a^2+c^2+2bc}{a^2+c^2-2bc}\right\}^{1/4} - \left(1+\frac{a^2}{bc}\right)\left\{\frac{a^2+c^2-2bc}{a^2+c^2+2bc}\right\}^{1/4}\right]^2 \tag{7.22}$$

Fig. 7 is a plot of $\mathrm{Im}\delta\omega_{D-}$ from (7.22), that is, of the growth rate of the destabilized diocotron root normalized to $\beta^2\Omega_o$. The growth rate is large near the boundary across which D becomes imaginary, but vanishes when $a = b$, that is when the inner edge of the beam is in contact

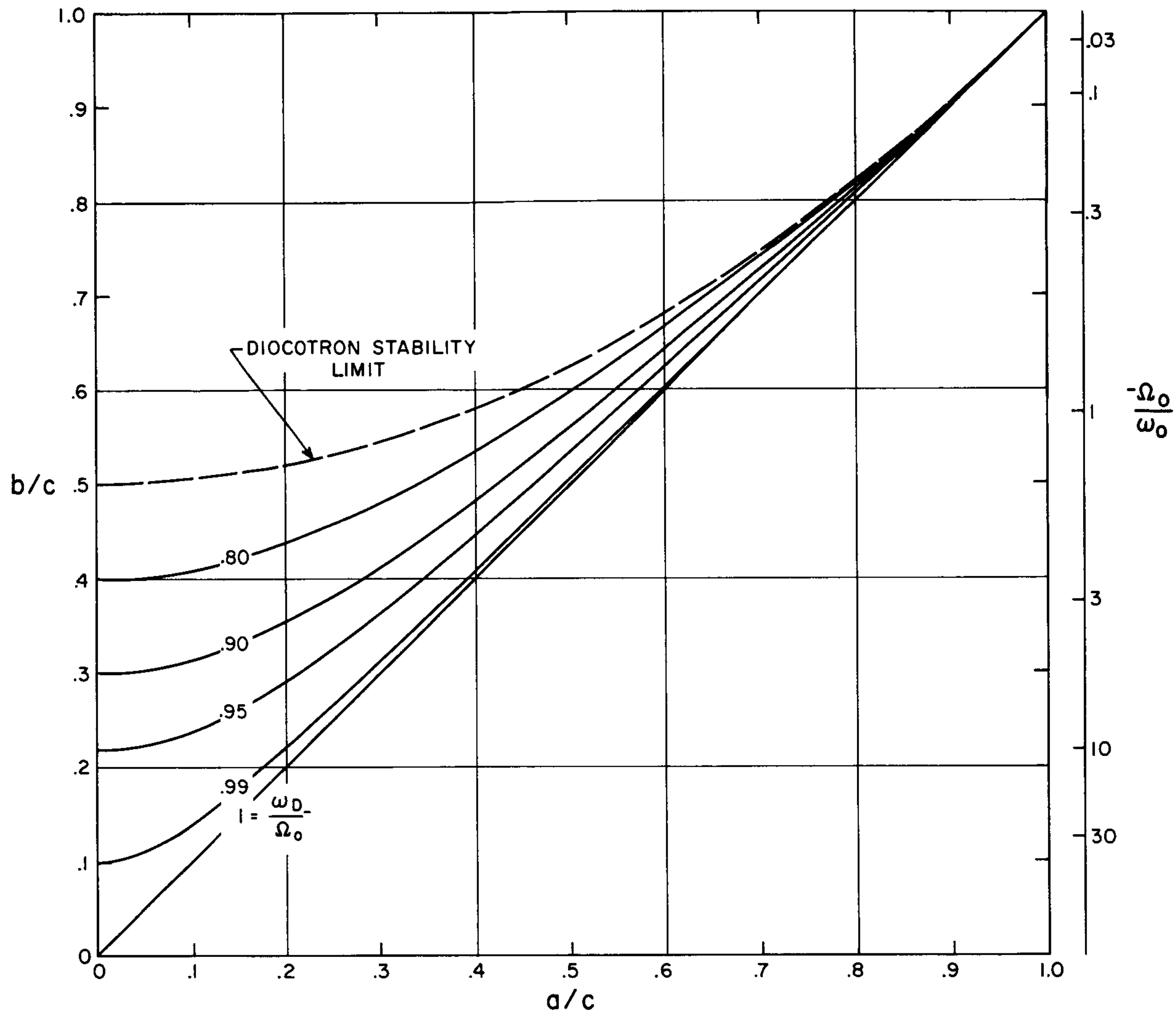

Fig. 6 This figure shows the real diocotron frequency which, when a small radiative perturbation is considered, yields a destablizing imaginary correction. The azimuthal mode number considered is $\ell = 1$, since this yields the largest destablizing correction. The calculation is only interesting when the diocotron frequency is real, that is, below the curve marked "Diocotron Stability Limit". The frequency in question is non-dimensionalized to Ω_o. For reference the ratio Ω_o/ω_o (which is a function only of b/c) is shown on a scale to the right.

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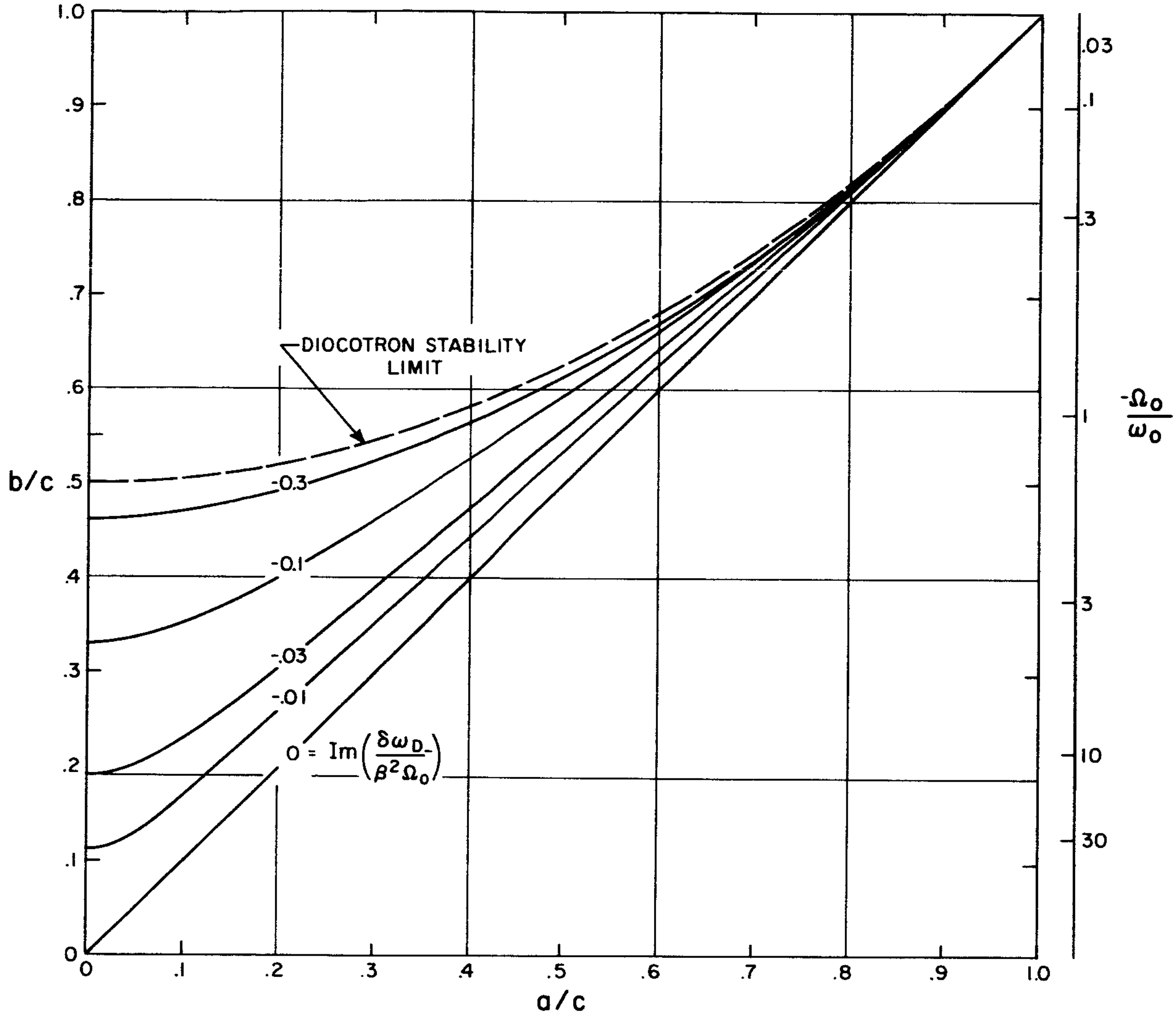

Fig. 7 For values of a/c and b/c within the diocotron stability limit, this figure shows the stabilizing imaginary correction to the real frequency plotted in Fig. 6. Ω_o being negative, $\delta\omega_{D-}$ is positive. $\delta\omega_{D-}$ is, as indicated proportional to the quantity β^2, assumed small. For reference, the ratio Ω_o/ω_o is shown on a scale to the right.

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with the conducting boundary. When $b \approx a$, (7.22) can be written

$$\mathrm{Im}\ \delta z_{D-} \approx \beta^2 \frac{\pi}{2} \frac{(1-a/b)^2}{(1-b^2/c^2)^2} \tag{7.23}$$

Since our ultimate object is to find stable configurations, the possibility of letting $a = b$ so that the growth rate indicated in (7.22) and (7.23), as well as the growth rates of all the higher modes vanish, is quite attractive. Clearly, it is not reasonable to suppose that the electron beam actually touches the conducting cylinder; but (7.23) is probably reasonable for $(b-a)$ on the order of the electron gyro-radius. The present analysis is not capable of giving reliable information on scales as small as this; a fuller treatment considering a finite electron temperature would be required. In the absence of such a treatment, it seems worthwhile to consider the idealized case in which $b = a$.

Mathematically, when $b = a$, $F_5' = 0$ from (6.6) and (6.9). Hence, from (6.10), $F_6 = F_5$. It is therefore convenient to redefine g_1 and g_2 as:

$$\left.\begin{aligned} g_1\,(a=b) &= \xi\, Q_1 \\ g_2\,(a=b) &= z\, Q_3 \end{aligned}\right\} \tag{7.24}$$

With these new definitions, the dispersion relation in the form (6.13) is still valid. On approximation to the diocotron case, using (7.24) we find instead of the quadratic (7.4) a linear equation yielding the single root

$$z_{D+} = -\frac{1}{2}\left[1 - \left(\frac{b}{c}\right)^{2\ell}\right] \tag{7.25}$$

the root z_{D-} being lost. This can be thought of as follows: the oscillation corresponding to z_{D-} requires a non-vanishing E_{Θ} at $r = b$. Placing the wall at $r = b$ therefore eliminates this mode. It can be shown that the radiative correction to (7.25) is always stabilizing. It follows that, insofar as it is reasonable to set $b = a$, perturbations of the diocotron roots are always stabilizing. This interesting possibility leads us to consider the case $a = b$ numerically in the next section to investigate the possible existence of unstable roots elsewhere in the complex frequency plane.

We conclude this section by returning to the case $\ell = 0$ which was excepted from the approximations of this section. Analysis (not given here) shows that there are in fact no azimuthally symmetric diocotron modes of oscillation at all; there is therefore nothing to correct for small radiative effects. The possibility remains for study, however, of unstable or stable roots elsewhere in the complex plane.

8. NUMERICAL STUDIES OF THE DISPERSION RELATION

No better method for numerical studies of the dispersion relation (6.13) is known than computing and plotting graphs showing M(z), and scanning them for possible zeros. This is particularly simple if the plots are constructed to show the lines in the complex z-plane on which the phase and the argument of M are constants; on such a graph, poles and zeros stand out very clearly. It has been possible to draw from a few such graphs general ideas about the behavior of M(z) in relation especially to its asymptotic character. These ideas in turn permit a definite statement about the location of unstable roots in the upper half of the complex frequency plane.

We pick two cases for detailed study. One (Case A) has $a < b$ in order to bring out the unstable root known to exist for small β near one of the real diocotron frequencies in such a case. In order to ensure that the diocotron frequencies are real for all ℓ we must have $a^2 + c^2 - 2bc > 0$. We fix a:b:c = 1:10:30 and choose different values of k_o to give different values of β. The actual values chosen for study are listed in Table A.

TABLE A

Case	$k_o a$	$k_o b$	$k_o c$	β	$-\Omega_o/\omega_o$	$V_o(a)/aE_o(a)$
A1	.001	.01	.03	.0400	4.000	3.039
A2	.01	.1	.3	.3910	3.910	3.037
A3	.03014	.3014	.9043	1.000	3.317	3.021

The second case has a = b. For this case only a single diocotron frequency exists, and it is always real. The radiative perturbation to this frequency is stabilizing. We fix a:b:c = 2:2:3 and choose different values of k_o to give different values of β. The actual values chosen are listed in Table B.

TABLE B

Case	$k_o a = k_o b$	$k_o c$	β	$-\Omega_o/\omega_o$	$V_o(\ell)/\ell E_o(\ell)$
B1	.02	.03	.0125	.6250	.2296
B2	.2	.3	.1245	.6225	.2296
B3	2.603	3.905	1.000	.3842	.2032

In both cases the modes studied were ℓ = 0, 1, and 2. For Case B the real diocotron frequency z_{D+} is $-.2778$ for $\ell = 1$ and $-.4012$ for $\ell = 2$.

The general behavior of M(z) is that it follows an inverse power law for very small z and, for very large z, separate exponential laws in two regions of which the first extends from somewhat below the real axis and includes the entire upper half plane, while the second extends over the region for which Im z is large and negative. It is important to observe that there is a region for moderate negative values of Im z for which no simple asymptotic expansion is available. In this region there are an infinite number of zeros; since they all necessarily correspond to damping, no effort has been expended to locate them accurately. For Case A the essential features of these expansions are given in the following formulae: for $|zk_oc| \ll 1$:

$$M(z) \sim z^{-(\ell+3)} \qquad (\ell \geq 1) \tag{8.1}$$

$$M(z) \sim z^{-2} \qquad \ell = 0 \tag{8.2}$$

For $|zk_oa| \gg 1$, and $\text{Im } z \gtrsim 0$,

$$M(z) \sim \left(\frac{2}{\pi k_o a}\right)^{5/2} \frac{a^2}{bc} z^{-3/2} \exp\left\{ i\,(zk_oa - \frac{1}{2}\ell\pi - \frac{3}{4}\pi)\right\} \tag{8.3}$$

For $zk_oa \gg 1$, and $\text{Im } zk_oa \ll -1$,

$$M(z) \sim \left(\frac{2}{\pi z k_o a}\right)^{7/2} \frac{\pi}{8} \cdot \frac{a^3}{bc^2} (2i\,\ell + k_oc) \exp\left\{ i\left[z(2k_oc - k_oa) - \frac{1}{2}\ell\pi + \frac{1}{4}\pi\right]\right\} \tag{8.4}$$

For Case B, the general behavior of M is similar in the same regions, and we omit the detailed formulae.

We shall first state the results of the numerical studies, and then illustrate them by exhibiting a very few of the many graphs produced showing the function $M(z)$. The results show that the conclusions reached in the previous section for small values of β are, in fact, applicable to all values of β. That is, when $a < b$, there is a single unstable root for each azimuthal mode number $\ell \geq 1$. This root tends, for small β, to the diocotron root z_{D-}. For $\ell = 0$ there are no unstable roots at all. When $a = b$ there are no unstable roots for any value of ℓ. We list in Table C the unstable roots calculated for the cases A1, A2, and A3, for the modes $\ell = 1$ and $\ell = 2$.

TABLE C

Mode	z_{D-}	Case	$\mathrm{Im}\,\delta z_{D-}$ *	Calculated Root Real Part	Calculated Root Imaginary Part
$\ell=1$	-3.4959	A1	6.4220×10^{-4}	-3.4965	6.3904×10^{-4}
		A2	6.4220×10^{-2}	-3.4655	5.5897×10^{-2}
		A3	.58351	-2.9024	.29479
$\ell=2$	-7.4996	A1	1.7824×10^{-6}	-7.4983	1.7754×10^{-6}
		A2	1.7824×10^{-2}	-7.3783	1.3291×10^{-2}
		A3	1.4714	-6.2836	.20117

*Calculated from (7.22) for comparison with the imaginary part of the calculated root.

It is noteworthy that the formula (7.22) provides an estimate of the growth rate of the unstable mode in question which is not in error by an order of magnitude even when $\beta = 1$, this in spite of the fact that it was derived on the assumption $\beta \ll 1$.

In Fig. 8 we exhibit four graphs showing, in the upper half of the complex z-plane lines on which the phase or modulus of M is constant. All the graphs shown refer to the Case A3, but all the other cases examined showed no important differences. Fig. 8a shows the mode $\ell = 0$, on a scale extending to $z = 100$. On this graph we see clearly the smooth way in which M makes the transition from behavior described by (8.2) to behavior described by (8.3). Figs. 8b, 8c, and 8d show the mode $\ell = 1$ on scales extending respectively to z = 5, 50, and 500. On the first of these we see both the zero corresponding to the root at $z = -2.9 + .29i$, as well as the pole at $z = -3.3$ expected from the discussion preceding (6.15). It is interesting to see how clearly zeros and poles show up on diagrams of this type. Figs. 8c and 8d show no further structure, but just the transition to the asymptotic form (8.2).

Having established the result that there is a single unstable mode of oscillation for each $\ell \geq 1$ whenever $a < b$, we conclude this section by exhibiting in some detail the exact character of this unstable mode. To do this, we establish the ratios of the constant multipliers of the various eigenfunctions discussed in Section 6. Figs. 9a and b illustrate the amplitudes (to an arbitrary scale) and the phases of the complex amplitudes E_Θ, E_r and c_oB. The jump in E_r at $r = b$ is $65.66\, e^{i83.05^\circ}$, and the jump in c_oB at $r = b$ has the same value since $\beta = 1$. The jump in E_r at $r = c$

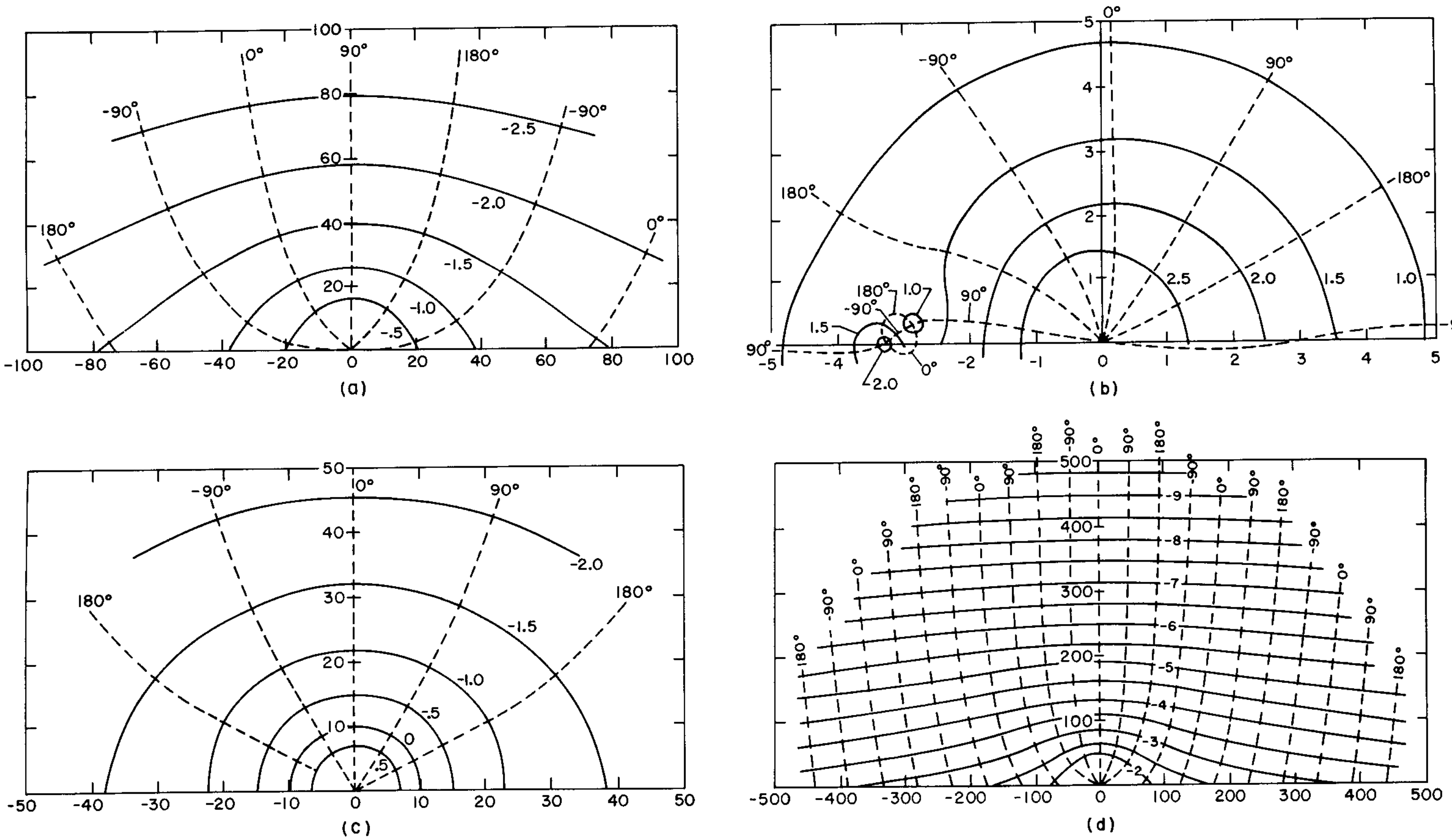

Fig. 8 Illustrates the upper half of the complex frequency plane for four representative cases. 8a refers to Case A3 for the mode $\ell = 0$, while 8b, c and d refer to Case A3 for the mode $\ell = 1$ on various scales. Note particularly the location of the zero and the pole in Fig. 8b. The numbers on the lines of constant modulus give the value of $\log_{10} |M|$; the numbers on the lines of constant phase give the value of arg M in degrees.

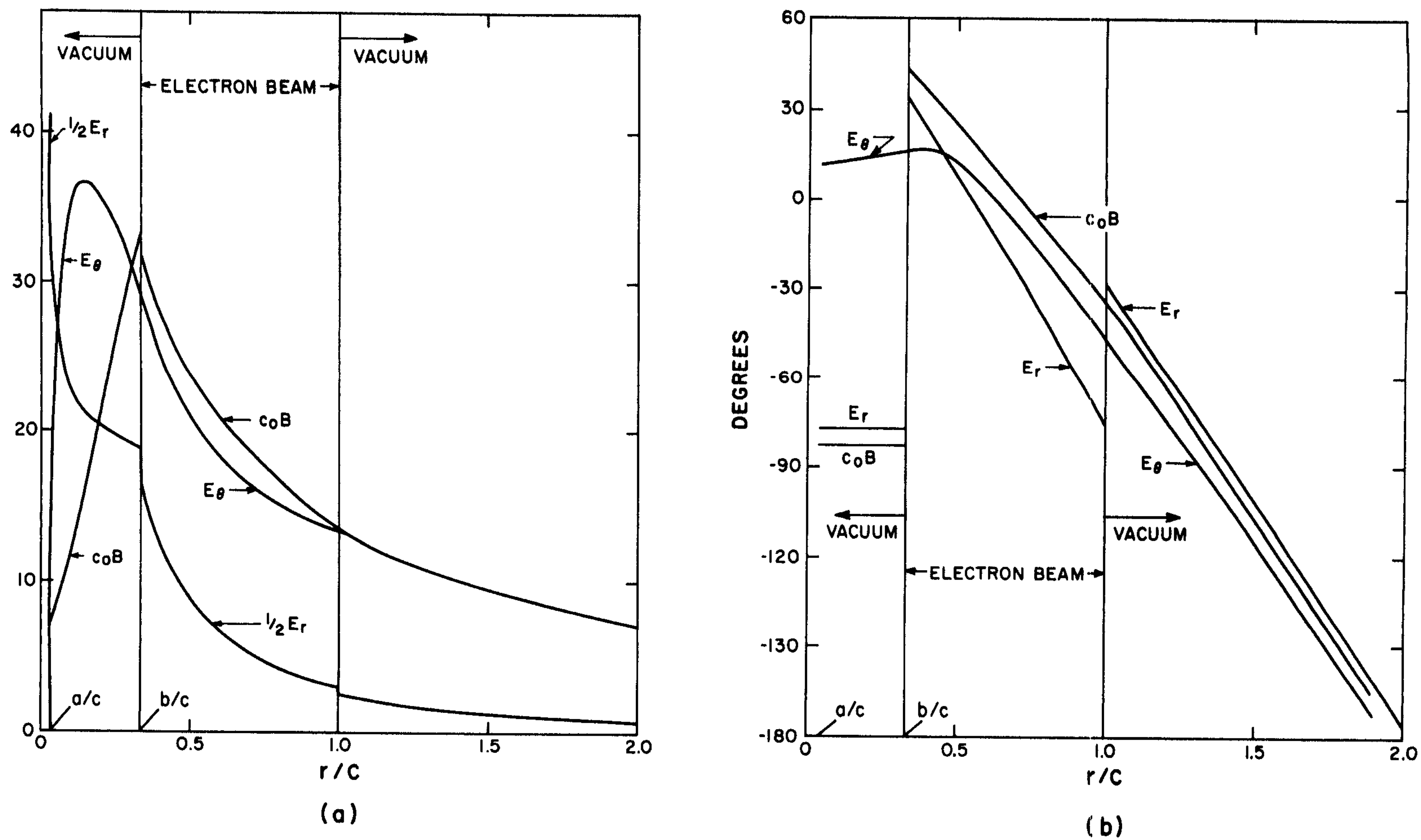

Fig. 9 Illustrates the structure of the unstable mode for the case A3, $\ell = 1$. The scale for the moduli is arbitrary.

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is 16.82 $e^{i21.67^{\circ}}$. It does not seem possible to give a simple explanation of the structure of this mode in terms of regions of high and low density leading or lagging regions of high and low velocity.

9. CONCLUSIONS

The dispersion relation (6.13) admits exactly one unstable eigenvalue for each $\ell \geq 1$, namely the one corresponding to one of the two real diocotron frequencies obtained when $\beta \rightarrow 0$. For small β, the growth rate corresponding to this mode is of order $\beta^2 \omega_o$ (or $\beta^2 \Omega_o$) multiplied by a geometrical factor which vanishes when $b = a$. Insofar as it is reasonable to suppose $b = a$, stability against the radiative effect considered is possible. The reasonableness of this supposition requires examination, but would require a treatment beyond the scope of this paper.

In physical terms this instability appears quite serious from the point of view of raising the potential of a space vehicle to very high values. It is not possible, however, to draw definite conclusions on this point yet, firstly on account of the important change in the geometry between the configuration studied and that proposed for use in space. Secondly, and probably more importantly, it may be possible to control the instability by appropriate selection of the admittance of the surface of the cylinder. Here we have only treated the simplest case of a perfectly conducting boundary. It may be useful to add a reactive component to the admittance. To illustrate this point we may consider yet another configuration, one discussed by Janes et al. [14] which in geometrical and magnetic configuration resembles the stellarator concept. [15] In this geometry it may be possible to attain potentials up to 10^9 volts in the laboratory. However since, in this case, the electron beam is entirely enclosed within conducting walls the impedance of free space (i. e. the radiation condition) is hardly important. The radiative stability of this configuration may therefore be expected to be entirely different from that of the configuration studied in this paper.

Finally, we discuss briefly the likely possibilities when the unperturbed distribution differs from the highly special one considered here. The best that can be said in this regard seems to be that our results should be representative of the class of equilibrium configurations having a single maximum in the electron density profile. No firmer basis for this statement exists at present than the knowledge that this type of argument is reasonably accurate in related shear flow problems.[9] It is hoped that this problem, as well as those connected with the relaxation of the five physical assumptions listed in Section 1 can be subjected to more quantitative analysis in the future.

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