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# BINDING ENERGY AND STABILITY OF SPHERICALLY SYMMETRIC MASSES IN GENERAL RELATIVITY*

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## ABSTRACT

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An analysis of non-adiabatic perturbations in a sequence of quasi-static general relativistic stellar models is used to prove that under certain assumptions about the entropy distribution a maximum or minimum in the binding energy implies some radial mode of oscillation is in neutral equilibrium. Also discussed is the derivation of a test for stability of radial modes of oscillation by considering virtual changes in the total energy under adiabatic perturbations.

Author

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# I. INTRODUCTION.

General relativistic equilibrium models of gas spheres have been calculated by several authors for various equations of state. We mention here the recent work of Iben (1963), Tooper (1964a,b), and Taub (1965). A given model is characterized by several parameters, such as the total energy M, the total baryon mass or "rest mass" $M_o$, the central baryon or "rest mass" density $(1/V)_c$, the central ratio of pressure to "rest mass" density $(PV)_c$ (proportional to the temperature for an ideal gas), and the value of the "gravitational potential" $2M/R$ at the surface. To completely specify a model, given the equation of state of the matter, it is necessary to know the values of two of the above parameters, say $M_o$ and $(1/V)_c$, and also the distribution of entropy as a function of radius.

At zero temperature (degenerate matter) there is only one independent parameter, which may be taken to be $M_o$. Models of this sort have been considered by Wheeler (1963), Misner and Zapolsky (1964), and Harrison (1965). At non-zero temperatures a one-parameter sequence of models may be constructed by letting the central density vary monotonically but keeping the number of baryons (or $M_o$) constant and at each central density choosing an entropy distribution. Such a sequence might correspond to a mass undergoing a generalized Kelvin-Helmholtz contraction in quasi-static equilibrium in which increase as well as decrease of the total energy is allowed. One finds that the total energy, or equivalently the binding energy, has a number of maxima and minima as a function of central density.

The stability of equilibrium models in general relativity has been discussed by Chandrasekhar (1964a,b) in terms of a normal mode analysis of radial oscillations. Chandrasekhar's stability criterion has been applied

to a series of isentropic models, assuming a gas-plus-radiation pressure equation of state, by Tooper (1964b) with the numerical result that the mass becomes unstable at the first maximum of the binding energy. Fowler (1964) and Wright (1964) have shown this analytically in the post-Newtonian approximation. The main purpose of this paper is to show that for any equation of state a sequence of general relativistic equilibrium models with constant number of baryons and constant non-thermal or "rest mass" energy per baryon at each mass element has the property that at a stationary point in the binding energy as a function of central density one and only one radial mode of oscillation has eigenfrequency $\omega^2 = 0$, if the change in specific entropy from one model to the next has the same sign at every mass element. The condition on the entropy change is satisfied, for example, if each model is isentropic, so the specific entropy has the same value at every mass point in a given model.

The last part of the paper is devoted to an analysis of the stability of an equilibrium model in terms of virtual changes of the total energy under adiabatic perturbations, extending to general relativity the work of Dyson (1961). The interesting point is that the correct stability integral is obtained only if the virtual changes in the energy are summed at infinity as if they had arrived there by some process of thermal diffusion and radiation.

## II. DEFINITIONS.

Our metric, in coordinates such that the baryon number inside a given r is independent of time and units such that $G = c = 1$, is

$$Ds^2 = Y^{-2}\,dt^2 - X^{-2}\,dr^2 - R^2\,d\Omega^2 \quad . \qquad (1)$$

The components of the energy-momentum tensor are taken to be $T_4^4 = E = (A+W)/V$, $T_1^1 = T_2^2 = T_3^3 \equiv -P$, $(Y/X)\, T_4^1 = K$. The specific volume V is conveniently normalized so the non-thermal energy per unit "mass", A, is equal or close to one. If any nuclear reactions are taking place, A will be a function of time. W is the thermodynamic internal energy per unit "mass". The justification for splitting E in this way is that except at extremely high temperatures or densities, the atomic composition of the matter is not in local thermodynamic equilibrium. Thus, nuclear mass and binding energies must be excluded from the thermal energy if simple thermodynamics is to be used. The energy flux K is introduced in order to be able to consider the transformation of one model in a sequence to another, and might in particular be due to thermal diffusion. The particular mechanism of energy loss is unimportant because even if it does involve non-thermal contributions to E and P, these will be negligible compared to the thermal energies and pressures and even negligible in their effects compared to K, in quasi-static models.

The quantity $1/V$ is an "invariant mass" density tied to the baryon density, and the total amount of "invariant mass" inside a given radius will be denoted by $M_o$,

$$M_o = \int_o^r \frac{4\pi R^2}{VX}\, dr \quad . \tag{2}$$

$M_o$ is a function only of r, and one may choose $r = M_o$. The most convenient definition of a "binding energy" for my purposes is as $M_o - M$, where M is the total energy. The true binding energy is $\int A dM_o - M$, but since the relation of binding energy to stability holds only if no nuclear reactions are

taking place and in this case V can be chosen so A = 1 through the whole star, the difference is unimportant.

The only way the total energy $M_T$ of the mass can change is through the flux K, since

$$\dot{M} = -4\pi PR^2 \dot{R} - 4\pi R^2 KZ/Y \tag{3}$$

and $P = 0$ at the surface of the mass. $Z = R'X = (1+U^2-2M/R)^{1/2}$, and $U = Y\dot{R}$. Let

$$\dot{M}_K = -4\pi R^2 KZ/Y \quad . \tag{4}$$

The $u^\nu T^\mu_{\nu;\mu} = 0$ equation may be written as

$$\dot{A} + \dot{W} + P\dot{V} = -Y \frac{\partial}{\partial M_o}\left(\frac{4\pi R^2 K}{Y^2}\right) \quad . \tag{5}$$

If S is the specific entropy and T the temperature, by thermodynamics

$$\dot{W} + P\dot{V} = T\dot{S} \quad . \tag{6}$$

Then (5) becomes

$$\dot{A} + T\dot{S} = \dot{Q} = \frac{1}{Z}\left[YZ \frac{\partial}{\partial M_o}\left(\frac{\dot{M}_K}{YZ}\right)\right] \quad . \tag{7}$$

$\dot{Q}$ is the rate at which energy as measured by a local observer is added to a mass element per unit invariant mass. The significance of (7) becomes clearer when it is compared with the integral for M; neglecting explicit time derivatives in the spirit of a quasi-static approximation,

$$M = \int (A+W)\, Z\, dM_o \quad . \tag{8}$$

In a quasi-static model $Z = \sqrt{1-2M/R}$, and $(A+W)Z$ is the rest mass, thermal, and gravitational potential energy per unit invariant mass of a mass element. Over a small time interval $\delta t$, $\delta Q = \dot{Q}\,\delta t$ is the non-adiabatic change in the internal energy per unit $M_o$ and $\delta QZ$ is the net non-adiabatic change in "local" energy, which includes the potential energy, per unit $M_o$. Equation (7) gives the non-adiabatic change in the total energy inside r, $\delta M_K = \dot{M}_K\,\delta t$, as

$$\delta M_K = YZ \int_o^r \frac{(\delta QZ)}{YZ}\, dM_o \quad . \tag{9}$$

The non-adiabatic change in local energy at a radius 1 is red-shifted by the amount $(YZ)_2/(YZ)_1$ in its contribution to the non-adiabatic change of the total energy inside a larger radius 2.

## III. PROOF AND DISCUSSION OF THEOREM.

The equilibrium equation, in a form convenient for making perturbations, is

$$-Z\,\frac{4\pi R^2}{(P+E)V}\,\frac{\partial P}{\partial M_o} - M/R^2 - 4\pi PR = \Phi = 0 \quad . \tag{10}$$

To first order in $\delta R$, with $r = R$ of the initial model,

$$\delta M = -4\pi PR^2\,\delta R + \delta M_K \quad , \tag{11}$$

$$\delta Z/Z = -\frac{Y'}{Y}\,\delta R - \delta M_K/(Z^2 R) \quad , \tag{12}$$

$$\delta V/V = \frac{1}{R^2}\,(R^2\,\delta R)' - \delta Z/Z \quad , \tag{13}$$

$$\delta P = -\Gamma_1 P \frac{\delta V}{V} + (\Gamma_2 - 1) \frac{T\delta S}{V} \quad , \tag{14}$$

$$\delta\left[(P+E)V\right] = \delta Q + V\delta P \quad . \tag{15}$$

(14) and (15) are the result of thermodynamics, and

$$\Gamma_1 = -\frac{V}{P} \left(\frac{\partial P}{\partial V}\right)_S \quad , \tag{16}$$

$$\Gamma_2 - 1 = -\frac{V}{T} \left(\frac{\partial T}{\partial V}\right)_S \quad . \tag{17}$$

The quantity $Y'/Y = P'/(P+E)$ evaluated for the initial equilibrium model. The hydrodynamic equations behind (10) - (13) are discussed by Misner and Sharp (1964, 1965).

A sinusoidal, adiabatic perturbation with $\delta R(r,t) = \delta R(r)\, e^{i\omega t}$ and $\delta A = \delta S = 0$ for all r gives the equation

$$\frac{d}{d\Omega}\left(\frac{\Gamma_1 P}{Y^3 Z}\frac{dg}{d\Omega}\right) - \left[\frac{4}{9}\frac{Y'}{Y} R\left(1 - \frac{1}{4}\frac{RY'}{Y}\right) + \frac{8\pi}{9}\frac{PR^2}{Z^2}\right]\frac{(P+E)}{Y^3 Z\,\Omega^2}\, g$$

$$= -\frac{\omega^2}{9}\frac{(P+E)\,R^2}{YZ^3\,\Omega^2}\, g = \frac{P+E}{Y^2 Z^3 R^2}\,\delta\Phi \quad , \tag{18}$$

in which $g = R^2 Y\,\delta R(r)$, $\Omega = R^3/3$, and $\delta\Phi$ is the first order variation of (10). (18) with boundary conditions $g/\Omega$ finite at $R = 0$ and $g$ finite at $P = 0$ is not quite a standard Sturm-Liouville problem, but as discussed by Ledoux and Walraven (1958) for the Newtonian case, the eigenvalues and eigenfunctions have the same properties.

Now consider two neighboring models in some continuous sequence with the same number of baryons, or the same $M_o$. One can define a displacement

$\delta R(r)$ between equivalent points in the two models, equivalent points having the same total baryon number inside them. Both models satisfy equation (10) and thus in the limit that the models converge, or that $\delta R(r) \to 0$, the difference between the left-hand sides of (10) for the two models, expanded to first order in $\delta R$, must vanish. That is, $\delta\Phi$ for one of the models, under a perturbation having the shape of $\delta R(r)$ in this limit, must vanish.

In order to connect up this result with equation (18), the perturbation must be an adiabatic one. From equation (9) and the fact that $\delta M = \delta M_K$ at the surface of the mass, we see that the difference of total energies of two adjacent models, or equivalently the difference of the binding energies, is given to first order by

$$\delta M_T = (YZ)_s \int_0^{r_s} \frac{\delta Q}{Y} \, dM_o \quad . \tag{19}$$

$r_s$ is the value of r at the surface. At a maximum or minimum in the binding energy, $\delta M_T$ is zero. If $\delta Q$ has the same sign at every mass element, this is only possible if $\delta Q$ is zero at every mass element. One case in which $\delta Q = \delta A + T\delta S$ does have the same sign everywhere is if $\delta A$ is zero (no nuclear reactions) and if the models are isentropic, so for any pair of models $\delta S$ has a constant value throughout the mass. The fact, then, that $\delta Q$ and $\delta S$ are zero at every mass element means that $\delta M_K$ is zero at every mass element and $\delta P$ is adiabatic. Furthermore, it is clear from equations (11) - (15) that all the perturbations are adiabatic in the strict sense needed to apply (18).

If the displacement between the two neighboring models is completely adiabatic, then the function $\delta R(r)$ is an eigenfunction of (18) with eigenvalue $\omega^2 = 0$. This means some radial mode of oscillation is in neutral

equilibrium at a stationary point in the binding energy as a function of central density for a series of isentropic models.

Calculations with equation of state

$$W = \frac{PV}{\Gamma_1 - 1} \quad , \tag{20}$$

$\Gamma_1$ a constant independent of V and P, show that if $\Gamma_1 > 4/3$, as the central density increases the binding energy increases to a maximum at which the fundamental mode becomes unstable, decreases to a minimum at which the first harmonic becomes unstable, and continues in rapidly damped oscillations with successively higher modes becoming unstable.

The fact that displacements between neighboring models take on the shape of a given radial mode of oscillation near the point where it becomes unstable is reflected in how the radius of the mass changes along the sequence of models, oscillating out of phase with the binding energy. As the central density increases, the radius decreases at the instability point of an even mode (even number of nodes) and increases at the instability point of an odd mode. If the binding energy is plotted against the value of $2M/R$ at the surface for the above models, the curve is a lop-sided spiral which rapidly converges to a limiting model when $PV \gg 1$, as is shown in Figure 1 for the case $\Gamma_1 = 2$.

The above behavior, I expect, is typical as long as $\Gamma_1$ does not increase sharply with density, which may cause an unstable mode to become stable again. In this event, if just the fundamental was unstable, the binding energy will have a minimum when the fundamental becomes stable and a maximum when it becomes unstable again. In the plot against $2M/R$, the curve will oscillate once before entering the spiral.

The requirement of isentropy may be relaxed somewhat without affecting the validity of the theorem as long as it is not possible for the entropy to increase in part of the mass and decrease in another part such that $\delta M_T$ is zero. However, the condition $\delta A = 0$ would seem to be a necessary condition for the stability to have anything to do with the binding energy.

The converse to the above theorem, that the marginal instability of a radial mode of oscillation of a model implies a maximum or minimum in the binding energy, is not true. All that one can say is that there exist sequences of models containing this particular model which are stationary in the binding energy at this model. There also exist sequences in which at the instability of the fundamental mode, say, the binding energy is still increasing as the central density increases. In the latter type of sequence, the subsequent maximum in the binding energy is not associated with the instability of any mode of oscillation.

Of course, only the instability of the fundamental mode is of direct physical interest, since unstable equilibrium models cannot be realized in nature.

## IV. VIRTUAL CHANGE IN TOTAL ENERGY.

A test for the stability of a mass may be obtained from equation (18) in the form of the integral

$$\int d\Omega \; \frac{(P+E)}{Y^3 Z} \left[ \frac{\Gamma_1 P}{P+E} \left(\frac{dg}{d\Omega}\right)^2 + \left[ \frac{4}{9} \; \frac{RY'}{Y} \left(1 - \frac{1}{4} \; \frac{RY'}{Y}\right) + \frac{8\pi}{9} \; \frac{PR^2}{Z^2} \right] \left(\frac{g}{\Omega}\right)^2 \right] \quad . \qquad (21)$$

If, for all functions $g(R)$ satisfying the boundary conditions $g/\Omega$ finite at $R = 0$ and $g$ finite when $P = 0$, the integral (21) is greater than zero,

the mass is stable; if for any such g the integral is less than zero the mass is unstable. In addition, for a mass in neutral equilibrium, the difference of (21) from zero is second order in the deviation of g from the eigenfunction of the fundamental mode.

To see how the integral (21) and a variational test for stability may be obtained from the virtual change in the total energy of the mass, $\delta_v M$, under a virtual adiabatic displacement, consider the integral (8), with V normalized so $A = 1$. $M_o$ is constant under the variations and to second order in $\delta R$

$$\delta_v M = \int \left[ \delta_1 W Z + (1+W)\, \delta_1 Z + \delta_2 W Z + \delta_1 W\, \delta_1 Z + (1+W)\, \delta_2 Z \right] dM_o \quad . \quad (22)$$

The second-order variations in (22) are evaluated by expanding the change in a quantity as

$$\delta f = \delta_1 f + \delta_2 f = \left(\frac{\partial f}{\partial R}\right)_{r,S} \delta R + \frac{1}{2} \left(\frac{\partial^2 f}{\partial R^2}\right)_{r,S} (\delta R)^2 \quad .$$

Z is taken equal to $(1-2M/R)^{1/2}$ in the variations, and

$$\delta_2 M = - 4\pi\, PR\, (\delta R)^2 + 2\pi\, \Gamma_1\, PR^2\, \delta R\, \frac{\delta V}{V} \quad . \quad (23)$$

The second-order variation in W is

$$\delta_2 W = - P\, \delta_2 V + \frac{1}{2}\, \Gamma_1\, P \left(\frac{\delta_1 V}{V}\right)^2 \quad ; \quad (24)$$

$\delta_2 V$ is evaluated from

$$Z\, \delta_2 V + \delta_1 Z\, \delta_1 V + V\, \delta_2 Z = \frac{\partial}{\partial M_o} \left[ 4\pi\, R (\delta R)^2 \right] \quad . \quad (25)$$

The result is that the integrand of (22) can be written in the form

$$\delta\left[(1+W)Z\right] = \frac{d}{dM_o}\left[\delta_1 M + \delta_2 M\right] + \frac{1}{2}\frac{d\Omega}{dM_o} YZ\left[-g\frac{d}{d\Omega}\left[\frac{\Gamma_1 P}{Y^3 Z}\frac{dg}{d\Omega}\right]\right.$$

$$\left. + \left[\frac{4}{9}\frac{RY'}{Y}\left(1 - \frac{1}{4}\frac{RY'}{Y}\right) + \frac{8\pi}{9}\frac{PR^2}{Z^2}\right]\frac{(P+E)}{Y^3\Omega^2 Z^2}g^2\right] . \qquad (26)$$

$\delta_1 M$ and $\delta_2 M$ are the first and second order actual, as opposed to virtual, adiabatic changes in M at a given mass point and are zero at the limits of integration. The first term in (26) is the actual adiabatic change of $(1+W)Z$ in the perturbation. The second term is the virtual change in $(1+W)Z$, $\delta_v\left[(1+W)Z\right]$, and is second order in $\delta R$. From the fact that (18) is essentially a Sturm-Liouville problem, the eigenfunctions $g_i(\Omega)$ of the radial modes of oscillation are orthogonal with respect to the weight function $(P+E)R^2/YZ^3\,\Omega^2$. The extra factor of YZ in $\delta_v\left[(1+W)Z\right]$ means that the integral (22) does not have the property that it is positive definite for all stable models. This is because (22) contains terms first order in the deviation of g from the eigenfunction of a radial mode of oscillation. The proper Sturm-Liouville integral (21) is, within a constant factor,

$$\int \frac{\delta_v\left[(1+W)Z\right]}{YZ}\, dM_o \quad .$$

The interesting point is the similarity with the integral (9). The correct evaluation of the virtual change in the total energy seems to be

$$\delta_v M = (YZ)_s \int_0^{r_s} \frac{\delta_v\left[(1+W)Z\right]}{YZ}\, dM_o \qquad (27)$$

instead of (22). This suggests the interpretation that it is the sum of the virtual changes in energy after they are carried to infinity, including the gravitational red shifts associated with this, that properly determines the stability of a mass. Or alternatively one might say that the virtual change in the local energy, $\delta_v\left[(1+W)Z\right]$, being a non-adiabatic change, has an effect on the total energy in the same way as a non-adiabatic thermal change, whose effect is given by (9).

A variational test for stability may be obtained directly from (27), following the method of Dyson (1961) for the Newtonian case. The equation

$$\frac{d}{d\Omega}\left[\frac{\Gamma_1 P}{Y^3 Z}\,\frac{dg}{d\Omega}\right] = \gamma\,\frac{(P+E)}{Y^3 Z\,\Omega^2}\left[\frac{\Omega}{Y}\,\frac{dY}{d\Omega}\left(1-\frac{1}{4}\,\frac{RY'}{Y}\right)+\frac{2\pi}{3}\,\frac{PR^2}{Z^2}\right] g$$

with the boundary conditions $g/\Omega$ finite at $R = 0$ and $g$ finite at $P = 0$ is a Sturm-Liouville problem with a lowest eigenvalue $\gamma = \gamma_0$. It is easy to verify that a necessary and sufficient condition that (21), equivalent to (27), be positive definite is that $\gamma_0 > 4/3$. A variational estimate of $\gamma_0$ can be obtained from the fact that $\gamma_0$ is the minimum over all $g$ of

$$\left[\int d\Omega\;\frac{\Gamma_1 P}{Y^3 Z}\left(\frac{dg}{d\Omega}\right)^2\right]\Big/\left[\int d\Omega\;\frac{(P+E)}{Y^3 Z}\left(\frac{\Omega}{Y}\,\frac{dY}{d\Omega}\left(1-\frac{1}{4}\,\frac{RY'}{Y}\right)+\frac{2\pi}{3}\,\frac{PR^2}{Z^2}\right)\left(\frac{g}{\Omega}\right)^2\right] .$$

However, the direct variational estimate of $\omega_0^2$ obtained from (18) (see Chandrasekhar (1964b)) is in most cases easier to evaluate accurately numerically.

## V. CONCLUSION.

Except for the appearance of the gravitational red-shift factor $YZ$, which is just a detail in the argument relating binding energy to stability,

the results of this paper are arrived at in the same way as they might be classically. This emphasizes what a great simplification spherical symmetry is in general relativity.

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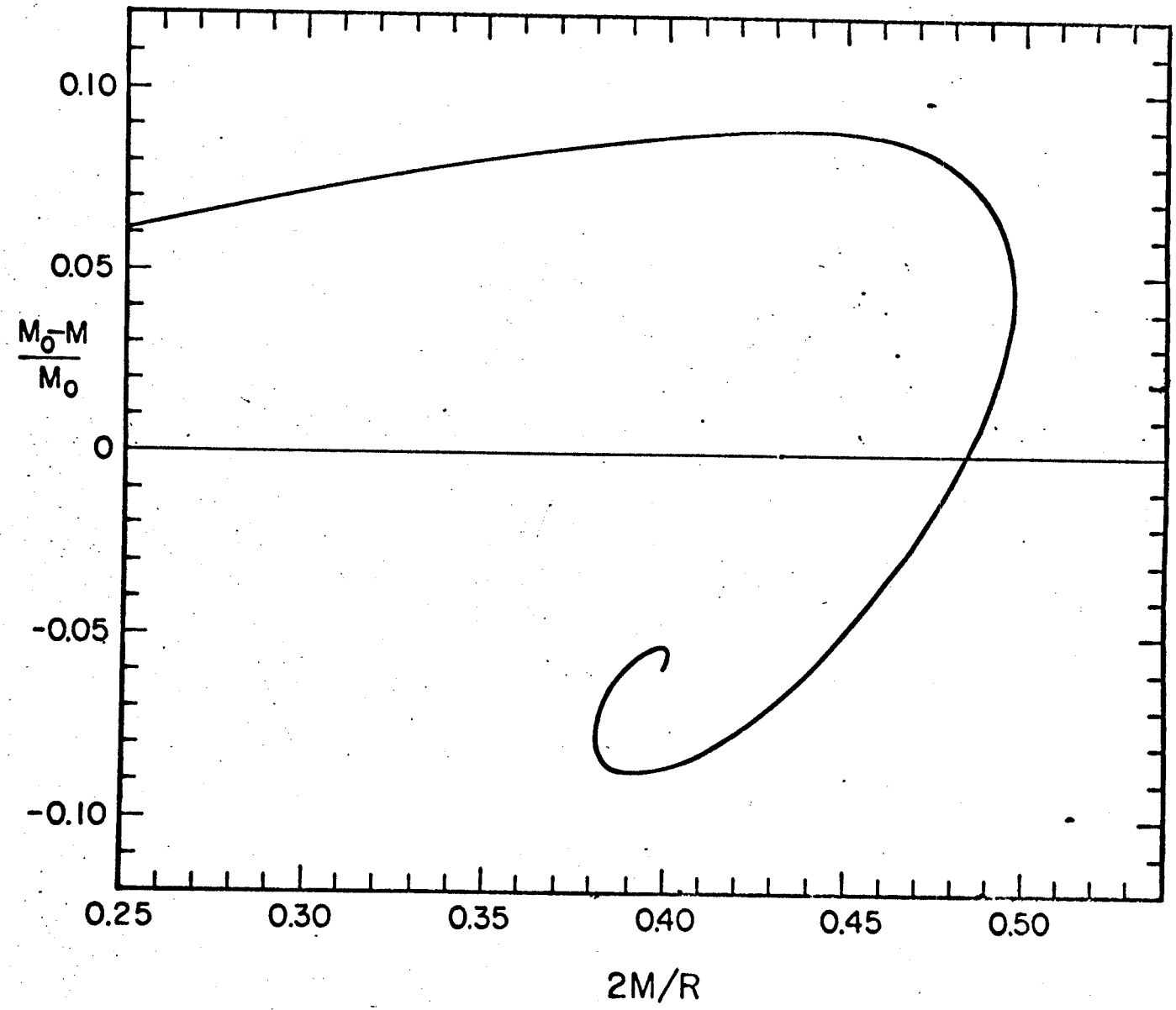
FIGURE CAPTION

Figure 1: The fractional binding energy is plotted against the surface value of $2M/R$, or $2GM/c^2R$ in ordinary units, for a sequence of isentropic models with $\Gamma_1 = 2$ (see text). The central value of $PV$ ranges from 0.1 at $2M/R = 0.25$ up to 250 at the last point plotted. The first maximum of the binding energy is at $(PV)_c = 0.320$, the first minimum at $(PV)_c = 5.07$, and the second maximum at $(PV)_c = 30$.


0.10
0.05
$\frac{M_0-M}{M_0}$
0
-0.05
-0.10
0.25
0.30
0.35
0.40
0.45
0.50
2M/R